# ON THE LOSS OF THE SEMIMARTINGALE PROPERTY AT THE HITTING TIME OF A LEVEL

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Abstract. This paper studies the loss of the semimartingale property of the process $g(Y)$ at the time a one-dimensional diffusion $Y$ hits a level, where $g$ is a difference of two convex functions. We show that the process $g(Y)$ can fail to be a semimartingale in two ways only, which leads to a natural definition of non-semimartingales of the *first* and *second kind*. We give a deterministic if and only if condition (in terms of $g$ and the coefficients of $Y$) for $g(Y)$ to fall into one of the two classes of processes, which yields a characterisation for the loss of the semimartingale property. A number of applications of the results in the theory of stochastic processes and real analysis are given: e.g. we construct an adapted diffusion $Y$ on $[0,\infty)$ and a *predictable* finite stopping time $\zeta$, such that $Y$ is a semimartingale on the stochastic interval $[0,\zeta)$, continuous at $\zeta$ and constant after $\zeta$, but is *not* a semimartingale on $[0,\infty)$.

## 1. Introduction

Continuous semimartingales form an important, general and well-studied class of stochastic processes. This paper deals with the phenomenon of the loss of the semimartingale property at the hitting time of a level as motivated and explained below.

**1.1.** The motivation for this work is best described by the following two examples.

**Example 1.1.** Let $B$ be an $(\mathcal{F}_t,\mathsf{P})$-Brownian motion starting from $x_0>0$ defined on some filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,\infty)},\mathsf{P})$. It is well-known that the process $\sqrt{|B|}$ is not a semimartingale (see the original reference [13] or the monograph [11, Th. 72]). A possible short argument is as follows. Let $X$ be a continuous semimartingale and $L^a_t(X)$ its local time at time $t\geq 0$ and level $a\in\mathbb{R}$. Recall that if $f$ is a strictly increasing function on $\mathbb{R}$, which moreover is the difference of two convex functions, then, for any $a\in\mathbb{R}$, it holds $L^{f(a)}_\cdot(f(X))=f'_+(a)L^a_\cdot(X)$ a.s., where $f'_+(a)$ is the right derivative of $f$ at the point $a$ (see [12, Ch. VI, Ex. 1.23]). If $X:=\sqrt{|B|}$ were a semimartingale, then, applying the statement above to $f(x)=x^2\operatorname{sgn}x$, we would get that $L^0_\cdot(|B|)\equiv 0$, which would contradict the well-known fact that the local time at zero of $|B|$ increases immediately after the time $\tau^B_0=\inf\{t\geq 0: B_t=0\}$.

Intuitively this can be summarized as follows: the semimartingale property of $\sqrt{|B|}$ fails *immediately after* $\tau^B_0$ because the increase in local time at zero of $|B|$ and the infinite slope of

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the function $x \mapsto \sqrt{x}$ at the origin make the process $\sqrt{|B|}$ accumulate an infinite amount of local time at zero immediately after $\tau_0^B$.

It is now natural to ask whether the square root of a nonnegative continuous semimartingale that does not accrue local time at zero may fail to be a semimartingale (for a different reason). This is also possible as the following example shows.

**Example 1.2.** Let $x_0 > 0$. Consider a squared Bessel process $Y$ of dimension $\delta \in (0,1)$ starting from $x_0^2$, i.e. it holds $dY_t = \delta\, dt + 2\sqrt{Y_t}\, dW_t$, where $W$ is a Brownian motion. It is well-known that $Y$ is a nonnegative semimartingale that a.s. hits 0 at a finite time, 0 is an instantaneously reflecting boundary point for $Y$, and $Y$ does not accrue local time at 0 (see e.g. [12, Ch. XI]). Let $\rho = (\rho_t)_{t\in[0,\infty)}$ be given by $\rho_t = \sqrt{Y_t}$, i.e. $\rho$ is a Bessel process of dimension $\delta \in (0,1)$ starting from $x_0 > 0$. It is known that $\rho$ is not a semimartingale. For completeness we present a formal proof of this fact in Appendix A. Here again the semimartingale property of $\rho$ fails *immediately after* $\tau_0^\rho = \inf\{t \geq 0 : \rho_t = 0\}$.

As we already observed the loss of the semimartingale property in both examples above occurs *immediately after* the hitting time of zero. Let us first discuss whether this happens in fact even *at* the hitting time of zero, i.e. whether the stopped processes $\sqrt{B^{\tau_0^B}}$ and $\rho^{\tau_0^\rho}$ are semimartingales. We shall see that they are semimartingales (see Corollaries 3.9 and 3.11), i.e. the loss of the semimartingale property in both examples above does not occur at the hitting time of zero.

The following natural question arises.

**Question I.** *Let $B$ be a Brownian motion starting from $x_0 > 0$. Does there exist a continuous strictly increasing function $g : [0,\infty) \to \mathbb{R}$, which is smooth on $(0,\infty)$, such that the process $g(B^{\tau_0^B})$ is not a semimartingale?*

In other words we are asking here if the loss of the semimartingale property can occur *at* $\tau_0^B$. The requirement for $g$ to be strictly increasing stems from the desire to construct a function "like $\sqrt{\cdot}$".

As we shall see, the answer to Question I is affirmative, and we will construct such examples below.

**1.2.** In this paper we consider a one-dimensional diffusion $Y$ with the state space $J = (l, r)$, $-\infty \leq l < r \leq \infty$, possibly exiting its state space at a finite time. By convention $Y$ is stopped after it reaches $l$ or $r$. The setting is formally described in Section 2. Denoting by $\zeta$ the exit time from $J$ (i.e. the hitting time of either $l$ or $r$), we study whether the process $g(Y)$ loses the semimartingale property at the time $\zeta$. A particular case of our discussion, when $g$ is equal to the identity, will answer the following question:

**Question II.** *Assuming that $Y$ exits $J$ only at finite endpoints[1], can $Y$ fail to be a semimartingale?*

[1]Note that if $Y$ were allowed to exit at an infinite endpoint, then $Y$ would clearly fail to be a semimartingale.

As we shall see, the answer to Question II is affirmative, and we will construct examples below. In particular, our construction gives rise to a *globally* defined continuous adapted process $Y = (Y_t)_{t\in[0,\infty)}$ and a *predictable* stopping time $\zeta$ such that $Y$ is a semimartingale on the stochastic interval $[0,\zeta)$, $Y$ is continuous at $\zeta$ and constant after $\zeta$, but it is *not* a semimartingale on $[0,\infty)$.

**1.3.** After finishing the paper we discovered the very deep and surprisingly general treatment [3], where one of the questions discussed is whether a function of a Markov process is a semimartingale. Theorem 4.6 in [3] gives a necessary and sufficient condition for this in a very general setting. The Brownian case is discussed in detail in Section 5 of [3], where explicit criteria are presented for a Brownian motion (Theorems 5.5 and 5.6), a reflecting Brownian motion (Theorem 5.8), and a killed Brownian motion (Theorem 5.9). At the end of Section 5 of [3], it is explained how the results for a Brownian motion can be used to imply the corresponding results for diffusions (via a state-space transformation and a random time-change), but the explicit statements are not presented.

In the present paper, the setting is far less general setting than that of Section 4 in [3]. As discussed above, we are interested only in the loss of the semimartingale property *at* the exit time $\zeta$. This allows us to assume from the outset that

$$g\colon J \to \mathbb{R} \text{ is a difference of two convex functions,}$$

which implies that $g(Y)$ is a continuous semimartingale on the stochastic interval $[0,\zeta)$, and investigate the behaviour of $g$ near the endpoints of $J$ that preserves the semimartingale property of $g(Y)$ globally, i.e. on $[0,\infty)$. Even though our setting is less general than the one in [3], the results obtained in this paper are *complementary* to the results in [3]. As explained in more detail below, we enrich the picture presented in [3] in several directions.

In Section 3 we present a necessary and sufficient condition for $g(Y)$ to be a semimartingale (Theorem 3.2), a sufficient one (Theorem 3.7), a necessary one (Theorem 3.12), and a discussion of the phenomena that lead to the loss of the semimartingale property at $\zeta$ (Theorem 3.15). It may be possible to establish our Theorem 3.2 from the general Theorem 4.6 in [3], but this way of proving Theorem 3.2 does not look straightforward. Furthermore, the authors of [3] recommend that one obtain results for diffusions from the corresponding results for Brownian motion, i.e. from the results of Section 5 in [3]. Thus, our Theorem 3.2 can be deduced from Theorem 5.9 in [3] via a state-space transformation and a random time-change. We, however, prove Theorem 3.2 directly. This requires an investigation of the convergence of certain additive functionals of diffusion processes, which is carried out in this paper. We hope that this classification of convergence obtained here is of interest in its own right.

The other main results of Section 3, Theorems 3.7, 3.12, and 3.15, do not have their analogues in [3] and thus do not follow from the results of [3]. The question arises of why we give a separate sufficient condition for $g(Y)$ to be a semimartingale (Theorem 3.7) and a separate necessary one (Theorem 3.12) in addition to a necessary and sufficient condition (Theorem 3.2). Even though Theorem 3.2 is a more precise result, it is often less convenient in specific situations. For example, the sufficient condition for $g(Y)$ to be a semimartingale in Theorem 3.7 is typically easier to verify than the necessary and sufficient condition in Theorem 3.2 (compare (3.8) and (3.4)). In specific situations we get some qualitative information (say, about the structure of certain examples) from Theorems 3.7 and 3.12 that is not easy to obtain from Theorem 3.2. For instance, if one wishes to construct an example demonstrating that the answer to Question II is affirmative, one requires the insight from Corollary 3.11 that the drift has to oscillate around zero near the finite endpoint, where $Y$ exits. Corollary 3.11 is an immediate consequence of Theorem 3.7 and does not follow from Theorem 3.2.

In Section 4 we construct examples answering Questions I and II. For each question we construct two examples: one for each of the two possible ways (characterised in Theorem 3.15) the loss of the semimartingale property can occur. In Section 5 we discuss in more detail the case where $Y$ is a Brownian motion stopped upon hitting zero. We start with two lemmas from real analysis that arise in the study of the Brownian case and are also of independent interest. Then we present a result, Theorem 5.4, where two different equivalent conditions for $g(Y)$ to be a semimartingale are given. One of them is a slight variation of the equivalent condition of Theorem 5.9 in [3] (simply put, it is observed that parts (ii) and (iii) of Theorem 5.9 in [3] imply part (i) of that theorem). The other one is new.

In Section 6 we consider the additive functional

$$\int_J L_t^y(Y)\,\nu(dy), \quad t \in [0,\zeta], \tag{1.1}$$

where $(L_t^y(Y);\, t \in [0,\zeta), y \in J)$ is the local time of the diffusion $Y$ and $\nu$ is an arbitrary positive measure on $J$. We describe the stopping time after which this additive functional is infinite, and present deterministic criteria for the convergence and divergence of (1.1) at this stopping time. As a particular case of this investigation, Lemma 5.10 in [3] is generalised to the diffusion setting and complemented by a criterion for a.s.-infiniteness of the additive functional. This characterisation is the reason why the idea behind the proof of the corresponding result in Section 6 differs from the one in [3, Lemma 5.10]: our treatment in Secton 6 uses the Ray-Knight theorem in the corresponding place. Finally, in Section 7 we prove the theorems from Section 3.

## 2. Setting and Notations

**2.1.** First we introduce some common notations used in the sequel. Let us consider an open interval $J = (l,r) \subseteq \mathbb{R}$.

- $\overline{J}$ denotes $[l,r] (\subseteq [-\infty,\infty])$.
- $\nu_L$ denotes the Lebesgue measure on $J$.
- $L^1_{\mathrm{loc}}(J)$ denotes the set of Borel functions $J \to [-\infty,\infty]$, which are locally integrable on $J$, i.e. integrable on compact subsets of $J$ with respect to $\nu_L$.
- For a positive measure $\nu$ on $J$, $L^1_{\mathrm{loc}}(l+,\nu)$ (resp. $L^1_{\mathrm{loc}}(r-,\nu)$) denotes the set of Borel functions $f\colon J \to [-\infty,\infty]$ such that for some $z \in J$, it holds $\int_{(l,z)} |f(y)|\,\nu(dy) < \infty$ (resp. $\int_{(z,r)} |f(y)|\,\nu(dy) < \infty$).
- $L^1_{\mathrm{loc}}(l+)$ and $L^1_{\mathrm{loc}}(r-)$ denote $L^1_{\mathrm{loc}}(l+,\nu_L)$ and $L^1_{\mathrm{loc}}(r-,\nu_L)$ respectively.

- For a function $x \mapsto f(x)$ on $J$, the notations "$f \in L^1_{\text{loc}}(l+, \nu)$" and "$f(x) \in L^1_{\text{loc}}(l+, \nu)$" are synonymous.
- For a locally finite signed measure $\nu_S$ on $J$, $|\nu_S|$ denotes the variation measure of $\nu_S$.

**2.2.** Let the state space be $J = (l, r)$, $-\infty \le l < r \le \infty$, and $Y = (Y_t)_{t \in [0,\infty)}$ be a $J$-valued solution of the one-dimensional SDE

$$dY_t = \mu(Y_t)\,dt + \sigma(Y_t)\,dW_t, \quad Y_0 = x_0, \tag{2.1}$$

on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, \mathsf{P})$, where $x_0 \in J$ and $W$ is an $(\mathcal{F}_t, \mathsf{P})$-Brownian motion. We allow $Y$ to exit its state space $J$ at a finite time in a continuous way. The exit time is denoted by $\zeta$. That is to say, $\mathsf{P}$-a.s. on $\{\zeta = \infty\}$ the trajectories of $Y$ do not exit $J$, while $\mathsf{P}$-a.s. on $\{\zeta < \infty\}$ we have: either $\lim_{t\uparrow\zeta} Y_t = r$ or $\lim_{t\uparrow\zeta} Y_t = l$. Then we need to specify the behaviour of $Y$ after $\zeta$ on $\{\zeta < \infty\}$. In what follows we assume that on $\{\zeta < \infty\}$ the process $Y$ stays after $\zeta$ at the endpoint of $J$ where it exits, i.e. $l$ and $r$ are by convention absorbing boundaries.

Throughout the paper it is assumed that the coefficients $\mu$ and $\sigma$ in (2.1) satisfy the Engelbert-Schmidt conditions

$$\sigma(x) \neq 0 \ \ \forall x \in J, \tag{2.2}$$

$$\frac{1}{\sigma^2}, \frac{\mu}{\sigma^2} \in L^1_{\text{loc}}(J). \tag{2.3}$$

Under (2.2) and (2.3) SDE (2.1) has a weak solution, unique in law, which possibly exits $J$ (see [5] or [8, Ch. 5, Th. 5.15]). Conditions (2.2) and (2.3) are reasonable weak assumptions: any locally bounded Borel function $\mu$ and locally bounded away from zero Borel function $\sigma$ on $J$ satisfy (2.2) and (2.3). In what follows we also need the scale function $s$ of $Y$ and its derivative $\rho$:

$$\rho(x) = \exp\left\{-\int_c^x \frac{2\mu}{\sigma^2}(y)\,dy\right\}, \quad x \in J, \tag{2.4}$$

$$s(x) = \int_c^x \rho(y)\,dy, \quad x \in \overline{J}, \tag{2.5}$$

for some $c \in J$. In particular, $s$ is an increasing $C^1$-function $J \to \mathbb{R}$ with a strictly positive derivative, which is absolutely continuous on compact intervals in $J$, while $s(r)$ (resp. $s(l)$) may take value $\infty$ (resp. $-\infty$).

## 3. Characterisation of the Semimartingale Property

In this section we study whether $g(Y)$ is a semimartingale for the possibly exiting diffusion $Y$ described in the previous section and a certain class of functions $g$ described below. Let us consider a function $g$ on the state space $J$ such that

$$g\colon J \to \mathbb{R} \text{ is a difference of two convex functions.} \tag{3.1}$$

In particular, the left derivative $g'_-$ and the right derivative $g'_+$ are well-defined everywhere on $J$ and are functions of finite variation on compact subsets of $J$. Furthermore the derivative $g'$ exists everywhere on $J$ except possibly on a countable set. So the second derivative $g''$ exists as a function $\nu_L$-a.e. on $J$. It follows from (3.1) that the second derivative of $g$ in the sense of distributions can be identified with a locally finite signed measure on $J$ (see § 3 in the appendix in [12]), which is typically denoted by $g''(dy)$ (see e.g. [12, Ch. VI, Th. 1.5]). An equivalent description of this object is as follows: $g''(dy)$ is the locally finite signed measure on $J$ satisfying $g''((a,b]) = g'_+(b) - g'_+(a)$, $l < a < b < r$. It follows that the Lebesgue decomposition of $g''(dy)$ with respect to $\nu_L$ takes the form

$$g''(dy) = g''(y)\,dy + g''_s(dy),$$

where the locally finite signed measure $g''_s(dy)$ on $J$ denotes the singular part of $g''(dy)$ with respect to $\nu_L$.

In what follows, given a function $g$ satisfying (3.1), we define a locally finite signed measure $\nu_g$ on $J$ by the formula

$$\nu_g(dy) := \left(\frac{g'\mu}{\sigma^2} + \frac{1}{2}g''\right)(y)dy + \frac{1}{2}g''_s(dy). \tag{3.2}$$

Below we use the following terminology:

$Y$ *exits* $J$ *at* $r$ means $\mathsf{P}\left(\zeta < \infty, \lim_{t\uparrow\zeta} Y_t = r\right) > 0$;

$Y$ *exits* $J$ *at* $l$ is understood in an analogous way.

We distinguish between the following four cases:

(A) $Y$ exits $J$ neither at $l$ nor at $r$;

(B) $Y$ exits $J$ at $l$, and there exists a finite limit

$$g(l) := \lim_{x\downarrow l} g(x);$$

$Y$ does not exit $J$ at $r$;

(C) $Y$ exits $J$ at $r$, and there exists a finite limit

$$g(r) := \lim_{x\uparrow r} g(x);$$

$Y$ does not exit $J$ at $l$;

(D) $Y$ exits $J$ at $l$ and at $r$, and there exist finite limits

$$g(l) := \lim_{x\downarrow l} g(x) \quad \text{and} \quad g(r) := \lim_{x\uparrow r} g(x).$$

In each of these cases $g(Y)$ is well-defined globally (i.e. on $[0,\infty)$) and finite, and hence the question whether $g(Y)$ is a semimartingale is well-posed.

**Remark 3.1.** By the Itô-Tanaka formula (see [12, Ch. VI, Th. 1.5]), condition (3.1) implies that

$$(g(Y_t))_{t\in[0,\zeta)} \text{ is a continuous semimartingale on } [0,\zeta). \tag{3.3}$$

In fact, (3.1) is equivalent to (3.3). In the Brownian case $\mu \equiv 0, \sigma \equiv 1$ (i.e. $Y$ is a Brownian motion absorbed at $l$ and $r$), this follows just as in the proofs of Theorems 5.5 and 5.6 in [3], and is stated right after the proof of Lemma 5.10 in [3]. In general it remains to note that (3.1) is equivalent to

$$g \circ s^{-1} \colon s(J) \to \mathbb{R} \text{ is a difference of two convex functions,}$$

because under (2.2) and (2.3) both $s$ and $s^{-1}$ are $C^1$-functions with derivatives absolutely continuous on compact subintervals in $J$, and refer to the discussion at the end of Section 5 in [3]. Thus, since condition (3.3) is necessary for $g(Y)$ to be a semimartingale globally (i.e. on $[0,\infty)$), assuming (3.1) and studying whether $g(Y)$ is a semimartingale amounts to studying whether the loss of the semimartingale property occurs *at* the time $\zeta$.

**Case (A).** There is nothing to study in this case: $g(Y)$ is always a semimartingale.

**Case (B).** First let us note that by Propositions B.3–B.5, case (B) amounts to the following:

(B.i) there is a finite limit $g(l) := \lim_{x \downarrow l} g(x)$;
(B.ii) $s(l) > -\infty$ and $\frac{s-s(l)}{\rho\sigma^2} \in L^1_{\text{loc}}(l+)$;
(B.iii) either $s(r) = \infty$, or:

$$s(r) < \infty \quad \text{and} \quad \frac{s(r)-s}{\rho\sigma^2} \notin L^1_{\text{loc}}(r-).$$

**Theorem 3.2.** *Assume* (3.1) *and case (B). Then* $g(Y)$ *is a semimartingale if and only if*

$$\frac{s-s(l)}{\rho} \in L^1_{\text{loc}}(l+, |\nu_g|), \tag{3.4}$$

*where the variation measure* $|\nu_g|$ *of the locally finite signed measure* $\nu_g$*, defined in* (3.2)*, equals*

$$|\nu_g|(dy) = \left| \frac{g'\mu}{\sigma^2} + \frac{1}{2} g'' \right| (y) dy + \frac{1}{2} |g''_s|(dy).$$

**Remark 3.3.** The proof of Theorem 3.2 will reveal that under (3.4), $g(Y)$ has the semimartingale decomposition

$$g(Y_t) = g(x_0) + A_t + M_t, \quad t \in [0,\infty), \tag{3.5}$$

where

$$A_t = \int_J L^y_{t\wedge\zeta}(Y)\, \nu_g(dy), \quad t \in [0,\infty), \tag{3.6}$$

$$M_t = \int_0^{t\wedge\zeta} (g'\sigma)(Y_u)\, dW_u, \quad t \in [0,\infty), \tag{3.7}$$

and the integrals in (3.6) and (3.7) are well-defined. The random field $\{L^y_t(Y) : y \in J, t \in [0,\zeta)\}$ in (3.6) denotes the local time of the semimartingale $Y$ defined on the stochastic interval $[0,\zeta)$ (see Section 6 for further details and references on local time of $Y$). Note also that the local martingale $M$ in (3.7) does not depend on the choice of $g'$ on any countable set. In particular, on the set where the left and the right derivatives of $g$ do not coincide we can define $g'$ arbitrarily.

When the measure $g''(dy)$ is absolutely continuous with respect to $\nu_L$, Theorem 3.2 implies the following characterisation.

**Corollary 3.4.** *If $g \in C^1(J, \mathbb{R})$ and $g'$ is absolutely continuous on compact intervals in $J$, then, in case (B), $g(Y)$ is a semimartingale if and only if*

$$\frac{s - s(l)}{\rho} \left| \frac{g'\mu}{\sigma^2} + \frac{1}{2} g'' \right| \in L^1_{\text{loc}}(l+).$$

**Remark 3.5.** Under the assumptions of Corollary 3.4, the signed measure $g''_s(dy)$ is a zero measure and the finite variation process in the semimartingale decomposition (3.5) takes the form

$$A_t = \int_0^{t\wedge\zeta} \left( g'\mu + \frac{1}{2} g''\sigma^2 \right) (Y_u)\, du, \quad t \in [0, \infty).$$

We now investigate when the process $Y$ itself is a semimartingale. To get a deterministic necessary and sufficient condition it is now enough to apply Theorem 3.2 or Corollary 3.4 with $g(x) = x$, $x \in J$.

**Corollary 3.6.** *Assume that $l > -\infty$, $Y$ exits $J$ at $l$, and $Y$ does not exit $J$ at $r$. Then $Y$ is a semimartingale if and only if*

$$\frac{s - s(l)}{\rho} \frac{|\mu|}{\sigma^2} \in L^1_{\text{loc}}(l+).$$

In specific examples it may be hard to check (3.4). The following result, Theorem 3.7, gives an easy-to-check sufficient condition for $g(Y)$ to be a semimartingale. In Theorem 3.12 below we present a necessary condition for the semimartingale property of $g(Y)$.

**Theorem 3.7.** *In addition to the assumptions of Theorem 3.2 suppose that, for some $a \in J$,*

(3.8) $\nu_g|_{(l,a)}$ *is either a positive measure or a negative measure.*

*Then $g(Y)$ is a semimartingale.*

**Remark 3.8.** (i) In view of Theorem 3.2, there is an equivalent reformulation of Theorem 3.7, which appears to be purely analytic: *under the assumptions of Theorem 3.2,* (3.8) *implies* (3.4). We note that our proof is probabilistic and raise the question of finding an analytic proof.

(ii) Observe that (3.4) does not imply (3.8). For instance, consider $J = (0, \infty)$, $\mu \equiv 0$, $\sigma \equiv 1$, $g(x) = \int_1^x (2 + \sin \frac{1}{\sqrt{y}})\, dy$, $x \in [0, \infty)$.

**Corollary 3.9.** *In addition to the assumptions of Theorem 3.2 suppose that, for some $a \in J$,*

$$\mu = 0 \quad \nu_L\text{-a.e. on } (l, a)$$

*and*

$$g \text{ is convex or concave on } (l, a).$$

*Then $g(Y)$ is a semimartingale.*

In particular, it immediately follows from Corollary 3.9 that $\sqrt{B^{\tau_0^B}}$ is a semimartingale (see the discussion after Examples 1.1 and 1.2). This can also be seen directly since, by Jensen's inequality, the process $\sqrt{B^{\tau_0^B}}$ is a supermartingale.

**Remark 3.10.** (i) Let $X$ be a continuous semimartingale satisfying $\mathsf{P}(X_t \geq l \ \forall t \geq 0) = 1$ for some $l > -\infty$, and $h\colon [l, \infty) \to \mathbb{R}$ a convex or concave function continuous at $l$ with a finite derivative $h'(l+)$. Then $h(X)$ is a semimartingale by the Itô-Tanaka formula because such a function $h$ can be extended to a convex or concave function on $\mathbb{R}$. However, if $|h'(l+)| = \infty$, the Itô-Tanaka formula cannot be used to conclude that $h(X)$ is a semimartingale (recall Examples 1.1 and 1.2, where the semimartingale property is lost for $h(\cdot) = \sqrt{\cdot}$, $l = 0$).

(ii) The statement in (i) demonstrates that the gist of Corollary 3.9 lies in the cases $|g'(l+)| = \infty$ or $l = -\infty$.

We now apply Theorem 3.7 to get a sufficient condition for $Y$ itself to be a semimartingale.

**Corollary 3.11.** *Assume that $l > -\infty$, $Y$ exits $J$ at $l$, $Y$ does not exit $J$ at $r$. Further suppose that, for some $a \in J$,*

$$\text{either} \quad \mu \geq 0 \quad \nu_L\text{-a.e. on } (l, a) \quad \text{or} \quad \mu \leq 0 \quad \nu_L\text{-a.e. on } (l, a).$$

*Then $Y$ is a semimartingale.*

In particular, it follows from Corollary 3.11 that $\rho^{\tau_0^\rho}$ is a semimartingale (see the discussion after Examples 1.1 and 1.2). Indeed, by Itô's formula, on the stochastic interval $[0, \tau_0^\rho)$ one has $d\rho_t = \frac{\delta-1}{2\rho_t}dt + dW_t$, hence Corollary 3.11 applies with $J = (0, \infty)$, $\sigma \equiv 1$, $\mu(y) = \frac{\delta-1}{2y} \leq 0$, $y \in J$.

It is interesting to note that even though Corollary 3.6 gives a more precise result than Corollary 3.11, the latter is sometimes more convenient. For instance, we can conclude from Corollary 3.11 (but not from Corollary 3.6) that for $Y$ to fail the semimartingale property, the drift $\mu$ has to oscillate around zero near the boundary point $l$. Such examples will be constructed below.

We now present a necessary condition for $g(Y)$ to be a semimartingale.

**Theorem 3.12.** *Under the assumptions of Theorem 3.2 let $g(Y)$ be a semimartingale. Then*

$$\frac{s - s(l)}{\rho}(g')^2 \in L^1_{\text{loc}}(l+). \tag{3.9}$$

Put differently, if (3.9) is violated, then $g(Y)$ is not a semimartingale. Let us note that in specific situations it may be easier to see that (3.9) is violated than that (3.4) is violated.

**Remark 3.13.** In the language of analysis, Theorem 3.12 can be recast as follows: *under the assumptions of Theorem 3.2,* (3.4) *implies* (3.9). Again we observe that our proof is probabilistic and that an analytic proof appears not to be straightforward. Note also that (3.9) does not in general imply (3.4) (see Example 4.1 below).

Finally, we characterise the phenomena that lead to the loss of the semimartingale property of $g(Y)$. As in [6] we will denote by $\operatorname{Var} A = (\operatorname{Var} A_t)_{t\in[0,\infty)}$ the variation process of a process $A = (A_t)_{t\in[0,\infty)}$. Let the assumptions of Theorem 3.2 hold (in particular, $\mathsf{P}(\zeta < \infty) > 0$) and let $g(Y)$ be a non-semimartingale. Decomposition (3.5) with $A$ and $M$ given by (3.6) and (3.7) still holds, but only on the stochastic interval $[0, \zeta)$ (also $A = (A_t)_{t\in[0,\zeta)}$ and $M = (M_t)_{t\in[0,\zeta)}$

are in general well-defined only on $[0,\zeta)$, $A$ has a locally finite variation on $[0,\zeta)$, $M$ is a local martingale on $[0,\zeta)$). We use this decomposition on the stochastic interval $[0,\zeta)$ in the following definition.

**Definition 3.14.** Let the assumptions of Theorem 3.2 hold and let $g(Y)$ be a non-semimartingale.

(i) We say that $g(Y)$ is a non-semimartingale *of the first kind* if $\mathsf{P}$-a.s. on $\{\zeta<\infty\}$ there are finite limits

$$M_\zeta=\lim_{t\uparrow\zeta}M_t \quad\text{and}\quad A_\zeta=\lim_{t\uparrow\zeta}A_t.$$

(ii) We say that $g(Y)$ is a non-semimartingale *of the second kind* if $\mathsf{P}$-a.s. on $\{\zeta<\infty\}$ one has

$$\limsup_{t\uparrow\zeta}M_t=-\liminf_{t\uparrow\zeta}M_t=\infty \quad\text{and}\quad \limsup_{t\uparrow\zeta}A_t=-\liminf_{t\uparrow\zeta}A_t=\infty.$$

We will now see that $g(Y)$ can lose the semimartingale property in these two ways only. Moreover, we have the following characterisation result.

**Theorem 3.15.** *Let the assumptions of Theorem 3.2 hold.*

*(i) $g(Y)$ is a non-semimartingale of the first kind if and only if* (3.9) *holds and* (3.4) *is violated. In this case the process $(M_{t\wedge\zeta})_{t\in[0,\infty)}$ is a continuous local martingale on $[0,\infty)$ (not only on $[0,\zeta)$), but $\operatorname{Var}A_\zeta=\infty$ $\mathsf{P}$-a.s. on $\{\zeta<\infty\}$.*

*(ii) $g(Y)$ is a non-semimartingale of the second kind if and only if* (3.9) *is violated.*

**Cases (C) and (D)** are treated similarly to case (B). For instance, the counterpart of Theorem 3.2 in case (D) is as follows: under (3.1), $g(Y)$ is a semimartingale if and only if

$$\frac{s-s(l)}{\rho}\in L^1_{\text{loc}}(l+,|\nu_g|) \quad\text{and}\quad \frac{s(r)-s}{\rho}\in L^1_{\text{loc}}(r-,|\nu_g|).$$

We omit further details.

## 4. Examples

**4.1. Answer to Question I.** Let $B$ be an $(\mathcal{F}_t,\mathsf{P})$-Brownian motion starting from $x_0>0$ defined on some filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,\infty)},\mathsf{P})$. Question I in the introduction asks whether it is possible to find a function $g\colon[0,\infty)\to\mathbb{R}$ satisfying

$$g\in C([0,\infty),\mathbb{R})\cap C^\infty((0,\infty),\mathbb{R}) \tag{4.1}$$

and

$$g \text{ is strictly increasing} \tag{4.2}$$

such that $(g(B_{t\wedge\tau_0^B}))_{t\in[0,\infty)}$ is not a semimartingale, where $\tau_0^B=\inf\{t\geq0:B_t=0\}$. Following the discussion at the end of Section 3 (see in particular Definition 3.14 and Theorem 3.15), two further natural subquestions arise:

(a) Can $g(B^{\tau_0^B})$ be a non-semimartingale of the first kind?

(b) Can $g(B^{\tau_0^B})$ be a non-semimartingale of the second kind?

The present setting here is a special case of the setting in Section 3 with $J = (0,\infty)$, $\mu \equiv 0$, $\sigma \equiv 1$, and we are in case (B) (note that condition (3.1) and the existence of a finite limit $g(0) := \lim_{x\downarrow 0} g(x)$ hold due to (4.1)). Conditions (3.4) and (3.9) take the form

$$x|g''(x)| \in L^1_{\text{loc}}(0+) \tag{4.3}$$

and

$$x(g'(x))^2 \in L^1_{\text{loc}}(0+) \tag{4.4}$$

respectively. Thus, question (a) above amounts to constructing a function $g\colon [0,\infty) \to \mathbb{R}$ satisfying (4.1), (4.2) and (4.4), such that (4.3) is violated; question (b) amounts to constructing a function $g$ satisfying (4.1) and (4.2), such that (4.4) is violated. The answers to both questions (a) and (b) are affirmative. We now construct both examples.

**Example 4.1** ($g(B^{\tau_0^B})$ is a non-semimartingale of the first kind)**.**
Let us consider the function $h\colon (0,\infty) \to \mathbb{R}$ given by

$$h(x) = \frac{1}{\sqrt{x}}\left(2 + \sin\frac{1}{x}\right), \quad x \in (0,\infty).$$

It is easy to see that $h$ satisfies

$$h \in C^\infty((0,\infty),\mathbb{R}), \tag{4.5}$$

$$h(x) > 0 \;\; \forall x \in (0,\infty), \tag{4.6}$$

$$h \in L^1_{\text{loc}}(0+), \tag{4.7}$$

$$xh^2(x) \in L^1_{\text{loc}}(0+), \tag{4.8}$$

$$x|h'(x)| \notin L^1_{\text{loc}}(0+). \tag{4.9}$$

Setting

$$g(x) = \int_1^x h(y)\,dy, \quad x \in [0,\infty)$$

(note that $g(0)$ is finite due to (4.7)), we get a function $g$ satisfying (4.1), (4.2), and (4.4) such that (4.3) is violated, which is what was required.

**Example 4.2** ($g(B^{\tau_0^B})$ is a non-semimartingale of the second kind)**.**
Let us set

$$a_n = \frac{1}{n} - \frac{1}{n^4}, \quad n = 2,3,\dots,$$

$$b_n = \frac{1}{n} + \frac{1}{n^4}, \quad n = 2,3,\dots,$$

$$E = \bigcup_{n=2}^{\infty} (a_n, b_n)$$

and define the strictly positive function

$$\overline{h}(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \in E, \\ \frac{1}{\sqrt{x}} & \text{if } x \in (0,\infty) \setminus E. \end{cases}$$

Since $\int_{a_n}^{b_n} \frac{dx}{x^2} = \frac{b_n - a_n}{a_n b_n} \sim \frac{\text{const}}{n^2}$ as $n \to \infty$, we get $\overline{h} \in L^1_{\text{loc}}(0+)$. It follows from $\int_{a_n}^{b_n} \frac{dx}{x^3} \geq \frac{1}{b_n} \int_{a_n}^{b_n} \frac{dx}{x^2} \sim \frac{\text{const}}{b_n n^2} \sim \frac{\text{const}}{n}$ as $n \to \infty$ that $x\overline{h}^2(x) \notin L^1_{\text{loc}}(0+)$. It is clear that such a function $\overline{h}$ can be smoothed in the neighbourhoods of the points $a_n$ and $b_n$, $n = 2, 3, \dots$, so that we get a function $h\colon (0,\infty) \to \mathbb{R}$ satisfying (4.5)–(4.7) and

$$xh^2(x) \notin L^1_{\text{loc}}(0+).$$

Setting

$$g(x) = \int_1^x h(y)\,dy, \quad x \in [0,\infty),$$

we get a function $g$ satisfying (4.1) and (4.2) such that (4.4) is violated.

**4.2. Answer to Question II.** Let us consider the setting and notation of Section 2. Question II in the introduction asks whether $Y$ can fail to be a semimartingale whenever $Y$ exits $J$ only at finite endpoints. Let us consider case (B) of Section 3 with $l > -\infty$ and $g(x) = x$, $x \in J$. Now two further natural subquestions arise:

(c) Can $Y$ be a non-semimartingale of the first kind?

(d) Can $Y$ be a non-semimartingale of the second kind?

The answers to both questions are affirmative. The examples are obtained from Examples 4.1 and 4.2 by setting $J := (g(0), g(\infty))$ and $Y := g(B^{\tau_0^B})$ (that is, $\mu = \frac{1}{2}g'' \circ g^{-1}$, $\sigma = g' \circ g^{-1}$).

## 5. Further Discussions in the Brownian Case

In this section we discuss in more detail the particular case, where $Y$ is a Brownian motion stopped upon hitting zero, i.e. the case $J = (0,\infty)$, $\mu \equiv 0$, $\sigma \equiv 1$.

**5.1. Two Lemmas from Real Analysis.** We will need the following result from real analysis, which is also of independent interest.

**Lemma 5.1.** *For some $a > 0$, let*

(5.1) $$g\colon (0,a) \to \mathbb{R} \text{ be a difference of two convex functions,}$$

(5.2) $$\int_{(0,u]} x\,|g''|(dx) < \infty$$

*for some $u \in (0,a)$. Then*

(5.3) $$\text{there exists a finite limit } g(0) := \lim_{x \downarrow 0} g(x),$$

(5.4) $$\int_{(0,u]} x(g'(x))^2\,dx < \infty.$$

Let us recall that $g''(dx)$ is the locally finite signed measure on $(0,a)$ satisfying $g''((x,y]) = g'_+(y) - g'_+(x)$, $0 < x < y < a$, and $|g''|(dx)$ is the variation measure of $g''(dx)$. Let us further note that statement (5.4) does not depend on the definition of the integrand on the (at most countable) set where $g'$ does not exist. For more details, see the discussion in the beginning of Section 3.

Let us observe that Lemma 5.1 is a refinement of the analytical statement implied by Theorems 3.2 and 3.12 in the Brownian case. Indeed, Remark 3.13 states that (5.1)–(5.3) imply (5.4). Note that (5.3) is assumed in Theorems 3.2 and 3.12 as a part of the description of case (B) in Section 3.

*Proof.* First we prove by contradiction that (5.1) and (5.2) imply (5.3). If not, there would exist a convex function $h$ on $(0,a)$ such that

$$\int_{(0,u]} x\, h''(dx) < \infty \tag{5.5}$$

and

$$\lim_{x \downarrow 0} h(x) = \infty \tag{5.6}$$

(note that for a convex function such a limit always exists but may be infinite). For $\varepsilon \in (0,u)$, integrating by parts, we get

$$\int_{(\varepsilon,u]} x\, h''(dx) = u h'_+(u) - \varepsilon h'_+(\varepsilon) - \int_{(\varepsilon,u]} h'_+(x)\, dx.$$

Since $h$ is convex on $(0,a)$, it is absolutely continuous on compact intervals in $(0,a)$, hence

$$\int_{(\varepsilon,u]} x\, h''(dx) = u h'_+(u) - \varepsilon h'_+(\varepsilon) - h(u) + h(\varepsilon). \tag{5.7}$$

As $\varepsilon \downarrow 0$ we now get a contradiction, as the limit of the left-hand side of (5.7) is finite by (5.5), while the limit of the right-hand side of (5.7) equals $\infty$ due to (5.6) and $-\varepsilon h'_+(\varepsilon) \geq 0$ for sufficiently small $\varepsilon > 0$.

It remains to prove the implication

$$(5.1)\text{–}(5.3) \Longrightarrow (5.4),$$

which follows from Theorems 3.2 and 3.12, as observed above. Such an argument is very indirect; we now present a short direct argument. Let $g$ satisfy (5.1)–(5.3). Clearly, (5.7) holds with $g$ instead of $h$. By (5.2) and (5.3), there is a finite $\lim_{\varepsilon \downarrow 0} \varepsilon g'_+(\varepsilon)$. Now using integration by parts in a different way, we obtain

$$\int_{(\varepsilon,u]} x (g'_+(x))^2\, dx = \frac{(u g'_+(u))^2 - (\varepsilon g'_+(\varepsilon))^2}{2} - \int_{(\varepsilon,u]} x^2 g'_+(x)\, g''(dx). \tag{5.8}$$

As $\varepsilon \downarrow 0$ the right-hand side, hence also the left-hand side, of (5.8) has a finite limit (here (5.2) and the existence of a finite $\lim_{\varepsilon \downarrow 0} \varepsilon g'_+(\varepsilon)$ are used). Since $x(g'_+(x))^2$ is a nonnegative function, statement (5.4) follows by monotone convergence (or by Fatou's lemma). ☐

Theorems 3.2 and 3.7 in the Brownian case imply another result from real analysis, again of interest in itself.

**Lemma 5.2.** *For some $a > 0$, let $g\colon (0,a) \to \mathbb{R}$ be a convex or concave function satisfying* (5.3). *Then, for any $u \in (0,a)$, it satisfies* (5.2).

We note that here assumption (5.3) cannot be dropped: consider, for instance, $g(x) = \frac{1}{x}$. The way of proving Lemma 5.2 via Theorems 3.2 and 3.7 is of course very indirect. We present a direct proof.

*Proof.* In the first step let us establish that $g'_+ \in L^1_{\text{loc}}(0+)$. Since $g$ is convex or concave on $(0,a)$, it is absolutely continuous on compact intervals in $(0,a)$. In particular, for $0 < \varepsilon < u < a$, we have

$$\int_{(\varepsilon,u]} g'_+(x)\,dx = g(u) - g(\varepsilon). \tag{5.9}$$

Again by convexity or concavity of $g$, $g'_+$ is monotone, hence $g'_+$ is either nonnegative or non-positive in a sufficiently small right neighborhood $(0,\delta)$ of zero. Now $g'_+ \in L^1_{\text{loc}}(0+)$ follows from (5.9) by letting $\varepsilon \downarrow 0$ and using monotone convergence and (5.3).

As in (5.7), we get

$$\int_{(\varepsilon,u]} x\, g''(dx) = ug'_+(u) - \varepsilon g'_+(\varepsilon) - g(u) + g(\varepsilon). \tag{5.10}$$

Since $g$ is convex or concave, $g''(dx)$ is a positive or negative measure. So, the left-hand side, hence also the right-hand side, of (5.10) has a limit as $\varepsilon \downarrow 0$, finite or infinite. By (5.3), there is a finite or infinite $\lim_{\varepsilon\downarrow 0} \varepsilon g'_+(\varepsilon)$. The latter limit can only be 0 (provided it exists), as otherwise $g'_+ \notin L^1_{\text{loc}}(0+)$. Hence

$$\int_{(0,u]} x\, g''(dx) = \lim_{\varepsilon\downarrow 0} \int_{(\varepsilon,u]} x\, g''(dx) \text{ is finite}$$

(the equality holds by the monotone convergence). We thus get (5.2). ☐

**5.2. Another Characterisation of the Semimartingale Property.** Let $B$ be a Brownian motion starting from $x_0 > 0$. In the following we consider the stopped process $B^{\tau_0^B}$ with $\tau_0^B := \inf\{t \geq 0 : B_t = 0\}$ and discuss the conditions on a Borel function $g\colon [0,\infty) \to \mathbb{R}$ under which the process $g(B^{\tau_0^B})$ is a semimartingale. Under the assumption that $g$ is continuous at 0 and the restricted function $g|_{(0,\infty)}$ is a difference of two convex functions, a necessary and sufficient condition is given in Theorem 3.2 above. With no assumption, a necessary and sufficient condition is given in Theorem 5.9 in [3]. Here we enrich the picture in two ways: firstly, we discuss the relations between the elementary conditions that form the necessary and sufficient condition of Theorem 5.9 in [3] (namely, parts (ii) and (iii) of [3, Th. 5.9] imply part (i) of that theorem); secondly, we present another necessary and sufficient condition for $g(B^{\tau_0^B})$ to be a semimartingale.

In order to formulate the result we introduce several conditions:

$$\text{the restriction } g|_{(0,\infty)} \text{ is a difference of two convex functions } (0,\infty)\to\mathbb{R}, \tag{5.11}$$

$$\text{there exists a finite limit } g(0) := \lim_{x\downarrow 0} g(x), \tag{5.12}$$

$$x \in L^1_{\mathrm{loc}}(0+, |g''|(dx)), \tag{5.13}$$

$$g = h_1 - h_2 \text{ with } h_i\colon [0,\infty)\to\mathbb{R} \text{ convex and continuous at } 0, \quad i=1,2. \tag{5.14}$$

**Remark 5.3.** Let us note that condition (5.14) is strictly stronger than (5.11) and (5.12). For instance, the functions $g$ constructed in Examples 4.1 and 4.2 satisfy (5.11) and (5.12), but for them, $g(B^{\tau_0^B})$ is not a semimartingale, hence, by Theorem 5.4 below, (5.14) fails.

**Theorem 5.4.** *Let $g\colon [0,\infty)\to\mathbb{R}$ be a Borel function. The following are equivalent:*
*(a) $g(B^{\tau_0^B})$ is a semimartingale;*
*(b)* (5.11) *and* (5.13) *hold;*
*(c)* (5.14) *holds.*

*Proof.* If (5.14) holds, then, by Corollary 3.9, $h_i(B^{\tau_0^B})$ are semimartingales, $i=1,2$ (alternatively, one can use Lemma 5.2 here). Thus, (c) $\Rightarrow$ (a). By [3, Th. 5.9], (a) is equivalent to (5.11)–(5.13). In particular, (a) $\Rightarrow$ (b).

It remains to prove that (b) $\Rightarrow$ (c). Assume (5.11) and (5.13). By Lemma 5.1, (5.12) holds. Let

$$g''(dx) = \nu_1(dx) - \nu_2(dx)$$

be the Jordan decomposition of the locally finite signed measure $g''(dx)$ on $(0,\infty)$, that is $\nu_i(dx)$ are locally finite positive measures on $(0,\infty)$ such that $\nu_1 \perp \nu_2$. In particular, we have

$$\nu_1(dx) + \nu_2(dx) = |g''|(dx),$$

so by (5.13)

$$x \in L^1_{\mathrm{loc}}(0+, \nu_i), \quad i=1,2. \tag{5.15}$$

For $i=1,2$, define the functions

$$k_i(x) = \begin{cases} \nu_i((1,x]) & \text{if } x\in[1,\infty), \\ -\nu_i((x,1]) & \text{if } x\in(0,1). \end{cases}$$

We prove that (5.14) is satisfied with functions $h_i = H_i$, where

$$H_i(x) = \int_1^x k_i(y)\,dy + a_i x + b_i, \quad x\in[0,\infty), \quad i=1,2,$$

for a suitable choice of constants $a_i, b_i$. Since $k_i$ are nondecreasing and right-continuous and $(H_i)'_+ = k_i + a_i$, we have that $H_i$ are convex functions on $(0,\infty)$. By construction it holds

$$(H_1 - H_2)'_+(x) = g'_+(x) - g'_+(1) + a_1 - a_2, \quad x\in(0,\infty).$$

Choosing $a_1$ and $a_2$ so that $a_1 - a_2 = g'_+(1)$, $b_1$ and $b_2$ so that $(H_1 - H_2)(1) = g(1)$, we obtain that $g = H_1 - H_2$ on $(0, \infty)$. It remains to prove that $\lim_{x\downarrow 0} H_i(x) < \infty$, $i = 1, 2$. To this end, it is enough to prove that $\int_0^1 \nu_i((y,1])\,dy < \infty$. For $i = 1, 2$, we have

$$\int_0^1 \nu_i((y,1])\,dy = \int_{(0,1]} \int_{(0,1]} I(y < x \le 1)\,\nu_i(dx)\,dy = \int_{(0,1]} x\,\nu_i(dx) < \infty$$

by (5.15). This concludes the proof. □

## 6. Finiteness of Additive Functionals of Diffusion Processes

In this section we study the finiteness of the process

$$\int_J L_t^y(Y)\,\nu(dy), \quad t \in [0, \zeta], \tag{6.1}$$

where $\nu$ is an arbitrary positive measure defined on the Borel $\sigma$-field $\mathcal{B}(J)$ (setting and notations in Section 2 apply), and $(L_t^y(Y); t \in [0,\zeta), y \in J)$ is an a.s. continuous in $t$ and càdlàg in $y$ version of the local time of $Y$ (in fact, it will be even a.s. jointly continuous in $(t, y)$; see [10, Prop. A.1]). The characterisation of the finiteness of the additive functional given in (6.1) plays a key role in the proofs of the results of Section 3. The occupation times formula (see [12, Ch. VI, Cor. 1.6]) implies that this question has been answered in [9] when the measure $\nu$ is absolutely continuous with respect to the Lebesgue measure $\nu_L$. In this section we give a deterministic characterisation of the finiteness of the additive functional in (6.1) for a general positive (possibly non-locally finite) measure $\nu$ on the interval $J$.

We proceed in two steps. First we reduce the study of the finiteness of (6.1) in general to the question of the convergence of the integral

$$\int_J L_\zeta^y(Y)\,\nu(dy), \tag{6.2}$$

where the measure $\nu$ is now locally finite on $J$. In the second step we formulate the answer to the latter problem in terms of a deterministic integrability criterion involving the scale function $s$ and its derivative $\rho$, given in (2.4)–(2.5), and the measure $\nu$.

Let us consider a general positive measure $\nu$ on $J$. With $B_\varepsilon(x) := (x - \varepsilon, x + \varepsilon)$ we set

$$D^\nu := \{l, r\} \cup \{x \in J : \forall \varepsilon > 0 \text{ it holds } \nu(B_\varepsilon(x)) = \infty\},$$

i.e. $D^\nu$ is the set of points in $J$ where local finiteness of $\nu$ fails, augmented by $\{l, r\}$. Clearly, $D^\nu$ is closed in $\overline{J}$. For a closed subset $E$ in $\overline{J}$ and $a, b \in \overline{J}$, we define the stopping times

$$\begin{aligned} \tau_E^Y &:= \inf\{t \in [0,\infty) : Y_t \in E\} \quad (\inf \emptyset := \infty), \\ \tau_a^Y &:= \tau_{\{a\}}^Y, \\ \tau_{a,b}^Y &:= \tau_a^Y \wedge \tau_b^Y. \end{aligned}$$

We start with the following result.

**Theorem 6.1.** P-*a.s. we have:*

$$\int_J L_t^y(Y)\,\nu(dy) < \infty, \quad t \in [0, \tau_{D^\nu}^Y), \tag{6.3}$$

$$\int_J L_t^y(Y)\,\nu(dy) = \infty, \quad t \in (\tau_{D^\nu}^Y, \zeta]. \tag{6.4}$$

**Remark 6.2.** Once Theorem 6.1 is established, it remains to study the convergence of the integral

$$\int_J L^y_{\tau^Y_{D^\nu}}(Y)\,\nu(dy).$$

If $x_0 \in D^\nu$, then there is nothing to study here because $\tau^Y_{D^\nu} \equiv 0$ and $\int_J L^y_0(Y)\,\nu(dy) = 0$. So assume now that $x_0 \notin D^\nu$, and define

$$\alpha = \sup([l, x_0) \cap D^\nu) \quad \text{and} \quad \beta = \inf((x_0, r] \cap D^\nu). \tag{6.5}$$

Then we have $\tau^Y_{D^\nu} = \tau^Y_{\alpha,\beta}$. Now if we consider $I := (\alpha, \beta)$ as a new state space for $Y$, then $\tau^Y_{\alpha,\beta}$ will be the new exit time, and we clearly have that $\nu$ is locally finite on $I$. This concludes the reduction of the study of the finiteness of the process in (6.1), with a general positive measure $\nu$, to the question of the convergence of the integral given in (6.2) with measure $\nu$, which is now locally finite on $J$.

*Proof of Theorem 6.1.* If $x_0 \in D^\nu$, then there is nothing to prove in (6.3). Let $x_0 \notin D^\nu$. A.s. on $\{t < \tau^Y_{D^\nu}\}$ the following holds: $[\inf_{u\le t} Y_u, \sup_{u\le t} Y_u] \subset (\alpha, \beta)$ with $\alpha$ and $\beta$ from (6.5), hence $\nu\left([\inf_{u\le t} Y_u, \sup_{u\le t} Y_u]\right) < \infty$, and the function $y \mapsto L^y_t(Y)$ is bounded as a càdlàg function with a compact support. Thus, statement (6.3) follows.

As for (6.4), let us first assume that $x_0 \notin D^\nu$. Then $\tau^Y_{D^\nu} = \tau^Y_{\alpha,\beta}$, hence $\{\tau^Y_{D^\nu} < t < \zeta\} = \{\tau^Y_\alpha < t < \zeta\} \cup \{\tau^Y_\beta < t < \zeta\}$. If $\mathsf{P}(\tau^Y_\alpha < t < \zeta) > 0$ (in particular, this means that $\alpha > l$), then (6.4) holds a.s. on $\{\tau^Y_\alpha < t < \zeta\}$ because $\alpha \in J \cap D^\nu$ and, by [2, Th. 2.7], the function $y \mapsto L^y_t(Y)$ is strictly positive in some neighbourhood of $\alpha$ a.s. on $\{\tau^Y_\alpha < t < \zeta\}$. Similarly, (6.4) holds a.s. on $\{\tau^Y_\beta < t < \zeta\}$. In the case $x_0 \in D^\nu$ statement (6.4) again follows from [2, Th. 2.7] by the same reasoning. □

It now remains to study the convergence of the integral in (6.2) under the assumption that the measure $\nu$ on $J$ is locally finite. The answer depends on the behaviour of $Y$. Theorems 6.3 and 6.4 below examine the cases $\mathsf{P}(A) = 1$ and $\mathsf{P}(B_r \cup B_l \cup C_r \cup C_l) = 1$ separately (the events $A$, $B_r$, $B_l$, $C_r$, $C_l$ are defined in Appendix B; see Propositions B.2 and B.3 for the description of these cases).

**Theorem 6.3.** *Let $\nu$ be a locally finite positive measure on the interval $J = (l, r)$. Assume that $s(r) = \infty$ and $s(l) = -\infty$. Then* P-*a.s. we have*

$$L^y_\zeta(Y) = \infty \text{ for every } y \in J, \tag{6.6}$$

*hence $\int_J L^y_\zeta(Y)\,\nu(dy) = \infty$* P-*a.s. whenever $\nu$ is a non-zero measure (i.e. $\nu(J) > 0$).*

Let us remark that the assumption $s(r) = \infty$ and $s(l) = -\infty$ of Theorem 6.3 is equivalent to $\mathsf{P}(A) = 1$ (see Propositions B.2 and B.3). In particular, in Theorem 6.3 we have $\zeta = \infty$ P-a.s.

The study of the remaining case $\mathsf{P}(B_r \cup B_l \cup C_r \cup C_l) = 1$ consists of the investigation of the convergence of (6.2) on the event $\{\lim_{t\uparrow\zeta} Y_t = l\}$ and on the event $\{\lim_{t\uparrow\zeta} Y_t = r\}$. In the following theorem we investigate the convergence of (6.2) on the event $\{\lim_{t\uparrow\zeta} Y_t = l\}$ (in particular, we need to assume $s(l) > -\infty$, which is, by Proposition B.3, equivalent to $\mathsf{P}(\lim_{t\uparrow\zeta} Y_t = l) > 0$).

**Theorem 6.4.** *For $\nu$ a locally finite positive measure on the interval $J = (l, r)$, assume that $s(l) > -\infty$.*

*(i) If*

$$\frac{s - s(l)}{\rho} \in L^1_{\text{loc}}(l+, \nu),$$

*then*

$$\int_J L^y_\zeta(Y)\,\nu(dy) < \infty \quad \mathsf{P}\text{-a.s. on } \left\{\lim_{t\uparrow\zeta} Y_t = l\right\}.$$

*(ii) If*

$$\frac{s - s(l)}{\rho} \notin L^1_{\text{loc}}(l+, \nu),$$

*then*

$$\int_J L^y_\zeta(Y)\,\nu(dy) = \infty \quad \mathsf{P}\text{-a.s. on } \left\{\lim_{t\uparrow\zeta} Y_t = l\right\}.$$

The investigation of the convergence of (6.2) on the event $\{\lim_{t\uparrow\zeta} Y_t = r\}$ is similar. This completes the study of the convergence of the integral in (6.2).

*Proofs of Theorems 6.3 and 6.4.* It is clear that Theorem 6.3 follows if we prove the equality in (6.6). By the Dambis-Dubins-Schwarz theorem, there exists a Brownian motion $B$ starting from $s(x_0)$ (possibly on an enlargement of the initial probability space) such that

$$s(Y_t) = B_{\langle s(Y), s(Y)\rangle_t} \quad \mathsf{P}\text{-a.s.}, \quad t \in [0, \zeta). \tag{6.7}$$

Since $s(r) = \infty$ and $s(l) = -\infty$, P-a.s. we have $\limsup_{t\uparrow\zeta} s(Y_t) = \infty$, $\liminf_{t\uparrow\zeta} s(Y_t) = -\infty$, hence $\langle s(Y), s(Y)\rangle_\zeta = \infty$ P-a.s. It can be deduced from the Itô-Tanaka formula that

$$L^y_t(Y) = \frac{1}{\rho(y)} L^{s(y)}_{\langle s(Y), s(Y)\rangle_t}(B), \quad (t, y) \in [0, \zeta) \times J, \quad \mathsf{P}\text{-a.s.} \tag{6.8}$$

Since P-a.s. we have $L^z_\infty(B) = \infty$ for any $z \in \mathbb{R}$ (see e.g. [12, Ch. VI, § 2]), the equality in (6.6) and Theorem 6.3 follow.

We prove Theorem 6.4 by reducing it to Lemma 6.5 below, which deals with an analogous problem for Brownian motion. Note first that (6.8) implies the following equality:

$$\int_J L^y_\zeta(Y)\,\nu(dy) = \int_J L^{s(y)}_{\langle s(Y), s(Y)\rangle_\zeta}(B)\,\frac{\nu(dy)}{\rho(y)} \quad \mathsf{P}\text{-a.s.} \tag{6.9}$$

Since $s(l) > -\infty$, we have $\mathsf{P}(L) > 0$, where $L := \{\lim_{t\uparrow\zeta} Y_t = l\}$. By the equality in (6.7) it follows that $\lim_{t\uparrow\zeta} B_{\langle s(Y), s(Y)\rangle_t} = s(l)$ P-a.s. on $L$, and hence

$$\langle s(Y), s(Y)\rangle_\zeta = \tau^B_{s(l)} \quad \mathsf{P}\text{-a.s. on } L, \tag{6.10}$$

where $\tau^B_{s(l)}$ is the first time the Brownian motion $B$ hits the level $s(l)$. Define $\widetilde{\nu}(dy) := \nu(dy)/\rho(y)$, $y \in J$, and let $\widetilde{\mu}$ be the pushforward measure of $\widetilde{\nu}$ via $s$: $\widetilde{\mu}(E) = \widetilde{\nu}(s^{-1}(E))$ for any Borel subset $E \subseteq s(J)$. Equalities (6.9) and (6.10) yield

$$\int_J L^y_\zeta(Y)\,\nu(dy) = \int_{s(J)} L^z_{\tau^B_{s(l)}}(B)\,\widetilde{\mu}(dz) \quad \mathsf{P}\text{-a.s. on } L.$$

Theorem 6.4 now follows from

$$\int_{(s(l),s(z))} (x - s(l))\,\widetilde{\mu}(dx) = \int_{(l,z)} \frac{s(y)-s(l)}{\rho(y)}\,\nu(dy), \quad z \in J,$$

and an application of Lemma 6.5. □

**Lemma 6.5.** *For some $l \in \mathbb{R}$, define $I := (l,\infty)$. Let $B$ be a Brownian motion starting from $x_0 \in I$ and $\nu$ a locally finite positive measure on $I$. Let $\tau^B_l$ denote the first time $B$ hits the level $l$.*

*(i) If $x - l \in L^1_{\text{loc}}(l+,\nu)$, then*

$$\int_I L^y_{\tau^B_l}(B)\,\nu(dy) < \infty \quad \mathsf{P}\text{-}a.s.$$

*(ii) If $x - l \notin L^1_{\text{loc}}(l+,\nu)$, then*

$$\int_I L^y_{\tau^B_l}(B)\,\nu(dy) = \infty \quad \mathsf{P}\text{-}a.s.$$

**Remark 6.6.** Lemma 6.5 is known and has a long history. On the one hand, Lemma 6.5 contains Lemma 5.10 in [3], which is complemented by a criterion for a.s.-infiniteness of the additive functional. That is why the proof below is different from that of Lemma 5.10 in [3]. On the other hand, Lemma 6.5 appeared in the literature already in this form. It can be traced back to [1, Lem. 1.4.1] (the discussion in [9, Sec. 4] gives a detailed account of the history of this result). The proof in [1, Lem. 1.4.1] is based on the Ray-Knight theorem and an application of Jeulin's [7] lemma (e.g. [1, Lem. 1.4.2]). Here we give a proof which replaces the application of Jeulin's lemma by a simple direct argument.

*Proof.* The mapping $x \mapsto L^x_{\tau^B_l}(B)$ is $\mathsf{P}$-a.s. a continuous function with compact support in $[l,\infty)$. So the finiteness of the integral $\int_I L^y_{\tau^B_l}(B)\,\nu(dy)$ reduces to the question

$$\text{whether} \quad \int_{(l,x_0)} L^y_{\tau^B_l}(B)\,\nu(dy) = \int_{(0,x_0-l)} L^{l+u}_{\tau^B_l}(B)\,\nu(l+du) \quad \text{is finite.}$$

Let $\overline{W}$ and $\widetilde{W}$ be independent Brownian motions starting from 0. We set $\eta_t = \overline{W}^2_t + \widetilde{W}^2_t$, i.e. $\eta = (\eta_t)_{t\in[0,\infty)}$ is a squared two-dimensional Bessel process starting from 0. It follows from the first Ray-Knight theorem that

$$\text{Law}\left(L^{l+u}_{\tau^B_l}(B); u \in [0, x_0 - l]\right) = \text{Law}\left(\eta_u; u \in [0, x_0 - l]\right).$$

So, the question is

$$\text{whether} \quad \int_{(0,x_0-l)} \eta_u\,\nu(l+du) = \int_{(l,x_0)} \eta_{y-l}\,\nu(dy) \quad \text{is finite.} \tag{6.11}$$

In what follows we prove that, for a Brownian motion $W$ starting from 0,

(A) $x - l \in L^1_{\text{loc}}(l+, \nu)$ implies that $\int_{(l,x_0)} W^2_{y-l}\,\nu(dy) < \infty$ $\mathsf{P}$-a.s.;

(B) $x - l \notin L^1_{\text{loc}}(l+, \nu)$ implies that $\int_{(l,x_0)} W^2_{y-l}\,\nu(dy) = \infty$ $\mathsf{P}$-a.s.

Together with (6.11) this will complete the proof of Lemma 6.5.

By Fubini's theorem we have $\mathsf{E}\int_{(l,x_0)} W^2_{y-l}\,\nu(dy) = \int_{(l,x_0)}(y-l)\,\nu(dy)$ and (A) follows.

In order to prove (B) we assume that

$$\mathsf{P}\left(\int_{(l,x_0)} W^2_{y-l}\,\nu(dy) < \infty\right) > 0. \tag{6.12}$$

Then there exists a large $M < \infty$ such that $\gamma := \mathsf{P}(E) > 0$, where

$$E := \left\{\int_{(l,x_0)} W^2_{y-l}\,\nu(dy) \le M\right\}.$$

For any positive $\delta$ and $u$, the probability $\mathsf{P}(W^2_u \ge \delta^2 u) = \mathsf{P}(|N(0,1)| \ge \delta)$ does not depend on $u$. Pick a sufficiently small $\delta > 0$ such that $\mathsf{P}(|N(0,1)| \ge \delta) \ge 1 - \frac{\gamma}{2}$, and note that, for any $y \in (l, x_0)$ we have

$$\mathsf{E}\left(W^2_{y-l} I_E\right) \ge \mathsf{E}\left(W^2_{y-l} I_{E \cap \{W^2_{y-l} \ge \delta^2 (y-l)\}}\right) \ge \frac{\delta^2\gamma}{2}(y-l).$$

By Fubini's theorem,

$$M \ge \mathsf{E}\left[I_E \int_{(l,x_0)} W^2_{y-l}\,\nu(dy)\right] = \int_{(l,x_0)} \mathsf{E}(W^2_{y-l} I_E)\,\nu(dy) \ge \frac{\delta^2\gamma}{2}\int_{(l,x_0)}(y-l)\,\nu(dy).$$

Hence (6.12) implies $x - l \in L^1_{\text{loc}}(l+, \nu)$, which proves (B), and the lemma follows. □

## 7. Proofs of Theorems from Section 3

In this section we will prove Theorems 3.2, 3.7, 3.12 and 3.15. We assume (3.1) and case (B) of Section 3.

**1.** Consider a sequence $(\alpha_n)_{n\in\mathbb{N}}$, $l < \alpha_n < x_0$, $\alpha_n \downarrow l$. By the Itô-Tanaka formula applied to the stopped process $g(Y^{\tau^Y_{\alpha_n}})$, $n \in \mathbb{N}$, we get

$$g(Y_t) = g(x_0) + \overline{A}_t + \overline{M}_t, \quad t \in [0, \zeta), \quad \mathsf{P}\text{-a.s.}, \tag{7.1}$$

where the locally finite measure $\nu_g$ on the interval $J$ is defined in (3.2) and

$$\overline{A}_t = \int_J L^y_t(Y)\,\nu_g(dy), \quad t \in [0,\zeta),$$
$$\overline{M}_t = \int_0^t (g'\sigma)(Y_u)\,dW_u, \quad t \in [0,\zeta).$$

We note that the process $\overline{M} = (\overline{M}_t)_{t\in[0,\zeta)}$ is a continuous local martingale on the stochastic interval $[0, \zeta)$ with

$$\langle \overline{M}, \overline{M}\rangle_t = \int_0^t (g'\sigma)^2(Y_u)\,du = \int_J L^y_t(Y)(g')^2(y)\,dy, \quad t \in [0,\zeta) \tag{7.2}$$

(the second equality follows from the occupation times formula), and the process $\overline{A} = (\overline{A}_t)_{t\in[0,\zeta)}$ has locally finite variation on $[0, \zeta)$.

With $\operatorname{Var}\overline{A} = (\operatorname{Var}\overline{A}_t)_{t\in[0,\zeta)}$ the variation process of $\overline{A}$, one has

$$\operatorname{Var}\overline{A}_t = \int_J L_t^y(Y)\,|\nu_g|(dy), \quad \mathsf{P}\text{-a.s.}, \quad t \in [0,\zeta), \tag{7.3}$$

where $|\nu_g|$ is the variation measure of $\nu_g$. We will now prove (7.3) by a pathwise argument, but first observe that the right-hand side of (7.3) is, clearly, $(\mathcal{F}_t)$-adapted and finite; finiteness $\mathsf{P}$-a.s. on $\{t<\zeta\}$ follows from the fact that $\mathsf{P}$-a.s. on $\{t<\zeta\}$ the function $y \mapsto L_t^y(Y)$ is càdlàg with compact support and the measure $|\nu_g|$ is locally finite on $J$. To prove (7.3), note that $\mathsf{P}$-a.s. on $\{t<\zeta\}$ there exists a compact interval $I \subset J$, which depends on $\omega$ and contains the support of $y \mapsto L_t^y(Y)$. Let $\omega$ be fixed. Since $|\nu_g|(I) < \infty$, there exists a Jordan decomposition $\nu_g = \nu_g^+ - \nu_g^-$: $\nu_g^+$ and $\nu_g^-$ are positive measures and $\nu_g^+(\cdot) = \nu_g(\cdot\cap P)$ and $\nu_g^-(\cdot) = -\nu_g(\cdot\cap(I\setminus P))$ for some Borel set $P$ in $I$. Furthermore, $|\nu_g| = \nu_g^+ + \nu_g^-$ on $I$. Note that

$$\overline{A}_t(\omega) = \int_I L_t^y(Y)(\omega)\,\nu_g^+(dy) - \int_I L_t^y(Y)(\omega)\,\nu_g^-(dy), \quad t \in [0,\zeta(\omega)), \tag{7.4}$$

is a decomposition of $\overline{A}(\omega)$ into a difference of two non-decreasing continuous functions. To show (7.3), it is sufficient to prove that the measures on $[0,\zeta(\omega))$ induced by these functions, i.e. the measures

$$\int_I dL_s^y(Y)(\omega)\,\nu_g^+(dy) \quad \text{and} \quad \int_I dL_s^y(Y)(\omega)\,\nu_g^-(dy), \tag{7.5}$$

are mutually singular (that is the decomposition in (7.4) is minimal). In fact it is easy to see that the former measure is concentrated on the set

$$\widetilde{P} = \{u \in [0,t] : Y_u(\omega) \in P\},$$

while the latter measure is concentrated on the corresponding set with $P$ replaced by $I \setminus P$. Indeed, by Fubini's theorem,

$$\int_{[0,\zeta(\omega))} I_{\widetilde{P}}(s) \int_I dL_s^y(Y)(\omega)\,\nu_g^-(dy) = \int_I \left(\int_{[0,\zeta(\omega))} I_{\widetilde{P}}(s)\,dL_s^y(Y)(\omega)\right)\nu_g^-(dy) = 0,$$

and similarly for the other statement. Thus, (7.3) follows.

**2.** Whenever

$$\mathsf{P}\text{-a.s. on } \{\zeta<\infty\} \text{ there exists a finite limit } \overline{M}_\zeta := \lim_{t\uparrow\zeta} \overline{M}_t, \tag{7.6}$$

we extend the process $(\overline{M}_t)_{t\in[0,\zeta)}$ to the process $M = (M_t)_{t\in[0,\infty)}$ by setting

$$M_t := \overline{M}_{t\wedge\zeta}, \quad t \in [0,\infty). \tag{7.7}$$

Let us prove that under (7.6) $M$ is a local martingale (now on the whole $[0,\infty)$). Indeed, there exists a sequence of stopping times $(\eta_n)_{n\in\mathbb{N}}$ such that $\eta_n \uparrow \zeta$ $\mathsf{P}$-a.s. and $M^{\eta_n}$ is a martingale for any $n \in \mathbb{N}$. For $m \in \mathbb{N}$, set

$$\xi_m = \inf\{t \in [0,\infty) \colon |M_t| \geq m\} \qquad (\inf\emptyset := \infty)$$

and note that $\xi_m \uparrow \infty$ $\mathsf{P}$-a.s. as $m \uparrow \infty$. Since for fixed $m \in \mathbb{N}$, the processes $M^{\eta_n\wedge\xi_m}$, $n \in \mathbb{N}$, are uniformly (in $n$) bounded martingales and $M_t^{\eta_n\wedge\xi_m} \to M_t^{\xi_m}$ $\mathsf{P}$-a.s. as $n \to \infty$ (note that $M$

is stopped at $\zeta$), the process $M^{\xi_m}$ is a martingale for any $m \in \mathbb{N}$. So, $M = (M_t)_{t\in[0,\infty)}$ is a local martingale.

**3.** Since we consider case (B) of Section 3, we have $\lim_{t\uparrow\zeta} Y_t = l$ $\mathsf{P}$-a.s. on $\{\zeta < \infty\}$, and there is a finite limit $g(l) := \lim_{x\downarrow l} g(x)$. Then it follows from (7.1) that condition (7.6) is equivalent to

$$\text{(7.8)}\qquad \mathsf{P}\text{-a.s. on } \{\zeta < \infty\} \text{ there exists a finite limit } \overline{A}_\zeta := \lim_{t\uparrow\zeta} \overline{A}_t.$$

Whenever (7.8) holds, we extend the process $(\overline{A}_t)_{t\in[0,\zeta)}$ to the process $A = (A_t)_{t\in[0,\infty)}$ by setting

$$\text{(7.9)}\qquad A_t := \overline{A}_{t\wedge\zeta}, \quad t \in [0,\infty).$$

Finally, we note that the condition

$$\text{(7.10)}\qquad \operatorname{Var} \overline{A}_\zeta < \infty \quad \mathsf{P}\text{-a.s. on } \{\zeta < \infty\}$$

implies (7.8) and under (7.10) the process $A = (A_t)_{t\in[0,\infty)}$ has locally finite variation (on the whole of $[0,\infty)$).

**4.** By applying Theorem 6.4 with the positive measure $\nu(dy) = (g')^2(y)dy$, we obtain from (7.2) the following alternative (additionally use the Dambis-Dubins-Schwarz theorem for continuous local martingales on stochastic intervals):

$(M_1)$ If (3.9) is satisfied, then

$$\langle \overline{M}, \overline{M}\rangle_\zeta < \infty \quad \mathsf{P}\text{-a.s. on } \{\zeta < \infty\},$$

hence (7.6) and (7.8) hold.

$(M_2)$ If (3.9) is violated, then

$$\langle \overline{M}, \overline{M}\rangle_\zeta = \infty \quad \mathsf{P}\text{-a.s. on } \{\zeta < \infty\},$$

hence

$$\text{(7.11)}\qquad \limsup_{t\uparrow\zeta} \overline{M}_t = -\liminf_{t\uparrow\zeta} \overline{M}_t = \infty \quad \mathsf{P}\text{-a.s. on } \{\zeta < \infty\},$$

$$\text{(7.12)}\qquad \limsup_{t\uparrow\zeta} \overline{A}_t = -\liminf_{t\uparrow\zeta} \overline{A}_t = \infty \quad \mathsf{P}\text{-a.s. on } \{\zeta < \infty\}.$$

(We note that (7.12) follows from (7.11) via (7.1).) Applying Theorem 6.4 once again with the measure $\nu = |\nu_g|$, we get from (7.3) another alternative:

$(A_1)$ (3.4) implies (7.10).

$(A_2)$ If (3.4) is violated, then

$$\operatorname{Var} \overline{A}_\zeta = \infty \quad \mathsf{P}\text{-a.s. on } \{\zeta < \infty\}.$$

**5.** We now assume that $g(Y)$ is a semimartingale, i.e.

$$g(Y_t) = g(x_0) + \widetilde{A}_t + \widetilde{M}_t, \quad t \in [0,\infty),$$

with a continuous process $\widetilde{A} = (\widetilde{A}_t)_{t\in[0,\infty)}$ of locally finite variation and a continuous local martingale $\widetilde{M} = (\widetilde{M}_t)_{t\in[0,\infty)}$. Then, for $t \in [0,\infty)$,

$$\widetilde{A}_t = \overline{A}_t \quad \text{and} \quad \widetilde{M}_t = \overline{M}_t \quad \mathsf{P}\text{-a.s. on } \{t < \zeta\},$$

hence (7.6) and (7.10) hold. By alternatives $(M_1)$, $(M_2)$ and $(A_1)$, $(A_2)$ above, (3.9) and (3.4) hold. This proves Theorem 3.12 and the "only if"-part of Theorem 3.2.

**6.** In order to prove the "if"-part of Theorem 3.2, we now assume that (3.4) holds. By $(A_1)$ and the reasoning in item 3, (7.10) and (7.6) (which is equivalent to (7.8)) are satisfied. Then, by items 2 and 3, $g(Y)$ is a semimartingale with the decomposition

$$g(Y_t) = g(x_0) + A_t + M_t, \quad t \in [0,\infty),$$

where $A$ and $M$ are given in (7.9) and (7.7).

Thus, Theorem 3.2 is proved. Theorem 3.15 can be proved similarly (again use the alternatives $(M_1)$, $(M_2)$ and $(A_1)$, $(A_2)$ and items 2 and 3).

**7.** It remains to prove Theorem 3.7. Let us assume that (3.8) is satisfied. Then $\mathsf{P}$-a.s. on $\{\zeta < \infty\}$, one has:

$$\text{there exists } \varepsilon > 0 \text{ such that } (\overline{A}_t)_{t\in(\zeta-\varepsilon,\zeta)} \text{ is monotone,} \tag{7.13}$$

hence, $\mathsf{P}$-a.s. on $\{\zeta < \infty\}$ there exist limits

$$\overline{A}_\zeta := \lim_{t\uparrow\zeta} \overline{A}_t \quad \text{and} \quad \overline{M}_\zeta := \lim_{t\uparrow\zeta} \overline{M}_t,$$

which are either both finite or both infinite (see (7.1)). By alternative $(M_1)$, $(M_2)$ above, either the limit $\lim_{t\uparrow\zeta} \overline{M}_t$ is finite or (7.11) holds. Then $\overline{M}_\zeta$ and, consequently, $\overline{A}_\zeta$ are finite. Thus, (7.8) holds.

Now it follows from the fact that $\overline{A}$ has locally finite variation on $[0,\zeta)$ and from (7.13) and (7.8) that (7.10) holds. By alternative $(A_1)$, $(A_2)$, we get (3.4), hence, by Theorem 3.2, $g(Y)$ is a semimartingale. This completes the proof.

## Appendix A. Bessel Process of Dimension $\delta \in (0,1)$ Is Not a Semimartingale

It is known that a Bessel process of dimension $\delta \in (0,1)$ is not a semimartingale. However, we could not find a direct reference for this. We think that it can be deduced from the general Theorem 7.9 in [3], but this does not look straightforward. So, we now present a direct proof.

Let $x_0 \geq 0$. Consider a squared Bessel process $Y$ of dimension $\delta \in (0,1)$ starting from $x_0^2$, i.e. $Y$ satisfies

$$Y_t = x_0^2 + \delta t + \int_0^t 2\sqrt{Y_s}\, dW_s, \quad t \geq 0, \tag{A.1}$$

with $W$ a Brownian motion. It is well-known that SDE (A.1) has a pathwise unique strong solution, which is nonnegative, and

$$\int_0^\infty I(Y_s = 0)\, ds = 0 \quad \text{a.s.} \tag{A.2}$$

(see [12, Ch. XI, § 1]). A Bessel process of dimension $\delta \in (0,1)$ starting from $x_0$ is by definition

$$\rho_t := \sqrt{Y_t}, \quad t \geq 0.$$

Assume $\rho = x_0 + M + A$ for a continuous local martingale $M$ and a continuous finite variation process $A$ with $M_0 = A_0 = 0$. In particular, $\rho$ has a version $(L_t^a(\rho); t \geq 0, a \in \mathbb{R})$ of local time, continuous in $t$ and càdlàg in $a$ The process $\int_0^\cdot I(\rho_s = 0)\, dM_s$ is a continuous local martingale starting from 0 with quadratic variation

$$\int_0^t I(\rho_s = 0)\, d\langle M, M\rangle_s = \int_0^t I(\rho_s = 0)\, d\langle \rho, \rho\rangle_s = \int_{\mathbb{R}} I_{\{0\}}(a) L_t^a(\rho)\, da = 0 \quad \text{a.s.}, \quad t \geq 0,$$

where the second equality follows from the occupation times formula (see [12, Ch. VI, Cor. 1.6]), i.e.

$$\int_0^t I(\rho_s = 0)\, dM_s = 0 \quad \text{a.s.}, \quad t \geq 0. \tag{A.3}$$

Since $Y = \rho^2$, we have

$$Y_t = x_0^2 + \int_0^t 2\rho_s\, dM_s + \int_0^t (2\rho_s\, dA_s + d\langle \rho, \rho\rangle_s), \quad t \geq 0. \tag{A.4}$$

Comparing decompositions (A.1) and (A.4) and using (A.3) and (A.2), we obtain

$$M_t = \int_0^t I(\rho_s \neq 0)\, dM_s = \int_0^t I(\rho_s \neq 0)\, dW_s = W_t \quad \text{a.s.}, \quad t \geq 0.$$

Then $\langle \rho, \rho\rangle_t = \langle M, M\rangle_t = t$, so by (A.1) and (A.4),

$$\int_0^t 2\rho_s\, dA_s = (\delta - 1)t, \quad t \geq 0,$$

whence

$$A_t = \int_0^t I(\rho_s = 0)\, dA_s + \int_0^t I(\rho_s \neq 0) \frac{\delta - 1}{2\rho_s}\, ds \quad \text{a.s.}, \quad t \geq 0. \tag{A.5}$$

By the occupation times formula, for the term $\int_0^t I(\rho_s \neq 0) \frac{\delta-1}{2\rho_s}\, ds$ to be finite, we necessarily have $L_t^0(\rho) = 0$ a.s., $t \geq 0$. Furthermore, $L_t^{0-}(\rho) = 0$ a.s., $t \geq 0$, because $\rho$ is nonnegative. By [12, Ch. VI, Th. 1.7],

$$\int_0^t I(\rho_s = 0)\, dA_s = \frac{1}{2}(L_t^0(\rho) - L_t^{0-}(\rho)) = 0 \quad \text{a.s.}, \quad t \geq 0.$$

Thus, using (A.5), we get that $\rho$ is a nonnegative global (i.e. on $[0, \infty)$) solution of the SDE

$$d\rho_t = I(\rho_t \neq 0) \frac{\delta - 1}{2\rho_t}\, dt + dW_t. \tag{A.6}$$

But, by [2, Th. 2.13], the latter SDE does not have a nonnegative global solution. Here is a description of what happens: the singular point 0 of SDE (A.6) has *right type 1*, which is one of *non-entrance types*, in the terminology of [2], that is, after $\rho$ reaches 0, which happens in finite time with probability 1, it cannot be continued in the positive direction (also see [2, Sec. 2.4]). The obtained contradiction completes the proof.

## Appendix B. Behaviour of One-Dimensional Diffusions

Now we state some well-known results about the behaviour of a one-dimensional diffusion $Y$ of (2.1) with coefficients satisfying (2.2) and (2.3). These results follow from the construction of solutions of (2.1) (see e.g. [5] or [8, Ch. 5.5] or [2, Ch. 2 and Ch. 4]), or can be deduced from the results in [4, Sec. 1.5].

**Proposition B.1.** *For any $a \in J$, with*

$$\tau_a^Y := \inf\{t \geq 0 : Y_t = a\} \qquad (\inf \emptyset := \infty),$$

*we have $\mathsf{P}(\tau_a^Y < \infty) > 0$.*

We consider the sets

$$A = \left\{\zeta = \infty,\ \limsup_{t\to\infty} Y_t = r,\ \liminf_{t\to\infty} Y_t = l\right\},$$

$$B_r = \left\{\zeta = \infty,\ \lim_{t\to\infty} Y_t = r\right\},$$

$$C_r = \left\{\zeta < \infty,\ \lim_{t\uparrow\zeta} Y_t = r\right\},$$

$$B_l = \left\{\zeta = \infty,\ \lim_{t\to\infty} Y_t = l\right\},$$

$$C_l = \left\{\zeta < \infty,\ \lim_{t\uparrow\zeta} Y_t = l\right\}.$$

**Proposition B.2.** *Either $\mathsf{P}(A) = 1$ or $\mathsf{P}(B_r \cup B_l \cup C_r \cup C_l) = 1$.*

**Proposition B.3.** *(i) $\mathsf{P}(B_r \cup C_r) = 0$ holds if and only if $s(r) = \infty$.*

*(ii) $\mathsf{P}(B_l \cup C_l) = 0$ holds if and only if $s(l) = -\infty$.*

In particular, we get that $\mathsf{P}(A) = 1$ holds if and only if $s(r) = \infty$, $s(l) = -\infty$.

**Proposition B.4.** *Assume that $s(r) < \infty$. Then either $\mathsf{P}(B_r) > 0$, $\mathsf{P}(C_r) = 0$ or $\mathsf{P}(B_r) = 0$, $\mathsf{P}(C_r) > 0$. Furthermore, we have*

$$\mathsf{P}\left(\lim_{t\uparrow\zeta} Y_t = r,\ \ Y_t > a\ \forall t \in [0,\zeta)\right) > 0$$

*for any $a < x_0$.*

**Proposition B.5** (Feller's test for explosions)**.** *We have $\mathsf{P}(B_r) = 0$, $\mathsf{P}(C_r) > 0$ if and only if*

$$s(r) < \infty \quad \textit{and} \quad \frac{s(r) - s}{\rho\sigma^2} \in L^1_{\mathrm{loc}}(r-).$$

Clearly, Propositions B.4 and B.5, which contain statements about the behaviour of one-dimensional diffusions at the endpoint $r$, have their analogues for the behaviour at $l$. Feller's test for explosions in this form is taken from [2, Sec. 4.1]. For a different (but equivalent) form see e.g. [8, Ch. 5, Th. 5.29].

Let us finally emphasize that the results stated in this appendix do not in general hold beyond (2.2) and (2.3).

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