ON GERBER-SHIU FUNCTIONS AND OPTIMAL DIVIDEND DISTRIBUTION FOR A LÉVY RISK-PROCESS IN THE PRESENCE OF A PENALTY FUNCTION

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Abstract. In this paper we consider an optimal dividend problem for an insurance company which risk process evolves as a spectrally negative Lévy process (in the absence of dividend payments). We assume that the management of the company controls timing and size of dividend payments. The objective is to maximize the sum of the expected cumulative discounted dividends received until the moment of ruin and a penalty payment at the moment of ruin which is an increasing function of the size of the shortfall at ruin; in addition, there may be a fixed cost for taking out dividends. We explicitly solve the corresponding optimal control problem. The solution rests on the characterization of the value-function as (i) the unique *stochastic solution* of the associated HJB equation and as (ii) the *pointwise smallest stochastic supersolution*. We show that the optimal value process admits a *dividend-penalty decomposition* as sum of a martingale (associated to the penalty payment at ruin) and a potential (associated to the dividend payments). We find also an explicit necessary and sufficient condition for optimality of a single dividend-band strategy, in terms of a particular Gerber-Shiu function. We analyze a number of concrete examples.

Keywords: Optimal control, Lévy process, De Finetti model, transaction costs, singular control, variational inequality, barrier policies, band policies, Gerber-Shiu function.

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Contents

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1. Optimal control of Lévy risk models

The spectrally negative Lévy risk model. Recall the classical Cramér-Lundberg model

$$X_t - X_0 = \eta\, t - S_t, \qquad S_t = \sum_{k=1}^{N_t} C_k - \lambda\, m\, t, \tag{1.1}$$

which is used in collective risk theory (e.g. Gerber [22]) to describe the surplus $X = \{X_t, t \in \mathbb{R}_+\}$ of an insurance company. Here, C_k are i.i.d. positive random variables representing the claims made, $N = \{N_t, t \in \mathbb{R}_+\}$ is an independent Poisson process with intensity λ modelling the times at which the claims occur, and $p\,t$, with $p = \eta + \lambda m$, represents the premium income up to time t, with profit rate $\eta > 0$ and mean $m < \infty$ of C_1.

In later years, the model (1.1) has been generalized to the "perturbed model"

$$X_t - X_0 := \sigma B_t + \eta\, t - S_t, \tag{1.2}$$

where B_t denotes an independent standard Brownian motion, which models small scale fluctuations of the risk process.

Since the jumps of X are all negative, the moment generating function $\mathbb{E}[\mathrm{e}^{\theta X_t}]$ exists for all $\theta \geq 0$ and $t \in \mathbb{R}_+$, and is log-linear in t, defining thus a function $\psi(\theta)$ satisfying:

$$\mathbb{E}[\mathrm{e}^{\theta(X_t - X_0)}] = \mathrm{e}^{t\psi(\theta)}, \ \psi(\theta) = \frac{\sigma^2}{2}\theta^2 + \eta\,\theta + \int_{(0,\infty)} (\mathrm{e}^{-\theta x} - 1 + \theta x)\nu(\mathrm{d}x), \tag{1.3}$$

where $\nu(\mathrm{d}x) = \lambda F_C(\mathrm{d}x)$, $x \in \mathbb{R}_+$, with F_C the distribution function of C_1, is the "Lévy measure" of the compound Poisson process S_t, and $\eta = \psi'(0)$ is the mean of $X_1 - X_0$.

The cumulant exponent $\psi(\theta)$ is well defined at least on the positive half-line, where it is strictly convex with the property that $\lim_{\theta\to\infty} \psi(\theta) = +\infty$. Moreover, ψ is strictly increasing on $[\Phi(0), \infty)$, where $\Phi(0)$ is the largest root of $\psi(\theta) = 0$. We shall denote the right-inverse function of ψ by $\Phi : [0, \infty) \to [\Phi(0), \infty)$.

An important generalization is to replace the process S in (1.2) by a general subordinator (a nondecreasing Lévy process, with Lévy measure $\nu(\mathrm{d}x), x \in \mathbb{R}_+$, which may have infinite mass). Under this model, the "small fluctuations" can arise either continuously, due to the Brownian motion, or due to the infinite jump-activity.

Taking S to be a pure jump-martingale with i.i.d. increments and negative jumps with Lévy measure $\nu(\mathrm{d}x)$, one arrives thus to a general integrable spectrally negative Lévy process $X = \{X_t,\ t \in \mathbb{R}_+\}$ i.e. (see Bertoin [13], Kyprianou [31], Sato [44]) a stochastic process that has stationary independent increments, no positive jumps and càdlàg paths with X_t integrable for any $t \in \mathbb{R}_+$, defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, where $\mathbf{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ is the natural filtration satisfying the usual conditions of right-continuity and completeness. The assumption that X_t has finite mean for any fixed $t \in \mathbb{R}_+$ is equivalent to the requirement that the Lévy measure ν satisfies the integrability condition

$$\nu_{1,\infty} := \int_{[1,\infty)} x\nu(\mathrm{d}x) < \infty.$$

To avoid degeneracies, we exclude the case that X has monotone paths. We denote by $\{\mathbb{P}_x, x \in \mathbb{R}\}$ the family of probability measures that correspond to the translations of X by a constant, that is, $\mathbb{P}_x[X_0 = x] = 1$.

An alternative characterization of spectrally negative Lévy processes is via the "q-harmonic homogeneous scale function" $W^{(q)}$, a non-decreasing function defined on the real line that is 0 on $(-\infty, 0)$, continuous on $\mathbb{R}_+$, with Laplace transform given by

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(y)\mathrm{d}y = (\psi(\theta) - q)^{-1}, \qquad \theta > \Phi(q). \tag{1.4}$$

Despite of the diversity of possible path behaviors displayed by spectrally negative Lévy processes, a wide variety of results may be elegantly expressed in a unifying manner via the homogeneous scale function $W^{(q)}$, bypassing thus "probabilistic complexity" via unified analytic methods. This paper further illustrates this aspect, by unveiling the way the scale function intervenes in a quite complex control problem.

De Finetti's dividend problem. Under the assumption that the increments of the surplus process have positive mean, the Lévy risk model has the unrealistic property that it converges to infinity with probability one.

In answer to this objection, De Finetti [18] introduced the risk process with dividends

$$U_t^\pi = X_t - D_t^\pi, \qquad t \geq 0, \tag{1.5}$$

where π is an "admissible" dividend control policy and D_t^π denotes the cumulative amount of dividends that has been transferred to a beneficiary up to time t, and where $U_{0-}^\pi = X_0 = x > 0$ is the initial capital.

Writing $\tau^\pi = \inf\{t \in \mathbb{R}_+ : U_t^\pi < 0\}$ for the time at which ruin occurs, the objective is to maximize the expected cumulative dividend payments until the time of ruin

$$v_*(x) := \sup_{\pi \in \Pi} \mathbb{E}_x \left[\int_{[0,\tau^\pi)} \mathrm{e}^{-qt} \mathrm{d}D_t^\pi \right],$$

with $\mathbb{E}_x[\cdot] = \mathbb{E}[\cdot | X_0 = x]$ and where Π denotes the set of all admissible strategies and $q > 0$ is the discount rate.

Note that ruin may be either exogeneous or endogeneous (i.e. caused by a claim or by a dividend payment). A dividend strategy is admissible if ruin is always exogeneous, or more precisely, an admissible dividend strategy $D^\pi = \{D_t^\pi, t \in \mathbb{R}_+\}$ is a right-continuous $\mathbf{F}$-adapted stochastic process that will satisfy that, at any time preceding ruin, a dividend payment is smaller than the size of the available reserves:

$$\text{for any } t \leq \tau^\pi, \qquad \begin{cases} \Delta D_t^\pi := D_t^\pi - D_{t-}^\pi \leq \left(X_t - D_{t-}^\pi\right) \vee 0, & \text{and} \\ D_t^{\pi(c)} - D_u^{\pi(c)} \leq p(t-u) & \text{for all } u \in [0,t), \text{ in the case } \nu_{0,1} < \infty, \end{cases} \tag{1.6}$$

where $D^{\pi(c)}$ denotes the continuous part of D^π and

$$p := \eta + \nu_{0,1} + \nu_{1,\infty}, \text{ with } \nu_{0,1} := \int_{(0,1)} x\nu(\mathrm{d}x). \tag{1.7}$$

The second line in Eqn. (1.6) states that, if the jump-part of X is of bounded variation, it is not admissible to pay dividends at a rate larger than the premium rate p at any time t that there are no reserves (i.e. $U_t^\pi = 0$), as this would lead to immediate ruin.

Single barrier policies. Recall first the simplest case when there are no transaction costs. One possible dividends distribution policy is the "barrier policy" π_b of transferring all surpluses above a given level b, which results in the optimal value:

$$v_b(x) := v_{\pi_b}(x) = \mathbb{E}_x \left[\int_{[0,\tau_b)} \mathrm{e}^{-qt} \mathrm{d}D_t^b \right] = \frac{W^{(q)}(x)}{W^{(q)\prime}(b)}, \qquad x \in [0,b],$$

and $v_b(x) = x - b + v_b(b)$ for $x > b$, where $\tau_b = \inf\{t \geq 0 : X_t < D_t^b\}$, and $D^b = D^{\pi_b}$ is a local time-type strategy, given explicitly in terms of X by $D_{0-}^b = 0$ and

$$D_t^b = \sup_{s \leq t} \left(X_s - b\right)^+, \qquad t \in \mathbb{R}_+,$$

with $x^+ = \max\{x, 0\}$. As this equation shows, a non-zero optimal barrier must be an inflection point of the scale function, if the latter is smooth.

Multiple bands policies. However, single barrier strategies might not be optimal cf. Gerber [20, 21]. The optimal strategy may be a "multi-bands strategy", involving several "continuation bands" $[a_i, b_i), i = 0, 1, \ldots$ with upper reflecting boundaries b_i, separated by "lump-sum dividend taking bands" $[b_i, a_{i+1}), i = 0, 1, \ldots$ of jumping to the next reflecting barrier below b_i, by paying all the excess as a lump-sum payment (see also Hallin [27], who formulated a system of time dependent integro-differential equations associated to multi-bands policies). Azcue & Muler [10] established the optimality of multi-bands strategies under the Cramér-Lundberg model in the presence of proportional and excess-of-loss reinsurance, adopting a viscosity approach. Recently, Albrecher & Thonhauser [2] proved the optimality of bands strategies, in the case that the reserves attract a fixed interest rate.

Gerber showed also that for exponential claims (and with no constraints on the dividends rate), the optimal policy involved only one barrier (and one continuation band); however, constructing examples where more than one band was necessary remained an open problem for a long time.

Optimality conditions for single barrier strategies. The interest in bands strategies was reawakened by Azcue & Muler [10], who produced the first example (with Gamma claims) in which a single constant barrier is not optimal. Let

$$b^* = \sup\left\{b > 0 : W^{(q)\prime}(b) \le W^{(q)\prime}(x) \ \text{ for all } x\right\} \tag{1.8}$$

denote the last global minimum of the derivative of the q-scale function.

Avram *et al.* [8] showed that

$$(\Gamma v_{b^*} - q v_{b^*})(x) \le 0, \qquad \text{for all } x > b^*, \tag{1.9}$$

where Γ denotes the infinitesimal generator of X, is a sufficient optimality condition for the single barrier strategy under a general spectrally negative Lévy model. In fact, the condition (1.8)–(1.9) is both necessary and sufficient, as follows by examining the variational inequality characterizing the problem – see [34, Lemmas 1, 2].

A simpler sufficient condition for the optimality of single band policies was obtained by Loeffen [34, 35] (with and without transaction costs), who showed that it is enough to check that the last local minimum of the q-scale function is also a global minimum. Even more direct optimality conditions in terms of the Lévy measure ν were provided by Kyprianou *et al.* [32], and Loeffen & Renaud [36], who showed respectively that log-convexity of the density and of the survival functions suffice (the second condition is more general). Note that the second result allowed also for an affine penalty function with slope less than unity, and that both results imply complete monotonicity of the Lévy density, and constitute therefore powerful generalizations of Gerber's unicity result [20, 21].

It turns out that b^* in (1.8) is always the right end point of the first continuation band. As already demonstrated in the rather terse Azcue & Muler example [10, pp. 274], left and right end points of subsequent bands can in principle be determined recursively (the former by ensuring the "smoothness" of the value function, and the latter similarly with b^*, by selecting last global maxima of updated value functions, adjusted by using the values of previous bands as stopping penalties). However, an explicit smoothness condition (6.11) seems not to have been reported previously.

Balancing dividends and ruin penalties. Several alternative objectives have been proposed recently, involving final penalties $w(x)$ at ruin, [17, 23, 49], or continuous payoffs until ruin [1, 16]. For example, the case where the insurance company is bailed out by the beneficiaries every time that there is a shortfall in the reserves was investigated in [8], and in Kulenko & Schmidli [30].

Our paper continues the investigation of the impact of a general final penalty and transaction costs on the optimal dividends policy. Assuming that the management of the company controls timing and size of dividend payments and is liable to pay a penalty that is a function of the shortfall at the moment of ruin, we solve the corresponding optimal control problem by constructing explicitly its solution. To show that the constructed function solves the stochastic optimal control problem, standard verification arguments that rely on the application of Itô's lemma cannot be employed, due to a lack of smoothness of the value function. In particular, it will follow from the form of the value-function and from results concerning the smoothness of scale functions (Kyprianou *et al.* [32], Lambert [33]) that, in general, the value-function is continuous but not C^1 on $\mathbb{R}_+\backslash\{0\}$ if X has bounded variation, and is

C^1 but not C^2 on $\mathbb{R}_+\backslash\{0\}$, if X has unbounded variation. The approach followed in this paper is probabilistic in nature and rests on the characterisation of the value-function as *stochastic solution* of the corresponding HJB equation, and on a *dual representation* of the value function as the point-wise minimum of stochastic supersolutions (Thm. 3.4), which yields as a consequence a comparison and local-verification result (Cor. 3.5). We also show (in Cor. 3.9) that the optimal value process admits a dividend-penalty decomposition as sum of a martingale (equal to the conditional expectation of the penalty payment at ruin) and a potential (related to the dividend payments).

A key point in our approach is the decomposition of the value function preceding and within a continuation band $[a, b]$

$$v_{a,b}(x) = \begin{cases} f(x), & x < a, \\ F(x) + W^{(q)}(x)\ G(a,b), & x \in [a,b], \end{cases} \tag{1.10}$$

into a *nonhomogeneous* solution $F(x)$, which we will call Gerber-Shiu function, and the product of the homogeneous scale function $W^{(q)}(x)$ by a "barrier-influence" function $G(a, b)$ defined in (5.2), which needs to be maximized at b and be smooth at a.

Note that that the function G in the decomposition (1.10) is only determined up to a constant, but becomes fixed once F has been selected – see (5.2).

To ensure smoothness at a, it seems then natural to use a "smooth Gerber-Shiu function" $F_f(x)$ associated to a given penalty $f(x), x \in (-\infty, a)$. Informally, $F_f(x)$ is the "smooth nonhomogeneous solution" of the Dirichlet problem on $\{x \geq a\}$ with boundary condition $f(x), x \in (-\infty, a)$. More precisely, it is defined in Defs. 4.1 and 4.2 in Sect. 4 by subtracting a multiple of the homogeneous scale function $W^{(q)}(x)$ out of the solutions of either the two-sided, or the reflected exit problem, such that the remaining part is *continuous* on $\mathbb{R}$ if f is continuous, and *continuously differentiable* on $\mathbb{R}$ if f is continuously differentiable on $\mathbb{R}_-$ and X has unbounded variation. This results in the explicit formula (4.6).

For exponential penalties $w(x) = \mathrm{e}^{xv}$, the Gerber-Shiu function takes a simple form (7.2), which may be used also as a generating function for the expected payoffs associated to polynomial penalties $x^k, k = 0, 1, ...$

The decomposition (1.10) with $F_f(x)$ chosen to fit the imposed penalty $f(x) = w(x)$ already determines the value function on the first continuation band (and the value function in the lump-dividend taking bands surrounding it) —see Prop. 5.1 and Thm. 6.5. It also leads to an explicit *necessary* and *sufficient* criterion for optimality of single dividend barrier policies —see Thm. 5.3 in Sect. 5, which is analogous to (1.9), modulo replacing the function $1/W^{(q)\prime}(b)$ by the two variables function $G(a, b)$.

Quite paradoxically, it is possible that beyond the lump-sum dividend taking band following the first continuation band, waiting for higher barriers $b_i, i \geq 2$, may become again optimal. The level a_2 where the second continuation band starts may be determined by examining the family of functions $G_2^{(a)}(b)$ defined in Eqn. (6.11), which are computed from a second Gerber-Shiu function, which uses the first value functions as stopping penalties, and so on, leading ultimately to all the optimal band levels —see Sect. 7.

Fixed transaction costs. It is interesting to consider also the effect of adding fixed transaction cost $K > 0$ that are not transferred to the beneficiaries when dividends are being paid. The objective of the beneficiaries becomes then to maximize $v_{\pi,K}(x)$:

$$v_*(x) = \sup_{\pi\in\Pi_K} v_{\pi,K}(x), \quad \text{where} \quad v_{\pi,K}(x) = \mathbb{E}_x\left[\int_{[0,\tau^\pi)} \mathrm{e}^{-qt}\mathrm{d}D_t^\pi - K\int_{[0,\tau^\pi)} \mathrm{e}^{-qt}\mathrm{d}N_t^\pi\right],$$

where $N^\pi = \{N_t^\pi, t \in \mathbb{R}_+\}$ is the stochastic process that counts the number of jumps of D^π in the interval $[0, t]$,

$$N_t^\pi = \#\{s \in [0,t] : \Delta D_s^\pi > 0\} \qquad t \in \mathbb{R}_+. \tag{1.11}$$

To avoid degeneracies the set Π_K is taken to be equal to the collection of admissible strategies for which any dividend payment is larger or equal to K. For any $\pi \in \Pi_K$ the range $R(D^{-1})$ is discrete and N^π_t is equal to the number of times a dividend has been paid out by time t. In the sequel we will drop the subscripts K and write $\Pi = \Pi_K$ and $v_\pi = v_{\pi,K}$ when no confusion is possible.

The introduction of a fixed transaction cost $K > 0$ has the usual effect of changing the optimal reflection boundaries b into strips $[b_-, b_+]$, so that when $U_t = b_+$, a lump-sum dividend $b_+ - b_-$ is paid, and the reserves process is diminished to the lower "entrance" point b_-. To emphasize this disappearance of reflection barriers, we will always use the term band when $K > 0$, and also when more than one barrier is present.

The typical optimal dividend strategy consists of "lump sum payments" [4], with π of the form $\pi = \{(J_k, T_k), k \in \mathbb{N}\}$, where $0 \le T_1 \le T_2 \le ...$ is an increasing sequence of $\mathbf{F}$-stopping times representing the times at which a dividend payment is made and $J_i \ge K$ is a sequence of positive $\mathcal{F}_{T_i}$-measurable random variables representing the sizes of the dividend payments. Then,

$$D^\pi_t = \sum_{k=1}^{N^\pi_t} J_k,$$

where $N^\pi_t = \#\{k : T_k \le t\}$ is the number of times that dividends have been paid by time t.

For single bands policies for example, the dividend distribution consists of the fixed amount $J_i = b_{i,+} - b_{i,-}$.

Contents. The remainder of the paper is organized as follows. In Sect. 2 the dividend-penalty problem is phrased and its optimal solution is presented, and Sect. 3 is devoted to the characterisation of the value-function as stochastic solution of the HJB, and as minimal stochastic supersolution. Sect. 4 is concerned with two stochastic boundary value problems associated to the value of dividend payments in the presence of a penalty, and Sects. 5 and 6 are devoted to single and two-bands strategies. In Sect. 7 the value function is constructed, and some examples are analyzed in detail in Sect. 8. Sects. 9 and 10 contain the proofs of the results in Sect. 3 and 7, respectively. A number of the proofs are presented in the Appendix.

2. The dividend-penalty control problem

Assume that the beneficiaries control the timing and size of dividend payments made by the company, and are liable to pay at the moment τ^π of ruin the penalty $-w(U^\pi_{\tau^\pi})$, which may be used to cover (part of) the claim that led to insolvency, where w is a penalty.

Def. 2.1. A *penalty* $w : \mathbb{R}_- \to \mathbb{R}_-$, with $\mathbb{R}_- = (-\infty, 0]$, is an increasing function that is left-continuous at 0 with a finite left-derivative $w'(0-)$, and satisfies the integrability condition

$$\int_{(1,\infty)} |w(-z)|\nu(\mathrm{d}z) < \infty. \tag{2.1}$$

The collection of penalties is denoted by $\mathcal{P}$. We shall also consider the class $\mathcal{R}$ of functions that contains $\mathcal{P}$.

Def. 2.2. We denote by $\mathcal{R}$ the set of Borel-measurable functions $w : \mathbb{R}_- \to \mathbb{R}$ that are left-continuous at 0, admit a finite left-derivative $w'(0-)$, and satisfy the integrability conditions

$$\text{(i)} \quad w_\nu(y) < \infty \ \forall y \in \mathbb{R}_+\backslash\{0\} \text{ and (ii)} \quad \int_0^\infty \int_y^\infty \mathrm{e}^{-\Phi(q)y} |w(y-z) - w(0)|\nu(\mathrm{d}z)\mathrm{d}y < \infty, \tag{2.2}$$

where the function $w_\nu : \mathbb{R}_+\backslash\{0\} \to \mathbb{R}$ is defined by

$$w_\nu(y) := \int_{(y,\infty)} \{w(y-z) - w(0)\}\, \nu(\mathrm{d}z), \qquad y \in \mathbb{R}_+\backslash\{0\}. \tag{2.3}$$

The beneficiaries seek to maximize the sum of the expected discounted cumulative dividends and an expected penalty payment by paying out dividends according to an admissible policy. The present value of the penalty payment discounted at rate $q > 0$, considered as function of the level of reserves, is called the "Gerber-Shiu penalty function" associated to the penalty w, and is given by

$$\mathcal{W}_w^\pi(x) := \mathbb{E}_x \left[\mathrm{e}^{-q\tau^\pi} w\left(U^\pi_{\tau^\pi}\right)\right], \qquad x \in \mathbb{R}_+.$$

Under condition (2.1), it holds that, for any level of initial capital $x \in \mathbb{R}_+$, $\mathcal{W}_w^\pi(x)$ is bounded uniformly over $\pi \in \Pi$ (see Lemma 9.3).

The objective of the beneficiaries of the insurance company is described by the following stochastic optimal control problem:

$$v_*(x) = \sup_{\pi\in\Pi} v_\pi(x), \qquad v_\pi(x) := \mathcal{W}_w^\pi(x) + \mathbb{E}_x\left[\int_{[0,\tau^\pi]} \mathrm{e}^{-qt}\mu_K(\mathrm{d}t)\right], \qquad x \in \mathbb{R}_+, \tag{2.4}$$

where Π denotes the set of admissible dividend policies π and μ_K is the (signed) random measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ defined by

$$\mu_K^\pi([0,t]) = D_t^\pi - KN_t^\pi, \tag{2.5}$$

with N_t^π equal to the counting process defined in Eqn. (1.11) and D_t^π equal to the cumulative amount of dividends that has been paid out by time t. We will restrict ourselves to the case of positive net income (or infinitesimal drift), $\eta := \mathbb{E}[X_1] > 0$. Here we note that we have $\mu_K(\{\tau^\pi\}) = 0$ for any admissible policy $\pi \in \Pi$. A solution to the stochastic control problem in Eqn. (2.4) consists of a pair (u, π^*) of a function $u : \mathbb{R}_+ \to \mathbb{R}$ and a policy $\pi^* \in \Pi$ such that $v_*(x) = u(x) = v_{\pi^*}(x)$ for all $x \in \mathbb{R}_+$.

3. Stochastic solution approach

In the literature two common approaches to solving stochastic control problems can be distinguished. In the *guess-and-verify* approach a candidate solution is constructed and, provided the candidate solution is sufficiently regular, optimality is subsequently verified using Itô's lemma. The *viscosity solution approach* commences with establishing that the value function solves the Hamilton-Jacobi-Bellman (HJB) equation in the viscosity sense and proceeds with deriving the form of the optimal policy—this is the approach taken in *e.g.* Azcue & Muler [10] where an analysis is presented of a dividend distribution problem with reinsurance under the Cramér-Lundberg model, using viscosity methods. A direct approach was developed in Schmidli [45] where a recursive algorithm is provided to find, in terms of solutions to certain integro-differential equations, the value function of the optimal dividend problem under the Cramér-Lundberg model in the absence of a penalty. We will follow below a stochastic approach to solve the optimal control problem in Eqn. (2.4), which will permit to by-pass the issues associated with the lack of regularity of the value-function in the setting of a general spectrally negative Lévy process. It is shown (Thm. 3.8) that the value-function of the stochastic control problem in Eqn. (2.4) is the unique function that is both a *stochastic supersolution* and a *stochastic subsolution* of the HJB equation associated to the dividend-penalty control problem. The concepts of stochastic super- and subsolution (described in Def. 3.2 below) bear similarity to the notions of viscosity super- and subsolution (see [10, Def. 3.2 and Cor. 5.2] for the definition and the existence and uniqueness result in the Cramér-Lundberg setting described above) but differ in one important aspect: whereas viscosity sub- and supersolutions are defined pointwise, stochastic sub- and supersolutions are defined in terms of some super- or sub-martingale property that these functions possess. In the course of showing that the value-function is a stochastic supersolution it is also established (Thm. 3.4) that the value function admits a global *dual representation* as the pointwise minimum of such stochastic supersolutions. Drawing on Thms. 3.4 and 3.8, we provide in Section 7 an explicit step-wise construction of the solution of the dividend-penalty control problem in terms of scale functions of the Lévy process X.

3.1. **Dynamic programming.** The analysis of the stochastic optimal control problem starts from the observation that the value function satisfies a dynamic programming equation.

Prop. 3.1. *(i) Extending v_* to the negative half-axis by $v_*(x) = w(x)$ for $x < 0$, we have for any $\tau \in \mathcal{T}$, the set of* **F***-stopping times,*

$$v_*(x) = \sup_{\pi\in\Pi} v_{\pi,\tau}(x), \qquad v_{\pi,\tau}(x) := \mathbb{E}_x\left[\mathrm{e}^{-q(\tau\wedge\tau^\pi)}v_*(U^\pi_{\tau\wedge\tau^\pi}) + \int_{[0,\tau\wedge\tau^\pi]}\mathrm{e}^{-qs}\mu^\pi_K(\mathrm{d}s)\right]. \tag{3.1}$$

(ii) For any fixed $\pi \in \Pi$ and $x \in \mathbb{R}_+$, the process $V^\pi = \{V^\pi_t, t \in \mathbb{R}_+\}$ given by

$$V^\pi_t = \mathrm{e}^{-q(\tau^\pi\wedge t)}v_*(U^\pi_{\tau^\pi\wedge t}) + \int_{[0,\tau^\pi\wedge t]}\mathrm{e}^{-qs}\mu^\pi_K(\mathrm{d}s), \tag{3.2}$$

is a $\mathbb{P}_x$-supermartingale.

The proof of Prop. 3.1(i) follows by straigtforward adaptation of classical arguments— see e.g. [10, pp.276-277], while that of Prop. 3.1(ii) is deferred to Appendix A. The corresponding HJB equation is given in terms of a gradient-type constraint, as follows:

$$\max\{\mathcal{L}g(x) - qg(x), 1 - \mathtt{d}_g(x)\} = 0, \qquad x > 0, \tag{3.3}$$

where

$$\mathtt{d}_g(x) = \begin{cases} \inf\limits_{y\in(0,x)} \dfrac{g(x) - g(x-y) + K}{y}, & \text{in the case } K > 0,\\ g'_+(x), & \text{in the case } K = 0,\end{cases} \tag{3.4}$$

with $g'_+(x)$ denoting the right-derivative of g at x, subject to the boundary condition

$$\begin{cases} g(x) = w(x), & \text{for all } x < 0,\\ g(0) = w(0), & \text{in the case } \{\sigma^2 > 0 \text{ or } \nu_{0,1} = \infty\},\end{cases} \tag{3.5}$$

where $\mathcal{L}$ denotes the infinitesimal generator of the Feller semi-group of X which acts on $f \in C^2_c(\mathbb{R}_+)$ as follows (cf. Sato [44, Thm. 31.5]):

$$\mathcal{L}f(x) = \frac{\sigma^2}{2}f''(x) + \eta f'(x) + \int_{\mathbb{R}_+\setminus\{0\}}[f(x-y) - f(x) + yf'(x)]\nu(\mathrm{d}y), \qquad x \in \mathbb{R}_+,$$

where f' denotes the derivative of f.

Remark. The boundary condition at $x = 0$ is of the form as stated in Eqn. (3.5) because of the fact that ruin is immediate (that is, $\tau^\pi = 0$ $\mathbb{P}_0$ a.s.) if and only if X has unbounded variation which corresponds precisely to the case $\{\sigma^2 > 0 \text{ or } \nu_{0,1} = \infty\}$.

3.2. **Stochastic solutions and dual representation.** The solution of the stochastic control problem in Eqn. (2.4) is based on a characterization of the optimal value function v_* as stochastic solution of the HJB equation.

Def. 3.2. (i) For given $a, b \in \mathbb{R}_+$ with $a < b$, a function $g : (-\infty, b] \to \mathbb{R}$ is a *local stochastic supersolution on the interval* $[a, b]$ of the HJB equation (3.3)—(3.5) if $g|_{[a,b]}$ is continuous,

$$\overline{M}^{g,T_{a,b}} := \left\{ \mathrm{e}^{-q(t\wedge T_{a,b})} g\left(X_{t\wedge T_{a,b}}\right),\ t \in \mathbb{R}_+ \right\} \text{ is a uniformly integrable (UI) } \mathbb{P}_x\text{-supermartingale,} \tag{3.6}$$

for any $x \in [a, b]$, and g satisfies the condition

$$g(x) - g(x - y) \geq y - K \qquad \text{for all } x, y \in [a, b] \text{ with } y < x, \tag{3.7}$$

and the boundary condition

$$\begin{cases} g(x) = v_*(x) & \text{for all } x \in (-\infty, a) \cup \{b\}, \\ g(a) = v_*(a) & \text{in case } \{\sigma^2 > 0 \text{ or } \nu_{0,1} = \infty\}. \end{cases} \tag{3.8}$$

The family of local stochastic supersolutions on the interval $[a, b]$ is denoted by $\mathcal{G}^+_{a,b}$.

(ii) A function $g : \mathbb{R} \to \mathbb{R}$ is a *stochastic supersolution* of the HJB equation in (3.3)—(3.5) if $g|_{\mathbb{R}_+}$ is continuous,

$$\overline{M}^g := \left\{ \mathrm{e}^{-q\left(t\wedge T_0^-\right)} g\left(X_{t\wedge T_0^-}\right),\ t \in \mathbb{R}_+ \right\} \text{ is a UI } \mathbb{P}_x\text{-supermartingale, for any } x \in \mathbb{R}_+, \tag{3.9}$$

and g satisfies the conditions in Eqns. (3.5) and (3.7). The family of stochastic supersolutions is denoted by $\mathcal{G}^+$.

(iii) A function $g : \mathbb{R} \to \mathbb{R}$ that is such that $g|_{\mathbb{R}_+}$ is continuous (in the case $K > 0$), and $g|_{\mathbb{R}_+}$ is continuous, $g(x)$ is right-differentiable at any $x > 0$, and $g'_+|_{(0,\infty)}$ is right-continuous (in the case $K = 0$), is a called a *stochastic subsolution* of the HJB equation in (3.3)—(3.5) if g satisfies the boundary condition stated in Eqn. (3.5) and we have

$$\underline{M}^g := \left\{ \overline{M}^g_{t\wedge H_{\overline{\mathcal{O}}_g}} - \overline{M}^g_0,\ t \in \mathbb{R}_+ \right\} \text{ is a UI } \mathbb{P}_x\text{-submartingale,} \tag{3.10}$$

with $H_{\overline{\mathcal{O}}_g} = \inf\{t \in \mathbb{R}_+ : X_t \notin \overline{\mathcal{O}}_g\}$, for any $x \in \mathcal{O}_g$ and any open interval $\mathcal{O}_g$ (with closure $\overline{\mathcal{O}}_g$) satisfying

$$\mathcal{O}_g \subset \mathcal{C}_g := \{x \in \mathbb{R}_+ : \mathtt{d}_g(x) > 1\}.$$

The family of stochastic subsolutions will be denoted by $\mathcal{G}^-$.

(iv) A function $g : \mathbb{R} \to \mathbb{R}$ is a *stochastic solution* of the HJB equation in (3.3)—(3.5) if we have $g \in \mathcal{G}^+ \cap \mathcal{G}^-$.

Rem. 3.3. For any stochastic subsolution g the corresponding set $\mathcal{C}_g$ is right-open, that is, for every $x > 0$ with $\mathtt{d}_g(x) > 1$ there exists an $\epsilon > 0$ such that for all $y \in [x, x + \epsilon]$ we have $\mathtt{d}_g(y) > 1$. In the case $K > 0$ this can be seen to hold by noting that for any $x > 0$ satisfying, for some $\delta > 0$, $g(x) - g(x - y) > (1 + \delta)y - K$ for all $y \in (0, x)$, the continuity of g on $[0, x + 1]$ implies that there exists an $\epsilon \in (0, 1)$ such that also the inequality $g(x + \epsilon) - g(x - y + \epsilon) > (1 + \delta/2)y - K$ holds for all $y \in (0, x + \epsilon)$. The argument for the case $K = 0$ is similar. Note that, since the set $\mathcal{C}$ is right-open, it is equal to a countable union of intervals with non-empty interior. As a consequence, the interior $\mathcal{D}^o_g = \mathbb{R}_+ \backslash \overline{\mathcal{C}}_g$ of the set

$$\mathcal{D}_g := \mathcal{C}^c_g = \mathbb{R}_+ \backslash \mathcal{C}_g$$

is a countable union of disjoint open intervals.

Thm. 3.4. *The value function v_* is the smallest stochastic supersolution of the HJB equation in* (3.3)—(3.5)*:*

$$v_*(x) = \min_{g \in \mathcal{G}^+} g(x) \qquad \text{for all } x \in \mathbb{R}_+. \tag{3.11}$$

In particular, for any $a, b \in \mathbb{R}_+$, $a < b$, Eqn. (3.11) *remains valid if $\mathcal{G}^+$ is replaced by $\mathcal{G}^+_{a,b}$.*

The proof of Thm. 3.4 is given in Sect. 9. As direct consequence of the dual representation in Eqn. (3.11), the dynamic programming equation and the definition of v_* we get a *local verification theorem.*

Cor. 3.5. *If there exist $b > a \geq 0$, $\pi \in \Pi$ and $g \in \mathcal{G}^+$ such that $g(x) = v_{\pi,\tau_a^\pi}(x)$ for all $x \in [a,b]$ where $\tau_a^\pi = \inf\{t \geq 0 : U_t^\pi < a\}$, then we have $v_*(x) = v_{\pi,\tau_a^\pi}(x)$ for all $x \in [a,b]$.*

This verification result will be used in the piecewise construction of the value-function, in Sections 5—7.

In the next result it is shown that *given* the optimal value function v_* a corresponding admissible optimal strategy π_* can be constructed.

Def. 3.6. To a stochastic solution $g \in \mathcal{G}^+ \cap \mathcal{G}^-$ of the stochastic control problem in Eqn. (2.4) is associated the policy $\pi(g) = \{D_t^{\pi(g)}, t \in \mathbb{R}_+\} \in \Pi$ defined in terms of the sets $\mathcal{C}_g$ and $\mathcal{D}_g$, the controlled process $U = U^{\pi(g)}$ and the level $y^*(v) := \sup\{u \in [0,v] : g(v) - g(v-u) + K = u\}$ (with $\sup\emptyset = 0$) as follows:

(a) In the case $K = 0$, let $D = D^{\pi(g)}$ be the increasing right-continuous $\mathbf{F}$-adapted process that satisfies

$$\begin{cases} U_t = X_t - D_t \in \overline{\mathcal{C}}_g, \text{ for any } t \in [0, \tau^{\pi(g)}), \\ \displaystyle\int_{[0,\tau^{\pi(g)})} \mathbf{1}_{\{s: X_s - D_{s^-} \notin \overline{\mathcal{D}}_g\}}(t)\mathrm{d}D_t = 0, \end{cases}$$

where $\mathbf{1}_A$ denotes the indicator of the set A.

(b) In the case $K > 0$, pay $\Delta D_t = y^*(U_t)$ as lump-sum dividend whenever we have $U_t \in \mathcal{D}_g$ and $y^*(U_t) > 0$.

(c) Otherwise, pay no dividends.

Rem. 3.7. The Skorokhod embedding lemma implies that the strategy $\pi(g) = \{D_t^{\pi(g)}, t \in \mathbb{R}_+\}$ in Def. 3.6(a) is explicitly given by

$$D_t^{\pi(g)} = \sup_{s \in [0, t\wedge\tau^{\pi(g)}]} (X_s - b(s)) \vee 0, \qquad \text{where} \qquad b(s) = b_{\iota(s)}, \qquad \iota(s) = \inf\{n \in \mathbb{N} : X_s - D_{s^-} < a_n\},$$

where we denoted the interior $\mathcal{D}_g^0$ of $\mathcal{D}_g$ by $\mathcal{D}_g^0 = \cup_n (b_n, a_n)$ with $0 \leq b_n < a_n$ such that the intervals (b_n, a_n) are disjoint. The condition in Def. 3.6(a) implies that the dividend strategy $\pi(g)$ is chosen such as to pay the minimal amount of dividends that will ensure that the process $U^{\pi(g)}$ takes values in the set $\overline{\mathcal{C}}_g$ (the closure of the set $\mathcal{C}_g$).

Thm. 3.8. *(i) The optimal strategy for the stochastic control problem in Eqn.* (2.4) *is given by $\pi_* := \pi(v_*)$, that is, $v_* = v_{\pi_*}$.*

(ii) The value function v_* is the unique stochastic solution of the HJB equation in (3.3)—(3.5) *satisfying the condition*

$$v_*(0) = \min_{g \in \mathcal{G}^+} g(0),$$

which reduces to $v_*(0) = w(0)$ in the case $\{\sigma^2 > 0$ or $\nu_{0,1} = \infty\}$.

The proof of Thm. 3.8 is given in Sect. 9. The set $\mathcal{D}$ will be identified explicitly in Sect. 7 in terms of the scale functions of the underlying Lévy process X.

From Thms. 3.4 and 3.8 a decomposition can be derived of the value-function v^* of the stochastic control problem in Eqn. (2.4) into the sum of the *classical Gerber-Shiu function* (which definition is recalled in Eqn. (4.7) below) and the *expected discounted dividend payments* adjusted for the additional penalty payment that will be incurred due to dividend payments that have been made (compared with the benchmark penalty payment in the case that no dividends had been paid out). The dynamic counterpart of this decomposition states that the discounted optimal value $\mathrm{e}^{-q(t\wedge T_0^-)} v_*(X_{t\wedge T_0^-})$, which is a supermartingale by Thm. 3.4, is equal to the sum of a *martingale* (the collection of $\mathcal{F}_t$-conditional expectations of the discounted penalty at ruin) and a *potential* (the conditional expectations of the remaining cumulative discounted dividend payments until the moment of ruin, again adjusted for the risk of incurring an additional penalty payment at ruin due to the dividend payments that have been made). This potential is explicitly expressed in terms of the interior $\mathcal{D}_*^o$ of the set

$$\mathcal{D}_* := \{x \in \mathbb{R}_+ : \mathtt{d}_{v_*}(x) = 1\} \tag{3.12}$$

where $\mathtt{d}_{v_*}$ is defined in Eqn. (3.4) and of collection $\{J^*(x), x \in \mathcal{D}_*^o\}$, with

$$J^*(x) = \psi'(0) + \int_0^\infty [v_*(x-z) - v_*(x) + z]\,\nu(\mathrm{d}z) - qv_*(x), \qquad x \in \mathcal{D}_*^o,$$

where v_* is extended to the negative half-line by $v_*(x) = w(x)$ for $x < 0$.

Cor. 3.9 (Dividend-penalty decomposition)**.** *(i) The process $S^* = \{S_t^*, t \in \mathbb{R}_+\}$ given by*

$$S_t^* = \mathrm{e}^{-q\left(t\wedge T_0^-\right)} v_*\left(X_{t\wedge T_0^-}\right)$$

admits the decomposition

$$S_t^* = \mathbb{E}\left[\mathrm{e}^{-qT_0^-} w\left(X_{T_0^-}\right)\middle|\mathcal{F}_t\right] + \left(\mathbb{E}\left[A^*_{T_0^-}\middle|\mathcal{F}_t\right] - A_t^*\right), \qquad t \in \mathbb{R}_+, \tag{3.13}$$

where $A^* = \{A_t^*, t \in \mathbb{R}_+\}$ is the increasing process given by

$$A_t^* = \int_0^{t\wedge T_0^-} \mathrm{e}^{-qs}\mathbf{1}_{\{X_{s^-}\in\mathcal{D}_*^o\}}[-J^*(X_{s^-})]\mathrm{d}s, \tag{3.14}$$

(ii) In particular, we have

$$v_*(x) = \mathcal{V}_w^{0,\infty}(x) + \mathbb{E}_x\left[A^*_{T_0^-}\right], \qquad x \in \mathbb{R}_+,$$

where $\mathcal{V}_w^{0,\infty}$ denotes the classical Gerber-Shiu function defined in Eqn. (4.7) below.

Proof. (i) Since the process S_t^* is a UI supermartingale (Thm. 3.4), the Doob-Meyer decomposition (e.g. [43, Thm. 7]) implies that S_t^* can be decomposed as $S_t^* = M_t^* - A_t^*$, where M^* is a local martingale and A^* is a increasing locally natural process. It follows that A^* is dominated by the random variable $A^*_{T_0^-}$ that is integrable since S^* is uniformly integrable. As a consequence, the process M^* is a UI martingale. Thus, we have for any $t \in \mathbb{R}_+$

$$S_t^* + A_t^* = M_t^* = \mathbb{E}\left[M^*_{T_0^-}\middle|\mathcal{F}_t\right] = \mathbb{E}\left[\mathrm{e}^{-qT_0^-} w\left(X_{T_0^-}\right) - A^*_{T_0^-}\middle|\mathcal{F}_t\right].$$

The proof that A^* is of the form stated in Eqn. (3.14) is given in Sect. 9.

(ii) The statement follows from part (i) by taking expectations under the measure $\mathbb{P}_x$ in Eqn. (3.13). ☐

Rem. 3.10. In Sect. 6 we encounter a generalisation of the control problem in Eqn. (2.4) that is used in the construction of its solution and is phrased as follows: assume that in addition to deciding the timing and size of dividend payments the management of the company also has the option to "wind up the company" at any stopping time τ before the ruin time τ^π. The beneficiaries receive the payment $f(U^\pi_\tau)$ if τ is before the ruin time τ^π, and are liable to pay a penalty payment $-f(U^\pi_{\tau^\pi})$ if ruin occurs before the company is wound up, for some pre-specified function $f : \mathbb{R} \to \mathbb{R}$. In this control problem the controls are pairs (τ, π) of dividend strategies $\pi \in \Pi$ and $\mathbf{F}$-stopping times $\tau \in \mathcal{T}$, and the value-function is given by

$$v_f^*(x) = \sup_{\tau\in\mathcal{T},\pi\in\Pi} \widetilde{v}_{\tau,\pi,f}(x), \qquad \widetilde{v}_{\tau,\pi,f}(x) := \mathbb{E}_x\left[\int_0^{\tau\wedge\tau_\pi} \mathrm{e}^{-qt}\mu_K^\pi(\mathrm{d}t) + \mathrm{e}^{-q(\tau\wedge\tau_\pi)} f(U^\pi_{\tau\wedge\tau_\pi})\right]. \tag{3.15}$$

As generalisation of Thm. 3.4 we then have the following representation of the value function v_f^*:

$$v_f^*(x) = \min_{g\in\mathcal{G}_f} g(x), \tag{3.16}$$

where $\mathcal{G}_f$ denotes the set of stochastic supersolutions of the control problem in Eqn. (3.15), which is equal to the collection of functions $g \in \mathcal{G}$ satisfying in addition the requirement

$$g(x) \geq f(x), \qquad \text{for all } x \in \mathbb{R}_+. \tag{3.17}$$

The corresponding local verification theorem (that generalises Cor. 3.5) states

> *If there exist $b > a \geq 0$, $\pi \in \Pi$, $\tau \in \mathcal{T}$ and $g \in \mathcal{G}_f$ such that $g(x) = v_{\pi,\tau,f}(x)$ for all $x \in [a, b]$, then we have $v_f^*(x) = v_{\pi,\tau,f}(x)$ for all $x \in [a, b]$.*

In particular, the dynamic programming equation in Prop. 3.1 implies that when the pay-off function f is specified by $f^{(a)} = v_* \mathbf{1}_{(\infty,a]}$ for some $a \in \mathbb{R}_+$, we have $v^*_{f^{(a)}} = v_*$.

4. Two related exit problems

The local verification theorem suggests to construct the optimal value-function step-wise, by identifying at each stage a policy that is "locally optimal". Such a strategy and the corresponding value-function are explicitly constructed in Sects. 5 and 6. In preparation for these sections the problem under consideration in the current section is the identification of the solutions of the two-sided exit problem (with two absorbing boundaries) and the mixed absorption/reflection exit problems, in terms of the q-scale function $W^{(q)}$. We note that these two solutions can both be expressed in terms of a common "continuous nonhomogeneous solution", which we chose to call Gerber-Shiu function – a non-standard terminology – see Defs. 4.1 and 4.2.

It is well known that, on $(0, \infty)$, $W^{(q)}$ is non-decreasing and everywhere right- and left-differentiable (with finite derivative). Throughout the paper, we will denote by $W^{(q)\prime}(x)$ the right-derivative at $x > 0$, which is right-continuous on $(0, \infty)$. We also recall that, if the Gaussian coefficient σ is positive, then $W^{(q)}|_{\mathbb{R}_+}$ is C^2 (see [32]), and $W^{(q)\prime}(0+) = \frac{2}{\sigma^2}$.

To state the stochastic representation of the solutions to the two-sided and mixed absorption/reflection problems some extra notation is needed. Given $a \in \mathbb{R}_+$, $b \in \mathbb{R}_+ \cup \{+\infty\}$, $a < b$, let T_b^+, T_a^- be the first entrance times of X into the sets (b, ∞) and $(-\infty, a)$,

$$T_b^+ = \inf\{t \in \mathbb{R}_+ : X_t > b\}, \qquad T_a^- = \inf\{t \in \mathbb{R}_+ : X_t < a\},$$

with $\inf \emptyset = +\infty$, and let $T_{a,b} = T_a^- \wedge T_b^+$ denote the two-sided exit time from the interval $[a, b]$. Also consider, for any $a, b \in \mathbb{R}$ with $a < b$, the first-passage time into (a, ∞)

$$\tau_a = \inf\{t \in \mathbb{R}_+ : Y_t^b < a\}$$

of the process $Y^b = \{Y_t^b, t \in \mathbb{R}_+\}$ that is equal to the process X reflected at the level b,

$$Y_t^b = X_t - \overline{X}_t^b \qquad \text{with} \qquad \overline{X}_t^b = \sup_{s \leq t} (X_t - b) \vee 0.$$

The solutions $\mathcal{V}_w^{a,b} : (a, b) \to \mathbb{R}$ and $\mathcal{U}_w^{a,b} : (a, b) \to \mathbb{R}$ to the two-sided exit problem and the reflected exit problem are given by

$$\mathcal{V}_w^{a,b}(x) = \mathbb{E}_x \left[\exp\{-qT_{a,b}\} \, w\left(X_{T_a^-}\right) \mathbf{1}_{\{T_a^- < T_b^+\}} \right] + \delta \, \mathbb{E}_x \left[\exp\{-qT_{a,b}\} \, \mathbf{1}_{\{T_a^- > T_b^+\}} \right], \tag{4.1}$$

$$\mathcal{U}_w^{a,b}(x) = \mathbb{E}_x \left[\exp\{-q\tau_a\} \, w\left(Y_{\tau_a}^b\right) \right] + \beta \, \mathbb{E}_x \left[\int_0^{\tau_a} \mathrm{e}^{-qs} \mathrm{d}\overline{X}_s^b \right], \tag{4.2}$$

for $q \in \mathbb{R}_+$, $\beta, \delta \in \mathbb{R}$, and any given Borel-measurable function $w : (-\infty, a] \to \mathbb{R}$ (the "pay-off") satisfying the integrability condition

$$\int_{(b,\infty)} |w(x - y)| \nu(\mathrm{d}y) < \infty \qquad \text{for all } x \in [a, b]. \tag{4.3}$$

Def. 4.1. Let $a < b < \infty$, $\delta, \beta \in \mathbb{R}$ and *pay-off* $w : (-\infty, a] \to \mathbb{R}$ be given. We will call $F : \mathbb{R} \to \mathbb{R}$ a *Gerber-Shiu function for payoff* w if $F|_{\mathbb{R}_+ \setminus \{0\}}$ is right-differentiable, with right-derivative at $x > 0$ denoted by $F'(x)$, and the following hold:

$$\mathcal{V}_w^{a,b}(x) = F(x-a) + W^{(q)}(x-a)\frac{\delta - F(b-a)}{W^{(q)}(b-a)}, \qquad x \in (a,b), \tag{4.4}$$

$$\mathcal{U}_w^{a,b}(x) = F(x-a) + W^{(q)}(x-a)\frac{\beta - F'(b-a)}{W^{(q)\prime}(b-a)}, \qquad x \in (a,b). \tag{4.5}$$

Of course, such a function F is not unique. In this section, we construct special Gerber-Shiu functions that are *continuous* on $\mathbb{R}$ for continuous payoffs w and *continuously differentiable* on $\mathbb{R}$ if X has unbounded variation and w is continuously differentiable (note that neither $\mathcal{V}_w^{a,b}$, $\mathcal{U}_w^{a,b}$, nor $W^{(q)}$ are continuous or continuously differentiable on $\mathbb{R}$ in general). Note that in the literature $\mathcal{V}_w^{0,\infty}$ is often called a Gerber-Shiu function. The proofs of results in this Section are deferred to Appendix B.

To each payoff w in the set $\mathcal{R}$ (which was defined in Def. 2.2) we associate a scale function F_w:

Def. 4.2. Let $q \in \mathbb{R}_+$ and $w \in \mathcal{R}$. The function $F_w : \mathbb{R} \to \mathbb{R}$ is given by $F_w(x) = w(x)$ for $x < 0$, is continuous on $\mathbb{R}_+$ and has Laplace transform given by

$$\int_0^\infty \mathrm{e}^{-\theta x} F_w(x)\mathrm{d}x = (\psi(\theta) - q)^{-1}\left[\frac{\sigma^2}{2}[w'(0-)] + \frac{\psi(\theta)}{\theta}w(0) - w_\nu^*(\theta)\right], \qquad \theta > \Phi(q), \tag{4.6}$$

where w_ν^* denotes the Laplace transform of w_ν.

Rem. 4.3. Key properties of the function F_w are collected in Appendix B.4.

The classical Gerber-Shiu function $\mathcal{V}_w^{0,\infty}(x)$ corresponding to penalty w can be explicitly expressed in terms of F_w, as follows (See Biffis & Kyprianou [15] for an equivalent representation of $\mathcal{V}_w^{0,\infty}$ in terms of $W^{(q)}$):

Prop. 4.4 (Classical Gerber-Shiu function)**.** *Let $w \in \mathcal{R}$. For any $x \in \mathbb{R}$ it holds*

$$\mathcal{V}_w^{0,\infty}(x) = \mathbb{E}_x\left[\exp\left\{-qT_0^-\right\} w(X_{T_0^-})\mathbf{1}_{\{T_0^- < \infty\}}\right] = F_w(x) - W^{(q)}(x)\kappa_w, \tag{4.7}$$

where

$$\kappa_w := \left[\frac{\sigma^2}{2}w'(0-) + \frac{q}{\Phi(q)}w(0) - w_\nu^*(\Phi(q))\right]. \tag{4.8}$$

In particular, the following martingale properties hold true:

$$\left(\mathrm{e}^{-q(t\wedge T_a^-)}W^{(q)}\left(X_{t\wedge T_a^-} - a\right), t \in \mathbb{R}_+\right) \qquad \textit{and} \tag{4.9}$$

$$\left(\mathrm{e}^{-q(t\wedge T_a^-)}F_w\left(X_{t\wedge T_a^-} - a\right), t \in \mathbb{R}_+\right) \qquad \textit{are } \mathbb{P}_x\textit{-martingales, for any } x, a \in \mathbb{R}. \tag{4.10}$$

Denoting the composition of a function f with a translation θ_a over $a \in \mathbb{R}$ by

$$_af := f \circ \theta_a := f(\cdot + a). \tag{4.11}$$

a regular Gerber-Shiu function is identified as follows:

Thm. 4.5. *Let $w \in \mathcal{R}$ and $a \in \mathbb{R}$.*

(i) The function $F_{_aw}$ is a Gerber-Shiu function for the payoff $_aw$.

(ii) If $_aw$ is continuous, then $F_{_aw}$ is continuous.

(iii) In the case $w \in C^1(\mathbb{R}_-)$ and $\{\sigma^2 > 0$ or $\nu_{0,1} = \infty\}$ we have $F_{_aw} \in C^1(\mathbb{R})$.

4.1. **Notation: generator and boundary condition.** To express explicity the dependence of the infinitesimal generator of the Feller-semigroup of X on the function w and the domain, we will denote by ${}_a\mathcal{L}^w_\infty$ the operator ${}_a\mathcal{L}^w_\infty : C^2([a,\infty)) \to D([a,\infty))$ for $a \in \mathbb{R}$ that is defined by

$$\begin{aligned} {}_a\mathcal{L}^w_\infty f(x) &= \frac{\sigma^2}{2} f''(x) + (\eta + \overline{\nu}_1(x-a))f'(x) - (q + \overline{\nu}(x-a))f(x) \\ &+ \int_{(0,x-a]} [f(x-y) - f(x) + f'(x)y]\,\nu(\mathrm{d}y) + \int_{(x-a,\infty)} w(x-y)\nu(\mathrm{d}y), \qquad x \geq a, \end{aligned} \tag{4.12}$$

where $\overline{\nu}(x) = \nu((x,\infty))$ and $\overline{\nu}_1(x) = \int_{(x,\infty)} y\nu(\mathrm{d}y)$. In case that X has bounded variation the operator ${}_a\mathcal{L}^w_\infty$ takes the following equivalent form:

$$\begin{aligned} {}_a\mathcal{L}^w_\infty f(x) &= pf'(x) + \int_{(0,x-a]} [f(x-y) - f(x)]\,\nu(\mathrm{d}y) + \int_{(x-a,\infty)} w(x-y)\nu(\mathrm{d}y) \\ &- (q + \overline{\nu}(x-a))f(x), \qquad x \geq a. \end{aligned} \tag{4.13}$$

The operator ${}_a\mathcal{L}^w_\infty$ coincides with $\mathcal{L}$, that is, $\mathcal{L}f(x) = {}_a\mathcal{L}^w_\infty g(x)$ for $x > a$, where $g = f|_{[a,\infty)}$ for functions f in the set $\{f \in C^2_c(\mathbb{R}) : f|_{(-\infty,a)} = w|_{(-\infty,a)}\}$.

5. Single dividend-band strategies

We will first consider the case of single dividend band strategies. The value $v_b(x) := v_{\pi_b}(x)$ associated to the single dividend band strategy π_b at a non-zero level b when X_0 is equal to x, is given by

$$v_b(x) = \mathbb{E}_x\left[\int_0^{\tau_b} \mathrm{e}^{-qt}\mu^b_K(\mathrm{d}t) + \mathrm{e}^{-q\tau_b} w(U^b_{\tau_b})\right],$$

where $\mu^b_K := \mu^{\pi_b}_K$, $U^b := U^{\pi_b}$ and $\tau^b = \tau^{\pi_b} = \inf\{t \in \mathbb{R}_+ : U^{b+}_t < 0\}$. In the following result v_b is explicitly expressed in terms of scale functions.

Prop. 5.1. *For $b_+ > b_- \geq 0$ and $x \in [0, b_+]$ it holds that*

$$v_b(x) = \begin{cases} w(x), & x < 0, \\ W^{(q)}(x)\ G(b_-, b_+) + F(x), & x \in [0, b_+], \\ x - b_+ + v_b(b_+) & x > b_+, \end{cases} \tag{5.1}$$

where $F = F_w$ and

$$G(b_-, b_+) := \begin{cases} \dfrac{b_+ - b_- - K - (F(b_+) - F(b_-))}{W^{(q)}(b_+) - W^{(q)}(b_-)}, & K > 0,\ b_+ > b_-, \\ \dfrac{1 - F'(b_+)}{W^{(q)\prime}(b_+)}, & K = 0,\ b_+ = b_-. \end{cases} \tag{5.2}$$

Proof. Consider the case $K > 0$. Taking note of the fact that no dividend payment takes place before X reaches the level b_+ it follows that $\{X_t, t \leq T_{0,b_+}\}$ and $\{U^{b+}_t, t \leq \tau^{\pi_b}\}$ have the same law. In view of the strong Markov property of X and the absence of positive jumps it follows then that for all $x \in [0, b_+]$ and with $v = v_b$:

$$\begin{aligned} v(x) &= \mathbb{E}_x\left[\mathrm{e}^{-qT^+_{b_+}}(v(b_-) + \Delta b - K)\mathbf{1}_{\{T^+_{b_+} < T^-_0\}}\right] + \mathbb{E}_x\left[\mathrm{e}^{-qT^-_0} w(U_{T^-_0})\mathbf{1}_{\{T^+_{b_+} > T^-_0\}}\right] \\ &= \frac{W^{(q)}(x)}{W^{(q)}(b_+)}[v(b_-) + \Delta b - K] + \left[F(x) - F(b_+)\frac{W^{(q)}(x)}{W^{(q)}(b_+)}\right], \end{aligned} \tag{5.3}$$

where $F = F_w$ and we used the form in Eqn. (4.4) of $\mathcal{V}^w_{a,b}$ for $w \equiv 0$ and $\delta = 1$, and the definition of a single dividend band strategy. Evaluating Eqn. (5.3) at $x = b_-$, solving the resulting linear equation for $v(b_-)$ and inserting the result in Eqn. (5.3) yields the stated form. The case $K = 0$ follows by a similar line of reasoning (using the form of Eqn. (4.5) with $\beta = 0$, $w \equiv 0$ which was established in [8]). $\square$

We next turn to the determination of the candidate optimal levels. The form of G suggests to define the level $b^* = (b^*_-, b^*_+)$ as a maximizer of $G(x, y)$ over all $x, y \geq 0$ in the case $K > 0$, and similarly, to define b^*_+ as a maximizer of $G(x, x)$ over all $x \geq 0$ in the case $K = 0$.

Rem. 5.2. Observe that in the case $K > 0$ the partial right-derivatives of $G(x, y)$ are given by

$$\frac{\partial G}{\partial x}(x,y) = \frac{W^{(q)\prime}(x)}{W^{(q)}[x,y]}[G(x,y) - G^{\#}(x)], \quad \frac{\partial G}{\partial y}(x,y) = -\frac{W^{(q)\prime}(y)}{W^{(q)}[x,y]}[G(x,y) - G^{\#}(y)], \tag{5.4}$$

where $W^{(q)}[x, y] := W^{(q)}(y) - W^{(q)}(x)$ and $G^{\#}(x)$ is given by

$$G^{\#}(x) := \frac{1 - F'(x)}{W^{(q)\prime}(x)}. \tag{5.5}$$

Therefore, an interior maximum (x^*, y^*) will satisfy $G(x^*, y^*) = G^{\#}(x^*) = G^{\#}(y^*)$, and a candidate optimum may be found by fixing $d = y - x$, and optimizing the left end-point $x(d)$ for fixed d (graphically, this would amount to determining the highest value of the function $G^{\#}$ where the "width" $y(d) - x(d)$ of the function $G^{\#}$ is d).

In the case of strictly positive K we fix therefore $d > 0$, we let

$$b^* = b^*(d) = \sup\{b \geq 0 : G(b, b + d) \geq G(x, x + d) \quad \forall x \geq 0\} \tag{5.6}$$

denote the last global maximum of $G(x, x + d)$.

We choose now d^* to be the last global maximum of $G(b^*(y), b^*(y) + y)$:

$$d^* = \sup\{d > 0 : G(b^*(d), b^*(d) + d) \geq G(b^*(y), b^*(y) + y) \quad \forall y \geq 0\}, \tag{5.7}$$

where $\inf \emptyset = +\infty$.

The candidate optimal levels are then defined as follows:

$$b^* = (b^*_-, b^*_+) \quad \text{with} \quad b^*_- = b^*(d^*), \quad b^*_+ = b^*(d^*) + d^*. \tag{5.8}$$

In the absence of transaction cost ($K = 0$), we set

$$b^*_+ = b^*_- = \sup\{b \geq 0 : G^{\#}(b) \vee G^{\#}(b-) \geq G^{\#}(x) \ \ \forall x \geq 0\}, \tag{5.9}$$

where we denote $G^{\#}(0-) = G^{\#}(0)$.

Thm. 5.3. *We have $b^*_+ < \infty$ and it holds*

$$v_*(x) = W^{(q)}(x) G^{\#}\left(b^*_+\right) + F(x), \qquad x \in [0, b^*_+], \tag{5.10}$$

where $F = F_w$. In particular, if $X_0 \in [0, b^*_+]$, it is optimal to adopt the strategy π_{b^*}.

Proof. b^*_+ is finite: On account of the facts that the map $x \mapsto G^{\#}(x)$ defined in (5.2) is right-continuous and monotone decreasing for all x sufficiently large (Prop. C.1), there exists an $x^* \in \mathbb{R}_+$ such that $\sup_{x \geq 0} G^{\#}(x) = G^{\#}(x^*) \vee G^{\#}(x^*-)$ where $G^{\#}(0-) := G^{\#}(0)$. In the case that K is strictly positive, G attains its maximum at some $(x^*, y^*) \in (R_+ \backslash \{0\})^2$, since $G(x, y)$ is continuous at any (x, y) with $y > x \geq 0$, is monotone decreasing for y sufficiently large and fixed x (Prop. C.1, Appendix C) and tends to minus infinity if $x \searrow y$ and tends to the constant κ_w in Eqn. (4.8) if $|x| + |y| \nearrow \infty$ such that $x < y$.

Verification of optimality: We claim that the function $h_* : \mathbb{R}_+ \to \mathbb{R}$ defined by the right-hand side of Eqn. (5.10) is a supersolution in the sense of Def. 3.2, and hence dominates the value-function v_*. In fact, since $h_*(x)$ is equal to the value $v_{b^*}(x)$ of the strategy π_{b^*} for any level x of initial reserve smaller or equal to b^*_+, Cor. 3.4 implies that

$h_*(x)$ is equal to the optimal value $v_*(x)$ for all $x \in [0, b_+^*]$. That h_* is a supersolution follows from the facts that we have that (a) $\mathrm{e}^{-q(t\wedge T_0^-)}h_*(X_{t\wedge T_0^-})$ is a martingale and (b) h_* satisfies the inequality

$$h_*(x) - h_*(y) \geq x - y - K \qquad \text{for any } 0 \leq y < x.$$

Fact (a) in turn follows from the martingale properties of F_w and $W^{(q)}$, while (b) follows on account of the definitions of b^* and $G^{\#}$. In the case $K = 0$, we have

$$h_*'(x) = W^{(q)\prime}(x)G^{\#}(b^*) - F_w'(x) \geq W^{(q)\prime}(x)G_*(x) - F_w'(x) = 1, \qquad x > 0.$$

Similarly, if $K > 0$ and $x > y$, $h_*(x) - h_*(y) = (W^{(q)}(x) - W^{(q)}(y))G(b_-^*, b_+^*) - F_w(x) + F_w(y)$ is bounded below by

$$(W^{(q)}(x) - W^{(q)}(y))G(y, x) - F_w(x) + F_w(y) = x - y - K.$$

The two displays imply that $h_*(x) - h_*(y) \geq x - y - K$ for any $x, y, K \geq 0$ with $x \geq y$. This completes the proof of Thm. 5.3. □

6. Two-bands strategies

When the level of the reserves is larger than b_+^* then it may be optimal to pay out the overflow over b_-^* as a lump-sum dividend payment, and then adopt the policy π_{b^*} —necessary and sufficient conditions for such a strategy to be optimal are given in Sect. 7. In this section we consider a complementary case in which it is optimal to have a *second* dividend band. The problem to find the optimal levels of the second dividend band differs from the single-band optimisation problem in two respects:

(i) at any time t prior to the time of ruin it is possible to make a lump sum payment to bring the reserves down to the level b_-^*, yielding a pay-off of $v_{b^*}(U_t) = U_t - b_+^* + v_{b^*}(b_+^*)$, and

(ii) it is not optimal to place a barrier at levels close to b_+^*.

The observation in item (i) in combination with the dynamic programming principle and Thm. 5.3 yields the representation

$$v_*(x) = \sup_{\pi\in\Pi, \tau\in\mathcal{T}} \mathbb{E}_x \left[\int_0^{\tau\wedge\tau_{b^*}^\pi} \mathrm{e}^{-qt}\mu_K^\pi(\mathrm{d}t) + \mathrm{e}^{-q(\tau_{b^*}^\pi\wedge\tau)} v_{b^*}\left(U^\pi_{\tau_{b^*}^\pi\wedge\tau}\right)\right], \tag{6.1}$$

where $\tau_{b^*}^\pi = \inf\{t \geq 0 : U_t^\pi < b_+^*\}$. In this section we consider the generic form of this optimal control problem (which will also turn up in the case the optimal strategy takes the form of a general multi-dividend bands strategy that is considered in Sect. 7), given by

$$V_*^f(x) = \sup_{\pi\in\Pi, \tau\in\mathcal{T}} V_{\tau,\pi}^f(x), \qquad V_{\tau,\pi}^f(x) = \mathbb{E}_x \left[\int_0^{\tau\wedge\tau^\pi} \mathrm{e}^{-qt}\mu_K^\pi(\mathrm{d}t) + \mathrm{e}^{-q(\tau^\pi\wedge\tau)} f\left(U^\pi_{\tau^\pi\wedge\tau}\right)\right], \tag{6.2}$$

where, as before $\tau^\pi = \inf\{t \geq 0 : U_t^\pi < 0\}$, and the function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f|_{\mathbb{R}_+} \text{ is given by } f(x) = x + c \text{ for } x \in \mathbb{R}_+, \text{ for some } c \in \mathbb{R}, \tag{6.3}$$

$$f'(0-) = 1, \tag{6.4}$$

$$_0\mathcal{L}_\infty^{\bar{w}} f(u) > 0 \text{ for some } u > 0, \text{ with } \bar{w} = f|_{\mathbb{R}_-}\,, \tag{6.5}$$

$$\limsup_{\epsilon\searrow 0}\, (\bar{V}^f(\epsilon) - f(0))/\epsilon < 1, \tag{6.6}$$

where $\bar{V}^f(x)$ is the value associated to the single band strategy $\pi_{\beta_f^*(0)}$ at the level $\beta_f^*(0)$ that is defined in Eqn. (6.13) below.

The following result shows that in the setting of the stochastic control problem in Eqn. (6.1) the conditions in Eqns. (6.3)—(6.6) are satisfied.

Lem. 6.1. *If we have $v_{\pi_{b^*}}(x) < v_*(x)$ for some $x > b^*_+$, then the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = v_{b^*}(b^*_+ + x)$ satisfies the stated conditions in Eqns.* (6.3)—(6.6).

The proof of this lemma is given at the end of this section. Next we specify candidate optimal policies for the control problem in Eqn. (6.2). Since, as suggested in item (ii) above, we expect that it is not optimal to place a barrier at levels sufficiently close to 0, and consider the strategy $(\tau_a^{\pi_b}, \pi^b)$ to pay out dividends according the single dividend-band strategy π_b at levels (b_-, b_+) and to stop at the first moment $\tau_a^{\pi_b} = \inf\{t \geq 0 : U_t^{\pi_b} < a\}$ that U^{π_b} falls below the level $a > 0$. Another strategy that is worth considering in case $K > 0$ is to refrain from paying dividends and to stop at the first moment that the reserves process exits the interval $[a, b_+]$; we denote this strategy by $(\pi^\emptyset, T_{a,b_+})$. The value associated to the strategies $(\tau_a^{\pi_b}, \pi^b)$ and $(\pi^\emptyset, T_{a,b_+})$ is given by

$$V^f_{a,b_-,b_+}(x) = \mathbb{E}_x\left[\int_0^{\tau_a^{\pi_b}} \mathrm{e}^{-qt}\mu_K^b(\mathrm{d}t) + \mathrm{e}^{-q\tau_a^{\pi_b}} f\left(U^b_{\tau_a^{\pi_b}}\right)\right], \quad V^{f,\emptyset}_{a,b_+}(x) = \mathbb{E}_x\left[\mathrm{e}^{-qT_{a,b_+}} f\left(X_{T_{a,b_+}}\right)\right],$$

where $\mu_K^b = \mu_K^{\pi_b}$. In the following result, which can be derived by a line of reasoning similar to the one used in the proof of Prop. 5.1, the functions V^f_{a,b_-,b_+} and $V^{f,\emptyset}_{a,b_+}$ are explicitly expressed in terms of scale functions and of the families of functions $(y,z) \mapsto G^{(a)}_f(y,z)$, $G^{(a)}_{f,\emptyset}(y,z)$, $a \geq 0$, that are defined as follows:

$$G^{(a)}_f(b_-, b_+) = \begin{cases} \dfrac{b_+ - b_- - K - (F^{(a)}(b_+ - a) - F^{(a)}(b_- - a))}{W^{(q)}(b_+ - a) - W^{(q)}(b_- - a)}, & K > 0, \\ G^{(a)}_{f,\#}(b_+) := \dfrac{1 - F^{(a)\prime}(b_+ - a)}{W^{(q)\prime}(b_+ - a)}, & K = 0, \end{cases} \tag{6.7}$$

$$G^{(a)}_{f,\emptyset}(b_+) = \frac{f(b_+) - F^{(a)}(b_+ - a)}{W^{(q)}(b_+ - a)}, \tag{6.8}$$

where $F^{(a)} = F_{{}_af}$ is the Gerber-Shiu function for pay-off ${}_af = f(a + \cdot)$.

Prop. 6.2. *For any b_-, b_+, a such that $b_+ \geq b_- \geq a \geq 0$ the following holds true:*

$$V^f_{a,b_-,b_+}(x) = \begin{cases} f(x), & x \in [0, a), \\ W^{(q)}(x-a)G^{(a)}_f(b_-, b_+) + F^{(a)}(x - a), & x \in [a, b_+], \\ x - b_+ + V^f_{a,b_-,b_+}(b_+), & x \in (b_+, \infty); \end{cases} \tag{6.9}$$

$$V^{f,\emptyset}_{a,b_+}(x) = \begin{cases} f(x), & x \notin [a, b_+], \\ W^{(q)}(x-a)G^{(a)}_{f,\emptyset}(b_+) + F^{(a)}(x - a), & x \in [a, b_+]. \end{cases} \tag{6.10}$$

Next we turn to the determination of the candidate optimal levels. Focussing first on the case where dividends are paid and fixing a for the moment, we define $\beta^*_f(a) = \left(\beta^*_{f,-}(a), \beta^*_{f,+}(a)\right)$ to be the pair that maximizes $G^{(a)}_f$, similarly as was done in the case of the single dividend-band strategy. Hence, in the case $K > 0$ we set $\beta^*_{f,+}(a) = \beta^*_f(a, \delta^*_f(a)) + \delta^*_f(a)$ where

$$\begin{cases} \beta^*_f(a,d) = \sup\left\{b \geq a : G^{(a)}_f(b, b+d) \geq G^{(a)}_f(x, x+d) \ \ \forall x \geq 0\right\}, \\ \delta^*_f(a) = \sup\left\{d > 0 : G^{(a)}_f\left(\beta^*_f(a,d), \beta^*_f(a,d) + d\right) \geq G^{(a)}_f\left(\beta^*_f(a,y), \beta^*_f(a,y) + y\right) \ \ \forall y \geq 0\right\}, \end{cases}$$

while in the case $K = 0$, we define the levels $\beta^*_{f,+}(a) = \beta^*_{f,-}(a)$ by

$$\beta^*_{f,+}(a) = \beta^*_{f,-}(a) = \beta^*_{f,\#}(a) := \sup\left\{b \geq a : G^{(a)}_{f,\#}(b) \vee G^{(a)}_{f,\#}(b-) \geq G^{(a)}_{f,\#}(x) \ \ \forall x \geq 0\right\}.$$

The candidate optimal specification α^*_f of a is given by

$$\alpha^*_f = \inf\left\{a \geq 0 : G^{(a)}_f\left(\beta^*_{f,-}(a), \beta^*_{f,-}(a) + d^*_f(a)\right) > 0\right\}, \tag{6.11}$$

in the case $K > 0$, and by

$$\alpha_f^* = \inf\left\{a \geq 0 : G_{f,\#}^{(a)}\left(\beta_{f,\#}^*(a)-\right) \vee G_{f,\#}^{(a)}\left(\beta_{f,\#}^*(a)\right) > 0\right\}, \tag{6.12}$$

in the case $K = 0$, and we have

$$\beta_f^* = \left(\beta_{f,-}^*, \beta_{f,+}^*\right), \qquad \beta_{f,-}^* = \beta_{f,-}^*\left(a_f^*\right), \qquad \beta_{f,+}^* = \beta_{f,+}^*\left(a_f^*\right). \tag{6.13}$$

Next we turn to the strategy to continue without paying dividends and stop upon exiting a finite interval. It will turn out that in the case $K = 0$ such a strategy will never be optimal. In the case $K > 0$ we define

$$\beta_{f,\emptyset}^*(a) = \sup\left\{b \geq a : G_{f,\emptyset}^{(a)}(b) \geq G_{f,\emptyset}^{(a)}(x) \quad \forall x \geq 0\right\}, \tag{6.14}$$

$$\alpha_{f,\emptyset}^* = \inf\left\{a \geq 0 : G_{f,\emptyset}^{(a)}\left(\beta_{f,\emptyset}^*(a)\right) > 0\right\}, \ \beta_{f,\emptyset}^* = \beta_{f,\emptyset}^*(\alpha_{f,\emptyset}^*). \tag{6.15}$$

The levels $\beta_{f,+}^*$, α_f^*, $\beta_{f,\emptyset}^*$ and $\alpha_{f,\emptyset}^*$ given above are finite and strictly positive:

Lem. 6.3. *Suppose that f satisfies the conditions in Eqns.* (6.3)—(6.6) *and denote $\bar{w} = f|_{\mathbb{R}_-}$.*

(i) *In the case $K = 0$ we have $0 < \alpha_f^* \leq \beta_{f,+}^* < \infty$ and $G_{f,\#}^{(\alpha_f^*)}(\beta_f^*-) \vee G_{f,\#}^{(\alpha_f^*)}(\beta_f^*) = 0$. Furthermore, we have ${}_0\mathcal{L}_\infty^{\bar{w}} f(u) \leq 0$ for all $u \in \left(0, \alpha_f^*\right)$.*

(ii) *In the case $K > 0$ it holds $0 < \alpha_{f,\emptyset}^* \leq \beta_{f,\emptyset}^* < \infty$ and $G_{f,\emptyset}^{(\alpha_{f,\emptyset}^*)}(\beta_{f,\emptyset}^*) = 0$. If we have in addition $\alpha_f^* < \infty$, then it holds $0 < \alpha_f^* < \beta_{f,+}^* < \infty$ and $G_f^{(\alpha_f^*)}\left(\beta_f^*\right) = 0$. Furthermore, we have ${}_0\mathcal{L}_\infty^{\bar{w}} f(u) \leq 0$ for all $u \in \left(0, \alpha_{f,\emptyset}^*\right)$.*

(iii) *If either (a) $K > 0$ or (b) $K = 0$ and X has unbounded variation, then we have $\alpha_f^* < \beta_{f,+}^*$.*

The proof of Lem. 6.3 is given in Sect. 10.

Rem. 6.4. The choice of α_f^* coincides with what would be found by applying the *principles of continuous* and *smooth fit* from the theory of optimal stopping (see [40, Ch. IV.9]), which state that in this case it can be expected that v_* be *continuous* and *continuously differentiable* at the level α^* if α^* is irregular for $(-\infty, \alpha^*)$ or if α^* is regular for $(-\infty, \alpha^*)$ for U^{π_*}, where π_* denotes the optimal strategy, respectively. Since in our case α^* is regular for $(-\infty, \alpha^*)$ if and only if X has unbounded variation, the heuristic yields that

$$\alpha^* \text{ satisfies } V'_{\alpha^*,\beta^*}(\alpha^*+) = f'(\alpha^*-) \text{ if } X \text{ has unbounded variation,}$$

and

$$\alpha^* \text{ satisfies } V_{\alpha^*,\beta^*}(\alpha^*) = f(\alpha^*) \text{ if } X \text{ has bounded variation.}$$

The equation in the first display equation is equivalent to the expression in Eqn. (6.11) on account of the form of $V_{a,b}$ and the facts (i) $F'_{a\,f}(0+) = f'(a-)$ for any $a > 0$ and (ii) $W^{(q)\prime}(0+) \in (0, \infty]$. The equation in the second display can also be equivalently expressed as Eqn. (6.11), in view of (i) the form of V^f_{a,b_-,b_+} in Eqn. (6.9) and (ii) the fact that $W^{(q)}(0)$ is strictly positive precisely if X has bounded variation.

We next state the solution of the optimal control problem in Eqn. (6.2) for small levels of the reserves.

Thm. 6.5. *Suppose that f satisfies the conditions in Eqns.* (6.3)—(6.6).

(i) *When we have either $K = 0$ or $\{K > 0$ and $\alpha_{f,\emptyset}^* \geq \alpha_f^*\}$, it holds*

$$V_*^f(x) = V^f_{\alpha_f^*,\beta_f^*}(x) \qquad \text{for any } x \in \left[0, \beta_{f,+}^*\right].$$

Moreover, for $X_0 \in \left[0, \beta_{f,+}^*\right]$ it is optimal to adopt the policy $\left(\tau_{\alpha^*}^{\pi_{\beta^*}}, \pi_{\beta^*}\right)$.

(ii) *In the case* $\{K > 0$ *and* $\alpha^*_{f,\emptyset} < \alpha^*_f\}$, *then we have*

$$V^f_*(x) = V^{f,\emptyset}_{\alpha^*_{f,\emptyset},\beta^*_{f,\emptyset}}(x) \qquad \text{for any } x \in \left[0, \beta^*_{f,\emptyset}\right].$$

Moreover, for $X_0 \in \left[0, \beta^*_{f,\emptyset}\right]$ *it is optimal to adopt the policy* $\left(T_{\alpha^*_{f,\emptyset},\beta^*_{f,\emptyset}}, \pi^\emptyset\right)$.

Furthermore we have

$$V^f_*(x) = \begin{cases} f(x), & x \in [0, a^*), \\ F^{(a^*)}(x - a^*), & x \in [a^*, b^*], \end{cases} \tag{6.16}$$

where $F^{(a^*)} = F_{a^* f}$ *and* $(a^*, b^*) = \left(\alpha^*_f, \beta^*_{f,+}\right)$ *in the cases* $K = 0$ *or* $\{K > 0$ *and* $\alpha^*_{f,\emptyset} \geq \alpha^*_f\}$, *and* $(a^*, b^*) = \left(\alpha^*_{f,\emptyset}, \beta^*_{f,\emptyset}\right)$ *in the case* $\{K > 0$ *and* $\alpha^*_{f,\emptyset} < \alpha^*_f\}$.

Proof of Thm. 6.5. (i) Analogously as in the proof of Thm. 5.3 it follows from the definition of β^*_f and the form of the function $V = V^f_{\alpha^*_f,\beta^*_f}$ given in Prop. 6.2 that we have the inequality

$$V(x) - V(y) \geq x - y - K \tag{6.17}$$

for all $x, y \geq 0$ satisfying $x \geq y \geq \alpha^*_f$. Taking note of the fact $V'(x) = 1$ for $x \in (0, \alpha^*_f)$, we see that the inequality in the previous display is in fact valid for all x and y satisfying $x \geq y \geq 0$.

Next we verify that V satisfies the inequality

$$V(x) \geq f(x). \tag{6.18}$$

for all $x \geq 0$. To see why this relation holds true, we first note that we have $V(0) = f(0)$, as a consequence of the form of V (Prop. 6.2) and the fact $\alpha^*_f > 0$ (Lem. 6.3). In the case $K = 0$, Eqn. (6.18) is hence a special case of Eqn. (6.17) (with $y = 0$). In the case $\{K > 0$ and $\alpha^*_{f,\emptyset} \geq \alpha^*_f\}$, the definitions of $\alpha^*_{f,\emptyset}$, $\beta^*_{f,\emptyset}$ and $G^{(a)}_{f,\emptyset}$, the monotonicity of the map $x \mapsto W^{(q)}(x)$ and the fact $\beta^*_{f,+} \leq \beta^*_{f,\emptyset}$ (Lem. 6.3(ii)) imply

$$F^{(a)}(x - a) \geq f(x) \text{ for all } x \in \left[0, \beta^*_{f,\emptyset}(a)\right] \text{ and } a \in \left[0, \alpha^*_{f,\emptyset}\right],$$

which yields the inequality in Eqn. (6.18), in view of the facts $V(x) = F^{(a)}(x - a)$ for $x \leq b := \beta^*_{f,+}$ and $V(x) = V(b) + x - b$ for $x > b$ (Prop. 6.2 and Lem. 6.3(i)).

In view of the observations

$$\mathrm{e}^{-q\left(t \wedge T_{0,\alpha^*_f}\right)} f\left(X_{t \wedge T_{0,\alpha^*_f}}\right) \qquad \text{is a } \mathbb{P}_x\text{-supermartingale, for all } x \in \left(0, \alpha^*_f\right), \text{ and} \tag{6.19}$$

$$\mathrm{e}^{-q\left(t \wedge T^-_{\alpha^*_f}\right)} F^{(\alpha^*_f)}\left(X_{t \wedge T^-_{\alpha^*_f}} - \alpha^*_f\right) \qquad \text{is a } \mathbb{P}_x\text{-martingale, for all } x \geq \alpha^*_f, \tag{6.20}$$

and the pasting lemma, it follows that

$$\mathrm{e}^{-q\left(t \wedge T^-_0\right)} F^{(\alpha^*_f)}\left(X_{t \wedge T^-_0} - \alpha^*_f\right) \qquad \text{is a } \mathbb{P}_x\text{-super martingale, for all } x \in \mathbb{R}_+.$$

Here, the supermartingale property in Eqn. (6.19) follows from Lem. 6.3(i), reasoning as in the proof of Lem. 7.13 below, while the martingale property in Eqn. (6.20) follows from Prop. 4.4.

This supermartingale property and the inequalities in Eqns. (6.17) and (6.18) imply that $F^{(\alpha^*_f)}(x - \alpha^*_f)$ is a stochastic supersolution for the stochastic control problem in Eqn. (6.2). Since we have $V^f_{\alpha^*_f,\beta^*_f}(x) = F^{(\alpha^*_f)}(x - \alpha^*_f)$ for $x \leq \beta^*_f$, the assertion follows from Rem. 3.10.

(ii) The line of reasoning is analogous to the one in part (i), and omitted. □

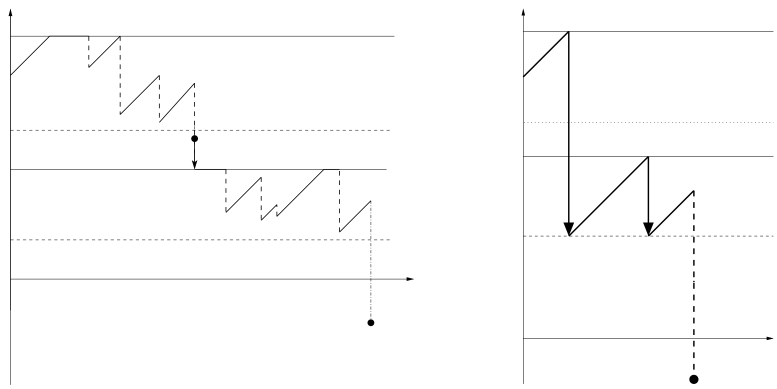

FIGURE 1. Illustrated in the figure on the left is a path of the risk process U^π in the absence of transaction cost ($K = 0$) for a three-bands strategy with the lowest level b_1^+ equal to zero. The figure on the right pictures a path of the risk process U^π in the case $K > 0$ and π is a two-bands strategy with $b_2^- = b_1^-$. The vertical dashed stretches represent the claims, while lump sum dividend payments are indicated by arrows. At the moment τ of ruin a penalty payment $w(U_\tau)$ is required that is a function of the shortfall U_τ

Proof of Lem 6.1. We verify that the conditions in Eqns. (6.3)—(6.6) are satisfied. Firstly, we note that the condition in Eqn. (6.3) holds since we have $v_{b^*}(x) = x - b_+^* + v_{b^*}(b_+^*)$ for $x > b_+^*$ and $v'_{b^*}(b_+^*-) = 1$. Furthermore, if the condition in Eqn. (6.5) does not hold then we have $v_{b^*} = v_*$ by Thm. 7.3 in Sect. 7 below. In view of the definition of b^* in Eqns. (5.8) and (5.9) (as *last* supremum) it follows that we have $G(b_-^*, b_+^* + c) - G(b_-^*, b_+^*) < 0$ for all $c > 0$. Hence, the identity in Eqn. (C.1) in the Appendix implies

$$F'_{v_{b^*}}(c) > 1 \text{ for all } c > 0, \tag{6.21}$$

so that $G_f^{(0)}(\beta_f^*(0)) < 0$. That the condition in Eqn. (6.6) is satisfied follows from the form of V^f_{a,b_-,b_+} in Eqn. (6.9) and the facts (from Thm. 4.5) $F_{b_+^* v_{b^*}}(0) = v_{b^*}(b_+^*)$, and $F'_{v_{b^*}}(0) = v'_{b^*}(b_+^*-) = 1$ (in the case $\{\sigma^2 > 0$ or $\nu_{0,1} = \infty\}$). $\square$

7. Multi dividend-band policies: the recursion for the dividend band levels

A flexible class of dividend strategies are the so-called multi dividend-bands strategies, that generalize the single and double dividend-bands strategies and are specified as follows:

Def. 7.1. The *multi dividend-bands strategy* $\pi_{\underline{a},\underline{b}}$, associated to sequences $\underline{a} = (a_n)_n$, $\underline{b}^- = (b_n^-)_n$, $\underline{b}^+ = (b_n^+)_n$ with $a_n, b_n^-, b_n^+ \in [0,\infty]$ satisfying the intertwining conditions

$$a_1 = 0 \le b_1^+ < a_2 \le b_2^+ < ... < a_n \le b_n^+ < \dots, \qquad b_n^- \le b_n^+,$$

is described as follows:

(i) When we have $U^{\underline{a},\underline{b}} := U^{\pi_{\underline{a},\underline{b}}} = y \in (b_n^+, a_{n+1})$, make a lump-sum payment $y - b_n^-$;

(ii) When we have $U^{\underline{a},\underline{b}} = b_n^+$ make a lump-sum payment $b_n^+ - b_n^-$, if $K > 0$, and pay the minimal amount to keep $U^{\underline{a},\underline{b}}$ below $b_n^- = b_n^+$ if $K = 0$;

(iii) While we have $U^{\underline{a},\underline{b}} \in [a_n, b_n^+)$ do not pay any dividends.

The strategy $\pi^{\underline{a},\underline{b}}$ is called an N-dividend-bands strategy if $b_N^+ < \infty = a_{N+1}$.

Analogously s in Rem. 3.7 it follows that in the case $K = 0$ a multi dividend-bands strategy $\pi_{\underline{a},\underline{b}}$ consists in paying out "the minimal amount to keep $U_t^{\underline{a},\underline{b}}$ below the boundary $b(t)$", where

$$b(t) = b^+_{\rho(t)} \qquad \text{with } \rho(t) = \min\{i \ge 1 : U_t^{\underline{a},\underline{b}} < a_i\}.$$

In this case the process $U^{\underline{a},\underline{b}}$ is equal to the process X is reflected at the level $b^+_{\rho(t)}$ and the corresponding cumulative dividend process $D^{\underline{a},\underline{b}}$ is equal to a local time of $U^{\underline{a},\underline{b}}$ at the boundary $b = (b(t))_{t\in\mathbb{R}_+}$. In the case of a positive fixed transaction cost K the "reflection boundaries" b^+_n widen to strips $[b^-_n, b^+_n]$ and the "local time" type payments are replaced by lump-sum payments $b^+_n - b^-_n$ where b^-_n may lie below a_{n-1} (see Figure 7).

7.1. **Optimality of single band policies.** A necessary and sufficient condition for the optimality of the single band policy $\pi_{\underline{b}^*}$ at levels $\underline{b} = b^*_1 = (b^*_-, b^*_+)$ defined in Eqns. (5.8)–(5.9) can be explicitly expressed in terms of the function $G^* : (b^*_-, \infty) \to \mathbb{R}$ given by

$$G^*(y) = G(b^*_-, y) = \begin{cases} \dfrac{y - b^*_- - K - (F(y) - F(b^*_-))}{W^{(q)}(y) - W^{(q)}(b^*_-)}, & \text{if } K > 0,\\ G^{\#}(x) = \dfrac{1 - F'(x)}{W^{(q)\prime}(x)}, & \text{if } K = 0. \end{cases} \tag{7.1}$$

We present next necessary and sufficient optimality conditions for single barrier policies, which generalize the results of [8]. These conditions are expressed in terms of the following function:

Def. 7.2. For $q \in \mathbb{R}_+$, $v \in \mathbb{R}_+$, the function $Z^{(q,v)} : \mathbb{R} \to \mathbb{R}$ is defined by $Z^{(q,v)}(x) = \mathrm{e}^{vx}$ for $x \le 0$ and, for $x > 0$, by

$$Z^{(q,v)}(x) = \mathrm{e}^{vx} + (q - \psi(v)) \int_0^x \mathrm{e}^{v(x-y)} W^{(q)}(y)\mathrm{d}y. \tag{7.2}$$

Recall that a function $f : (a, \infty) \to \mathbb{R}_+\backslash\{0\}$, $a \in \mathbb{R}$, is *completely monotone* if $(-1)^{k-1} f^{(k)}(x) \ge 0$ for all $k \in \mathbb{N}$ and $x > a$, where $f^{(k)}$ denotes the kth derivative with respect to x.

Thm. 7.3. *(i) The single-band policy $\pi_{\underline{b}^*}$ at level $\underline{b}^* = b^*_1$ is optimal for the stochastic control problem* (2.4) *if and only if*

$$_{b^*_+}(\mathcal{L}^w_\infty v_{b^*_+} - q\, v_{b^*_+})(x) \le 0,$$

or, alternatively, if and only if $\Xi^* : (\Phi(q), \infty) \to \mathbb{R}$ is completely monotone, where

$$\Xi^*(\theta) = -\frac{\mathrm{e}^{\theta b^*_+}}{\theta} \int_{(b^*_+,\infty)} \mathrm{e}^{-\theta z} Z^{(q,\theta)\prime}(z) G^*(\mathrm{d}z). \tag{7.3}$$

(ii) In particular, if G^* is non-increasing on (b^*_+, ∞), then the strategy π_{b^*} is optimal.

Thm. 7.3(ii) yields a useful very simple sufficient optimality condition:

Cor. 7.4. *(i) The unimodality of the function G^* implies the optimality of single dividend-band policies.*
(ii) In particular, in the case $K = 0$ and if $G^{\#}$ is monotone decreasing, then a "lump sum" strategy π_0 is optimal.

The proof of Thm. 7.3 is given in Sect. 10.2.

Rem. 7.5. In the absence of transaction cost ($K = 0$), the function Ξ^* in (7.3) can be equivalently expressed as

$$\Xi^*(\theta) = G^{\#}(b^*_+)L_0(\theta) + \frac{(\psi(\theta) - q)}{\theta^2}\, \mathbb{E}[F'(b^*_+ + \mathbf{e}_\theta) - F'(b^*_+)],$$

where $\mathbf{e}_\theta$ denotes an independent exponential random variable with mean θ^{-1} and $L_0 : \mathbb{R}_+\backslash\{0\} \to \mathbb{R}$ is given by

$$\theta \mapsto L_0(\theta) := \frac{\psi(\theta) - q}{\theta^2}\, \mathbb{E}[W^{(q)\prime}(b^*_+ + \mathbf{e}_\theta) - W^{(q)\prime}(b^*_+)].$$

In particular, if the penalty is zero and there are no transaction cost ($w = K = 0$), the necessary and sufficient optimality condition simplifies to the complete monotonicity of $L_0(\theta)$ on the interval $(\Phi(q), \infty)$. This observation appears new even in this particular case.

Rem. 7.6. The integral in Eqn. (7.3) is a Stieltjes integral with respect to the function G^*, which is equal to a difference of two monotone real-valued functions (cf. Lem. B.2(vi)).

Rem. 7.7 (Lump sum strategy)**.** In the absence of transaction cost ($K = 0$), the "lump sum" strategy π_0 is to "pay out all the reserves to the beneficiaries and subsequently pay all the premiums as dividends, until the moment of ruin." Note that π_0 is a single dividend-band strategy at level 0. In the case that X is given by the Cramér-Lundberg model, the first jump (claim) arrives after an exponential time $\mathbf{e}_\lambda$ with finite mean λ^{-1}, and the value v_0 of the liquidation strategy is equal to

$$
\begin{aligned}
v_0(x) &= \mathbb{E}_x\left[x + p\int_0^{\mathbf{e}_\lambda} \mathrm{e}^{-qt}\mathrm{d}t + \mathrm{e}^{-qT}w(\Delta X_{\mathbf{e}_\lambda})\right]\\
&= \mathbb{E}_x\left[x + \frac{p}{q}(1-\mathrm{e}^{-q\mathbf{e}_\lambda}) + \mathrm{e}^{-q\mathbf{e}_\lambda}(w(\Delta X_{\mathbf{e}_\lambda}) - w(0)) + w(0)\mathrm{e}^{-q\mathbf{e}_\lambda}\right]\\
&= \left[x + \frac{p}{\lambda+q} + \frac{1}{\lambda+q}w_\nu(0) + \frac{\lambda}{\lambda+q}w(0)\right] = \left[x + \frac{p + w_\nu(0) + \lambda w(0)}{\lambda+q}\right],
\end{aligned}
$$

where $\Delta X_{\mathbf{e}_\lambda} = X(\mathbf{e}_\lambda) - X(\mathbf{e}_\lambda-)$, and $w_\nu : \mathbb{R}_+\backslash\{0\} \to \mathbb{R}$ is defined by (2.3). If X_0 is zero and X has infinite activity, ruin occurs immediately if strategy π_0 is followed, that is, in this case $\tau^{\pi_0} = 0$, $\mathbb{P}_0$-a.s. and $v_0(x) = x + w(0)$.

Hence, the value of the lump-sum strategy is equal to $v_0(x) = (x+\gamma_w)\mathbf{1}_{\{x\geq 0\}} + w(x)\mathbf{1}_{\{x<0\}}$ where

$$
\gamma_w = v_0(0) = \begin{cases} \frac{1}{q+\overline{\nu}}\left[p + w_\nu(0) + \overline{\nu}\, w(0)\right], & \text{in the case } \overline{\nu} := \nu(\mathbb{R}_+) < \infty,\\ w(0), & \text{in the case } \overline{\nu} = \infty. \end{cases}
$$

If $G^{\#}$ is monotone decreasing, it attains it maximum over $\mathbb{R}_+$ at zero and the function Ξ is completely monotone, so that π_0 is optimal (Thm. 7.3(ii)).

Rem. 7.8 (Single barrier strategies at positive levels)**.** In the absence of transaction cost ($K = 0$) we will call a penalty $w \in \mathcal{R}$ *severe* if (i) $w(0) \leq \gamma_w := v_0(0)$, and (ii) $w(x+y) - w(y) \leq x$ for all $x, y \in \mathbb{R}_-$. Condition (i) states that the penalty payment for ruin occurring without shortfall is not smaller than the expected value minus transaction cost of liquidation (i.e. the sum of the expected premium income until the moment of ruin and the expected penalty payment), while condition (ii) implies that the additional penalty payment for an additional shortfall of size x is at least x.

We have the following explicit condition for optimality of a single barrier-strategy at a positive level:

Cor. 7.9. *In the case $\{K = 0$ and $b_1^* > 0\}$, if ν admits a convex density ν' and the penalty w is* severe*, then the strategy $\pi_{b_1^*}$ is optimal.*

The proof of Cor. 7.9 is given in Appendix C.

7.2. **Optimality of two-bands policies.** When a single dividend-band strategy is not globally optimal for the stochastic control problem in Eqn. (2.4), it is not optimal to pay out a lump-sum dividend at all levels above b_+^* and will be optimal to postpone paying dividends when the reserves process is in a certain region of (b_*^+, ∞). In this section we consider the case that it is optimal to adopt a two-bands policy with only one additional band. The candidate optimal two-bands strategy $\pi_{\underline{a}^*, \underline{b}^*}$ is fixed at the levels $\underline{a}^* = (0, a_2^*)$ and $\underline{b}^* = (b_1^*, b_2^*)$ where the levels $b_1^* = (b_-^*, b_+^*)$ associated to the first band have been defined in Eqns. (5.8)–(5.9), and where the levels associated to the second band are given by

$$
\{a_2^*, b_2^*\} = b_{1,+}^* + \begin{cases} \left\{\alpha_{w^*}^*, \left(\beta_{w^*,-}^*, \beta_{w^*,+}^*\right)\right\}, & \text{in the cases } K = 0 \text{ and } \{K > 0 \text{ and } \alpha_{w^*,\emptyset}^* \geq \alpha_{w^*}^*\};\\ \left\{\alpha_{v_{b_1^*},\emptyset}^*, \left(b_-^*, \beta_{w^*,\emptyset}^*\right)\right\}, & \text{in the case } \{K > 0 \text{ and } \alpha_{w^*,\emptyset}^* < \alpha_{w^*}^*\} \end{cases}
$$

where we denote

$$w^* := {}_{b^*_{1,+}}v_{b^*_1}$$

and the levels $\alpha^*_{w^*}$, $\alpha^*_{w^*,\emptyset}$, $\beta^*_{w^*,-}$, $\beta^*_{w^*,+}$ and $\beta^*_{w^*,\emptyset}$ have been defined in Eqns. (6.11)–(6.14).

Necessary and sufficient conditions for the two-bands policy $\pi_{\underline{a}^*,\underline{b}^*}$ to be (globally) optimal are expressed in terms of the functions Ξ^* defined in Eqn. (7.3) and the function

$$\Xi^{**} = \begin{cases} \Xi_{a^*_2,b^*_2}(w^*) & \text{in the cases } K=0 \text{ and } \{K>0 \text{ and } \alpha^*_{w^*,\emptyset} \geq \alpha^*_{w^*}\}, \\ \Xi^{\emptyset}_{a^*_2,b^*_2}(w^*) & \text{in the case } \{K>0 \text{ and } \alpha^*_{w^*,\emptyset} < \alpha^*_{w^*}\}, \end{cases}$$

where for any levels a, b_- and b_+ with $a \leq b_- \leq b_+$ and any $f \in \mathcal{P}$ the functions $\Xi_{a,b_-,b_+}(f)$ and $\Xi^{\emptyset}_{a,b_+}(f)$ are given by

$$\Xi_{a,b_-,b_+}(f) : \theta \mapsto -\frac{\mathrm{e}^{\theta b_+}}{\theta} \int_{(b_+,\infty)} \mathrm{e}^{-\theta z} Z^{(q,\theta)\prime}(z) G^{(a)}_{f,b_-}(\mathrm{d}z),$$

$$\Xi^{\emptyset}_{a,b_+}(f) : \theta \mapsto -\frac{\mathrm{e}^{\theta b}}{\theta} \int_{(b,\infty)} \mathrm{e}^{-\theta z} Z^{(q,\theta)\prime}(z) G^{(a)}_{f,\emptyset}(\mathrm{d}z),$$

with, for any $z \geq b_-$, $G^{(a)}_{f,b_-}(z) := G^{(a)}_f(b_-, z)$, where the functions $G^{(a)}_{f,\emptyset}$ and $G^{(a)}_f$ have been defined in Eqns. (6.8) and (6.7).

Before stating the optimality condition for this two-bands policy, we first state a condition for (global) optimality of the policies $\left(\tau^{\pi_{\beta^*_f}}_{\alpha^*_f}, \pi_{\beta^*_f}\right)$ and $\left(T_{\alpha^*_{f,\emptyset},\beta^*_{f,\emptyset}}, \pi^{\emptyset}\right)$ in the auxiliary stochastic optimal control problem in Eqn. (3.15).

Thm. 7.10. *Suppose that f satisfies the conditions in Eqns.* (6.3)—(6.6).

(i) *Suppose that we have either $K=0$ or $\{K>0$ and $\alpha^*_{f,\emptyset} \geq \alpha^*_f\}$. Then the strategy $(\tau^{\pi_{\beta^*_f}}_{\alpha^*_f}, \pi_{\beta^*_f})$ is optimal for the stochastic optimal control problem in Eqn.* (3.15) *if and only if the function $\Xi_{\alpha^*_f,\beta^*_{f,-},\beta^*_{f,+}}(f)$ is completely monotone.*

(ii) *Suppose that we have $\{K>0$ and $\alpha^*_{f,\emptyset} < \alpha^*_f\}$. Then the strategy $\left(T_{\alpha^*_{f,\emptyset},\beta^*_{f,\emptyset}}, \pi^{\emptyset}\right)$ is optimal for the stochastic optimal control problem in Eqn.* (3.15) *if and only if the function $\Xi_{\alpha^*_{f,\emptyset},\beta^*_{f,\emptyset}}(f)$ is completely monotone.*

The relationship between the stochastic control problems in Eqns. (2.4) and (3.15) (cf. discussion at the beginning of Sect. 6) immediately yields necessary and sufficient conditions for the two-bands strategy $\pi_{\underline{a}^*,\underline{b}^*}$ to be an optimal policy (from Thm. 7.10) and the form of the optimal strategy that should be applied when reserves are below $b^*_{2,+}$ (from Thm. 6.5).

Cor. 7.11. *(i) The two-bands strategy $\pi_{\underline{a}^*,\underline{b}^*}$ at finite levels $\underline{a} = (0, a^*_2)$ and $\underline{b} = (b^*_1, b^*_2)$ is optimal for the stochastic control problem in Eqn.* (2.4) *if and only if Ξ^* is* not *completely monotone and Ξ^{**} is completely monotone.*

(ii) If Ξ^* is not completely monotone then the levels a^*_2 and $b^*_{2,+}$ are finite, and it is optimal to adopt the two-bands strategy $\pi_{\underline{a}^*,\underline{b}^*}$ when $X_0 \in [0, b^*_{2,+}]$, and it holds (with $F^{(a^*_{2,+})}_* = F_{a^*_{2,+}v_*}$)

$$v_*(x) = \begin{cases} W^{(q)}(x)\frac{1-F'_w(b^*_{1,+})}{W^{(q)\prime}(b^*_{1,+})} + F_w(x), & x \in [0, b^*_{1,+}], \\ x - b^*_{1,+} + v_*(b^*_{1,+}), & x \in (b^*_{1,+}, a^*_{2,+}), \\ F^{(a^*_{2,+})}_*(x - a^*_{2,+}), & x \in [a^*_{2,+}, b^*_{2,+}]. \end{cases} \tag{7.4}$$

The proof of Thm. 7.10 is given in Sect. 10.2.

7.3. **Large levels of the reserves.** From the form of the infinitesimal generator $\mathcal{L}$ it can be deduced that the value-function is affine for large levels of the reserves.

Prop. 7.12. *Suppose that either $K = 0$ or $\{K > 0$ and $\nu(\mathbb{R}_+) < \infty\}$. Then, for sufficiently large levels of the reserves, it is optimal to immediately pay out a lump-sum dividend, and for some $y \in \mathbb{R}_+$, the function v_* restricted to $[y, \infty)$ takes the form*

$$v_*(x) = x - y + v_*(y) \text{ for any } x - y \in \mathbb{R}_+.$$

The proof rests on the following observation:

Lem. 7.13. *Suppose that the function $\ell_y : [y, \infty) \to \mathbb{R}$ defined by $\ell_y(x) = x - y + v_*(y)$ satisfies*

$$({}_y\mathcal{L}^{v_*}_\infty \ell_y)(x) \leq 0 \qquad \forall x > y.$$

Then $\{e^{-q(t\wedge T_y^-)}\ell_y(X_{t\wedge T_y^-}), t \in \mathbb{R}_+\}$ is a $\mathbb{P}_x$-supermartingale for all $x \geq y$.

Proof. An application of Itô's lemma (which is justified since $\ell_y \in C^2([y, \infty))$) shows

$$e^{-q(t\wedge T_y^-)}\ell_y(X_{t\wedge T_y^-}) - \int_0^{t\wedge T_y^-} e^{-qs}(\mathcal{L}\ell_y)(X_s)\mathrm{d}s \text{ is a } \mathbb{P}_x\text{-martingale, for } x \geq y, \tag{7.5}$$

Here we used that we have $\int_0^{T_y^-} \mathbf{1}_{\{X_s=y\}}\mathrm{d}s = 0$, $\mathbb{P}_x$-a.s. for all $x \geq y$, Lem. 7.13 and Eqn. (7.5) imply the asserted supermartingale property. □

Proof of Prop. 7.12. We claim that in the cases $K = 0$ and $\{K > 0$ and $\nu(\mathbb{R}_+) < \infty\}$ we have the inequality in Lem. 7.13 for all $y \in \mathbb{R}_+$ sufficiently large. The assertion in Prop. 7.12 follows by combining Prop. 9.5, Lem. 7.13 and the shifting lemma, Lem. 9.6 below.

We next show that the criterion in Lem. 7.13 is satisfied. Observe that, for any $x \in \mathbb{R}_+$,

$$\begin{aligned} ({}_y\mathcal{L}^{v_*}_\infty \ell_y)(x) &= \eta\ell_y'(x) - q\ell_y(x) + \int_{\mathbb{R}_+\setminus\{0\}} [\ell_y(x-z) - \ell_y(x) + z\ell_y'(x)]\nu(\mathrm{d}z) \\ &= \eta - q(x - y + v_*(y)) + \int_{(x-y,\infty)} [v_*(x-z) - v_*(y) + z + y - x]\nu(\mathrm{d}z). \end{aligned} \tag{7.6}$$

Since we have $v_*(x-z)-v_*(y) \leq x-z-y+K$ for all $z \in (x-y, x)$ on account of Lem. 9.1 below, and since w is non-positive, it follows that the integral term in (7.6) is bounded above by $K\nu(x-y, x)+\int_{[x,\infty)}(z+y-x-v_*(y))\nu(\mathrm{d}z)$. Moreover, since v_* is bounded by an affine function (Lem. 9.3(ii)) and $\nu_{1,\infty}$ is finite, the integral tends to zero when $x = y + a$ and y tend to infinity such that $a = x - y$ is fixed to be equal to a positive constant. As we have $v_*(y) \to \infty$ as $y \to \infty$ (Lem. 9.3) and $K\nu(a, \infty)$ is bounded under the assumptions (a) and (b), it follows that $({}_y\mathcal{L}^{v_*}_\infty \ell_y)(x)$ is strictly negative for all y sufficiently large and all $x > y$. □

7.4. **General solution of the stochastic control problem.** Repeatedly solving optimal control problems that are of the same form as the one in Eqn. (3.15) but with suitablly updated choices of reward functions f and successively applying Thm. 6.5, would suggest that the candidate optimal policy is given by a multi-dividend-bands strategy $\pi_{\underline{a}^*,\underline{b}^*}$ with levels $\underline{a}^*$, $\underline{b}^*$ specified in the recursive construction given below, and with corresponding value function $V_{\underline{a}^*,\underline{b}^*} = v_{\pi_{\underline{a}^*,\underline{b}^*}}$ given by

$$V_{\underline{a}^*,\underline{b}^*}(x) := \begin{cases} W^{(q)}(x)\, C_1^* + F_w(x), & x \in [a_{i-1}^*, b_{i,+}^*], i = 1, \\ x - b_{i,-}^* + V_{\underline{a}_*,\underline{b}_*}(b_{i,-}^*), & x \in (b_{i,+}^*, a_{i+1}^*), i \geq 1, \\ F_{f_i}(x - a_i^*), & x \in [a_i^*, b_{i,+}^*], i > 1, \end{cases} \tag{7.7}$$

where $f_i : \mathbb{R}_- \to \mathbb{R}$ is given by $f_i(x) = V_{\underline{a}_*,\underline{b}_*}(a^*_{i-1} + x), i > 1$, and

$$C^*_1 = G(b^*_{1,-}, b^*_{1,+}) \vee G(b^*_{1,-}, b^*_{1,+}-).$$

The recursive procedure for obtaining the candidate optimal levels $\underline{a}^*$, $\underline{b}^*_-$ and $\underline{b}^*_+$ is given as follows:

	Recursion to find the candidate optimal band levels
[0.]	**Set** $i \leftarrow 1$, $\underline{a}^* \leftarrow \{0\}$, $\underline{b}^* \leftarrow \{b^*\}$, $f \leftarrow {}_{b^*_+}v^*_b$ and $\Xi \leftarrow \Xi^*(f)$, where $\Xi^*(f)$ is given by Eqn. (7.3).
[1.]	**If** Ξ is completely monotone, **set** $\underline{a}^* \leftarrow \underline{a}^* \cup \{\infty\}$. **Return** $\{\underline{a}, \underline{b}\}$.
[2.]	**Else if** $K = 0$ or **if** $\{K > 0$ and $\alpha^*_{f,\emptyset} \geq \alpha^*_f\}$ **define** $\left(a^*_{i+1}, b^*_{i+1}\right) \leftarrow \left(b^*_{i,+} + \alpha^*_f, b^*_{i,+} + \beta^*_f\right)$, where the levels α^*_f and β^*_f are defined in Eqns. (6.11) and (6.13) above. **Else if** $\{K > 0$ and $\alpha^*_{f,\emptyset} < \alpha^*_f\}$ **define** $\left(a^*_{i+1}, b^*_{i+1}\right) \leftarrow \left(b^*_{i,+} + \alpha^*_{f,\emptyset}, \left\{b^{**}_{i,-}, b^*_{i,+} + \beta^*_{f,\emptyset}\right\}\right)$. with $b^{**}_{i,-} = \inf\left\{b^*_{i,-} : V_{\underline{a}^*,\underline{b}^*}(b^*_{i,+} + \beta^*_{f,\emptyset}) - V_{\underline{a}^*,\underline{b}^*}(b^*_{i,-}) = \beta^*_{f,\emptyset} + b^*_{i,+} - b^*_{i,-} - K\right\}$. where the levels $\alpha^*_{f,\emptyset}$ and $\beta^*_{f,\emptyset}$ are defined in Eqn. (6.14) above.
[3.]	**Set** $\underline{a}^* \leftarrow \underline{a} \cup \{a^*_{i+1}\}$, $\underline{b}^* \leftarrow \underline{b} \cup \{b^*_{i+1}\}$, $f \leftarrow {}_{b^*_{i+1,+}}V_{\underline{a}^*,\underline{b}^*}$, $\Xi \leftarrow \Xi_{\underline{a}^*,\underline{b}^*}(f)$, $i \leftarrow i + 1$.
[4.]	**Go to** step 1.

Rem. 7.14. There may exist a limit point $\gamma_* = \lim_{i\to\infty} b^*_{i,+} = \lim_{i\to\infty} a^*_i$ of the band levels. In that case the procedure will converge to the value-function $V_{\underline{\tilde{a}}^*,\underline{\tilde{b}}^*}$ corresponding to the levels $\underline{\tilde{a}}^* = (a^*_i)$, $\underline{\tilde{b}}^* = (b^*_i)$, and needs to be re-started as follows:

[0.′] **Set** $i \leftarrow 1$, $\underline{a}^* \leftarrow \underline{\tilde{a}}^*$, $\underline{b}^* \leftarrow \underline{\tilde{b}}^*$, $f \leftarrow {}_{\gamma^*}V_{\underline{\tilde{a}}^*,\underline{\tilde{b}}^*}$, $\Xi \leftarrow \Xi_{\underline{\tilde{a}}^*,\underline{\tilde{b}}^*}(f)$.

In the following result it is confirmed that the constructed candidate policy $\pi_{\underline{a}^*,\underline{b}^*}$ is indeed optimal:

Thm. 7.15. *The multi dividend-bands strategy $\pi_{\underline{a}^*,\underline{b}^*}$ is an optimal strategy for the control problem in Eqn.* (2.4) *and the optimal value function is given by $v^* = V_{\underline{a}^*,\underline{b}^*}$.*

Rem. 7.16. In Shreve *et al.* [46, p. 74] an explicit example is given of an optimal control problem in a diffusion setting in which a multi-dividend-bands strategy is optimal with countably many bands. Azcue & Muler [11] provide an example of an optimal strategy with infinitely many bands below a finite level, for the classical De Finetti's dividend problem with bounded dividend rates in the setting of a compound Poisson process. It is an open problem to construct an explicit example in which a multi-dividend-bands strategy with countably many bands is optimal in the dividend-penalty problem.

8. Examples

8.1. **Exponential and polynomial boundary conditions.** In the case that the penalty w is exponential, $w(x) = -e^{xv}$ for $x \in \mathbb{R}_-$, or polynomial, $w(x) = x^k$ for $x \in \mathbb{R}_-$ and $k \in \mathbb{N}$, the solutions of the two-sided and mixed absorbing/reflected exit problems from Sect. 4 can be expressed in terms of the function $Z^{(q,v)}$ specified in Def. 7.2 and closely related functions Z_k that are defined as follows:

Def. 8.1. With n the largest integer such that $\int_{-\infty}^{-1} |x|^n \nu(\mathrm{d}x) < \infty$, the related family of functions $Z_k : \mathbb{R} \to \mathbb{R}$, $k = 0, \ldots n$, is defined by

$$Z_k(x) = \frac{\partial^k}{\partial v^k}\bigg|_{v=0+} Z^{(q,v)}(x). \tag{8.1}$$

Note that, for any $q, v \in \mathbb{R}_+$, $Z^{(q,v)}|_{\mathbb{R}_+}$ is C^1, as a consequence of the continuity of $W^{(q)}|_{\mathbb{R}_+}$.

Let $e_v(x) := \mathrm{e}^{vx}\mathbf{1}_{\mathbb{R}_-}(x)$ denote an exponential pay-off, and $e_{v,a} :=_{-a} e_v$ the translated version. In the case $\delta = \beta = 0$ and with pay-off $w = e_{v,a}$ the functions $\mathcal{U}^w_{a,b}$ and $\mathcal{V}^w_{a,b}$ are given as follows (with proof given in Appendix B.4):

Prop. 8.2. *For $q \in \mathbb{R}_+$ and $v \in \mathbb{R}_+$, $Z^{(q,v)}$ is a Gerber-Shiu function with payoff $e_{v,a}$. In particular, the following hold true:*

$$\mathcal{V}^{a,b}_{e_{v,a}}(x) = \mathbb{E}_x\left[\mathrm{e}^{-qT_{a,b}+v(X_{T_{a,b}}-a)}\mathbf{1}_{\{T_a^- < T_b^+\}}\right] = Z^{(q,v)}(x-a) - W^{(q)}(x-a)\frac{Z^{(q,v)}(b-a)}{W^{(q)}(b-a)}, \tag{8.2}$$

$$\mathcal{U}^{a,b}_{e_{v,a}}(x) = \mathbb{E}_x\left[\mathrm{e}^{-q\tau_a+v(Y^a_{\tau_a}-a)}\right] = Z^{(q,v)}(x-a) - W^{(q)}(x-a)\frac{Z^{(q,v)\prime}(b-a)}{W^{(q)\prime}(b-a)}. \tag{8.3}$$

For use in the sequel we record the special case of the kth moment of the overshoot

$$m_k(x) := \mathbb{E}_x\left[\mathrm{e}^{-qT_{a,b}}(X_{T_{a,b}}-a)^k\mathbf{1}_{\{T_a^- < T_b^+\}}\right], \qquad \tilde{m}_k(x) := \mathbb{E}_x\left[\mathrm{e}^{-q\tau_a}(Y^a_{\tau_a}-a)^k\right], \tag{8.4}$$

which follows as a direct consequence of Prop. 8.2. If $E[|X_1|^k] < \infty$, then $\psi^{(r)}(0)$ and $m_r(x)$ are finite for $r = 1, \ldots, k$, and it follows that $m_k(x)$ and $\tilde{m}_k(x)$ are equal to the kth derivative of (8.2) and (8.3) with respect to v at $v = 0$. This implies the following form of $m_k(x)$ and $\tilde{m}_k(x)$:

Cor. 8.3. *Let $k \in \mathbb{N}$. Suppose that $\int_{-\infty}^{-1} |x|^k \nu(\mathrm{d}y) < \infty$. Then, for $x \in [a,b]$, $m_k(x)$ and $\tilde{m}_k(x)$ are finite, and are given by*

$$m_k(x) = Z_k(x-a) - W^{(q)}(x-a)\frac{Z_k(b-a)}{W^{(q)}(b-a)}, \quad \tilde{m}_k(x) = Z_k(x-a) - W^{(q)}(x-a)\frac{Z_k'(b-a)}{W^{(q)\prime}(b-a)}.$$

In particular, Z_k is a Gerber-Shiu function with payoff $w(x) = (x-a)^k$.

8.2. **General computations for processes with rational Laplace exponent.** The determination of the optimal policy start with the identification of the last global maximum of the barrier influence function G. For example, in the presence of an exponential penalty $w(x) = c\mathrm{e}^{vx}$ or a linear penalty $w(x) = cx + c_0$, we must compute the extrema of the functions

$$G^{(v)}(x) := \frac{1 - cZ^{(q,v)\prime}(x)}{W^{(q)\prime}(x)}, \qquad G_1(x) := \frac{1 - cZ_1'(x) - c_0 q W^{(q)}(x)}{W^{(q)\prime}(x)}, \tag{8.5}$$

respectively.

Therefore, our first step will be computing the homogeneous and generating scale functions $W^{(q)}(x)$, $Z^{(q,v)}(x)$, for processes with rational Laplace exponent. We will assume the typical case

$$W^{(q)}(x) = \sum A_i \mathrm{e}^{\zeta_i(q)x},$$

with $A_i \in \mathbb{R}$ and the roots $\zeta_i(q)$ of the Cramér-Lundberg equation $\psi(\zeta) = q$ being distinct.

This implies

$$Z^{(q,v)}(x) = \mathrm{e}^{vx}(1 + (q-\psi(v))\int_0^x \mathrm{e}^{-vy}W^{(q)}(y)\mathrm{d}y) = \mathrm{e}^{vx} + (q-\psi(v))\sum_i A_i\frac{\mathrm{e}^{\zeta_i(q)x} - \mathrm{e}^{vx}}{\zeta_i(q) - v} \tag{8.6}$$

$$= (\psi(v) - q)\sum_i \frac{A_i}{v - \zeta_i(q)}\mathrm{e}^{\zeta_i(q)x}, \tag{8.7}$$

where we have used that $\sum \frac{A_i}{v-\zeta_i(q)} = \frac{1}{\psi(v)-q}$. In particular,

$$Z^{(q)}(x) = q\sum_i A_i \frac{\mathrm{e}^{\zeta_i(q)x}}{\zeta_i(q)}, \tag{8.8}$$

$$Z_1(x) = \overline{Z}^{(q)}(x) - \psi'(0)\overline{W}^{(q)}(x) = q\sum_i A_i \frac{\mathrm{e}^{\zeta_i(q)x}}{\zeta_i^2(q)} - \psi'(0)\sum_i A_i \frac{\mathrm{e}^{\zeta_i(q)x}}{\zeta_i(q)}, \tag{8.9}$$

and

$$Z^{(q,v)}(x) = Z^{(q)}(x) + \sum_i A_i \mathrm{e}^{\zeta_i(q)x} \frac{v}{v-\zeta_i(q)}\left(\frac{\psi(v)}{v} - \frac{q}{\zeta_i(q)}\right).$$

The simplest examples may be completely analyzed by studying the sign of the functions that are given by $D^{\#}(x) = -G^{\#\prime}(x)W^{(q)\prime}(x)^2$, and $D^*(x) = -G^{*\prime}(x)W^{(q)\prime}(x)^2$, which determine the critical point b^* (in particular whether it is 0), and the eventual unimodality after b^*, which implies optimality of the single barrier policy. To alleviate notation, we will omit the $\#, *$ in this section, since the function considered can always be inferred from the absence/presence of transaction costs.

For exponential and affine penalties, we must compute therefore

$$\begin{aligned}
&D^{(v)}(x) = -G^{(v)\prime}(x)W^{(q)\prime}(x)^2 = W^{(q)\prime\prime}(x) + c(W^{(q)\prime}(x)Z^{(q,v)\prime\prime}(x) - W^{(q)\prime\prime}(x)Z^{(q,v)\prime}(x)),\\
&D_1(x) = -G_1'(x)W^{(q)\prime}(x)^2\\
&= W^{(q)\prime\prime}(x) - c(Z_1'(x)W^{(q)\prime\prime}(x) - Z_1''(x)W^{(q)\prime}(x)) - c_0(Z(x)'W^{(q)\prime\prime}(x) - Z''(x)W^{(q)\prime}(x))\\
&= W^{(q)\prime\prime}(x) - c(Z^{(q)}(x)W^{(q)\prime\prime}(x) - Z^{(q)\prime}(x)W^{(q)\prime}(x)) + (c\psi'(0) - c_0 q)(W^{(q)}(x)W^{(q)\prime\prime}(x) - (W^{(q)\prime}(x))^2),
\end{aligned}$$

which results in

$$\begin{aligned}
D^{(v)}(x) &= W^{(q)\prime\prime}(x)(1 - cZ^{(q,v)\prime}(x)) + cZ^{(q,v)\prime\prime}(x)W^{(q)\prime}(x)\\
&= \sum_j A_j\zeta_j(q)^2\mathrm{e}^{\zeta_j(q)x} + c(\psi(v)-q)\sum_j\sum_{k>j}\frac{\zeta_j(q)\zeta_k(q)(\zeta_j(q)-\zeta_k(q))^2}{(v-\zeta_j(q))(v-\zeta_k(q))}A_jA_k\mathrm{e}^{(\zeta_j(q)+\zeta_k(q))x},\\
D_1(x) &= \sum_j A_j\zeta_j(q)^2\mathrm{e}^{\zeta_j(q)x} - cq\sum_j\sum_{k>j}\frac{(\zeta_j(q)+\zeta_k(q))}{\zeta_j(q)\zeta_k(q)}(\zeta_j(q)-\zeta_k(q))^2A_jA_k\mathrm{e}^{(\zeta_j(q)+\zeta_k(q))x}\\
&\quad + (c\psi'(0) - c_0q)\sum_j\sum_{k>j}(\zeta_j(q)-\zeta_k(q))^2A_jA_k\mathrm{e}^{(\zeta_j(q)+\zeta_k(q))x}.
\end{aligned}$$

(Note that the coefficients of c and $c\psi'(0) - c_0q$ are the intervening Wronskians, and that the function $D^{(v)}(x) - W^{(q)\prime\prime}(x)$ is a generating function for the corresponding functions obtained with polynomial penalties).

Let us record also the form of the function $({}_b\mathcal{L}^f_\infty v - qv)(x)$ in case v is the linear value function following a continuation interval $x \geq b > a$,

$$\begin{aligned}
&({}_b\mathcal{L}^f_\infty v - qv)(x) = p - (\lambda+q)(x - b + v(b))\\
&+\left(\int_0^{x-b}(x-y-b+v(b))\nu(\mathrm{d}y) + \int_{x-b}^{x-a} v(x-y)\nu(\mathrm{d}y) + \int_{x-a}^{\infty} f(x-y)\nu(\mathrm{d}y)\right),
\end{aligned}$$

where $v(x) = v(x,a,b) = F(x) + G(a,b)W(x)$, is the value in the preceding continuation band, and $f(x)$ denotes the value function in previous bands (which is equal to the penalty w on the negative half-line), which is the cornerstone in the determination of the optimal dividend band policy. The three integrals correspond respectively to jumps above the barrier, in the preceding continuation band, and in the "preceding stopping domain".

8.3. **Cramér-Lundberg model with exponential jumps.** We analyze now the Cramér-Lundberg model (1.1) with exponential jump sizes with mean $1/\mu$, jump rate λ, and Laplace exponent $\psi(s) = ps - \lambda s/(\mu + s)$. The homogeneous scale function is:

$$W^{(q)}(x) = A_+ e^{\zeta^+(q)x} - A_- e^{\zeta^-(q)x},$$

where $A_\pm = p^{-1}\frac{\mu+\zeta^\pm(q)}{\zeta^+(q)-\zeta^-(q)}$, and $\zeta^+(q) = \Phi(q)$, $\zeta^-(q)$ are the largest and smallest roots of the polynomial $(\psi(s) - q)(s + \mu) = ps^2 + s(p\mu - \lambda - q) - q\mu$:

$$\zeta^\pm(q) = \frac{q + \lambda - \mu p \pm \sqrt{(q + \lambda - \mu p)^2 + 4pq\mu}}{2p}.$$

Hence, we find

$$\begin{aligned}
Z^{(q)}(x) &= q\left(\frac{A_+}{\zeta^+(q)} e^{\zeta^+(q)x} - \frac{A_-}{\zeta^-(q)} e^{\zeta^-(q)x}\right) = \mu^{-1}\left(\zeta^+(q)A_- e^{\zeta^-(q)x} - \zeta^-(q)A_+ e^{\zeta^+(q)x}\right)\\
&= \frac{(q - \zeta^-(q))e^{\zeta^+(q)x} + (\zeta^+(q) - q)e^{\zeta^-(q)x}}{\zeta^+(q) - \zeta^-(q)},\\
Z^{(q,v)}(x) &= Z^{(q)}(x) + \lambda \frac{v}{v+\mu} \frac{e^{\zeta^+(q)x} - e^{\zeta^-(q)x}}{\zeta^+(q) - \zeta^-(q)},\\
D^{(v)}(x) &= W^{(q)\prime\prime}(x) + c\frac{\psi(v) - q}{(v - \zeta^+(q))(v - \zeta^-(q))} A_+A_-(\zeta^+(q) - \zeta^-(q))^2(-(\zeta^+(q)\zeta^-(q)))e^{(\zeta^+(q)+\zeta^-(q))x}\\
&= \alpha_+ e^{\zeta^+(q)x} - \alpha_- e^{\zeta^-(q)x} + c\alpha_v e^{(\zeta^+(q)+\zeta^-(q))x},
\end{aligned}$$

where

$$\begin{aligned}
\alpha_+ &= A_+(\zeta_+(q))^2 > 0, \qquad \alpha_- = A_-(\zeta_-(q))^2 > 0, \qquad C = (\mu + \zeta_+(q))(\mu + \zeta_-(q)) = \frac{\lambda\mu}{p} > 0,\\
\alpha_v &= \frac{p}{v+\mu}\frac{C}{p^2}\frac{q\mu}{p} = \frac{\lambda q\mu^2}{p^3}\frac{1}{v+\mu} > 0.
\end{aligned}$$

Then, differentiating $v \mapsto Z^{(q,v)}(x)$, $v \mapsto \alpha_v$ or by (8.9) and using that $(\zeta^+(q) + \zeta^-(q))/(\zeta^+(q)\zeta^-(q)) = \psi'(0)/q - 1/\mu$, we find

$$\begin{aligned}
Z_1(x) &= \lambda\mu^{-1}\frac{e^{\zeta^+(q)x} - e^{\zeta^-(q)x}}{\zeta^+(q) - \zeta^-(q)} = C_+ e^{\zeta^+(q)x} + C_- e^{\zeta^-(q)x},\\
D_1(x) &= \alpha_+ e^{\zeta^+(q)x} - \alpha_- e^{\zeta^-(q)x} + \alpha_1 e^{(\zeta^+(q)+\zeta^-(q))x},
\end{aligned}$$

where $C_\pm = \pm\lambda\mu^{-1}(\zeta^+(q) - \zeta^-(q))^{-1}$ and

$$\begin{aligned}
\alpha_1 &= A_+A_-(\zeta^+ - \zeta^-)^2(cq\frac{\zeta^+ + \zeta^-}{\zeta^+\zeta^-} - c\psi'(0) + c_0 q)\\
&= \frac{C}{p^2}(c_0 q - c\frac{q}{\mu}) = \frac{\lambda q}{p^3}(c_0\mu - c).
\end{aligned}$$

Let us recall now that in the absence of penalty and costs ($w(x) = K = 0$), the function $W^{(q)\prime}(x) = G(x)^{-1}$ is unimodal [8] with global minimum at

$$b^* = \frac{1}{\zeta^+(q) - \zeta^-(q)}\begin{cases}\log\frac{\zeta^-(q)^2(\mu+\zeta^-(q))}{\zeta^+(q)^2(\mu+\zeta^+(q))}, & \text{in the case } W^{(q)\prime\prime}(0) < 0 \Leftrightarrow (q+\lambda)^2 - p\lambda\mu < 0,\\ 0, & \text{in the case } W^{(q)\prime\prime}(0) \ge 0 \Leftrightarrow (q+\lambda)^2 - p\lambda\mu \ge 0.\end{cases}$$

(Since $W^{(q)\prime\prime}(0) \sim \zeta^+(q)^2(\mu + \zeta^+(q)) - \zeta^-(q)^2(\mu + \zeta^-(q))/(\zeta^+(q)) - \zeta^-(q)) = (q+\lambda)^2 - p\lambda\mu$, the optimal strategy is always the barrier strategy at level b^*).

We show next that the functions $G^{(v)}$ and G_1 continue to be unimodal when w is exponential or affine and $K=0$, as a consequence of the Lemma 8.4 below, and hence single barrier policies continue to be optimal, in view of Lem. 7.4 (in the case of affine penalties this has already been established in [36, 9]).

Lem. 8.4. *Let $\alpha_i, \lambda_i \in \mathbb{R}$, $i=1,2,3$ satisfy $\alpha_1 > 0 > \alpha_3$, and $\lambda_1 > \lambda_2 > \lambda_3$. Then the function $f(x) := \sum_{i=1}^3 \alpha_i \mathrm{e}^{\lambda_i x}$ has a unique root c^* of $f(c^*)=0$, and it holds $f'(c^*)>0$, and*

$$f(x) > 0 \qquad \text{for all } x > c^*.$$

Furthermore, if $h:\mathbb{R}_+ \to \mathbb{R}$ is such that $h'(x) = k(x)f(x)$ for $x>0$, where $k:\mathbb{R}_+ \to \mathbb{R}_+\backslash\{0\}$, then h is unimodal.

Proof. The function $g(x) := \mathrm{e}^{-\lambda_3 x} f(x)$ tends to $+\infty$ and to $\alpha_3 < 0$ as $x \to \pm\infty$. If we have $\alpha_2 \geq 0$, g is strictly convex and strictly increasing. In the case $\alpha_2 < 0$, g attains a minimum at the unique root of g'. In both cases the equation $g(c)=0$ admits a unique root c, and it holds $g'(c) > 0$. Hence we have that c is a unique root of $f(c)=0$, with $f'(c)>0$ and with $f(x)>0$ for $x>c$. In particular, h has a unique stationary point where it attains a maximum, so that it is unimodal. □

Let us next characterize the optimal level b^*.

(1) For $K=0$ and in the case of an exponential penalty, $b^*_{v,+}=0$ iff

$$G^{(v)\prime}(0) \leq 0 \Leftrightarrow (q+\lambda)^2 - \lambda\mu p \geq -c\lambda q \frac{\mu^2}{v+\mu},$$

as follows from the expression for $D^{(v)}(x)$. Similarly, in the case of linear penalty, it holds $b^*_{1,+}=0$ iff

$$G_1'(0) \leq 0 \Leftrightarrow (q+\lambda)^2 - \lambda\mu p \geq \lambda q(c - c_0\mu),$$

in view of the expression for $D_1(x)$. If b^*_+ is positive, it is a stationary point, and hence solves the equation

$$G^{(v)\prime}(b) = 0 \Leftrightarrow 0 = D^{(v)}(b) = \alpha_+ \mathrm{e}^{\zeta^+(q)b} - \alpha_- \mathrm{e}^{\zeta^-(q)b} + c\alpha_v \mathrm{e}^{(\zeta^+(q)+\zeta^-(q))b},$$

if the penalty w is exponential and

$$G_1'(b) = 0 \Leftrightarrow 0 = D_1(b) = \alpha_+ \mathrm{e}^{\zeta^+(q)b} - \alpha_- \mathrm{e}^{\zeta^-(q)b} + \alpha_1 \mathrm{e}^{(\zeta^+(q)+\zeta^-(q))b},$$

if w is an affine penalty.

(2) Suppose next $K>0$. Then b^*_+ is strictly positive as a consequence of the positive transaction cost K, and the optimal levels (b^*_-, b^*_+) are given by $(b^*_-, b^*_- + d^*)$ where (b,d) maximizes over $(b,d) \in \mathbb{R}_+ \times \mathbb{R}_+\backslash\{0\}$ the function

$$\widetilde{G}^{(v)} : (b,d) \mapsto \frac{d - K - B_+ \mathrm{e}^{\zeta^+(q)b}(\mathrm{e}^{\zeta^+(q)d}-1) + B_- \mathrm{e}^{\zeta^-(q)b}(\mathrm{e}^{\zeta^-(q)d}-1)}{A_+ \mathrm{e}^{\zeta^+(q)b}(\mathrm{e}^{\zeta^+(q)d}-1) - A_- \mathrm{e}^{\zeta^-(q)b}(\mathrm{e}^{\zeta^-(q)d}-1)}$$

if w is an exponential penalty, and the function

$$\widetilde{G}_1 : (b,d) \mapsto \frac{d - K - C_+ \mathrm{e}^{\zeta^+(q)b}(\mathrm{e}^{\zeta^+(q)d}-1) + C_- \mathrm{e}^{\zeta^-(q)b}(\mathrm{e}^{\zeta^-(q)d}-1)}{A_+ \mathrm{e}^{\zeta^+(q)b}(\mathrm{e}^{\zeta^+(q)d}-1) - A_- \mathrm{e}^{\zeta^-(q)b}(\mathrm{e}^{\zeta^-(q)d}-1)}$$

if w is an affine penalty.

The following result sums up the form of the optimal dividend policy:

Lem. 8.5. *Consider a Cramér-Lundberg process (1.1) with exponential jump sizes with mean $1/\mu$, and fixed cost $K \geq 0$. The optimal dividend policy is given by a* single dividend-band strategy π_{b^*} *for the following Gerber-Shiu penalties w:*

a) Exponential penalties: $w(x) = c\mathrm{e}^{xv}$, $c<0$.

(i) *In the case $\{K=0$ and $(q+\lambda)^2 - \lambda\mu p \geq -c\lambda q\frac{\mu^2}{v+\mu}\}$, then $b^*=0$.*

(ii) *In the case* $\{K=0$ *and* $(q+\lambda)^2-\lambda\mu p < -c\lambda q\frac{\mu^2}{v+\mu}\}$, *then* b^* *is the unique solution* $b\in\mathbb{R}_+\backslash\{0\}$ *of the equation* $D^{(v)}(b)=0$.

(iii) *In the case* $K>0$, *we have* $b_+^*=b_-^*+d^*$ *where* b_-^* *and* d^* *maximize over* $b\geq 0$, $d>0$, *the function* $\widetilde{G}^{(v)}$.

b) Affine penalties: $w(x)=cx+c_0$, $c\geq 0$.

(i) *In the case* $\{K=0$ *and* $(q+\lambda)^2-\lambda\mu p\geq\lambda q(c-c_0\mu)\}$, *then we have* $b^*=0$.

(ii) *In the case* $\{K=0$ *and* $(q+\lambda)^2-\lambda\mu p<\lambda q(c-c_0\mu)\}$, *then* b^* *is the unique solution* $b\in\mathbb{R}_+\backslash\{0\}$ *of the equation* $D_1(b)=0$.

(iii) *In the case* $K>0$, *we have* $b_+^*=b_-^*+d^*$ *where* $b_{1,-}^*\geq 0$ *and* $d^*>0$ *maximize over* (b,d), *the function* $\widetilde{G}_1$.

8.4. **Cramér-Lundberg model with Erlang jumps.** Suppose next that X is given by the Cramér-Lundberg model (1.1) with the Erlang (n,μ) jump sizes. The corresponding Laplace exponent is equal to $\psi(s)=ps+\frac{\lambda\mu^n}{(\mu+s)^n}-\lambda$, and by Laplace inversion it follows that its q-scale function is given by

$$W^{(q)}(x)=\sum_{j=0}^{n}A_j\mathrm{e}^{\zeta_j(q)x},\qquad x>0,$$

where $\zeta_0(q)>0>\zeta_1(q)>-\mu>\zeta_2(q)>\ldots$ are the $n+1$ roots of the Cramér-Lundberg equation $\psi(\zeta)=q$, and

$$A_j=\frac{(\zeta_j(q)+\mu)^n}{p\prod_{k\neq j}(\zeta_j(q)-\zeta_k(q))}.$$

Let $K=0$ and $w(x)=c\mathrm{e}^{vx}$ an exponential penalty ($c<0$), and denote by b the point where $G^{(v)}$ attains its maximum. In general a single dividend-band strategy may not be optimal. A necessary and sufficient criterion for optimality of π_b is the complete monotonicity of the function $\Xi_v:(\Phi(q),\infty)\to\mathbb{R}_+$ given by

$$\Xi_v(s)=\frac{\psi(s)-q}{s}\cdot\mathrm{e}^{sb}\int_b^\infty\mathrm{e}^{-sz}\left(W^{(q)\prime}(z)G^*(b)-[1-F'(z)]\right)\mathrm{d}c=I(s)\cdot W^{(q)\prime}(b)^{-1}\,I_v(s),$$

where

$$\begin{aligned}
I(s) &= s^{-1}\left[ps+\frac{\lambda\mu^n}{(\mu+s)^n}-\lambda-q\right],\\
I_v(s) &= I_0(s)-c\sum_{j>i}\frac{(\zeta_j(q)-\zeta_i(q)^2(v-\zeta_i(q)-\zeta_j(q))}{(s-\zeta_j(q))(s-\zeta_i(q))(v-\zeta_j(q))(v-\zeta_i(q))}A_jA_i\mathrm{e}^{(\zeta_i(q)+\zeta_j(q))b},\\
I_0(s) &= \int_0^\infty\mathrm{e}^{-sx}[W^{(q)\prime}(b+x)-W^{(q)\prime}(b)]\mathrm{d}x=\sum_{j=0}^{n}A_j\frac{\zeta_j(q)^2}{s(s-\zeta_j(q))}\mathrm{e}^{\zeta_j(q)b}.
\end{aligned}$$

If in addition there is no penalty ($w=0$), the expressions simplify. If b denote the minimum of $W^{(q)\prime}$, π_b is optimal precisely if $\Xi_0:(\Phi(q),\infty)\to\mathbb{R}_+$ is completely monotone, where

$$\Xi_0(s)=I(s)\cdot I_0(s).$$

The Azcue-Muler example. Let us consider the example in Azcue and Muller [10], with pure Erlang claims of order $n=2$, with $\mu=1$, $\lambda=10$, $p=\frac{107}{5}$, $q=\frac{1}{10}$, $\theta=\frac{7}{100}$ and Laplace exponent $\psi(s)-q=ps+\lambda(\frac{\mu}{\mu+s})^2-\lambda-q=\frac{p}{(\mu+s)^2}(s+\zeta_1)(s+\zeta_2)(s-\zeta_0)$, with $\zeta_0\approx 0.0396$, $\zeta_1\approx 0.0794$, $\zeta_2\approx 1.4882$. In addition we consider a linear penalty $w(x)=cx$, $c\in\mathbb{R}_+$. We will analyze below four particular cases $c\in\{0,0.2,0.6,1.0\}$. In cases $c\in\{0.6,1.0\}$ the optimal strategy is a single dividend band strategy at level b_0, while in the cases $c\in\{0,0.2\}$ it is optimal to adopt a two-band stratregy with $b_0=0$ (in the case $c=0$ we thus recover the form of the optimal strategy found in [10]) . The parameters of the three optimal strategies are summarised in the following table (with v_1 denoting the difference of the value function and the identity $x\mapsto x$ at the end of the non-empty continuation band):

	b_0	v_1	a_1	b_1
$c=0$	0	≈ 2.44	≈ 1.83	≈ 10.45
$c=0.2$	0	≈ 1.72	≈ 1.90	≈ 10.47
$c=0.6$	≈ 10.96	≈ 1.71	∞	∞
$c=1.0$	≈ 11.37	≈ 1.30	∞	∞

In the cases $c \in \{0.6, 1\}$ a plot of the function G_1 defined in (8.5) reveals that G_1 is monotone decreasing on the right of the level at which attains its unique global maximum which implies the optimal strategy is a single-dividend band strategy at this level (Thm. 7.3). In the cases $c \in \{0, 0.2\}$ a plot of G_1 shows that this function attains its global maximum at 0 but also attains a second local maximum at some strictly positive level. The optimal value function in these cases is given by

$$v(x) = \begin{cases} x + v_0, & a_0 = 0 \le x < a_1, \\ F_1(x - a_1), & x \in [a_1, b_1], \\ x + v_1, & x > b_1. \end{cases}$$

Here $v_1 = -b_1 + F_1(b_1 - a_1)$ and $v_0 = \frac{p-20c}{q+\lambda} = \frac{214-200c}{101}$ is the value of the strategy (at zero) of paying all premiums as dividends until the moment the first claim arrives, which is also the moment of ruin, and $F_1(x)$ is given by

$$F_1(x) = p(a_1 + v_0)W^{(q)}(x) - \int_0^x W^{(q)}(x-y)[f_{\nu,a_1}(y)]\mathrm{d}y,$$

with

$$f_{\nu,a}(y) = \int_0^a (a - z + v_0)k(y+z)]\mathrm{d}z + c\int_a^\infty (a-z)k(y+z)\mathrm{d}z,$$

where $k(y) = \lambda\mu^2 y \mathrm{e}^{-\mu y}$ denotes the Lévy density at y.

The function v is the value function of a two-band strategy at levels (b_0, a_1, b_1) with $b_0 = 0$. The unknowns a_1, b_1 are determined by the optimality equations $F_1'((b_1 - a_1)-) = 1$ and $F_1''((b_1 - a_1)-) = 0$ which yield the following system of two non-linear equations for a_1 and b_1,

$$\begin{aligned} 1 &= p(a_1+v_0)W^{(q)\prime}(b_1-a_1) - p^{-1}f_{\nu,a_1}(b_1) - \int_0^{b_1-a_1} W^{(q)\prime}(b_1-a_1-y)f_{\nu,a_1}(y)\mathrm{d}y \\ 0 &= p(a_1+v_0)qW^{(q)\prime\prime}(b_1-a_1) - p^{-1}f_{\nu,a_1}'(b_1) - W^{(q)\prime}(0)f_{a_1,\nu}(b_1) - \int_0^{b_1-a_1} W^{(q)\prime\prime}(b_1-a_1-y)f_{\nu,a_1}(y)\mathrm{d}y, \end{aligned}$$

with $W_+^{(q\prime)}(0) = \frac{101}{10}\cdot\frac{25}{107^2}$. The two-band strategies at the levels $(a_1, b_1) = (1.83, 10.45)$ $[c=0]$ and $(a_1, b_1) = (1.90, 10.47)$ $[c=0.2]$ are indeed optimal since it holds $({}_{b_1}\mathcal{L}^w_\infty v - qv)(y) \le 0$ for all $y > b_1$ and $({}_0\mathcal{L}^w_\infty v - qv)(y) \le 0$ for all $y \in (0, a_1)$.

9. Proofs of the stochastic solution approach

This section is devoted to the proofs of Thms. 3.4 and 3.8 and Cor. 3.9.

9.1. Properties of the value function.

Before proceeding to the proofs of Thms. 3.4 and 3.8 and Cor. 3.9, we collect a number of properties of the value function v_* for later reference.

Lem. 9.1. *(i) For every $x, y \ge 0$, with $x \ge y$, it holds*

$$(x - y - K) \le v_*(x) - v_*(y). \tag{9.1}$$

(ii) The function $x \mapsto v^*(x)$ is continuous on $\mathbb{R}_+$.

Proof of Lem. 9.1. (i) Let $x > y$. Denote by $\pi_\epsilon(y)$ an ϵ-optimal strategy for the case $U_0 = y$. Then a possible strategy is to immediately pay out $x - y$ and subsequently to adopt the strategy $\pi_\epsilon(y)$, so that the following holds:

$$v_*(x) \geq x - y - K + v_{\pi_\epsilon}(y) \geq v_*(y) - \epsilon + x - y - K.$$

Since this inequality holds for any $\epsilon > 0$, the lower bound in Eqn. (9.1) follows.

(ii) To prove the stated continuity we first establish an upper bound for the difference $v_*(x) - v_*(y)$ with $x > y$. Let $\tilde{\pi}_\epsilon(x)$ denote an ϵ-optimal strategy for the case $U_0 = x$ for a given $\epsilon > 0$. Then a possible strategy is to refrain from paying any dividends until the first time that the reserves hit the level x, and to subsequently follow the policy $\tilde{\pi}_\epsilon$. Hence the following bound holds:

$$v_*(y) \geq \frac{W^{(q)}(y)}{W^{(q)}(x)}(v_{\tilde{\pi}_\epsilon}(x) - F_w(x)) + F_w(y).$$

Rearranging and letting ϵ tend to zero yields the upper-bound

$$v_*(x) - v_*(y) \leq \left(1 - \frac{W^{(q)}(y)}{W^{(q)}(x)}\right)[v_*(x) - F_w(x)] + F_w(x) - F_w(y), \qquad x \geq y. \tag{9.2}$$

In the case $K = 0$ the bounds in Eqns. (9.1) and (9.2) yield that v_* is continuous on $\mathbb{R}_+$. In the case $K > 0$ continuity of v_* on $\mathbb{R}_+$ follows by combining the upper bound in Eqn. (9.2) with a different lower bound that we derive next.

For fixed $\epsilon > 0$ and given initial reserves $U_0 = y$ for some $y > x$, a possible strategy is to adopt $\tilde{\pi}_\epsilon(x)$ until the first moment that the reserves U fall below $\delta := y - x$, and to follow then a waiting strategy π_w (of not paying any dividends). Then we have, with $\pi = \tilde{\pi}_\epsilon(x)$,

$$\begin{aligned} v_*(y) - v_*(x) &\geq \mathbb{E}_y\left[\int_0^{\tau_\delta^\pi} \mathrm{e}^{-qt}\mu_K^\pi(\mathrm{d}t) + \mathrm{e}^{-q\tau_\delta^\pi} w\left(U^\pi_{\tau_\delta^\pi}\right)\mathbf{1}_{\{\tau_\delta^\pi = \tau_0^\pi\}} + \mathrm{e}^{-q\tau_\delta^\pi} v_{\pi_w}\left(U^\pi_{\tau_\delta^\pi}\right)\mathbf{1}_{\{\tau_\delta^\pi < \tau_0^\pi\}}\right] - v_*(x) \\ &= v_\pi(x) - v_*(x) + \mathbb{E}_y\left[\mathrm{e}^{-q\tau_\delta^\pi}\left(w\left(U^\delta_{\tau_\delta^\pi}\right) - w\left(U^\delta_{\tau_d^\pi} - \delta\right)\right)\mathbf{1}_{\{\tau_\delta^\pi = \tau_0^\pi\}}\right] + f_\epsilon(x,y) \\ &\geq -\epsilon + f_\epsilon(x,y), \qquad y \geq x, \end{aligned}$$

where we denoted $\tau_\delta^\pi = \inf\{t \geq 0 : U_t^\pi < \delta\}$ and

$$f_\epsilon(x,y) = \mathbb{E}_y\left[\mathrm{e}^{-q\tau_\delta^\pi}\left(v_{\pi_w}\left(U^\pi_{\tau_\delta^\pi}\right) - w\left(U^\pi_{\tau_\delta^\pi} - \delta\right)\right)\mathbf{1}_{\{\tau_\delta^\pi < \tau_0^\pi\}}\right],$$

and where we used the monotonicity of w. We claim that $f_\epsilon(x,y)$ tends to zero when $\delta = y - x$ tends to 0. Given this claim and the bound in Eqn. (9.2) it follows that we have (since ϵ was arbitrary)

$$\liminf_{|x-y|\to 0}[v_*(y) - v_*(x)] \geq 0. \tag{9.3}$$

Similarly, it follows $\limsup_{|x-y|\to 0}[v_*(y) - v_*(x)] \leq 0$. Combining the two limits yields that $v_*(x)$ is continuous at each $x \in \mathbb{R}_+$.

Finally, we turn to the proof of the claim that $f_\epsilon(x,y)$ tends to zero. We have the estimate

$$f_\epsilon(x,y) \leq \left(\sup_{x\in[0,\delta]} v_{\pi_w}(x) - w(-\delta)\right)\mathbb{E}_y[\mathrm{e}^{-q\tau_\delta^\pi}\mathbf{1}_{\{\tau_\delta^\pi < \tau_0^\pi\}}]. \tag{9.4}$$

If X has unbounded variation, then we have $v_{\pi_w}(0) = w(0)$, so that the left-continuity of w at zero, the right-continuity of $\mathcal{V}_w^{0,\infty}(y)$ at $y = 0$ and the fact $v_{\pi_w} = \mathcal{V}_w^{0,\infty}$ combined with the inequality in Eqn. (9.4) imply $f_\epsilon(x,y) \to 0$ when $\delta = y - x \to 0$. If X has bounded variation, $v_{\pi_w}(0)$ is (in general) not equal to $w(0)$, and we show that the second factor in Eqn. (9.4) tends to zero if $\delta \to 0$. Note that the policy $\tilde{\pi}_\epsilon(x)$, being element of $\Pi = \Pi_K$, consists of only finitely many dividends payments almost surely. Denoting the times of the dividend payments by $\tau_1, \tau_2, \ldots$, and the values of $U^{\tilde{\pi}_\epsilon(x)}$ at those times by $U_1, U_2, \ldots$, the strong Markov property of X implies

$$\begin{aligned}\mathbb{E}_y[\mathrm{e}^{-q\tau^\pi_\delta}\mathbf{1}_{\{\tau^\pi_\delta<\tau^\pi_0\}}] &= \sum_i \mathbb{E}_y[\mathrm{e}^{-q\tau^\pi_\delta}\mathbf{1}_{\{\tau^\pi_\delta<\tau^\pi_0,\tau^\pi_\delta\in[\tau_i,\tau_{i+1})\}}]\\ &\leq \sum_i \mathbb{E}_y[\mathrm{e}^{-q\tau_i}\mathbf{1}_{\{\tau_i<\tau^\pi_0\}}\mathbb{E}_{U_i}[\mathrm{e}^{-qT^-_\delta}\mathbf{1}_{\{T^-_\delta<T^-_0\}}]].\end{aligned}$$

As X has bounded variation, we have $\mathbb{P}_x(X(T^-_\delta) < \delta) = 1$ for all $x \in [\delta,\infty)$ so that it follows that, for any $x \in [\delta,\infty)$, the probability $\mathbb{P}_x(T^-_d < T^-_0) = \mathbb{P}_x(0 < X(T^-_\delta) < \delta)$ tends to zero as δ tends to zero. By the bounded convergence theorem it follows that the right-hand side of the previous display converges to zero when δ tends to 0. This completes the proof of the claim in Eqn. (9.3) □

In the following result additional regularity of the value-function v^* is established in the case $K = 0$.

Lem. 9.2. *In the case $K = 0$, $v^*(x)$ is right-differentiable at any $x > 0$, and $x \mapsto v^{*\prime}_+(x)$ is right-continuous on $(0,\infty)$.*

Proof. Let $x > 0$ be arbitrary. Following the procedure in Sect. 7.4 a stochastic supersolution $\widetilde{g}$ for the HJB equation in Eqn. (3.3) can be constructed that is equal to the value-function of some admissible strategy on the interval $[0, x+1]$. Cor 3.5 then implies that v^* is equal to $\widetilde{g}$ on this interval $[0, x+1]$. In view of the form of $\widetilde{g}$ (see Eqn. (7.7)) and the fact that the q-scale function $W^{(q)}$ and Gerber-Shiu function F_w are right-differentiable at any $x > 0$ with a derivative that is right-continuous (Lem B.2), the proof of the assertion is complete. □

Lem. 9.3. *(i) For any $q > 0$, and $w \in \mathcal{P}$, there exists a $C \in \mathbb{R}_+\backslash\{0\}$ such that the following bound holds true:*

$$\sup_{x\in\mathbb{R}_+}\sup_{\pi\in\Pi}\mathbb{E}_x\left[\mathrm{e}^{-q\tau}w(U^\pi_\tau)\right] \geq -C.$$

Furthermore, for any $x \in \mathbb{R}_+$, we have

$$\mathbb{E}_x\left[\sup_{t\in\mathbb{R}_+,\pi\in\Pi}\left\{\mathrm{e}^{-qt}U^\pi_t\mathbf{1}_{\{t<\tau^\pi\}} + \int_0^t \mathrm{e}^{-qs}\mathrm{d}D^\pi_s + \int_0^t \mathrm{e}^{-qs}X_s\mathrm{d}s\right\}\right] < \infty, \tag{9.5}$$

with $\overline{X}_t = \sup_{s\leq t} X_s$ denoting the supremum of X_s over the $s \in [0,t]$.

(ii) v_* is dominated by an affine function: for any $x \in \mathbb{R}_+$, $v_*(0) - K \leq v_*(x) - x \leq \frac{1}{\Phi(q)}$, and the process $V^\pi = \{V^\pi_t, t \in \mathbb{R}_+\}$ defined in Eqn. (3.2) is UI under $\mathbb{P}_x$.

Proof. (i) The following bounds hold true:

$$\sup_{t\in\mathbb{R}_+}\mathrm{e}^{-qt}U^\pi_t\mathbf{1}_{\{t<\tau^\pi\}} \leq \sup_{t\in\mathbb{R}_+}\mathrm{e}^{-qt}X_t \leq \sup_{t\in\mathbb{R}_+}\int_t^\infty q\mathrm{e}^{-qs}\overline{X}_s\mathrm{d}s. \tag{9.6}$$

Since the running supremum $\overline{X}_{\eta(q)}$ at an independent exponential random time with mean q^{-1} follows an exponential distribution with parameter $\Phi(q)$ (e.g. [13, Cor. VII.2]), the expectation under $\mathbb{P}_x$ of the expression on the rhs of Eqn. (9.6) is bounded by $x + 1/\Phi(q)$.

The compensation formula applied to the Poisson point process $(\Delta X_t, t \in \mathbb{R}_+)$, the monotonicity of w and the fact that $w(0)$ is non-positive yield that the following inequalities holds true, for any $x \in \mathbb{R}_+$:

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-q\tau^\pi}w(U^\pi_{\tau^\pi})\right] &\geq w(-1) + \mathbb{E}_x\left[\mathrm{e}^{-q\tau^\pi}w(U^\pi_{\tau^\pi})\mathbf{1}_{\{U^\pi_{\tau^\pi}<-1\}}\right]\\ &= w(-1) + \int_0^\infty\int_0^\infty w(y-z)\mathbf{1}_{\{y-z<-1\}}\nu(\mathrm{d}z)\tilde{R}^q_x(\mathrm{d}y)\\ &\geq w(-1) + \int_1^\infty\int_0^\infty w(-z)\tilde{R}^q_x(\mathrm{d}y)\nu(\mathrm{d}z) \geq w(-1) + \frac{1}{q}\int_1^\infty w(-z)\nu(\mathrm{d}z),\end{aligned} \tag{9.7}$$

where $\tilde{R}^q_x(\mathrm{d}y)$ denote the q-potential measure of U^π under $\mathbb{P}_x$,

$$\tilde{R}^q_x(\mathrm{d}y) = \int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(U^\pi_t \in \mathrm{d}y, t < \tau^\pi).$$

The rhs of (9.7) is bounded below, since the bound in Eqn. (2.1) holds as w is element of $\mathcal{P}$.

(ii) In the case $K = 0$ integration by parts, the non-negativity of w and the condition (1.6) of "no exogeneous ruin" imply that

$$\begin{aligned} v_\pi(x) &\leq \mathbb{E}_x\left[\int_{[0,\tau^\pi)} \mathrm{e}^{-qt}\mathrm{d}D^\pi_t\right] = \mathbb{E}_x\left[\int_0^{\tau^\pi} q\mathrm{e}^{-qs}D^\pi_s\mathrm{d}s + \mathrm{e}^{-q\tau^\pi}D^\pi_{\tau^\pi}\right] \\ &\leq \mathbb{E}_x\left[\int_0^{\tau^\pi} q\mathrm{e}^{-qs}X_s\mathrm{d}s + \mathrm{e}^{-q\tau^\pi}X_{\tau^\pi-}\right] \leq \mathbb{E}_x\left[\int_0^\infty q\mathrm{e}^{-qs}\overline{X}_s\mathrm{d}s\right] = x + \frac{1}{\Phi(q)}, \end{aligned}$$

where we used $\overline{X}_{\eta(q)} \sim \mathrm{Exp}(\Phi(q))$. In the case $K > 0$, then the above bound remains valid since the value $v_*(x)$ decreases if the transaction cost K increases. The lower bound for the value-function in (ii) follows from Eqn. 9.1 in Lem. 9.1 (with $x = 0$). The uniform integrability of V^π follows by virtue of the fact that V^π is dominated by an integrable random variable, in view of Eqn. (9.5) and the bounds in Lem. 9.3(ii). □

9.2. **Controlled representation of the value function.** The proof of the dual representation in Thm. 3.4 is based on an alternative representation of v_* as the point-wise minimum of a class of "controlled" supersolutions of the stochastic control problem.

Def. 9.4. For any $y \in \mathbb{R}_+$ and $z \in \mathbb{R}_+ \cup \{\infty\}$ with $y < z$, a Borel-measurable function $H : \mathbb{R} \to \mathbb{R}$ is called a *controlled supersolution on the interval* $[y, z]$ *(in the case* $z < \infty$*) or on* $[y, \infty)$ *(in the case* $z = \infty$*)* for the stochastic control problem in Eqn. (2.4) if we have

$$\mathrm{e}^{-q(\tau^\pi_{y,z}\wedge t)}H(U^\pi_{\tau^\pi_{y,z}\wedge t}) + \int_0^{\tau^\pi_{y,z}\wedge t} \mathrm{e}^{-qs}\mu^\pi_K(\mathrm{d}s) \text{ is a UI } \mathbb{P}_x\text{-super martingale, for any } x \in [y, z),\ \pi \in \Pi,$$

where $\tau^\pi_{y,z} = \inf\{t \geq 0 : U^\pi_t \notin [y, z]\}$, with boundary condition

$$\begin{cases} H(x) \geq v_*(x) & \text{for } x \in (-\infty, y), \text{ and for } x = z \text{ in the case } z < \infty, \\ H(y) \geq v_*(y) & \text{in the case } \sigma^2 > 0 \text{ or } \nu_1 = \infty. \end{cases}$$

The family of such functions will be denoted by $\mathcal{H}_{y,z}$.

Prop. 9.5. *For any* $y \in \mathbb{R}_+$ *and* $z \in \mathbb{R}_+ \cup \{\infty\}$ *with* $y < z$ *the value-function* v_* *restricted to* $[y, z]$ *(in the case* $z < \infty$*) or to* $[y, \infty)$ *(in the case* $z = \infty$*) admits the following representation:*

$$v_*(x) = \min_{H\in\mathcal{H}_{y,z}} H(x) \qquad \textit{for all } x \in [y, z).$$

The proof of the representation of the value function v_* in Prop. 9.5 rests on the fact that for any admissible policy $\pi \in \Pi$ and any "uncontrolled" supermartingale there exists a corresponding "controlled" supermartingale.

Lem. 9.6 (Shifting lemma)**.** *(i) Let* $g \in \mathcal{G}^+$*. If*

(9.8) $$\overline{M}^g = \left\{\overline{M}^g_t = \mathrm{e}^{-q(t\wedge T_0^-)}g\left(X_{t\wedge T_0^-}\right), t \in \mathbb{R}_+\right\} \textit{ is a UI } \mathbb{P}_x\textit{-supermartingale for all } x \in \mathbb{R}_+,$$

then, for any $\pi \in \Pi$ *and* $x \in \mathbb{R}_+$*,* $\widetilde{M}^{g,\pi} = \{\widetilde{M}^{g,\pi}_t, t \in \mathbb{R}_+\}$ *is a UI* $\mathbb{P}_x$*-supermartingale, where*

(9.9) $$\widetilde{M}^{g,\pi}_t = \mathrm{e}^{-q(t\wedge\tau^\pi)}g(U^\pi_{t\wedge\tau^\pi}) + \int_0^{\tau^\pi\wedge t} \mathrm{e}^{-qs}\mu^\pi_K(\mathrm{d}s).$$

(ii) Let $g \in \mathcal{G}^{+}_{a,b}$ *for some* $a, b \in \mathbb{R}_+$, $a < b$. *If the stopped process*

$$\overline{M}^{g,T_{a,b}} = \left\{ \overline{M}^{g}_{t\wedge T_{a,b}}, t \in \mathbb{R}_+ \right\} \text{ is a UI } \mathbb{P}_x\text{-supermartingale,}$$

then, for any $\pi \in \Pi$ *and* $x \in \mathbb{R}_+$

$$\widetilde{M}^{g,\pi,\tau^{\pi}_{a,b}} = \left\{ \widetilde{M}^{g,\pi}_{t\wedge\tau^{\pi}_{a,b}}, t \in \mathbb{R}_+ \right\} \text{ is a UI } \mathbb{P}_x\text{-supermartingale,}$$

where $\tau^{\pi}_{a,b} = \inf\{t \geq 0 : U^{\pi}_t \notin [a,b]\}$.

(iii) Let $g \in \mathcal{G}^+ \cap \mathcal{G}^-$. *If the process* $\overline{M}^g$ *given in Eqn. (9.8) is a UI* $\mathbb{P}_x$*-martingale for all* $x \in \mathbb{R}_+$, *then the process* $\widetilde{M}^{g,\pi_*}$ *with* $\pi_* = \pi(g_*)$ *is a UI* $\mathbb{P}_x$*-martingale for all* $x \in \mathbb{R}_+$.

Proof of Prop. 9.5. Fix $x, y \in \mathbb{R}_+$ and $z \in \mathbb{R}_+ \cup \{\infty\}$, and let H be any element of $\mathcal{H}_{y,z}$, and $\pi \in \Pi$ any admissible policy. The supermartingale property with boundary condition in Def. 9.4, and the uniform integrability yield the following:

$$\begin{aligned} H(x) &\geq \lim_{t\to\infty} \mathbb{E}_x \left[\mathrm{e}^{-q(\tau^{\pi}_{y,z}\wedge t)} H(U^{\pi}_{\tau^{\pi}_{y,z}\wedge t}) + \int_0^{\tau^{\pi}_{y,z}\wedge t} \mathrm{e}^{-qs} \mu^{\pi}_K(\mathrm{d}s) \right] \\ &\geq \mathbb{E}_x \left[\mathrm{e}^{-q\tau^{\pi}_{y,z}} v_*(U^{\pi}_{\tau^{\pi}_{y,z}}) + \int_0^{\tau^{\pi}_{y,z}} \mathrm{e}^{-qs} \mu^{\pi}_K(\mathrm{d}s) \right]. \end{aligned}$$

Taking the supremum over $\pi \in \Pi$ and using the dynamic programming equation (Prop 3.1) show that $H(x) \geq v_*(x)$. Since $H \in \mathcal{H}_{y,z}$ was arbitrary, it holds thus

$$\inf_{H\in\mathcal{H}_{y,z}} H(x) \geq v_*(x).$$

The inequality in the display is in fact an equality since v_* is a member of $\mathcal{H}_{y,z}$, by virtue of the fact that V^{π} is a UI supermartingale in view of Prop. 3.1, Lem. 9.3 and Doob's optional stopping theorem. □

9.3. **Proof of the shifting lemma 9.6.** The proof of the shifting lemma is based on the following auxiliary result:

Lem. 9.7. *Let* $a > 0$ *and* $x \in [0, a]$ *be given and suppose that the function* $g : \mathbb{R} \to \mathbb{R}$ *is such that* $g|_{\mathbb{R}_-} \in \mathcal{P}$, $g|_{\mathbb{R}_+}$ *is continuous and* g *is right-differentiable at* $a > 0$. *If* $M = \{M_t, t \in \mathbb{R}_+\}$ *with*

$$M_t = \mathrm{e}^{-q(t\wedge T_{0,a})} g(X_{t\wedge T_{0,a}})$$

is a $\mathbb{P}_x$*-martingale, then* $Z = \{Z_t, t \in \mathbb{R}_+\}$ *with*

$$Z_t = \mathrm{e}^{-q(t\wedge\tau_0)} g\left(Y^a_{t\wedge\tau_0}\right) - g(Y^a_0) - g'_+(a) \int_0^{t\wedge\tau_0} \mathrm{e}^{-qs} \mathrm{d}\overline{X}^a_s$$

is a $\mathbb{P}_x$*-martingale, where* $g'_+(a)$ *denotes the right-derivative of* g *at* a, $\overline{X}^a = \sup_{s\leq t}(X_s - a) \vee 0$, *and* $Y^a = \overline{X}^a - X$.

The proof of this result rests on an application of Itô's lemma and a density argument. Details are omitted since these follow straightforwardly from [39, Prop. 1].

Proof of Lem. 9.6, part (i). Fix arbitrary $X_0 = x \in \mathbb{R}_+$ and $\pi \in \Pi$ and $s, t \in \mathbb{R}_+$ with $s < t$, and denote

$$\widetilde{M}^{g,\pi}_t = A_t + B_t = \left(\mathrm{e}^{-qt} g(U^{\pi}_t) \mathbf{1}_{\{t<\tau^{\pi}\}}\right) + \left(\mathrm{e}^{-q\tau^{\pi}} w(U^{\pi}_{\tau^{\pi}}) \mathbf{1}_{\{\tau^{\pi}\leq t\}} + \int_0^{\tau^{\pi}\wedge t} \mathrm{e}^{-qs} \mu^{\pi}_K(\mathrm{d}s) \right). \tag{9.10}$$

Since g is continuous, $\widetilde{M}^{g,\pi}_t$ is $\mathcal{F}_t$-measurable. Also, the collection of random variables $\{A_t, t \in \mathbb{R}_+\}$ is bounded below by a finite constant (since g is bounded, in view of the inequality in Eqn. (3.7)) and is locally bounded above with localisation denoted by $(\tilde{T}_m)$ (since U^{π}_t has no positive jumps and g is continuous), and the collection $\{B_t, t \in \mathbb{R}_+\}$ is dominated by an integrable random variable (by Lemma 9.3).

Consider the sequence $(\pi_n)_{n\in\mathbb{N}}$ of strategies defined by $\pi_n = \{D^{\pi_n}_t, t \in \mathbb{R}_+\}$ with

$$D^{\pi_n}_u = \begin{cases} \sup\{D^\pi_v : v \le u, v \in \mathbb{T}_n\}, & u < \tau^\pi, \\ D^{\pi_n}_{\tau^\pi -}, & u \ge \tau^\pi, \end{cases} \qquad \mathbb{T}_n := \left(\left\{t_k := t - (t-s)\frac{k}{2^n}, k \in \mathbb{N}\cup\{0\}\right\} \cup \{0\}\right) \cap \mathbb{R}_+.$$

Note that the dividend process D^{π_n} satisfies $D^{\pi_n} \le D^\pi$ and is constant on the interval $[\tau^\pi, \infty)$. The remainder of the proof rests on the following martingale property:

Lem. 9.8. *For every $n \in \mathbb{N}$, $M^{(n)} = \{\widetilde{M}^{g,\pi_n}_{u\wedge\tau^\pi} : u \in \mathbb{T}_n\}$ is a $\mathbb{P}_x$-supermartingale.*

Given this result we have the following inequalities:

$$\mathbb{E}\left[\widetilde{M}^{g,\pi}_t \middle| \mathcal{F}_{s\wedge\tau^\pi}\right] \overset{(a)}{\le} \liminf_{n\to\infty} \mathbb{E}\left[\widetilde{M}^{g,\pi_n}_t \middle| \mathcal{F}_{s\wedge\tau^\pi}\right] \overset{(b)}{\le} \liminf_{n\to\infty} \widetilde{M}^{g,\pi_n}_{s\wedge\tau^\pi} \overset{(c)}{=} \widetilde{M}^{g,\pi}_{s\wedge\tau^\pi} \overset{(d)}{=} \widetilde{M}^{g,\pi}_s.$$

This series of (in)equalities can be seen to hold for the following reasons: (a) On account of the form of π_n, it follows $D^{\pi_n} \nearrow D^\pi$ as n tends to infinity, which, combined with the Monotone Convergence Theorem (MCT) and an integration-by-parts, implies $\int_0^{\tau^\pi\wedge t} \mathrm{e}^{-qs}\mathrm{d}D^{\pi_n}_s \nearrow \int_0^{\tau^\pi\wedge t} \mathrm{e}^{-qs}\mathrm{d}D^\pi_s$. Also, we have in the case $K > 0$, $\int_0^{\tau^\pi\wedge t} \mathrm{e}^{-qs}\mathrm{d}N^{\pi_n}_s \nearrow \int_0^{\tau^\pi\wedge t} \mathrm{e}^{-qs}\mathrm{d}N^\pi_s$. Thus, by continuity of the function g we have

$$\widetilde{M}^{g,\pi_n}_{t\wedge\tau^\pi} \longrightarrow \widetilde{M}^{g,\pi}_{t\wedge\tau^\pi} \qquad \text{as } n \to \infty. \tag{9.11}$$

Since, in Eqn. (9.10), A_t is bounded below and B_t is dominated, Lebesgue's dominated convergence theorem and Fatou's lemma imply that the inequality (a) holds true. Inequality (b) follows by the supermartingale property in Lem. 9.8 and the fact that the grid $\mathbb{T}_n$ contains both s and t for each $n \in \mathbb{N}$. Equality (c) is a consequence of the pointwise convergence in Eqn. (9.11) (which also holds with t replaced by s) while (d) follows since we have $\widetilde{M}^{g,\pi}_s = \widetilde{M}^{g,\pi}_{s\wedge\tau^\pi}$ (by definition of the process $\widetilde{M}^{g,\pi}$ in Eqn. (9.9)). Since s and t are arbitrary, it follows that $\widetilde{M}^{g,\pi}$ is a $\mathbb{P}_x$-supermartingale. □

Next we turn to the proof of the Lemma 9.8.

Proof of Lem. 9.8. Denoting $T_i := \tilde{T}_m \wedge \tau^\pi \wedge t_i$ (where $\tilde{T}_m$ is the localization used in part (i) of the proof of Lem. 9.6) and $M = M^{(n)}(\cdot \wedge \tilde{T}_m)$, $D = D^{\pi_n}(\cdot \wedge \tilde{T}_m)$, we can write

$$M_t - M_s = \sum_{i=1}^{2^n} Y_i + \sum_{i=1}^{2^n} Z_i,$$

where $Z_i = \mathrm{e}^{-qT_i}(g(X_{T_i} - D_{T_i}) - g(X_{T_i} - D_{T_{i-1}}) + \Delta D_{T_i} - K)\mathbf{1}_{\{\Delta D_i > 0\}}$ with $\Delta D_i = D_{T_i} - D_{T_{i-1}}$ and

$$Y_i = \mathrm{e}^{-qT_i} g\left(X_{T_i} - D_{T_{i-1}}\right) - \mathrm{e}^{-qT_{i-1}} g\left(X_{T_{i-1}} - D_{T_{i-1}}\right).$$

The strong Markov property of X and the definition of U imply

$$\begin{aligned} \mathbb{E}[Y_i|\mathcal{F}_{T_{i-1}}] &= \mathrm{e}^{-qT_{i-1}}\mathbb{E}\left[\mathrm{e}^{-q(T_i - T_{i-1})} g\left(U_{T_{i-1}} + X_{T_i} - X_{T_{i-1}}\right) - g\left(U_{T_{i-1}}\right) \middle| \mathcal{F}_{T_{i-1}}\right] \\ &= \mathrm{e}^{-qT_{i-1}}\mathbb{E}_{U_{T_{i-1}}}\left[\mathrm{e}^{-q\tau_i} g\left(X_{\tau_i}\right) - g(X_0)\right], \end{aligned} \tag{9.12}$$

where we denoted $\tau_i = T_i \circ \theta_{T_{i-1}}$ with θ the translation-operator. The right-hand side of Eqn. (9.12) is non-positive as a consequence of the supermartingale property in Eqn. (9.8) and the discrete time version of Doob's stopping theorem. Furthermore, in view of Eqn. (3.7) it follows that all the Z_i are non-positive (in the case $T_i = \tau^\pi$ the positivity can be seen to hold since we have, by construction, $\Delta D^{\pi_n}(\tau_n^+) = 0$ where $\tau_n^+ = \sup\{v \le \tau^\pi : v \in \mathbb{T}_n\}$).

Hence, the tower-property of conditional expectation yields

$$\mathbb{E}[M_t - M_s|\mathcal{F}_s] \le \sum_{i=1}^{2^n} \mathbf{1}_{\{T_{i-1} > s\}} \mathbb{E}\left[\mathbb{E}[Y_i|\mathcal{F}_{T_{i-1}}]|\mathcal{F}_s\right] \le 0,$$

and we deduce that $M = M^{(n)}(\cdot \wedge \tilde{T}_m)$ is a supermartingale. Letting $m \to \infty$ and applying Fatou's lemma and the Lebesgue's Dominated Convergence Theorem (in view of the decomposition in Eqn. (9.10)), we may remove the localisation $\widetilde{T}_m$, and conclude that $M^{(n)}$ is a supermartingale. □

Proof of Lem. 9.6, part (ii). Apply the reasoning of part (i) to the process $\widetilde{M}^{g,\pi,\tau_{a,b}}$. □

Proof of Lem. 9.6, part (iii). The proof is a modification of the proof of part (i). Note that the set of different epochs $\tilde{\mathbb{T}}$ at which lump-sum dividend payments occur is countable:

$$\tilde{\mathbb{T}} = \{\tilde{T}_i : \Delta D_{\tilde{T}_i} > 0\} \text{ with } \tilde{T}_i = \inf\{t > \tilde{T}_{i-1} : X_t^{\pi(g)} - D_{t-}^{\pi(g)} \in \mathcal{D}_g\},\ i \in \mathbb{N} \text{ with } \tilde{T}_0 = 0$$

and $\inf \emptyset = \infty$. Indeed, in the case $K > 0$, ruin will occur if more than $C(k,l)/K$ dividend payments are made in any time-interval $[k,l]$, where $C(k,l) := \sup_{t\in[k,l]} U_t$ is finite $\mathbb{P}_x$-a.s. for any $x, u, v \in \mathbb{R}_+$, $u < v$. Hence, as no dividend payments take place at or after the moment of ruin, it follows that the sequence $\tilde{\mathbb{T}}$ is in fact discrete. In the case $K = 0$, the form of the strategy $\pi(g)$ implies that the sequence $(U_{\tilde{T}_i})_i$ is decreasing with $U_{\tilde{T}_i} - U_{\tilde{T}_{i-1}} > 0$ on the set $\{\tilde{T}_i < \infty\}$. In particular, it follows that that $\tilde{\mathbb{T}}$ is countable.

Write $D = D^{\pi(g)}$ and $M = \widetilde{M}^{g,\pi}$, fix $t \in \mathbb{R}_+$ and denote $T_i = \tilde{T}_i \wedge t$. We have

$$M_t = \sum_{i\geq 1} Y_i + \sum_{i\geq 0} Z_i,$$

where we denoted $Z_i = \mathrm{e}^{-qT_i}(g(X_{T_i} - D_{T_i}) - g(X_{T_i} - D_{T_i-}) + \Delta D_i - K)\mathbf{1}_{\{\Delta D_i > 0\}}$ and

$$Y_i = \mathrm{e}^{-qT_i} g(X_{T_i} - D_{T_i-}) - \mathrm{e}^{-qT_{i-1}} g(X_{T_{i-1}} - D_{T_{i-1}}) - \int_{(T_{i-1},T_i)} \mathrm{e}^{-qs}\mathrm{d}D_s, \tag{9.13}$$

with $\Delta D_i = D_{T_i} - D_{T_{i-1}}$. By definition of the strategy $\pi(g)$ we have $Z_i = 0$ for all i.

In the case $K > 0$ the integral term in Eqn. (9.13) vanishes and we have $D_{T_{i-1}} = D_{T_i-}$ for $i \geq 0$. By reasoning as in part (i) we find that the equality in Eqn. (9.12) holds. By combining Eqn. (9.12) with the fact that g is a stochastic subsolution, with Doob's optional stopping theorem and with the definition of T_i we have

$$\mathbb{E}[Y_i|\mathcal{F}_{T_{i-1}}] = \mathrm{e}^{-qT_{i-1}}\mathbb{E}_{U_{T_{i-1}}}\left[\mathrm{e}^{-q\tau_i} g\left(X_{\tau_i}\right) - g(X_0)\right] = 0,$$

where we denoted $\tau_i = T_i \circ \theta_{T_{i-1}}$. The tower-property hence yields $\mathbb{E}[M_t - M_s|\mathcal{F}_s] = 0$ for any $s \leq t$, so that M is a martingale.

In the case $K = 0$, the definition of $\pi(g)$ implies that the process $\{U_{T_{i-1}+t}, t < T_i - T_{i-1}\}$ conditional on $\mathcal{F}_{T_{i-1}}$ has the same law as the process $\{Y_t^b, t < \tau_b(a)\}$ with $X_0 = b = U_{T_{i-1}}$ and $\tau_b(a) = \inf\{t \geq 0 : Y^b < a\}$, conditional on $U_{T_{i-1}}$, where Y^b is independent of $U_{T_{i-1}}$. The strong Markov property of Y^a implies

$$\mathbb{E}[Y_i|\mathcal{F}_{T_{i-1}}] = \mathrm{e}^{-qT_{i-1}}\mathbb{E}_{U_{T_{i-1}}}\left[\mathrm{e}^{-q\tau_b(a)} g(Y^b_{\tau_b(a)}) - g(Y_0) - \int_{(0,\tau^b(a))} \mathrm{e}^{-qs}\mathrm{d}\overline{X}^b_s\right].$$

This expectation is zero in view of Lem. 9.7 and the fact that $g'_+(a) = 1$. Again, an application of the tower-property yields $\mathbb{E}[M_t - M_s|\mathcal{F}_s] = 0$ for any $s \leq t$, so that M is a martingale. □

Proof of the dual representation (Thm. 3.4). The identity in Eqn. (3.11) follows from Prop. 9.5 in view of the observations that

(a) $\mathcal{G}^+$ is contained in $\mathcal{H}_{0,\infty}$ and
(b) v_* is an element of the set $\mathcal{G}^+$.

Observation (a) follows on account of Lem. 9.6(i), while observation (b) is a direct consequence of Prop. 3.1 (taking π equal to the waiting strategy π_w of not paying any dividends) and Lemmas 9.1 and 9.2. The proof of part (i) follows by a line of reasoning that is analogous to that of part (ii), using the facts $\mathcal{G}^+_{a,b} \subset \mathcal{H}_{a,b}$ (Lem. 9.6(ii)) and $v_* \in \mathcal{G}^+_{a,b}$. □

Proof of existence and uniqueness (Thm. 3.8). That v_* is a stochastic supersolution follows from Thm. 3.4. To complete the proof we will next show that we have

(a) v_* is a stochastic subsolution of the HJB in Eqn. (3.3),
(b) v_* is the unique stochastic solution of the HJB in Eqn. (3.3), and
(c) $v_* = v_{\pi_*}$.

(a) Proof that v_* is a stochastic subsolution: We give a proof by contradiction. Note first that Lemmas 9.1 and 9.2 imply that v_* is continuous on $\mathbb{R}_+$ $(K > 0)$ and is continuous on $\mathbb{R}_+$, right-differentiable with right-continuous derivative $v'_{*,+}$ $(K = 0)$. Also it is clear that v_* satisfies the boundary condition in Eqn. (3.5) (in view of its definition). Denote $\overline{M} = \overline{M}^{v_*}$ and $\underline{M} = \underline{M}^{v_*}$ with $\overline{M}^{v_*}$ and $\underline{M}^{v_*}$ defined in Eqns (3.9) and (3.10), respectively, and recall that v_* is a stochastic supersolution (by Thm. 3.4). To show that v_* is a stochastic subsolution it remains to verify that v_* satisfies the requirement in Eqn. (3.10) in the definition of stochastic subsolution, which we establish by a proof by contradiction.

Assume hence that, for some open interval $\widetilde{O} = (a, b) \subset \mathcal{C}$, we have that the process $\underline{M} = \{\underline{M}_t, t \in \mathbb{R}_+\}$, with $\underline{M}_t = \overline{M}_{t\wedge T}$ and $T = T_{a,b}$, is a $\mathbb{P}_z$-supermartingale for all $z \in (a, b)$ that is *not* a $\mathbb{P}_{z_0}$-martingale for some $z_0 \in (a, b)$. Consider the function $\widetilde{g}_{a,b} : \mathbb{R}_+ \to \mathbb{R}$ given by

$$\widetilde{g}_{a,b}(x) = \mathbb{E}_x\left[\underline{M}_{T(a,b)}\right] = v_*(b)\mathbb{E}_x\left[\mathrm{e}^{-qT_b^+}\mathbf{1}_{\{T_b^+<T_a^-\}}\right] + \mathbb{E}_x\left[\mathrm{e}^{-qT_a^-}v_*(X_{T_a^-})\mathbf{1}_{\{T_b^+>T_a^-\}}\right]. \tag{9.14}$$

Since $\underline{M}$ is a zero-mean strict $\mathbb{P}_{z_0}$-supermartingale, it follows

$$v_*(z_0) > \widetilde{g}_{a,b}(z_0). \tag{9.15}$$

By definition of the set $\mathcal{C}$ and right-continuity it follows that, for any pair (a', b') with $a < a'$, $b' < b$, we have $\epsilon(a', b') = \inf_{x\in(a',b')} \mathtt{d}_{v_*}(x) - 1 > 0$. In particular, since we have $\widetilde{g}_{a',b'}(a') = v_*(a')$, $\widetilde{g}_{a',b'}(a') = v_*(b')$ it follows

$$\widetilde{g}_{a',b'}(b') > \widetilde{g}_{a',b'}(a') + (b' - a')(1 + \epsilon(a', b')) - K. \tag{9.16}$$

Denote $e_1(x) = \mathbb{E}_x\left[\mathrm{e}^{-qT_{b'}^+}\mathbf{1}_{\{T_{b'}^+<T_{a'}^-\}}\right]$ and

$$e_2(x) = \mathbb{E}_x\left[\mathrm{e}^{-qT_{a'}^-}\widetilde{g}_{a',b'}(X_{T_{a'}^-})\mathbf{1}_{\{T_{b'}^+>T_{a'}^-\}}\right], \qquad e_3(x) = \widetilde{g}_{a',b'}(a')\left(1 - \mathbb{E}_x\left[\mathrm{e}^{-qT_{b'}^+}\mathbf{1}_{\{T_{b'}^+>T_{a'}^-\}}\right]\right).$$

Since we have $e_1(x) \nearrow 1$, $e_2(x) \searrow 0$ and $e_3(x) \searrow 0$ as $x \nearrow b'$, and $\epsilon(a', b') > 0$, there exist $a' = a_0, b' = b_0$ and $\eta > 0$ with

$$\begin{cases} z_0 \in (b_0 - \eta, b_0),\ b_0 - \eta > a_0 > a \text{ and } b_0 < b \text{ and} \\ (b_0 - a_0)(1 + \epsilon(a_0, b_0))e_1(x) + e_2(x) - e_3(x) \geq x - a_0 \qquad \text{for all } x \in (b_0 - \eta, b_0). \end{cases} \tag{9.17}$$

Denoting $\widetilde{g} = \widetilde{g}_{a_0,b_0}$ and $\epsilon_0 = \epsilon(a_0, b_0)$ and combining Eqns. (9.14), (9.16) and (9.17) yields

$$\begin{aligned} \widetilde{g}(x) &> (\widetilde{g}(a_0) + (b_0 - a_0)(1 + \epsilon_0) - K)\mathbb{E}_x[\mathrm{e}^{-qT_{b_0}^+}\mathbf{1}_{\{T_{b_0}^+<T_{a_0}^-\}}] + \mathbb{E}_x[\mathrm{e}^{-qT_{a_0}^-}\widetilde{g}(X_{T_{a_0}^-})\mathbf{1}_{\{T_{b_0}^+>T_{a_0}^-\}}] \\ &= \widetilde{g}(a_0) + (b_0 - a_0)(1 + \epsilon_0) - K)e_1(x) + e_2(x) - e_3(x) \\ &\geq \widetilde{g}(a_0) + x - a_0 - K, \end{aligned} \tag{9.18}$$

for all $x \in (b_0 - \eta, b_0)$. Hence, $\widetilde{g}$ is a local supersolution for the HJB in Eqn. (3.3) with the property $\widetilde{g}(z_0) < v_*(z_0)$. But this yields a contradiction with Thm. 3.4. Thus, it follows that v_* is a stochastic subsolution.

(b) Proof of uniqueness: Let h be a stochastic solution. As v_* is the pointwise smallest supersolution (Thm. 3.4), it follows $v_* \leq h$. We will now verify that also the opposite inequality, $h \leq v_*$, holds true. Denote by $\pi(h)$ the policy

corresponding to h given in Def. 3.6. Since the processes $\widetilde{M}^{v_*,\pi(h)}$ and $\widetilde{M}^{h,\pi(h)}$, which were defined in Eqn. (9.9), are a UI supermartingale and a UI martingale (by Lem. 9.6, parts (i) and (iii), respectively), Doob's optional stopping theorem implies

$$v_*(x) - h(x) \geq \lim_{t\to\infty} \mathbb{E}_x \left[\widetilde{M}^{v_*,\pi(h)}_{t\wedge\tau^{\pi(h)}} - \widetilde{M}^{h,\pi(h)}_{t\wedge\tau^{\pi(h)}}\right] = 0, \qquad x \in \mathbb{R}_+,$$

where we used $\mathbb{P}_x(\tau^{\pi(h)} < \infty) = 1$ for all $x \in \mathbb{R}_+$ together with the boundary condition

$$\widetilde{M}^{v_*,\pi(h)}_{\tau^{\pi(h)}} = \widetilde{M}^{h,\pi(h)}_{\tau^{\pi(h)}} = \mathrm{e}^{-q\tau^{\pi(h)}} w(U^{\pi(h)}_{\tau^{\pi(h)}}).$$

This completes the proof of (b).

(c) Proof of identity: Since v^* is a stochastic solution, the shifting lemma, Lem. 9.6(iii), implies that the process $\mathrm{e}^{-q(t\wedge\tau_{\pi_*})} v_*(U^{\pi_*}_{t\wedge\tau_{\pi_*}}) + \int_{[0,t\wedge\tau_{\pi_*})} \mathrm{e}^{-qs}\mu^{\pi_*}_K(\mathrm{d}s)$ is a UI $\mathbb{P}_x$-martingale for all $x \in \mathbb{R}_+$. In particular, the uniform integrability and the boundary condition imply the identity

$$v_*(x) = \mathbb{E}_x\left[\mathrm{e}^{-q\tau^{\pi_*}} w(U^{\pi_*}_{\tau_{\pi_*}}) + \int_{[0,\tau_{\pi_*})} \mathrm{e}^{-qt}\mu^{\pi_*}_K(\mathrm{d}t)\right] = v_{\pi_*}(x), \qquad x \in \mathbb{R}_+.$$

□

9.4. **Proof of the dividend-penalty decomposition.** Cor. 3.9 is a direct consequence the following result:

Lem. 9.9. *Let $S^* = \{S^*_t, t \in \mathbb{R}_+\}$ be the stochastic process with $S^*_t = \mathrm{e}^{-q(t\wedge T_0^-)} v_*(X_{t\wedge T_0^-})$ and let $\mathcal{D}^o_*$ be the interior of the set $\mathcal{D}_*$ defined in Eqn. (3.12). The Doob-Meyer decomposition of S^* is given by $S^* = M^* - A^*$ where $A^* = \{A^*_t, t \in \mathbb{R}_+\}$ is an increasing and locally natural process given by*

$$A^*_t = \int_0^{t\wedge T_0^-} \mathbf{1}_{\{u\in\mathbb{R}_+ : X_{u^-}\in\mathcal{D}^o_*\}}(s) J^*(X_{s^-})\mathrm{d}s, \qquad t \in \mathbb{R}_+$$

and M^* is a martingale.

The proof is based on an auxiliary result concerning the form of v_* restricted to the set $\mathcal{D}^o_*$. Recall that, since the set $\mathcal{D}^o_* \subset \mathbb{R}_+$ is open, it is of the form $\mathcal{D}^o_* = \cup_n(a_n, b_n)$ for some $a_n, b_n \in \mathbb{R}_+$ that are such that $a_n < b_n$ and (a_n, b_n) are disjoint intervals.

Lem. 9.10. *The value function v_* satisfies*

$$v_*(x) = x - a_n + v_*(a_n) \qquad \text{for all } x \in [a_n, b_n), \tag{9.19}$$

where $a_n, b_n \in \mathbb{R}_+$, $a_n < b_n$, are such that we have $\mathcal{D}^o_* = \cup_n(a_n, b_n)$.

Proof. In the case $K = 0$ the statement holds since by definition of the set $\mathcal{D}^o_*$, we have $v'_{*,+}(x) = 1$ for all $x \in (a_n, b_n)$ and v_* is continuous at a_n (where $v'_{*,+}(x)$ denotes the right-derivative of v_* at x).

In the case $K > 0$ the stated linearity follows by combining the following two facts: (a) for any $x \in (a_n, b_n)$ it is optimal to immediately make a lump-sum payment of size $y^*(x) > 0$ (Thm. 3.8) and (b) we then have $v_*(z) = z - (x - y^*(x)) + v_*(x - y^*(x)) - K$ for all $z \in [a_n, x]$. We next establish point (b): On the one hand, Lem. 9.1(i) implies that v^* satisfies the inequality $v_*(z) \geq z - (x - y^*(x)) + v_*(x - y^*(x)) - K$ for all $z \in [a_n, x]$. On the other hand, the definition of $y^*(x)$ and again Lem. 9.1(i) imply

$$\begin{aligned} y^*(x) - K + v(x - y^*(x)) &= v_*(x) \\ &\geq x - z_1 + y(z_1) - K + v(z_1 - y^*(z_1)) = x - z + v_*(z_1) \\ &\Leftrightarrow v_*(z_1) \leq z - (x - y^*(x)) + v_*(x - y^*(x)) - K \end{aligned}$$

Combination of the two inequalities yields the statement in point (b), and completes the proof. □

Proof of Lem. 9.9. Let $x \in \mathbb{R}_+$ be given and recall that S^* is a UI $\mathbb{P}_x$-supermartingale by Thm. 3.4, and denote by $S^* = M^* - A^*$ the Doob-Meyer decomposition of S^*. We identify the Doob-Meyer decompositions of an approximating sequence $(\widetilde{S}^{\epsilon_n})_n$ of processes and find the form of A^* by passing to the limit.

For given $\epsilon > 0$ denote by $\mathcal{D}^\epsilon$ the strict subset of $\mathcal{D}^o_*$ given by $\mathcal{D}^\epsilon = \cup_n(a_n + \epsilon\delta_n, b_n - \epsilon\delta_n)$ where $\delta_n = b_n - a_n$ (recall that $\mathcal{D}^o_* = \cup_n(a_n, b_n)$ where $a_n < b_n$ and the intervals (a_n, b_n) are disjoint). Consider the sequence of stopping times $(T^\epsilon_i)_{i\in\mathbb{N}}$ given by

$$T^\epsilon_1 = 0, \qquad T^\epsilon_{2i} = \inf\{t \geq T^\epsilon_{2i-1} : X_t \in \mathcal{D}^\epsilon\} \wedge T^-_0, \qquad T^\epsilon_{2i+1} = \inf\{t \geq T^\epsilon_{2i} : X_t \notin \mathcal{D}^o_*\} \wedge T^-_0.$$

By right-continuity of the paths of X, we have $T^\epsilon_{2i} < T^\epsilon_{2i+1}$ on the set $\{T^\epsilon_{2i} < T^-_0\}$. By Doob's optional stopping theorem, the process $\widetilde{S}^\epsilon = \{\widetilde{S}^\epsilon_t, t \in \mathbb{R}_+\}$ given by

$$\widetilde{S}^\epsilon_t = \sum_{i\in\mathbb{N}} \left(S^*_{t\wedge T^\epsilon_{2i+1}} - S^*_{t\wedge T^\epsilon_{2i}} \right) \tag{9.20}$$

is a UI $\mathbb{P}_x$-supermartingale, with increasing process in the Doob-Meyer decomposition denoted by $\widetilde{A}^\epsilon$. The difference $S^* - \widetilde{S}^\epsilon = \{S^*_t - \widetilde{S}^\epsilon_t, t \in \mathbb{R}_+\}$ is a UI $\mathbb{P}_x$-super-martingale that can be represented by

$$S^*_t - \widetilde{S}^\epsilon_t = \sum_{i\in\mathbb{N}} \left(S^*_{t\wedge T^\epsilon_{2i}} - S^*_{t\wedge T^\epsilon_{2i-1}} \right). \tag{9.21}$$

By a line of reasoning analogous to the one used in Lem. 10.2 it follows

$$\mathbb{E}_x[S^*_t - \widetilde{S}^\epsilon_t] \longrightarrow 0 \qquad \text{as } \epsilon \searrow 0, \text{ for any } t \in \mathbb{R}_+ \tag{9.22}$$

We claim that the process $\widetilde{S}^\epsilon + \widetilde{A}^\epsilon$ is a UI $\mathbb{P}_x$-martingale and $\widetilde{A}^\epsilon$ is an increasing locally natural process, where the process $\widetilde{A}^\epsilon = \left\{\widetilde{A}^\epsilon_t, t \in \mathbb{R}_+\right\}$ is given by

$$\widetilde{A}^\epsilon_t := \sum_i \int_{t\wedge T^\epsilon_{2i-1}}^{t\wedge T^\epsilon_{2i}} \mathrm{e}^{-qs}[-J^*(X_{s^-})]\mathrm{d}s \tag{9.23}$$

Proof of claim: An application of Itô's lemma to the processes $\overline{M}^{(i)} = \left\{\overline{M}^{(i)}_t, t \in \mathbb{R}_+\right\}$, $i \in \mathbb{N}$, given by

$$\overline{M}^{(i)}_t = (S^*_{t\wedge T^\epsilon_{2i+1}} - S^*_{t\wedge T^\epsilon_{2i}}) - \int_{t\wedge T^\epsilon_{2i}}^{t\wedge T^\epsilon_{2i+1}} \mathrm{e}^{-qs} J^*(X_{s^-})\mathrm{d}s, \qquad t \in \mathbb{R}_+,$$

(which is justified since $v_*|_{(a_n,b_n)}$ is a C^2-function by Lem. 9.10) yields that $\overline{M}^{(i)}$ is a $\mathbb{P}_x$-martingale for all $x \in \mathbb{R}_+$ and $i \in \mathbb{N}$. The process $\widetilde{A}^\epsilon$ is increasing (as $\widetilde{S}^\epsilon$ is a supermartingale) and is locally natural since $\widetilde{A}^\epsilon_{T^-_0}$ is integrable (as $\widetilde{S}^\epsilon$ is a uniformly integrable) and $t \mapsto \widetilde{A}^\epsilon_t$ is continuous.

Since $\mathcal{D}^\epsilon$ increases to $\mathcal{D}^o_*$ and we have

$$\int_0^{T^-_0\wedge t} \mathrm{e}^{-qs}\mathbf{1}_{\{u\in\mathbb{R}_+:X_{u^-}\in\mathcal{D}^\epsilon\}}(s)[-J^*(X_{s^-})]\mathrm{d}s \leq \widetilde{A}^\epsilon_t \leq \int_0^{T^-_0\wedge t} \mathrm{e}^{-qs}\mathbf{1}_{\{u\in\mathbb{R}_+:X_{u^-}\in\mathcal{D}^o_*\}}(s)[-J^*(X_{s^-})]\mathrm{d}s, \tag{9.24}$$

the monotone convergence theorem implies that the LHS of Eqn. (9.24) tends to the RHS if $\epsilon \searrow 0$, so that $\widetilde{A}^\epsilon_t$ converges to the RHS of Eqn. (9.24) as $\epsilon \searrow 0$. Since the process $S^* - \widetilde{S}^\epsilon$ in Eqn. (9.21) is a UI $\mathbb{P}_x$-super-martingale and the increasing process in its Doob-Meyer decomposition is given by $A^* - \widetilde{A}^\epsilon$, it follows that $A^*_t - \widetilde{A}^\epsilon_t$ is non-negative, so that in view of Eqn. (9.22) and the fact $\mathbb{E}_x[A^*_t - \widetilde{A}^\epsilon_t] = \mathbb{E}_x[S^*_t - \widetilde{S}^\epsilon_t]$, the difference $A^*_t - \widetilde{A}^\epsilon_t$ tends to zero $\mathbb{P}_x$-a.s. as ϵ tends to zero. Thus we deduce that A^*_t is equal to the RHS of Eqn. (9.24) for any $t \in \mathbb{R}_+$ $\square$

10. Proofs of optimality of single and multi dividend-band strategies

10.1. **Martingale pasting.** The optimality of the value-function g of a candidate-optimal policy satisfying the bound in Eqn. (3.7) will follow from Thm. 3.4 once the supermartingale property and uniform integrablity of the process

$$S = \left\{ \mathrm{e}^{-q(t\wedge T_0^-)} g\left(X_{t\wedge T_0^-}\right), t \in \mathbb{R}_+ \right\} \tag{10.1}$$

are established. In the following result it is shown that, provided that the function g is sufficiently regular, the verification of the supermartingale property can be carried out *locally*:

Lem. 10.1 (Pasting lemma)**.** *Let $(C_i)_{i=0}^{n-1}$ be a finite partition of $\mathbb{R}_+$ (that is, $C_0 := 0$, $C_n := +\infty$ and $C_{i-1} < C_i$ for all $i = 0, \ldots, n$), and let $(\tau_i)_i$ be the sequence of stopping times*

$$\tau_i = \inf\{t \in \mathbb{R}_+ : X_t \notin [C_{i-1}, C_i)\}, \qquad i = 1, \ldots, n.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a Borel-measurable function such that $g|_{\mathbb{R}_+\setminus\{0\}}$ is continuous if X has bounded variation, and is C^1 if X has unbounded variation. If

$$S^{\tau_i} = \{S_{t\wedge\tau_i}, t \in \mathbb{R}_+\} \text{ are } \mathbb{P}_x\text{-UI supermartingales, for any } x \in [C_{i-1}, C_i) \text{ and } i = 1, \ldots, n, \tag{10.2}$$

then the process S given in Eqn. (10.1) *is a $\mathbb{P}_x$-supermartingale for any $x \in \mathbb{R}_+$.*

Proof. For the ease of presentation we will restrict ourselves to the case of a partition of the form $[0, a) \cup [a, \infty)$ for some $a > 0$. The general case follows by a similar line of reasoning.

Fix $t > 0$ and $x \in \mathbb{R}_+$. Suppose first that X has bounded variation. Then a is irregular for $(-\infty, a)$ for X, so that the following set of stopping times forms a discrete set:

$$T_0 = 0, \qquad T_{2i} = (T_a^+ \wedge T_0^-) \circ \theta_{T_{2i-1}}, \qquad T_{2i-1} = T_a^- \circ \theta_{T_{2i-2}}, \qquad i \in \mathbb{N}, \tag{10.3}$$

where θ denotes the translation operator. The strong Markov property of X implies that on the event $\{s \leq T_{i-1}, T_{i-1} < \infty\}$, $i \in \mathbb{N}$, we have:

$$\begin{aligned} \mathbb{E}\left[S_{t\wedge T_i} - S_{t\wedge T_{i-1}} | \mathcal{F}_s\right] &= \mathbb{E}\left[\mathbb{E}[S_{t\wedge T_i} - S_{t\wedge T_{i-1}} | \mathcal{F}_{T_{i-1}}] | \mathcal{F}_s\right] \\ &= \mathbb{E}\left[\mathbf{1}_{\{t>T_{i-1}\}} \mathrm{e}^{-qT_{i-1}} \mathbb{E}_{X_{T_{i-1}}} \left[\mathrm{e}^{-qR_v} g(X_{R_v}) - g(X_0)\right]\Big|_{v=T_{i-1}} \Big| \mathcal{F}_s\right], \end{aligned}$$

where $R_v = (\tau' \wedge t) \circ \theta_v$ where τ' is set equal to $T_{0,a}$ if $X_0 = x \in [0, a)$ and to T_a^- if $X_0 = x \geq a$. The expectation on the right-hand side is non-positive and finite in view of Doob's optional stopping theorem and the assumption in Eqn. (10.2).

The stated supermartingale property then follows by virtue of the fact that the terms in the following sum have non-positive finite conditional expectations under $\mathbb{E}[\cdot|\mathcal{F}_s]$:

$$S_t - S_s = \sum_j \mathbf{1}_{\{T_{j-1}<s\leq T_j\}} \left\{ (S_{T_j\wedge t} - S_{T_j\wedge s}) + \sum_{i>j} (S_{t\wedge T_i} - S_{t\wedge T_{i-1}}) \right\}, \qquad s < t.$$

Suppose next that X has unbounded variation. Denote by $(T_i)_{i\in\mathbb{N}\cup\{0\}}$ the sequence of subsequent passage times into the sets $[a-\epsilon, a+\epsilon]$ and $\mathbb{R}\backslash[a-2\epsilon, a+2\epsilon]$:

$$T_0 = 0, \qquad T_{2i-1} = H_{[a-\epsilon,a+\epsilon]} \circ \theta_{T_{2i-2}}, \qquad T_{2i} = T_{a-2\epsilon,a+2\epsilon} \circ \theta_{T_{2i-1}} \qquad i \in \mathbb{N},$$

where, for any Borel set A, $H_A = \inf\{t \in \mathbb{R}_+ : X_t \in A\}$ (see Figure 2). Decompose S as $S - S_0 = S^{(1)} + S^{(2)}$ with

$$S_t^{(1)} = \sum_{i\geq 1} \left[S_{t\wedge T_{2i}} - S_{t\wedge T_{2i-1}}\right], \qquad S_t^{(2)} = \sum_{i\geq 1} \left[S_{t\wedge T_{2i-1}} - S_{t\wedge T_{2i-2}}\right].$$

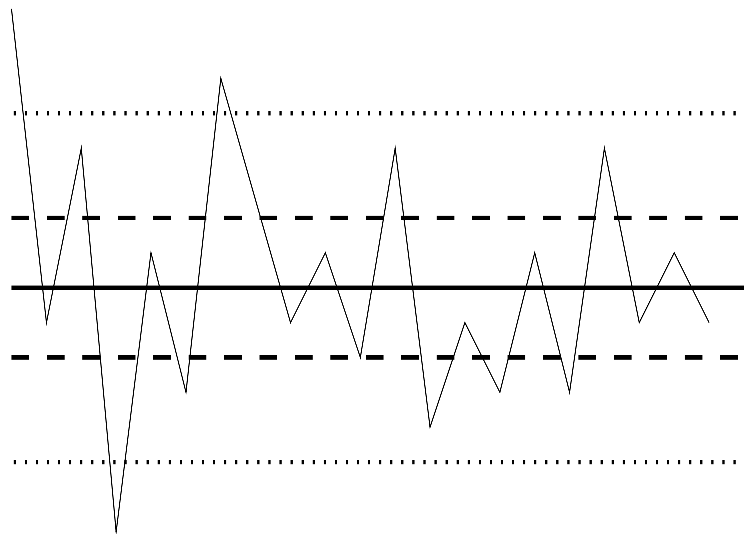

FIGURE 2. The martingale increments commence when X enters the inner band (dashed) and stop when X leaves the outer band (dotted).

The sum $S^{(1)}$ of increments of S during the periods that X spends in the band $[a-2\epsilon, a+2\epsilon]$ vanishes in expectation as $\epsilon \searrow 0$, as shown in the following result the proof of which is given in the below:

Lem. 10.2. *For any t and $x \in \mathbb{R}_+$, $\mathbb{E}_x\left[\left|S_t^{(1)}\right|\right] \to 0$ as $\epsilon \searrow 0$.*

By the line of the reasoning given in the first part of the proof it follows that $S^{(2)}$ is a supermartingale for every $\epsilon > 0$, so that also S is a supermartingale in view of Lem. 10.2. □

Proof of Lem. 10.2. Write $S^{(1)} = \Sigma^{(1)} + \Sigma^{(2)}$ where

$$\Sigma_t^{(1)} = \sum_{i\geq 1} \mathrm{e}^{-q(t\wedge T_{2i-1})}[g(X_{t\wedge T_{2i}}) - g(X_{t\wedge T_{2i-1}})], \quad \Sigma_t^{(2)} = \sum_{i\geq 1} g(X_{t\wedge T_{2i}})[\mathrm{e}^{-q(t\wedge T_{2i})} - \mathrm{e}^{-q(t\wedge T_{2i-1})}].$$

In view of the fact that $g(x) \leq ax + b$ for some constants $a, b > 0$ it follows that the following estimate holds for fixed $t > 0$:

$$\left|\Sigma_t^{(2)}\right| \leq (a\overline{X}_{t\wedge\tau_\pi} + b) \int_0^{t\wedge\tau_\pi} \mathrm{e}^{-qs}\mathbf{1}_{\{X_s\in(a-2\epsilon,a+2\epsilon)\}}\mathrm{d}s.$$

On account of the fact that the potential measure of X is absolutely continuous, the left-hand side tends to zero as $\epsilon \searrow 0$ $\mathbb{P}_x$-a.s. for any $x \in \mathbb{R}_+$. The dominated convergence theorem implies that this convergence also holds in $\mathbb{P}_x$-expectation. For the term $\Sigma^{(1)}$ the strong Markov property applied at T_{2i-1} and Def. 4.1(i) imply that following identity holds true:

$$\mathbb{E}_x\left[\Sigma_t^{(1)}\right] = \mathbb{E}_x\left[\sum_{i\geq 1} \mathrm{e}^{-q(t\wedge T_{2i-1})} L(X_{t\wedge T_{2i-1}} - a - 2\epsilon)\right], \tag{10.4}$$

with

$$L(x) = F(x) - \tilde{g}(x) + \frac{W^{(q)}(x)}{W^{(q)}(4\epsilon)}(\tilde{g}(4\epsilon) - F(4\epsilon)),$$

where $F = F_{\tilde{g}}$ denotes the Gerber-Shiu function corresponding to payoff $\tilde{g} := {}_{a-2\epsilon}g$. The triangle inequality, continuous differentiability of $\tilde{g}$ and F and the fact that $W^{(q)}$ is increasing yield the following estimate:

$$|L(x)| \leq 4\epsilon \times 2C(\epsilon) \text{ for all } x \in [0, 4\epsilon], \text{ with } C(\epsilon) = \max_{x\in[0,4\epsilon]} |F'(x) - \tilde{g}'(x)|. \tag{10.5}$$

Observe that the number of terms in the sum $\Sigma^{(1)}$ is bounded by $1+D_t^-(\epsilon)+U_t^+(\epsilon)$ where $D_t^-(\epsilon)$ and $U_t^+(\epsilon)$ denote the numbers of down-crossings of the band $(a-2\epsilon, a-\epsilon)$ and up-crossings of $(a+\epsilon, a+2\epsilon)$ by X before time t. Thus the expectation of $|\Sigma_t^{(1)}|$ can be bounded as follows:

$$\mathbb{E}_x\left[\left|\Sigma_t^{(1)}\right|\right] \leq 8\epsilon\, \mathbb{E}_x[1+D_t^-(\epsilon)+U^+(\epsilon)]\, C(\epsilon). \tag{10.6}$$

Since X is a submartingale, the up-crossing lemma implies that the expected number of up-crossings $U_t^+(\epsilon)$ of the band $(c,d)=(a+\epsilon, a+2\epsilon)$ by time t does not grow faster than ϵ^{-1}:

$$\epsilon \cdot \mathbb{E}_x[U_t^+(\epsilon)] \leq \mathbb{E}_x[(X_t-d)^+] - \mathbb{E}_x[(X_0-c)^+].$$

Thus, it follows that $\epsilon \cdot \mathbb{E}_x[U_t^+(\epsilon)]$ remains bounded as $\epsilon \to 0$. As the number of down-crossings $D_t^-(\epsilon)$ is bounded by two added to the number of up-crossings $U_t^-(\epsilon)$ of the band $(a-2\epsilon, a-\epsilon)$, $\epsilon \cdot \mathbb{E}_x[D_t^-(\epsilon)]$ also remains bounded. Since $C(\epsilon)$ tends to zero as $\epsilon \to 0$, on account the facts that F and $\tilde{g}$ are $C^1(\mathbb{R}_+)$ and $F'(0)=\tilde{g}'(0)$ (cf. Eqn. (B.6), recalling that X is assumed to have unbounded variation), it thus follows from Eqn. (10.6) that $\mathbb{E}_x[|\Sigma_t^{(1)}|]$ tends to 0 as ϵ tends to zero, and the proof is complete. □

10.2. **Single band and two-bands policies.** The following auxiliary result provides a key-step for obtaining necessary and sufficient optimality conditions for single barrier policies:

Lem. 10.3. *(i) For $\theta > \Phi(q)$, the Laplace transform $g^*(\theta) := \int_0^\infty \mathrm{e}^{-\theta x} g(x)\mathrm{d}x$ of the function*

$$g: \mathbb{R}_+\backslash\{0\} \to \mathbb{R} \qquad x \mapsto g(x) := {}_{b_+}(\mathcal{L}^w v_b - q\, v_b)(x) \tag{10.7}$$

is equal to $-\Xi(\theta)$ where

$$\Xi(\theta) = -\frac{\mathrm{e}^{\theta b_+}}{\theta}\int_{(b_+,\infty)} \mathrm{e}^{-\theta z} Z^{(q,\theta)\prime}(z) G_{b_-}(\mathrm{d}z),$$

where $G_{b_-}(x) := G(b_-, x)$.

(ii) The function g is non-positive if and only if the function $\theta \mapsto \Xi(\theta+\Phi(q))$ is completely monotone.

Proof of Theorem 7.3, part (i). We claim that the strategy π_{b^*} is an optimal policy for the stochastic control problem in Eqn. (2.4) if and only if the following condition for v_{b^*} is satisfied:

$$ {}_{b_+^*}\mathcal{L}_\infty^{\overline{w}} v_{b^*}(x) - q v_{b^*}(x) \quad \leq \quad 0, \qquad\qquad \text{for all } x > b_+^* \text{ and with } \overline{w} = v_{b^*}, \tag{10.8}$$

where the operator ${}_{b_+^*}\mathcal{L}_\infty^{\overline{w}}$ is defined in (4.12). Given this claim the assertion in (i) directly follows on account of Lem. 10.3.

Proof of claim: To verify that the condition in Eqn. (10.8) is sufficient we show that v_{b^*} is a stochastic supersolution. Then the (local) verification theorem in Cor. 3.5 implies that v_{b^*} is equal to the value-function v_*. The supersolution property of v_{b^*} follows from the pasting lemma (Lem. 10.1) and the facts

(a) $\exp\{-q(t\wedge T_{b_+^*}^-)\}v_{b^*}\left(X(t\wedge T_{b_+^*}^-)\right)$ is a $\mathbb{P}_x$-supermartingale for all $x \geq b_+^*$ (by Prop. 7.12 and Thm. 5.3),
(b) $\exp\{-q(t\wedge T_0^-)\}v_{b^*}\left(X(t\wedge T_0^-)\right)$ is a $\mathbb{P}_x$-martingale for all $x \in [0, b_+^*]$ (by the form of v_{b^*} in Eqn. 5.10 and the martingale properties of $W^{(q)}$ and F in Eqns.(4.9) and (4.10)) and
(c) v_{b^*} is continuous at b_+^* (if X has bounded variation) and C^1 at b_+^* (if X has unbounded variation) in view of the form of v_{b^*} in Eqn. 5.10.

To see that the condition (10.8) is also necessary, suppose that the condition in Eqn. (10.8) is not satisfied. Since $x \mapsto ({}_{b_+^*}\mathcal{L}_\infty^{\overline{w}} v_{b^*} - q v_{b^*})(x)$ is right-continuous for $x \geq b_+^*$, it follows that there exists an open interval (α,β) contained in (b_+^*, ∞) such that $({}_{b_+^*}\mathcal{L}_\infty^{\overline{w}} v_{b^*} - q v_{b^*})(x) > 0$. Define a strategy $\tilde{\pi}$ as follows: whenever U_t does not take a value in the interval (α,β) operate according to π_{b^*}, and while the reserve process U_t takes a value in the interval (α,β), do not pay any dividends. Then $S_t := \mathrm{e}^{-q(t\wedge T_{\alpha,\beta})}[v_{\tilde\pi}(X_{t\wedge T_{\alpha,\beta}}) - v_{b^*}(X_{t\wedge T_{\alpha,\beta}}))$ is a $\mathbb{P}_x$-supermartingale for any $x \in (\alpha, \beta)$, and the following holds true (cf. Eqn. (7.5)):

$$\mathbb{E}_x[S_t - S_0] = -\mathbb{E}_x\left[\int_0^{t\wedge T_{\alpha,\beta}} \mathrm{e}^{-qs}({}_{b_+^*}\mathcal{L}_\infty^{\overline{w}} v_{b^*} - qv_{b^*})(X_s)\mathrm{d}s\right] < 0 \qquad \text{for any } x \in (\alpha, \beta).$$

This identity implies that $v_{\tilde\pi}(x)$ is strictly larger than $v_{b^*}(x)$ for any $x \in (\alpha, \beta)$. □

Proof of Theorem 7.3, part (ii). The statement follows by combining Thm. 5.3(ii) with the following observation. □

Lem. 10.4. *If $x \mapsto G^*(x)$ is decreasing on (b_+^*, ∞), then $\Xi(\theta)$ is completely monotone on $(\Phi(q), \infty)$.*

Lem. 10.4 will be proved in Sect. C.1.

Proof of Lem. 10.3. (i) Taking the Laplace transform in c in Eqn. (C.1) and using the form of the Laplace transform of $W^{(q)}$ yields that, for $\theta > \Phi(q)$,

$$\begin{aligned} g^*(\theta)\cdot\frac{\theta}{\psi(\theta)-q} &= \int_{[0,\infty)} \mathrm{e}^{-\theta c} W^{(q)\prime}(b_+ + c)[G^*(b_+ + c) - G^*(b_+)]\mathrm{d}c \\ &= \int_{[0,\infty)}\int_{[z,\infty)} \mathrm{e}^{-\theta c} W^{(q)\prime}(b_+ + c)\mathrm{d}c\, G^*(b_+ + \mathrm{d}z) \\ &= \mathrm{e}^{\theta b_+}\int_{[b_+,\infty)}\int_{[z,\infty)} \mathrm{e}^{-\theta c} W^{(q)\prime}(c)\mathrm{d}c\, G^*(\mathrm{d}z) = \frac{\mathrm{e}^{\theta b_+}}{\psi(\theta)-q}\int_{[b_+,\infty)} \mathrm{e}^{-\theta z} Z^{(q,\theta)\prime}(z)\, G^*(\mathrm{d}z), \end{aligned}$$

by a change of the order of integration, which is justified by Fubini's theorem, and the form (B.10) of $Z^{(q,\theta)\prime}(z)$. Comparison with Ξ defined in (C.1) shows that $g^*(\theta) = -\Xi(\theta)$ for $\theta > \Phi(q)$. Here the last three (outer) integrals are Stieltjes integrals with respect to G^*.

(ii) The second assertion follows since a function $f : (c, \infty) \to \mathbb{R}$ with $c > 0$ is completely monotone if and only if it is the Laplace transform of a non-negative measure supported on $\mathbb{R}_+$. □

Proof of Thm. 7.10. (i) In this case it can be shown as in the proof of Lem. 10.3 that the complete monotonicity of the function $\Xi_{\alpha_f^*,\beta_{f,-}^*,\beta_{f,+}^*}(f)$ is equivalent to the condition

$${}_0\mathcal{L}_\infty^w V^f_{\alpha^*,\beta_f^*}(x) - qV^f_{\alpha^*,\beta_f^*}(x) \le 0 \qquad \text{for all } x > \beta_{f,+}^*.$$

That this is a necessary and sufficient condition for $V^f_{\alpha_*,\beta_*}$ to be identically equal to V^f_* follows by a line of reasoning analogous to the one employed in the proof of Thm. 7.3(i).

(ii) The proof is analogous to that of part (i), and is omitted. □

Proof of Lem. 6.3. (i) Consider the function $\overline{G} : \mathbb{R}_+ \to \mathbb{R}$ defined by $\overline{G}(a) = \sup_{b\ge 0} G^{(a)}_{f,\#}(b)$. The fact $\alpha_f^* > 0$ is a consequence of the intermediate value theorem and the following three assertions concerning $\overline{G}$:

(a) $\overline{G}(0) < 0$,
(b) there exists an $a_0 > 0$ such that $\overline{G}(a_0) > 0$ and
(c) the function $a \mapsto \overline{G}(a)$ is continuous at $a \in [0, a_0]$.

Assertion (a) follows from the definitions of $\beta_{f,-}^*(0)$ and $\beta_{f,+}^*(0)$, and the form of V^f_{a,b_-,b_+} (Prop. 6.2), and the facts $F(0) = f(0) = c$ with c given in Eqn. (6.3) and $F'(0+) = f'(0-) = 1$ in conjunction with the condition in Eqn. (6.6).

To verify assertion (b) we show that for some a_0 and b with $a_0 < b$, $G^{(a_0)}_{f,\#}(b)$ is strictly positive. In view of the form of $G^{(a_0)}_{f,\#}$ we thus need to show the existence of a pair a_0, b satisfying $F^{(a_0)\prime}(b-a_0) < 1$.

The condition in Eqn. (6.5) and the right-continuity of the map $J : \mathbb{R}_+\backslash\{0\} \to \mathbb{R}$ defined by

$$J(y) := {}_0\mathcal{L}^{\bar{w}}_{\infty} f(y) = \psi'(0) - q(y + \bar{w}(0)) + \int_{(y,\infty)} [\bar{w}(y-z) - \bar{w}(0) + z - y]\nu(\mathrm{d}z), \qquad y \in \mathbb{R}_+,$$

imply that the following statement holds true:

(10.9) There exists an interval $I = [u_-, u_+]$, with $0 < u_- < u_+$, such that $J(y) > 0$ for all $y \in I$.

For the choice $a_0 = \inf\{y \geq 0 : J(y) > 0\}$, it follows in view of the representation $F^{(a_0)\prime}(b-a_0) = 1 - \int_0^{b-a_0} J(b-z)W^{(q)}(\mathrm{d}z)$ (see Lem. B.2(iv), Eqn. (B.5)) that we have $F^{(a_0)\prime}(b-a_0) > 1$ for some b sufficiently small.

To see assertion (c) fix $a \geq 0$, and note that $V^f_{a,\beta^*(a)}(x) = W^{(q)}(x)\overline{G}(a) + F^{(a)}(x-a)$ for $x \in [a, \beta^*_+(a)]$. Analogously as in Thm. 5.3 it can be shown that we have

$$V^f_{a,\beta^*(a)}(x) = \sup_{\pi\in\Pi} \mathbb{E}_x \left[\int_0^{\tau^\pi_a} \mathrm{e}^{-qt}\mathrm{d}D^\pi_t + \mathrm{e}^{-q\tau^\pi_a} f(U^\pi_{\tau^\pi_a}) \right].$$

Let a_1, a_2 be such that $a_2 < a_1 < \min\{\beta^*(a_1), \beta^*(a_2)\}$ and fix $x_0 \in (a_1, \min\{\beta^*(a_1), \beta^*(a_2)\})$. To show the continuity of $\overline{G}(a)$ we will show that $V^f_{a_1,\beta^*(a_1)}(x_0) - V^f_{a_2,\beta^*(a_2)}(x_0) \to 0$ when $a_2 - a_1 \to 0$.

The triangle inequality implies that we have

$$\left| V^f_{a_1,\beta^*(a_1)}(x_0) - V^f_{a_2,\beta^*(a_2)}(x_0) \right| \leq \sup_{\pi\in\Pi} \mathbb{E}_{x_0} \left[\int_{\tau^\pi_{a_1}}^{\tau^\pi_{a_2}} \mathrm{e}^{-qt}\mathrm{d}D^\pi_t + |\mathrm{e}^{-q\tau^\pi_{a_2}} f(U^\pi_{\tau^\pi_{a_2}}) - \mathrm{e}^{-q\tau^\pi_{a_1}} f(U^\pi_{\tau^\pi_{a_1}})| \right]. \tag{10.10}$$

Since $\mathbb{P}_{x_0}(U_{\tau^\pi_{a_1}} \in [a_2, a_1)) = \mathbb{P}_{x_0}(\tau^\pi_{a_1} < \tau^\pi_{a_2})$ converges to zero if $a_1 - a_2 \searrow 0$, it follows that also the random variable under the expectation tends to zero if $a_1 - a_2 \searrow 0$. Since this random variable is dominated uniformly for all $\pi \in \Pi$, Lebesgue's dominated convergence theorem implies that the right-hand side of Eqn. (10.10) tends to zero when $a_1 - a_2 \searrow 0$. To see that the random variable is dominated recall that f is affine and and note that we have

$$\mathrm{e}^{-q\tau^\pi_{a_1}} D^\pi_{\tau^\pi_{a_1}} \vee \mathrm{e}^{-q\tau^\pi_{a_2}} D^\pi_{\tau^\pi_{a_2}} \vee \int_{[\tau^\pi_{a_1}, \tau^\pi_{a_2}]} \mathrm{e}^{-qt}\mathrm{d}D^\pi_t \leq \int_{[0,\infty)} \mathrm{e}^{-qt}\mathrm{d}D^\pi_t \leq \int_0^\infty q\mathrm{e}^{-qt} D_t\mathrm{d}t \leq \int_0^\infty q\mathrm{e}^{-qt}\overline{X}_t\mathrm{d}t$$

which has $\mathbb{P}_{x_0}$-expectation $x_0 + \Phi(q)^{-1}$, and

$$\left| \mathrm{e}^{-q\tau^\pi_a} X_{\tau^\pi_a} \right| \leq \mathrm{e}^{-q\tau^\pi_a}(\overline{X}_{\tau^\pi_a} - \underline{X}_{\tau^\pi_a}),$$

where $\underline{X}_t = \inf_{0\leq s\leq t} X_s$, which has $\mathbb{P}_{x_0}$-expectation that is bounded by the finite number $2x_0 + \mathbb{E}[\overline{X}_{\eta(q)} - \underline{X}_{\eta(q)}]$ where $\eta(q)$ denotes an independent exponential random time.

Finally, note that the finiteness of $\beta^*_{f,+}(\alpha^*_f)$ follows by a line of reasoning that is analogous to the one that was used in the proof of Thm. 5.3, while we have $\beta^*_{f,+}(\alpha^*_f) > \alpha^*_f$ by definition of $\beta^*_{f,+}(\alpha^*_f)$.

(ii) Observe that in the case $\alpha^*_f < \infty$ we have $\alpha^*_f \leq \beta^*_{f,-} < \beta^*_{f,+} < \infty$, where the first strict inequality is a direct consequence of the fact that it will never be optimal to pay a lump-sum dividend smaller than the transaction cost $K > 0$. The proof of the rest of the assertions in (ii) is analogous to that of part (i), and is omitted.

(iii) In the cases $K > 0$ or $\{K = 0$ and $\sigma^2 > 0$ or $\nu_{0,1} = \infty\}$ the equality $\alpha^* = \beta^*_+(\alpha^*)$ would imply that $V_{\alpha^*,\beta^*} \equiv f$— however, if there exists a u such that ${}_0\mathcal{L}^f_\infty(u) > 0$, there exist α, β such that $V_{\alpha,\beta}(x) > f(x)$ for $x \in (\alpha, \beta)$, which would yield a contradiction. $\square$

10.3. **Optimality of multi-dividend-bands policies.** Denote by $\underline{v}_* = (v_{i,j})_{(i,j)}$, $\underline{a}^* = (a^*_{i,j})_{(i,j)}$ and $\underline{b}^* = (b^*_{i,j})_{(i,j)}$ the sequence of value-functions and band levels generated by the algorithm in Sect. 7, where the index (i,j) refers to the ith iteration of the algorithm in the jth run of the algorithm (i.e. it has been restarted $j-1$ times, cf. Rem. 7.14). In particular, we have that $v_{i,j}$ is given by

$$v_{i,j}(x) = \begin{cases} V_{\underline{a}^*,\underline{b}^*}(x) & x \in [0, b^*_{i,j,+}], \\ x - b^*_{i,j,+} + v_{i,j}(b^*_{i,j,+}) & x > b^*_{i,j,+}. \end{cases} \tag{10.11}$$

The following result concerns the optimality of multi-dividend bands strategies and implies in particular Thm. 7.15:

Prop. 10.5. *(i) For a given pair (i,j) of iteration and run, $v_{i,j}$ is equal to the value-function $v_{\underline{a}_{i,j},\underline{b}_{i,j}}$ of the multi-dividend-bands strategy $\pi_{\underline{a}_{i,j},\underline{b}_{i,j}}$ at levels $\underline{a}^*_{i,j} = (0, a^*_{1,1}, \ldots, a^*_{i-1,j}, \infty)$ and $\underline{b}^*_{i,j} = (b^*_{1,1}, \ldots, b^*_{i,j})$.*

(ii) For each pair (i,j), $v_{(i,j)}(x) = v_*(x)$ for all $x \le b^*_{i,j,+}$.

(iii) The optimal value function v_* is equal to the value function $V_{\underline{a}^*,\underline{b}^*}$ of the strategy $\pi_{\underline{a}^*,\underline{b}^*}$.

Proof. (i) The strong Markov property of the process $U = U^{\pi_{\underline{a}_{i,j},\underline{b}_{i,j}}}$ applied at the stopping time $\tau^- = \tau^\pi_{a^*_{i-1,j}}$ implies

$$v_{i,j}(x) = \mathbb{E}_x\left[\int_0^{\tau^-} \mathrm{e}^{-qt}\mu^\pi_K(\mathrm{d}t) + v_{i-1,j}(U_{\tau^-})\right], \tag{10.12}$$

where $\pi = \pi_{\underline{a}_{i,j},\underline{b}_{i,j}}$. The form of $v_{i,j}$ follows by induction, starting from the expression for a single dividend band strategy and using the form of the value-function of the auxiliary stochastic control problem in Eqn. (3.15) (subsequently applied with pay-off functions $f(x) = v_{\pi_{\underline{a}^*_{k,l},\underline{b}^*_{k,l}}}(b^*_{k,l,+} + x)$ for all pairs (k,l) such that (l,k) is smaller than $(j, i-1)$ (in the lexico-graphical order).

(ii) The statement follows by induction (in k). Indeed, note that, from Cor. 7.11, it follows $v_{(2,1)}(x) = v_*(x)$ for all $x \le b^*_{2,1,+}$. Furthermore, that the induction step holds is verified as follows: Assuming that $v_{(k-1,l)}(x) = v_*(x)$ for all $x \le b^*_{k-1,l,+}$ for some pair (k,l), Thm. 6.5 with $f = {}_{b^*_{k-1,l,+}}v_*$ and the relation in Eqn. (10.12) imply that the previous line is valid with $(k-1,l)$ replaced by (k,l).

(iii) Since $v_{i,j}(x) = V_{\underline{a}^*,\underline{b}^*}(x)$ for all $x \le a^*_{i-1,j}$ (from Eqn. (10.11)), it follows by virtue of part (ii) that $v_*(x) = V_{\underline{a}^*,\underline{b}^*}(x)$ for all $x \le a^*_{i-1,j}$. Observing that the sequence $(a_{i,j})_{i,j}$ is strictly increasing and ultimately tends to infinity (cf. Step 1 of the algorithm and Lem. 6.3(i,ii)), we deduce $v_*(x) = V_{\underline{a}^*,\underline{b}^*}(x)$, for any fixed $x \in \mathbb{R}_+$. □

Appendix

Appendix A. Proof of Dynamic Programming Equation

Proof of Lem. 3.1 (ii). Fix arbitrary $\pi \in \Pi$, $x \in \mathbb{R}_+$ and $s, t \in \mathbb{R}_+$ with $s < t$. It is clear that V^π_t is $\mathcal{F}_t$-measurable and integrable on account of Lem. 9.3. Fix arbitrary $\pi \in \Pi$, $x \in \mathbb{R}_+$. Define by $W^\pi = \{W^\pi_s, s \in \mathbb{R}_+\}$ the following value-process:

$$W^\pi_s = \operatorname*{ess.sup}_{\tilde\pi \in \Pi_s} J^{\tilde\pi}_s, \qquad J^{\tilde\pi}_s = \mathbb{E}\left[\int_0^{\tau^{\tilde\pi}} \mathrm{e}^{-qu}\mu^{\tilde\pi}_K(\mathrm{d}u) + \mathrm{e}^{-q\tau^{\tilde\pi}} w(U^{\tilde\pi}_{\tau^{\tilde\pi}}) \middle| \mathcal{F}_s\right], \tag{A.1}$$

where $\Pi_s \subset \Pi$ denotes the set of strategies

$$\Pi_s = \left\{\tilde\pi = (\pi, \overline{\pi}) = \{D^{\pi,\overline{\pi}}_u, u \in \mathbb{R}_+\} : \overline{\pi} \in \Pi\right\}, \qquad D^{\pi,\overline{\pi}}_u = \begin{cases} D^\pi_u, & u \in [0,s); \\ D^\pi_s + D^{\overline{\pi}}_{u-s}(U^\pi_s), & u \ge s, \end{cases}$$

where $D^{\overline{\pi}}(x)$ denote the process of cumulative dividends of the strategy $\overline{\pi}$ corresponding to initial capital $X_0 = x$.

It follows that V^π is a supermartingale as direct consequence of the following $\mathbb{P}$-a.s. relations:

(a) $V^\pi_s = W^\pi_s$,
(b) $W^\pi_s \geq \mathbb{E}[W^\pi_t|\mathcal{F}_s]$,

where W^π is the process defined in Eqn. (A.1).

Proof of (b): The identity follows by classical arguments. Since the family of random variables $\{J^{\tilde\pi}_t, \tilde\pi \in \Pi_t\}$ is directed upwards, it follows from Neveu [38] that there exists a sequence $\pi_n \in \Pi_t$ such that $J^{\tilde\pi_n}_t \uparrow W^\pi_t$. Since $\Pi_t \subset \Pi_s$ it follows that W^π_s dominates $J^{\pi_n}_s = \mathbb{E}[J^{\pi_n}_t|\mathcal{F}_s]$, so that monotone convergence implies that we have

$$W^\pi_s \geq \lim_n \mathbb{E}[J^{\pi_n}_t|\mathcal{F}_s] = \mathbb{E}[W^\pi_t|\mathcal{F}_s].$$

Proof of (a): The form of $D^{\tilde\pi}$ implies that, conditional on U^π_s, $\{D^{\tilde\pi}_u - D^{\tilde\pi}_s, u \geq s\}$ is independent of $\mathcal{F}_s$. On account of the Markov property of X it also follows that conditional on U^π_s, $\{U^{\tilde\pi}_u - U^{\tilde\pi}_s, u \geq s\}$ is independent of $\mathcal{F}_s$. As a consequence, we have the following identity on the set $\{s < \tau^\pi\}$

$$\begin{aligned}
\mathbb{E}\left[\int_0^{\tau^{\tilde\pi}} \mathrm{e}^{-qu}\mu^{\tilde\pi}_K(\mathrm{d}u) + \mathrm{e}^{-q\tau^{\tilde\pi}} w(U^{\tilde\pi}_{\tau^{\tilde\pi}})\middle|\mathcal{F}_s\right] &= \mathrm{e}^{-qs}\mathbb{E}_{U^\pi_s}\left[\int_0^{\tau^{\overline\pi}} \mathrm{e}^{-qu}\mu^{\overline\pi}_K(\mathrm{d}u) + \mathrm{e}^{-q\tau^{\overline\pi}} w(U^{\overline\pi}_{\tau^{\overline\pi}})\right] \\
&\quad + \int_0^s \mathrm{e}^{-qu}\mu^\pi_K(\mathrm{d}u) \\
&= \mathrm{e}^{-qs} v_{\overline\pi}(U^\pi_s) + \int_0^s \mathrm{e}^{-qu}\mu^\pi_K(\mathrm{d}u).
\end{aligned}$$

In particular, $\mathbb{P}_x$-a.s. the following representation holds true:

$$J^{\tilde\pi}_s = \mathrm{e}^{-q(s\wedge\tau^\pi)} v_{\overline\pi}(U^\pi_{s\wedge\tau^\pi}) + \int_0^{s\wedge\tau^\pi} \mathrm{e}^{-qu}\mu^\pi_K(\mathrm{d}u),$$

which yields the following $\mathbb{P}_x$-a.s. representation for W^π_s:

$$W^\pi_s = \int_0^{s\wedge\tau^\pi} \mathrm{e}^{-qu}\mu^\pi_K(\mathrm{d}u) + \mathrm{e}^{-q(s\wedge\tau^\pi)} \operatorname*{ess.sup}_{\tilde\pi=(\pi,\overline\pi)\in\Pi_s} v_{\overline\pi}(U^\pi_{s\wedge\tau^\pi}). \tag{A.2}$$

In view of the definitions of Π_s and v_*, the essential supremum in Eqn. (A.2) is $\mathbb{P}$-a.s. equal to $v_*(U^\pi_{s\wedge\tau^\pi})$, which implies that, $\mathbb{P}$-a.s., $W^\pi_s = V^\pi_s$. $\square$

Appendix B. Properties of Gerber-Shiu functions

We collect below a number of key properties of the function F_w for pay-offs w in the set $\mathcal{R}$ (which was defined in Def. 2.2).

Lem. B.1. *Let $w \in \mathcal{R}$. Then the following hold true:*

(i) The function F_w can be expressed in terms of $W^{(q)}$ as follows:

$$F_w(x) = \frac{\sigma^2}{2} w'(0-)W^{(q)}(x) + w(0)Z^{(q)}(x) - \int_0^x W^{(q)}(x-y)w_\nu(y)\mathrm{d}y, \quad x \geq 0. \tag{B.1}$$

(ii) The value of F_w at $x = 0$ matches $w(0)$: $F_w(0) = w(0)$.

(iii) The following asymptotics hold true:

$$\frac{F_w(x)}{W^{(q)}(x)} \sim \kappa_w, \qquad \text{as } x \to \infty,$$

where κ_w is defined in Eqn. (4.8).

(iv) If X has paths of bounded variation, the Laplace transform of F_w simplifies as follows:

$$\int_0^\infty \mathrm{e}^{-\theta x} F_w(x)\mathrm{d}x = (\psi(\theta) - q)^{-1}[pw(0) - \widetilde{w}^*_\nu(\theta)],$$

where $\widetilde{w}^*_\nu$ is the Laplace transform of the function $\widetilde{w}_\nu : \mathbb{R}_+\backslash\{0\} \to \mathbb{R}$ given by $\widetilde{w}_\nu(x) = \int_{(x,\infty)} w(x-y)\nu(\mathrm{d}y)$.

Note that representation (B.1) and the continuity of $W^{(q)}|_{\mathbb{R}_+}$ imply that the Laplace-transform inverse in Eqn. (4.6) admits a continuous version on $\mathbb{R}_+$, which justifies Def. 4.2.

Proof. (i) The identity follows by term-wise inverting the Laplace transform (4.6), using the form (1.4) of the Laplace transform of $W^{(q)}$.

(ii) This follows directly from Eqn. (B.1) and the facts that $Z^{(q)}(0) = 1$ and $\sigma^2 W^{(q)}(0) = 0$.

(iii) Since $W^{(q)}(x) \sim \mathrm{e}^{\Phi(q)x}/\psi'(\Phi(q))$ as $x \to \infty$, the statement follows from Eqn. (B.1).

(iv) If X has bounded variation then we have $\overline{\nu}_1 := \int_0^\infty \overline{\nu}(x)\mathrm{d}x < \infty$. Hence, for any $x > 0$, $\widetilde{w}_\nu$ is finite and satisfies $w_\nu(x) = \widetilde{w}_\nu(x) - w(0)\overline{\nu}(x)$. □

Restricting to penalties w from the set $\mathcal{P}$ (which was defined in Def. 2.1) we have a number of additional properties. Recall that we denote the right-derivative of F at $x \geq 0$ by $F'(x)$.

Lem. B.2. *Let $w \in \mathcal{P}$.*

(i) The function $w_\nu : \mathbb{R}_+\backslash\{0\} \to \mathbb{R}$ defined in Eqn. (2.3) is increasing and right-continuous, and satisfies the following integrability condition:

$$\int_0^x |w_\nu(y)|\mathrm{d}y < \infty \qquad \text{for any } x > 0. \tag{B.2}$$

(ii) The function $J_w : \mathbb{R}_+\backslash\{0\} \to \mathbb{R}$ given by $J_w(x) = ({}_0\mathcal{L}^w_\infty p_w - qp_w)(x)$ for any $x > 0$, with $p_w(x) = w'(0-)x + w(0)$ and the map ${}_0\mathcal{L}^w_\infty$ defined in Eqn. (4.12), is right-continuous, and is equal to the following expression:

$$J_w(y) = [\psi'(0) - m_\nu(y)]w'(0-) + w_\nu(y) - q(w'(0-)y + w(0)), \qquad y > 0, \tag{B.3}$$

where w_ν is given in (2.3) and the map $m_\nu : \mathbb{R}_+\backslash\{0\} \to (-\infty, 0)$ is given by $m_\nu(x) = \int_x^\infty (x - z)\nu(\mathrm{d}z)$.

(iii) $F_w(x)$ is left- and right-differentiable at any $x > 0$ with right-derivative at $x > 0$ given by

$$F'_w(x) = \frac{\sigma^2}{2}w'(0-)W^{(q)\prime}(x) + w(0)qW^{(q)}(x) - \int_{[0,x)} w_\nu(x - y)W^{(q)}(\mathrm{d}y), \tag{B.4}$$

where the first term is zero if $\sigma^2 = 0$. In the case $\{\sigma^2 > 0$ or $\nu_{0,1} = \infty\}$, then $F_w|_{\mathbb{R}_+\backslash\{0\}} \in C^1(\mathbb{R}_+\backslash\{0\})$.

(iv) The following alternative representation of $F'_{w,+}(x)$ holds true:

$$F'_w(x) = w'(0-) - \int_{[0,x)} J_w(x - y)W^{(q)}(\mathrm{d}y), \qquad x > 0. \tag{B.5}$$

In particular, $x \mapsto F'_w(x)$ is right-continuous on $(0, \infty)$.

(v) The right-derivative at $x = 0$ of F_w takes the following form:

$$F'_w(0) = \begin{cases} w'(0-), & \text{in the case } \{\sigma^2 > 0 \text{ or } \nu_{0,1} = \infty\}, \\ -J_w(0+)W^{(q)}(0) = \frac{q}{p}w(0) - \frac{1}{p}w_\nu(0), & \text{in the case } \{\sigma^2 = 0 \text{ and } \nu_{0,1} < \infty\}, \end{cases} \tag{B.6}$$

where p and $\nu_{0,1}$ were defined in Eq. (1.7).

(vi) The map $F_w : \mathbb{R}_+\backslash\{0\} \to \mathbb{R}$ is equal to a difference of monotone functions.

Proof. (i) The integrability condition (B.2) follows from the condition (2.2) (as we have the inclusion $\mathcal{P} \subset \mathcal{R}$). The right-continuity and monotonicity of w_ν follow on account of the dominated convergence theorem and the monotonicity and right-continuity of w.

(ii) The representation in Eqn. (B.3) follows directly from the form of the operator ${}_0\mathcal{L}^w_\infty$ given in Eqn. (4.12). The function J_w inherits the right-continuity from w_ν, on account of in view of Eqn. (B.3) and the continuity of m_ν.

(iii) Recall that $W^{(q)}(x)$ is right- and left-differentiable at any $x > 0$ (with finite derivatives and with right-derivative at x denoted by $W^{(q)\prime}(x)$). The final term on the rhs of Eqn. (B.1) is also right-differentiable with derivative equal to the third term on the rhs of Eqn. (B.4), on account of the dominated convergence theorem, the monotonicity and right-continuity of w_ν and the right-differentiability of $W^{(q)}$. An analogous line of reasoning shows that $F'_w|_{\mathbb{R}_+\backslash\{0\}}$ is in fact continuous in case that we have $\sigma^2 > 0$ or $\nu_1 = \infty$, as it holds that $W^{(q)}|_{\mathbb{R}_+\backslash\{0\}}$ is C^1 in that case.

(iv) The equality of (B.4) and (B.5) can be verified by taking Laplace transforms, using that the Laplace transforms of $W^{(q)}$ and m_ν are given by Eqn. (1.4) and by the following expression:

$$m^*_\nu(\theta) = \theta^{-2}\int_0^\infty [\mathrm{e}^{-\theta z} - 1 + \theta z]\nu(\mathrm{d}z) = \theta^{-2}\left[\psi(\theta) - \theta\psi'(0) - \tfrac{\sigma^2}{2}\theta^2\right].$$

(v) If X has bounded variation, then $w_\nu(0+)$ exist and is finite. On account of the monotonicity of w_ν, the continuity of $W^{(q)}|_{\mathbb{R}_+}$ and the fact $W^{(q)}(0) = p^{-1}$, the expression in Eqn. (B.6) follows by taking the limit of x to 0 in Eqn. (B.4). If X has unbounded variation, the form of $F'_w(0+)$ follows by the fact that the convolution in Eqn. (B.5) vanishes as x tends to zero. This fact is a consequence of the following two observations: (1) Let $\eta > 0$ and $\delta > 0$ be such that, for all $z \in (0,\delta)$, $|\Delta w(-z) - w'(0-)| \leq \eta$, where $\Delta w(z) = \frac{w(z)-w(0)}{z}$. Then the form of w_ν implies that the following estimate holds true:

$$|w_\nu(x)| \leq \int_{[\delta,\infty)} |w(-y) - w(0-)|\nu(\mathrm{d}y) + \eta|m_\nu(x)|, \qquad x > 0. \tag{B.7}$$

(2) For any $a, b \geq 0$, define the function $K : \mathbb{R}_+\backslash\{0\} \to \mathbb{R}$ by $K(x) := \int_0^x (a - bm_\nu(x-y))W^{(q)}(\mathrm{d}y)$. As K is increasing and has a Laplace transform $K^*(\theta) = (\psi(\theta) - q)^{-1}\theta(a - bm^*_\nu(\theta))$ that satisfies $K^*(\theta) \sim c/\theta$ as θ tends to infinity for some constant c, a Tauberian theorem implies that $K(x)$ tends to zero as x tends to zero. The stated fact now follows by combining the observations (1) and (2) with the fact $W^{(q)}(0) = 0$.

(vi) The statement follows on account of the representation in Eqn. (B.4) and the facts that w_ν is monotone and non-positive and that $W^{(q)\prime}|_{\mathbb{R}_+\backslash\{0\}}$ is equal to the difference of two monotone functions (which holds as $W^{(q)}$ is log-concave, cf [37, Lemma 6]). □

In the case of exponential boundary condition w we record the following additional properties:

Rem. B.3. The family of functions $Z^{(q,v)}$ contains as member the function $Z^{(q,0)} = Z^{(q)}$, which corresponds to the case of a boundary condition equal to 1. Further, from (8.1) we read off that $Z_0 = Z^{(q)}$, and if $E[|X_1|] < \infty$, that $Z_1(x)$ is given by

$$Z_1(x) = x + q\overline{W}^{(q,1)}(x) - \psi'(0)\overline{W}^{(q)}(x), \tag{B.8}$$

where $\overline{W}^{(q,1)}(x) = \int_0^x (x-y)W^{(q)}(y)\mathrm{d}y$. More generally, if $E[|X_1|^k] < \infty$, then $\psi^{(r)}(0)$ is finite for $r = 1,\dots,k$, and the following representation holds true by an application of the Leibniz rule:

$$Z_k(x) = x^k + q\overline{W}^{(q,k)}(x) - \sum_{n=1}^k \binom{k}{n}\psi^{(n)}(0)\overline{W}^{(q,k-n)}(x) \tag{B.9}$$

with $\psi^{(n)}(0)$ being the nth right-derivative of ψ at zero and

$$\overline{W}^{(q,n)}(x) = \int_0^x (x-y)^n W^{(q)}(y)\mathrm{d}y.$$

Rem. B.4. *If w is an exponential, $w = e_v$, the function F_w reduces to the function $Z^{(q,v)}$ defined in Eqn. (7.2). Indeed, the Laplace transforms $F^*_{e_v}$ and $(Z^{(q,v)})^*$ of $F_{e_v}|_{\mathbb{R}_+}$ and $Z^{(q,v)}|_{\mathbb{R}_+}$ are both equal to*

$$F^*_{e_v}(\theta) = (Z^{(q,v)})^*(\theta) = (\psi(\theta) - q)^{-1}\frac{\psi(\theta) - \psi(v)}{\theta - v}.$$

Rem. B.5. (i) For $v \geq 0$, the function $x \mapsto Z^{(q,v)}(x)$ is strictly increasing on $\mathbb{R}_+$. In particular, for $x > 0$ and $v > \Phi(q)$, $Z^{(q,v)\prime}(x)$ is equal to

$$Z^{(q,v)\prime}(x) = (\psi(v) - q) \int_x^\infty \mathrm{e}^{v(x-y)} W^{(q)}(\mathrm{d}y). \tag{B.10}$$

which can be derived from Eqns. (1.4) and (7.2) by integration by parts.

(ii) The map $v \mapsto v^{-1} Z^{(q,v)\prime}(x)$ is completely monotone on $(\Phi(q), \infty)$, for any $x > 0$. This follows since $v \mapsto v^{-1} Z^{(q,v)}(x)$ is the Laplace transform of some measure on $\mathbb{R}_+$ which we show now. From the definition of $Z^{(q,v)}$ we find that the derivative $Z^{(q,v)\prime}(x)$ at $x > 0$ satisfies

$$Z^{(q,v)\prime}(x) = v Z^{(q,v)}(x) + (q - \psi(v)) W^{(q)}(x).$$

Inserting the forms of the Laplace transforms of $W^{(q)}|_{\mathbb{R}_+}$ and $Z^{(q,v)}|_{\mathbb{R}_+}$ (given in Eqn. (1.4) and Rem. (4.3), respectively), we find

$$\int_0^\infty \mathrm{e}^{-\theta x} Z^{(q,v)\prime}(x) \mathrm{d}x = \frac{q}{\psi(\theta) - q} + \frac{\theta v}{\psi(\theta) - q} \left[\frac{\sigma^2}{2} + \int_{\mathbb{R}_+ \backslash \{0\}} \frac{\mathrm{e}^{-\theta y} - \mathrm{e}^{-vy}}{v - \theta} \overline{\nu}(y) \mathrm{d}y \right]. \tag{B.11}$$

Observing that we have

$$\int_{\mathbb{R}_+ \backslash \{0\}} \frac{\mathrm{e}^{-\theta y} - \mathrm{e}^{-vy}}{v - \theta} \overline{\nu}(y) \mathrm{d}y = \int_{\mathbb{R}_+ \backslash \{0\}} \int_{\mathbb{R}_+ \backslash \{0\}} \mathrm{e}^{-\theta s - vt} \overline{\nu}(s+t) \mathrm{d}t \mathrm{d}s,$$

and inverting the Laplace transform in Eqn. (B.11) yields the expression

$$v^{-1} Z^{(q,v)\prime}(x) = \frac{q}{v} W^{(q)}(x) + \frac{\sigma^2}{2} W^{(q)\prime}(x) + \int_{\mathbb{R}_+ \backslash \{0\}} \int_{[0,x]} \mathrm{e}^{-vt} \overline{\nu}(x - y + t) W^{(q)}(\mathrm{d}y)\, \mathrm{d}t, \qquad x > 0.$$

By inspection we see that, for any $x > 0$, the function $v \mapsto v^{-1} Z^{(q,v)\prime}(x)$ is the Laplace transform of a measure on $[0, \infty)$, which implies the stated complete monotonicity.

(iii) If, for some $v_0 > 0$, $\mathbb{E}[\mathrm{e}^{-v_0 X_1}]$ is finite, $\psi(v)$ and $v \mapsto Z^{(q,v)}(x)$ can be analytically extended into a neighbourhood of $v = 0$, and $Z^{(q,v)}(x)$ can be expanded in terms of Z_k, $k \in \mathbb{N}$, as follows:

$$Z^{(q,v)}(x) = \sum_{k=0}^\infty \frac{v^k}{k!} Z_k(x).$$

Proof of Prop. 8.2. Note that, by changing measure and inserting form in Eqn. (4.4) of $\mathcal{V}^w_{a,b}$ for $w \equiv 1$, the following expression can be derived for $v \geq 0$:

$$\begin{aligned}
&\mathbb{E}_x[\mathrm{e}^{-qT_{a,b} + v(X_{T_{a,b}} - a)} \mathbf{1}_{\{T_a^- < T_b^+\}}] \\
&= \mathrm{e}^{(x-a)v} \mathbb{E}_x[\mathrm{e}^{-qT_{a,b} + \psi(v) T_{a,b} + v(X_{T_{a,b}} - x) - \psi(v) T_{0,a}} \mathbf{1}_{\{T_a^- < T_b^+\}}] \\
&= \mathrm{e}^{(x-a)v} \mathbb{E}_x^v[\mathrm{e}^{-(q - \psi(v)) T_{a,b}} \mathbf{1}_{\{T_a^- < T_b^+\}}] \\
&= \mathrm{e}^{(x-a)v} \left[Z_v^{(q-\psi(v))}(x-a) - \frac{Z_v^{(q-\psi(v))}(b-a)}{W_v^{(q-\psi(v))}(b-a)} W_v^{(q-\psi(v))}(x-a) \right],
\end{aligned}$$

where $W_v^{(r)}, Z_v^{(r)}$ are the r-scale functions under $\mathbb{P}^v$, the Cramér-Esscher change of measure of $\mathbb{P}$ with Radon-Nikodym derivative defined by $\frac{\mathrm{d}\mathbb{P}^v}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = \exp(vX_t - \psi(v)t)$. Using the identity (from [7])

$$W^{(q)}(x) = \mathrm{e}^{xv} W_v^{(q-\psi(v))}(x), \qquad v \geq 0, q \geq 0,$$

we find (8.2). The identity (8.3) follows by a similar line of reasoning. The uniqueness follows from Thm. 4.5.
□ □

Proof of Prop. 4.4. Writing $\mathcal{V}_w^{0,\infty}(x) = w(0)\mathbb{E}_x[\mathrm{e}^{-qT_0^-}]+\mathbb{E}_x[\mathrm{e}^{-qT_0^-}(w(X_{T_0^-})-w(0))]$ and applying the compensation formula to the Poisson point process $(\Delta X_t, t \in \mathbb{R}_+)$ yield the following expressions for any $x \in \mathbb{R}_+$:

$$\mathcal{V}_w^{0,\infty}(x) - w(0)\mathcal{V}_{e_0}^{0,\infty}(x) = \int_0^\infty \int_y^\infty (w(y-z)-w(0))\nu(\mathrm{d}z)U^q(x,\mathrm{d}y) \tag{B.12}$$

$$= W^{(q)}(x)w_\nu^*(\Phi(q)) - \int_0^x W^{(q)}(x-y)w_\nu(y)\mathrm{d}y, \tag{B.13}$$

where $U^q(x,\mathrm{d}y)$ is the q-potential measure of X under $\mathbb{P}_x$ killed upon entering $(-\infty,0)$,

$$U^q(x,\mathrm{d}y) = [W^{(q)}(x)\mathrm{e}^{-\Phi(q)y} - W^{(q)}(x-y)]\mathrm{d}y, \qquad y > 0,$$

and $\mathcal{V}_{e_0}^{0,\infty}(x) = \mathbb{E}_x[\mathrm{e}^{-qT_0^-}]$ is expressed in terms of the scale function $W^{(q)}$ by

$$\mathcal{V}_{e_0}^{0,\infty}(x) = Z^{(q)}(x) - \frac{q}{\Phi(q)}W^{(q)}(x).$$

The two integrals in above display are finite in view of the integrability condition (2.2) and the fact that $W^{(q)}|_{\mathbb{R}_+}$ is continuous. Thus, Eqn. (4.7) follows from Eqn. (B.1) (since the term $\frac{\sigma^2}{2}w'(0-)W^{(q)}(x)$ cancels).

The martingale property in Eqn. (4.10) follows from Eqn. (4.7) and the strong Markov property of X, and the fact that

$$\left(\mathrm{e}^{-q(t\wedge T_a^-)}W^{(q)}(X_{t\wedge T_a^-}-a), t\in\mathbb{R}_+\right) \qquad \text{is a } \mathbb{P}_x\text{-martingale for any } x\in\mathbb{R}.$$

□

Proof of Theorem 4.5: An application of the compensation formula yields the following representation of $\mathcal{U}_w^{a,b}(x)$:

$$\mathcal{U}_w^{a,b}(x) - w(0)\mathcal{U}_{e_{0,a}}^{a,b}(x) = \int_a^b \int_y^\infty (w(y-z)-w(0))\nu(\mathrm{d}z)R_{a,b}^q(x,\mathrm{d}y),$$

where $\mathcal{U}_{e_{0,a}}^{a,b}$ is given in (8.3), and q-resolvent measure $R_{a,b}^q(x,\mathrm{d}y)$ of Y^b killed upon entering $(-\infty,a)$ which is given by ([41, Thm. 1])

$$R_{a,b}^q(x,\mathrm{d}y) = \frac{W^{(q)}(x-a)}{W^{(q)\prime}(b-a)}W^{(q)}(b-\mathrm{d}y) - W^{(q)}(x-y)\mathrm{d}y, \qquad x,y\in[a,b]. \tag{B.14}$$

Combining these expressions with Lem. B.2(iii) and taking note of the fact that the term $\frac{\sigma^2}{2}{}_aw'(0-)W^{(q)}(x)$ cancels yields that Eqn. (4.5) holds with $F = F_{{}_aw}$. □

Appendix C. Proofs of optimality of single dividend-band strategies

C.1. **Key representation.** The following result provides an explicit connection between the function the boundary influence function G and the infinitesimal generator of X:

Prop. C.1. *Let $c > 0$ and $b_+ \geq b_- \geq 0$ (with $b_+ \neq b_-$ in the case $K > 0$). (i) The following identity holds true:*

$$W^{(q)\prime}(b_+ + c)[G(b_-, b_+ + c) - G(b_-, b_+)] = \int_{[0,c]} ({}_{b_+}\mathcal{L}_\infty^{v_b} v_b)(b_+ + c - y)W^{(q)}(\mathrm{d}y) \tag{C.1}$$

$$= 1 - F'_{{}_{b_+}v_b}(c). \tag{C.2}$$

(ii) If $G(b_-, b_+ + c) \leq G(b_-, b_+)$, then $F'_{{}_{b_+}v_b}(c) \geq 1$.

(iii) The functions $y \mapsto G(b^-, y)$ and $y \mapsto G^\#(y)$ are decreasing for all y sufficiently large.

The proof of Prop. C.1 is based on the following representation which is itself a consequence of the shifting lemma and the pasting lemma:

Lem. C.2. *For any $c>0$ and any $b_+\geq b_-\geq 0$, (with $b_+\neq b_-$ if $K>0$) the following identity holds true for any $x\leq b_++c$:*

$$\mathbb{E}_x\left[\mathrm{e}^{-q(t\wedge\tau_{b+c})}v_b(U^{b+c}_{t\wedge\tau_{b+c}})+\int_0^{t\wedge\tau_{b+c}}\mathrm{e}^{-qs}\mathrm{d}D^{b+c}_s\right]-v_b(x) \tag{C.3}$$

$$=\quad \mathbb{E}_x\left[\int_0^{t\wedge\tau_{b+c}}\mathrm{e}^{-qs}({}_{b_+}\mathcal{L}^{\overline{w}}_\infty v_b)(U^{b+c}_{s-})\mathbf{1}_{\{U^{b+c}_{s-}>b_+\}}\mathrm{d}s\right]. \tag{C.4}$$

with $\overline{w}=v_{v_b}$, and we denoted $\tau_{b+c}=\tau^{\pi(b_-,b_++c)}$, $D^{b+c}=D^{\pi(b_-,b_++c)}$ and $U^{b+c}=U^{\pi(b_-,b_++c)}$.

Proof of Prop. C.1. First consider the case $K=0$. Denoting the q-resolvent of Y^{b_++c} killed upon entering $(-\infty,0)$ by

$$R^q_{0,b_++c}(x,\mathrm{d}y)=\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(Y^{b_++c}_t\in\mathrm{d}y,t<\tau_0)\mathrm{d}t$$

and letting $t\to\infty$ in (C.3) the dominated convergence theorem implies that for $x\in(0,b_++c)$

$$v_{b+c}(x)-v_b(x)\quad=\quad\mathbb{E}_x\left[\int_0^{\tau_{b+c}}\mathrm{e}^{-qs}[{}_{b_+}\mathcal{L}^{\overline{w}}_\infty v_b](U^{b+c}_{s-})\mathbf{1}_{\{U^{b+c}_{s-}>b_+\}}\mathrm{d}s\right] \tag{C.5}$$

$$=\quad\int_{[b_+,b_++c]}[{}_{b_+}\mathcal{L}^{\overline{w}}_\infty v_b](y)R^q_{0,b_++c}(x,\mathrm{d}y), \tag{C.6}$$

where $\overline{w}=v_b$. Inserting the explicit expressions (5.1) and (B.14) for v_b,v_{b+c} and $R^q_{0,b_++c}(x,\mathrm{d}y)$, we find that

$$W^{(q)}(x)[G^{\#}(b_++c)-G^{\#}(b_+)]=W^{(q)}(x)\int_{[b_+,b_++c]}[{}_{b_+}\mathcal{L}^{\overline{w}}_\infty v_b](y)\frac{W^{(q)}(b_++c-\mathrm{d}y)}{W^{(q)\prime}(b_++c)},\qquad x\in(0,b_+),$$

with $G^{\#}$ defined in Eqn. (7.1) and where we used that $W^{(q)}(x)=0$ for $x<0$. Changing coordinates in the integral and using that $W^{(q)}(x)$ is strictly positive at any $x>0$ yields the first equality in Eqn. (C.1). The second equality in Eqn. (C.1) follows by the representation in Eqn. (B.5) and the fact $v_b'(b_+-)=1$. The case $b_+=0$ follows by approximation, taking the limit of b_+ to zero. The proof of the case $K>0$ is similar and omitted.

The statement in (ii) is a direct consequence of Eqn. (C.1). The ultimate monotonicity of $y\mapsto G(b_-,y)$ and $y\mapsto G^{\#}(y)$ follows from the fact that ${}_{b_+}\mathcal{L}^w_\infty v_b(x)$ tends to minus infinity when $x\to\infty$. □

Proof of Lem. 10.4. If the function G^* is decreasing, then the function Ξ is completely monotonicity in view of the form of Ξ given in Eqn. (7.3), the complete monotonicity of $\theta^{-1}\mathrm{e}^{\theta(b-x)}Z^{(q,\theta)\prime}(x)$ (cf. Rem. B.5(ii)) and the following facts:

(i) A function $f:(c,\infty)\to\mathbb{R}_+$, $c>0$, is completely monotone if f is the Laplace transform of a measure supported on $[0,\infty)$.

(ii) If $f(\theta)$ is the Laplace transform of the measure μ supported on $[0,\infty)$ then, for any $c>0$, $\mathrm{e}^{-\theta c}f(\theta)$ is the Laplace transform of the translated measure $y\mapsto\mathbf{1}_{\{y\geq c\}}\mu(\mathrm{d}(y-c))$.

(iii) If $\theta\mapsto f_x(\theta)$, $x>b$, $b\in\mathbb{R}$, is a collection of Laplace transforms of measures μ_x on $[0,\infty)$ and m is a measure supported on the interval $[b,\infty)$, then $\theta\mapsto\int_{[b,\infty)}f_x(\theta)m(\mathrm{d}x)$ is equal to the Laplace transform of the measure supported on $[0,\infty)$ given by $\int_{[b,\infty)}\mu_x(\mathrm{d}y)m(\mathrm{d}x)$.

□

Proof of Cor. 7.9. In view of Cor. 3.5, it suffices to verify that Eqn. (10.8) is satisfied.

We need to show that $J(x)\leq 0$ for all $x>0$ where $J:\mathbb{R}_+\backslash\{0\}\to\mathbb{R}$ is given by $J(x):=({}_{b_+^*}\mathcal{L}^{\tilde{w}}_\infty v_{b^*})(b_+^*+x)$ with $\tilde{w}=v_{b^*}$. In view of the forms of the operator ${}_{b_+^*}\mathcal{L}^w_\infty$ and of $v_{b_+^*}(x)$ for $x>b_+^*$, it follows that $J(x)$ is given by the following expression:

$$J(x)\quad=\quad\psi'(0)-q(x+v(b))+\int_0^\infty[v(b-y)-v(b)+y]\nu'(x+y)\mathrm{d}y,\qquad x>0, \tag{C.7}$$

where we denoted $b = b_+^*$ and $v = v_{b^*}$.

The assertion that $J(x) \leq 0$ for any $x > 0$ then follows once we show that

(i) J is concave on $\mathbb{R}_+\backslash\{0\}$,
(ii) $J(0+) = 0$ and
(iii) $J'(0+) \leq 0$.

To show (i) note that under condition (a) the integrand in (C.7) is non-positive for all y. Indeed, for $y \in (0, b)$, $[v(b-y) - v(b) + y] \leq 0 \Leftrightarrow v(b) - v(b-y) \geq y$ (as $K = 0$), and for $y \geq b$ we have that $w(b-y) - v(0) - b + y \leq 0$ and $v(0) - v(b) + b \leq 0$ which yields that $w(b-y) - v(b) \leq y$ for $y \geq b$. As ν' is convex, and a mixture of convex functions with positive weights is again convex, we deduce that J is concave on $\mathbb{R}_+\backslash\{0\}$.

Given (ii) statement (iii) follows since if $J'(0+)$ were positive, $(J(x) - J(0))/x = J(x)/x$ would be positive which would be in contradiction with Eqn. (C.8) below.

To see that (ii) holds, note that, from (C.1) with $b_- = b_-^*$ and $b_+ = b_+^*$,

$$0 \geq \int_{[0,c]} J(c-y) W^{(q)}(\mathrm{d}y) \qquad \text{for all } c > 0 \text{ sufficiently small.} \tag{C.8}$$

Thus, we deduce that $J(0+) \leq 0$.

To complete the proof we next verify that $J(0+) = 0$. First consider the case that σ^2 is strictly positive: The observations that, for any $b > 0$, $\mathrm{e}^{-q(t\wedge T_{0,b})} v_b(X_{t\wedge T_{0,b}})$ is a martingale and $v_b \in C^2$ together with Itô's lemma yield that $({}_0\mathcal{L}_\infty^w v_b)(x) = 0$ for all $x \in (0, b_+)$ which in turn implies that $J(0) = ({}_0\mathcal{L}_\infty^w v_b)(b_+) = 0$ on account of the continuity of $x \mapsto ({}_0\mathcal{L}_\infty^w v_{b^*}(x)$ at $x = 0$.

Consider next the case $\{\sigma^2 = 0$ and $\nu_{0,1} < \infty\}$. It follows by taking Laplace transforms in Eqn. (4.13) that $({}_0\mathcal{L}_\infty^w v_b)(x) = 0$ for Leb-a.e. $x \in (0, b_+)$. Let $x_n \in (0, b_+^*)$ be a sequence tending to b satisfying $({}_0\mathcal{L}_\infty^w v_b)(x_n) = 0$. On account of Fatou's lemma, the convexity of ν', the continuity of $W^{(q)}|_{\mathbb{R}_+}$ and $W^{(q)\prime}|_{\mathbb{R}_+\backslash\{0\}}$ and the fact $v(b-y) - v(b) \leq y$ for all $y \leq b$, we deduce $J(0+) \geq 0$:

$$0 = \lim_n ({}_0\mathcal{L}_\infty^w v_b)(x_n) \leq \psi'(0) - qv(b) + \int_0^\infty (v(b-y) - v(b) + y)\nu'(y)\mathrm{d}y = J(0+).$$

Hence also in the case that X has bounded variation it holds that $J(0+) = 0$.

The case $\{\sigma^2 = 0$ and $\nu_{0,1} = \infty\}$ follows by approximation: we claim that by adding a small Brownian component with variance $\sigma^2 > 0$ to X and then letting $\sigma^2 \to 0$, it follows that also in this case $J(0+) = 0$.

To verify this claim we show that $J(0+) \geq 0$. If $\sigma \searrow 0$, the continuity theorem implies that the scale functions $W^{(q)(\sigma)}$ and $F_w^{(\sigma)}$ of the perturbed process $X^{(\sigma)} := X + \sigma B$ (where B is a Brownian motion independent of X) and the corresponding derivatives $W^{(q)(\sigma)\prime}$ and $F_w^{(\sigma)\prime}$ converge pointwise to the corresponding (derivatives of) scale functions of X at any point of continuity. Denote by $J^{(\sigma)}(x)$ the expression on the rhs of (C.7) with the function v replaced by the function $v^{(\sigma)}$ corresponding to the perturbed process $X^{(\sigma)}$. An application of Fatou's lemma, which is justified on account of the bounds in Lem. 9.1 and Lem. 9.3, then yields that

$$0 = \lim_{\sigma\searrow 0} J^{(\sigma)}(x) \leq J(x), \qquad \text{for any } x > 0.$$

The proof is complete. □

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