GENERALISED ARBITRAGE-FREE SVI VOLATILITY SURFACES

GAOYUE GUO, ANTOINE JACQUIER, CLAUDE MARTINI, AND LEO NEUFCOURT

Abstract. In this article we propose a generalisation of the recent work of Gatheral-Jacquier [11] on explicit arbitrage-free parameterisations of implied volatility surfaces. We also discuss extensively the notion of arbitrage freeness and Roger Lee's moment formula using the recent analysis by Roper [18]. We further exhibit an arbitrage-free volatility surface different from Gatheral's SVI parameterisation.

1. Introduction

European option prices are usually quoted in terms of the corresponding implied volatility, and over the last decade a large number of papers (both from practitioners and academics) has focused on understanding its behaviour and characteristics. The most important directions have been towards (i) understanding the behaviour of the implied volatility in a given model (see [2], [3] [8], [12] for instance) and (ii) deciphering its behaviour in a model-independent way, as in [17], [18] or [19]. These results have provided us with a set of tools and methods to check whether a given parameterisation is free of arbitrage or not. In particular, given a set of observed data (say European calls and puts for different strikes and maturities), it is of fundamental importance to determine a methodology ensuring that both interpolation and extrapolation of this data are also arbitrage-free. Such approaches have been carried out for instance in [4], in [7] and in [20]. Several parameterisations of the implied volatility surface have now become popular, in particular [13], [15] and [9], albeit not ensuring absence of arbitrage.

In the recent paper [11], Gatheral and Jacquier proposed a new class of SVI implied volatility parameterisation, originally proposed in [9]. In particular they provide explicit sufficient and—in a certain sense—almost necessary conditions ensuring that such a surface is free of arbitrage. This new class depends on the maturity and can hence be used to model the whole volatility surface, and not a single slice. It also depends on the at-the-money total implied variance θ_t, and on a positive function φ such that the total variance w as a function of time-to-maturity t and log-(forward)-moneyness k is given by $w(k,t) \equiv \theta_t \mathrm{SVI}_\rho(k\varphi(\theta_t))$, where SVI_ρ is the classical (normalised) SVI parameterisation from [11], and ρ an asymmetry parameter (essentially playing the role of the correlation between spot and volatility in stochastic volatility models).

In this work, we generalise their framework to volatility surfaces parameterised as $w(k,t) \equiv \theta_t \Psi(k\varphi(\theta_t))$ for some (general) function Ψ. We obtain (Sections 3 and 4) necessary and sufficient conditions coupling the functions Ψ and φ, so that the corresponding implied volatility surface is free of arbitrage. This allows us to obtain (i) the exact set of admissible functions φ in the symmetric ($\rho = 0$) SVI case, and (ii) a constraint-free parameterisation of Gatheral-Jacquier functions satisfying [11, Corollary 1]. Along the road (Section 5) we also refine the notion of arbitrage freeness used in [11], in connection with the recent

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analysis by Roper [18]. More precisely, we get a sufficient condition for a real function to represent the slice of an arbitrage-free implied volatility surface at a given maturity. We provide a purely analytical 'ticket proof' of Roger Lee moment formula [17] in our (restricted) case. In the last section (Section 6), we exhibit examples of non-SVI arbitrage-free implied volatility surface after formulating an extension of Roper's theorem in implied volatility to convex non-smooth smiles.

Notations: In this paper, we consider European option prices with maturity $t \geq 0$ and strike $K \geq 0$, written on an underlying stock $(S_t)_{t\geq 0}$. Without loss of generality we shall always assume that $S_0 = 1$ and that interest rates are null, and hence the log (forward) moneyness reads $k := \log(K)$. We shall denote the Black-Scholes value for a European Call option $\mathrm{C}_{\mathrm{BS}}(K, w)$, for a strike K and total variance w, and more generally $\mathrm{C}(K, t)$ for (any) European Call prices with strike K and maturity t. For any $k \in \mathbb{R}$ and t$\geq$ 0, the corresponding implied volatility is denoted by $\sigma(k, t)$ and the total variance w is defined by $w(k, t) := \sigma(k, t)^2 t$. With a slight abuse of language (commonly accepted in the finance jargon), we shall refer to the two-dimensional map $(k, t) \mapsto w(k, t)$ as the (implied) volatility surface. Finally, for two functions g and h, we shall say that $g(z) \sim h(z)$ at $z = 0$ whenever $\lim_{z\to 0} g(z)/h(z) = 1$.

2. Gatheral-Jacquier SVI parameterisation

Extending the pioneering work of [9], Gatheral and Jacquier introduced in [11] the following parameterisation for the total implied variance w:

$$w(k,t) = \frac{\theta_t}{2}\left\{1 + \rho k \varphi(\theta_t) + \sqrt{(k\varphi(\theta_t)+\rho)^2 + (1-\rho^2)}\right\}, \tag{2.1}$$

with $\theta_t > 0$ for $t > 0$ and φ is a smooth function from $\mathbb{R}_+^*$ to $\mathbb{R}_+$ and $\rho \in (-1, 1)$. The main result in their paper (Corollary 5.1) is the following theorem, which provides sufficient conditions for the implied volatility surface w to be free of static arbitrage:

Theorem 2.1. *The surface* (2.1) *is 'free of static arbitrage' if the following conditions are satisfied:*

(1) $\partial_t\theta_t \geq 0$ *for all* $t > 0$*;*
(2) $\varphi(\theta) + \theta\varphi'(\theta) \geq 0$ *for all* $\theta > 0$*;*
(3) $\varphi'(\theta) < 0$ *for all* $\theta > 0$*;*
(4) $\theta\varphi(\theta)(1 + |\rho|) < 4$ *for all* $\theta > 0$*;*
(5) $\theta\varphi(\theta)^2(1 + |\rho|) \leq 4$ *for all* $\theta > 0$.

A few remarks are in order here:

(1) The notion of 'static arbitrage freeness' is defined in [11] as equivalent to the positivity of the numerator and the denominator of Dupire's equation. These properties are necessary, yet the right statement is that there exists an inhomogeneous diffusion process associated to the implied volatility through Dupire's equation. To this end, the local volatility defined through Dupire's equation should have adequate continuity and growth properties, which should be discussed at least. An alternative route is to follow Roper conditions in [18], which add some limiting behaviour to Gatheral and Jacquier conditions; we shall come back on this issue in detail in Section 5;
(2) the conditions in Theorem 2.1 are sufficient, but may be far from necessary;
(3) the full characterisation of the functions φ guaranteeing absence of (static or not) arbitrage in the symmetric SVI case $\rho = 0$ is left open.

(4) it would be useful to 'parameterise' the set of functions φ satisfying the conditions of Theorem 2.1. This could lead to easy-to-implement calibration algorithms among the whole admissible class, without being tied to a particular family as in [11].

In this paper, we try to settle all these points. We also state our results in a slightly more general framework, which is not tied to the specific shape of the SVI model, i.e. we consider implied volatility surfaces of the form

$$w(k,t) = \theta_t \Psi(k\varphi(\theta_t)), \quad \text{for all } k \in \mathbb{R}, t \geq 0. \tag{2.2}$$

We shall make the following assumptions on the parameters:

Assumption 2.2.

(i) $\theta \in \mathcal{C}^1\left(\mathbb{R}_+^* \to \mathbb{R}_+\right)$, $\lim_{t\searrow 0} \theta_t = 0$ and θ is not constant;
(ii) $\varphi \in \mathcal{C}^1\left(\mathbb{R}_+^* \to \mathbb{R}_+^*\right)$;
(iii) $\Psi \in \mathcal{C}^2\left(\mathbb{R} \to \mathbb{R}_+^*\right)$, $\Psi(0) = 1$ and Ψ is not constant.

The time-dependent function θ models the at-the-money total variance; the assumption on its behaviour at the origin is thus natural. A constant function Ψ corresponds to deterministic time-dependent volatility, a trivial case we rule out here. Likewise, were θ assumed to be constant, it would be null everywhere, which we shall also not consider. The main goal here is to provide sufficient conditions on the triplet (θ, φ, Ψ) that will guarantee absence of static arbitrage. Note that when the correlation $|\rho|$ is strictly small than 1, the SVI parameterisation (2.1) corresponds to the case $\Psi(z) = \frac{1}{2}(1 + \rho z + \sqrt{z^2 + 2\rho z + 1})$. In the sequel, we shall refer to this case as the SVI case. In the following sections 3 and 4 we stick to the definitions of arbitrage freeness as used in [11]. In Section 5 the cleaner formulation of [18] is applied.

3. Elimination of calendar spread arbitrage

We first concentrate here on determining (necessary and sufficient) conditions on the triplet (θ, φ, Ψ) in order to eliminate calendar spread arbitrage.

3.1. **The first coupling condition.**

The following definition is now well-known, and further details (and a proof) can be found in [10] or [11]

Definition 3.1. A volatility surface w is free of calendar spread arbitrage if and only if $\partial_t w(k,t) \geq 0$, for all $k \in \mathbb{R}$ and $t > 0$.

The quantity $\partial_t w(k,t)$ is nothing else than the numerator of the local volatility expressed in terms of the implied volatility, i.e. Dupire's formula (see [10]). Define now the functions $F : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ by

$$F(z) := z\frac{\Psi'(z)}{\Psi(z)}, \qquad f(u) := u\frac{\varphi'(u)}{\varphi(u)}. \tag{3.1}$$

These functions will play a major role in our analysis, and Assumption 2.2 (iii) implies in particular that $F(z) \sim \Psi'(0)z$ at the origin and $F(0) = 0$. Note that Ψ and φ can be recovered through the identities

$$\Psi(z) = \exp\left(\int_0^z \frac{F(u)}{u}\right), \qquad \varphi(u) = \varphi(r)\exp\left(\int_r^u \frac{f(v)}{v}\right),$$

for some arbitrary constant $r > 0$. The following proposition gives new conditions for absence of calendar spread arbitrage.

Proposition 3.2. (**First coupling condition**) *Let $\theta_\infty := \lim_{t\nearrow\infty}\theta_t$. The surface w in (2.2) is free of calendar spread arbitrage if and only if the following two conditions hold:*

(i) θ is not decreasing;
(ii) $1 + F(z)f(u) \geq 0$ for any $z \in \mathbb{R}$ and $u \in (0, \theta_\infty)$.

Proof. By Definition 3.1, the surface defined by (2.2) is free of calendar spread arbitrage if and only if

$$\partial_t w(k,t) = \theta'_t\Psi(z) + \theta_t\Psi'(z)k\varphi'(\theta_t)\theta'_t \geq 0, \quad \text{for all } k \in \mathbb{R}, t > 0, \tag{3.2}$$

where we define $z := k\varphi(\theta_t)$. Since $\Psi(z)$ is strictly positive by Assumption 2.2 (iii), the inequality (3.2) is equivalent to $\theta'_t(1 + F(z)f(\theta_t)) \geq 0$ for all $z \in \mathbb{R}$ and $t > 0$, where the functions F and f are defined in (3.1). For $z = 0$ we get $\theta'_t \geq 0$ for all $t > 0$. Otherwise (ii) is necessary and sufficient for the surface to be free of calendar spread arbitrage. □

Remark 3.3. We do not assume here that θ_∞ is infinite. In most popular stochastic volatility models with or without jumps, θ_∞ is infinite. In fact, as studied in [16], as soon as the random family $\left(t^{-1}\log(S_t)\right)_{t>0}$ satisfies a large deviations principle as t tends to infinity with a continuous rate function and speed t^{-1}, then $\theta_\infty = \infty$. Rogers & Tehranchi [19] showed that for a non-negative martingale $(S_t)_{t\geq 0}$ satisfying $S_t > 0$ almost surely for all $t \geq 0$, the equality $\theta_\infty = \infty$ is equivalent to the almost sure equality $\lim_{t\nearrow\infty} S_t = 0$ (where the limit exists by the martingale convergence theorem). However, it may occur that $\theta_\infty < \infty$. As a corollary of coupling properties of stochastic volatility models, Hobson [14] provides instances where such a phenomenon appears. Simple examples are the SABR [13] model with $\beta = 1$ and models with bubbles.

Remark 3.4. Condition (ii) in Proposition 3.2 can be stated in a more compact way:

$$1 - \sup F_+ \sup f_- \geq 0 \qquad \text{and} \qquad 1 - \sup F_- \sup f_+ \geq 0,$$

where $f_+ := \max(f, 0)$ and $f_- := \max(-f, 0)$.

Motivated by the celebrated moment formula in [17], which forces the function Ψ to be at most linear at (plus/minus) infinity, let us propose the following definition:

Definition 3.5. We say that the function Ψ is (strictly) asymptotically linear if and only if Ψ' has a finite (non-zero) limit α_+ (resp. α_-) when $z \to +\infty$ (resp. $z \to -\infty$).

Using this definition we obtain a necessary condition on the behaviour of the function φ in (2.2).

Proposition 3.6. *If Ψ is strictly asymptotically linear and if there is no calendar spread arbitrage, then the map $u \mapsto u\varphi(u)$ is non-decreasing.*

Proof. Using (3.1), if Ψ is strictly asymptotically linear, then $\lim_{z\to\pm\infty} z\Psi'(z)/\Psi(z) = \lim_{z\to\pm\infty} F(z) = 1$, so that absence of calendar spread arbitrage implies $1 + f(u) \geq 0$ for any $u \in (0, \theta_\infty)$ by Proposition 3.2 (ii). Since φ is a strictly positive function by Assumption 2.2 (ii), the proposition follows from the definition of the function f in (3.1). □

Note that if $\lim_{z\to\pm\infty}\Psi'(z)=0$ then the limit of the function F at (plus or minus) infinity does not necessarily exist. Whenever it does, since the function $z\mapsto\Psi(z)/z$ is decreasing as $z\to\pm\infty$, the limit can take any value in $(-\infty,1)$.

3.2. **Application to SVI.** In the SVI case (2.1), we have $\Psi'(z)\equiv\frac{1}{2}\left(\rho+\frac{z+\rho}{\sqrt{z^2+2\rho z+1}}\right)$, so that Ψ is strictly asymptotically linear when $|\rho|<1$ with $\alpha_+=\rho+1$ and $\alpha_-=\rho-1$. Therefore Proposition 3.6 applies, and a necessary condition is that $u\mapsto u\varphi(u)$ is not decreasing. In [11, Theorem 5.1], this condition and the condition that φ is not increasing are shown to be sufficient. In the case of the symmetric SVI model (where $\rho=0$), the following corollary relates our conditions and that of [11].

Corollary 3.7. *In the symmetric SVI case, the necessary condition of Proposition 3.6 is also sufficient.*

Proof. Let us consider the the symmetric case $\rho=0$. We compute explicitly

(3.3)
$$\Psi(z)=\frac{1+\sqrt{1+z^2}}{2},\qquad \Psi'(z)=\frac{1}{2}\frac{z}{\sqrt{1+z^2}},\qquad \Psi''(z)=\frac{z^2}{\sqrt{1+z^2}\left(1+\sqrt{1+z^2}\right)},\quad\text{for all } z\in\mathbb{R}.$$

Therefore

$$F(z)=\frac{z^2}{\sqrt{1+z^2}\left(1+\sqrt{1+z^2}\right)}\quad\text{and}\quad F'(z)=\frac{\left(z^2+2\left(1+\sqrt{1+z^2}\right)\right)z}{(1+z^2)^{3/2}\left(1+\sqrt{1+z^2}\right)^2}.$$

It is then clear that the even function F is strictly increasing on $\mathbb{R}_+^*$ and strictly decreasing on $\mathbb{R}_-^*$ with a global minimum attained at the origin for which $F(0)=0$. In light of Remark 3.4, we have $\sup F_+=1$ and $\sup F_-=0$. By Proposition 3.2 there is hence no calendar spread arbitrage if and only if $f(u)\geq -1$, which is equivalent to $u\mapsto u\varphi(u)$ being non-decreasing. □

4. Elimination of butterfly arbitrage

We now consider the butterfly arbitrage which, probably not surprisingly, is more subtle to handle.

4.1. **The second coupling condition.** We define the operator $\mathcal{L}$ acting on $\mathcal{C}^{2,1}$ functions by

$$\mathcal{L}w(k,t):=\left(1-\frac{k\partial_k w(k,t)}{2w(k,t)}\right)^2-\frac{\partial_k w(k,t)^2}{4}\left(\frac{1}{w(k,t)}+\frac{1}{4}\right)+\frac{\partial^2_{kk}w(k,t)}{2},\tag{4.1}$$

for all $k\in\mathbb{R}$ and $t>0$. The operator does not act on the second component of the function. We however keep this notation for clarity. When the function w represents the total implied variance, $\mathcal{L}w$ is nothing else that the denominator of the local variance expressed in terms of the implied volatility, i.e. Dupire's formula, see for instance [10, Chapter 1]. Recall the following definition from [11, Definition 4.1]: [1]

Definition 4.1. A volatility surface w is free of butterfly arbitrage if and only if $\mathcal{L}w(k,t)\geq 0$ for all $k\in\mathbb{R}$ and $t>0$.

We now reformulate the butterfly arbitrage condition in our setting. For any $u>0$ define the set

$$Z_+(u):=\left\{z\in\mathbb{R}:\frac{1}{4u}\left(\frac{\Psi'(z)^2}{\Psi(z)}-2\Psi''(z)\right)+\frac{\Psi'(z)^2}{16}>0\right\},\tag{4.2}$$

where we use the convention $\inf_{\emptyset}=\infty$.

[1] The authors of [11] have informed us that a revised version of the preprint [11] includes an additional condition ensuring that the corresponding density integrates to one. Section 5 below precisely deals with this, so we leave it as it is for now.

Proposition 4.2. Second coupling condition, general formulation *The surface w given in* (2.2) *is free of butterfly arbitrage if and only if*

$$(u\varphi(u))^2 \leq \inf_{z\in Z_+(u)} \frac{\left(1-\frac{z\Psi'(z)}{2\Psi(z)}\right)^2}{\frac{1}{4u}\left(\frac{\Psi'(z)^2}{\Psi(z)}-2\Psi''(z)\right)+\frac{\Psi'(z)^2}{16}}, \quad \textit{for all } u\in(0,\theta_\infty).$$

Proof. From (2.2), we clearly have $\partial_k w(k,t) = \theta_t \Psi'(z)\varphi(\theta_t)$ and $\partial^2_{kk} w(k,t) = \theta_t\Psi''(z)\varphi(\theta_t)^2$ for all $k\in\mathbb{R}$ and $t>0$. Therefore

$$\begin{aligned}
\mathcal{L}w(k,t) &= \left(1-\frac{k\partial_k w(k,t)}{2w(k,t)}\right)^2 - \frac{\partial_k w(k,t)^2}{4}\left(\frac{1}{w(k,t)}+\frac{1}{4}\right)+\frac{\partial^2_{kk}w(k,t)}{2}\\
&= \left(1-\frac{k\theta_t\Psi'(z)\varphi(\theta_t)}{2\theta_t\Psi(z)}\right)^2 - \frac{(\theta_t\Psi'(z)\varphi(\theta_t))^2}{4}\left(\frac{1}{\theta_t\Psi(z)}+\frac{1}{4}\right)+\frac{\theta_t\Psi''(z)\varphi(\theta_t)^2}{2}\\
&= \left(1-\frac{z\Psi'(z)}{2\Psi(z)}\right)^2 - \frac{\theta_t\varphi(\theta_t)^2}{4}\left(\frac{(\Psi')^2(z)}{\Psi(z)}-2\Psi''(z)\right)-\frac{(\theta_t\varphi(\theta_t))^2}{16}(\Psi')^2(z)\\
&= \left(1-\frac{z\Psi'(z)}{2\Psi(z)}\right)^2 - (\theta_t\varphi(\theta_t))^2\left\{\frac{1}{4\theta_t}\left(\frac{(\Psi')^2(z)}{\Psi(z)}-2\Psi''(z)\right)+\frac{(\Psi')^2(z)}{16}\right\}. \quad (4.3)
\end{aligned}$$

Using the definition of the set $Z_+(u)$ in (4.2), the proposition follows. □

4.2. **Asymptotically linear Ψ.** We now consider the case where Ψ is strictly asymptotically linear. Define the sets

$$\overline{Z}_+ := \left\{z\in\mathbb{R} : \left(\frac{\Psi'(z)^2}{\Psi(z)}-2\Psi''(z)\right)>0\right\}, \quad \text{and} \quad \overline{Z}_- := \mathbb{R}\setminus\overline{Z}_+, \quad (4.4)$$

as well as the constant

$$M_\infty := \lim_{u\nearrow+\infty} u\varphi(u). \quad (4.5)$$

The following proposition is a reformulation of Proposition 3.6 in the asymptotically linear case. It provides sufficient and necessary conditions for the surface (2.2) to be free of calendar spread arbitrage.

Proposition 4.3. *Assume Ψ is strictly asymptotically linear and there is no calendar spread arbitrage. There is no butterfly arbitrage if and only if the following two conditions hold:*

(i)

$$M_\infty^2 \leq \inf_{z\in\overline{Z}_-\bigcap Z_+(\theta_\infty)} \frac{\left(1-\frac{z\Psi'(z)}{2\Psi(z)}\right)^2}{\frac{1}{4\theta_\infty}\left(\frac{\Psi'(z)^2}{\Psi(z)}-2\Psi''(z)\right)+\frac{\Psi'(z)^2}{16}},$$

where M_∞ is defined in (4.5)

(ii) for any $u\in(0,\theta_\infty)$,

$$(u\varphi(u))^2 \leq \inf_{z\in\overline{Z}_+} \frac{\left(1-\frac{z\Psi'(z)}{2\Psi(z)}\right)^2}{\frac{1}{4u}\left(\frac{\Psi'(z)^2}{\Psi(z)}-2\Psi''(z)\right)+\frac{\Psi'(z)^2}{16}}.$$

Moreover $\overline{Z}_+$ is not empty and not bounded from above.

Proof. Assume that Ψ is strictly asymptotically linear and that there is no calendar spread arbitrage. Then $u \mapsto u\varphi(u)$ is non-decreasing on $\mathbb{R}_+^*$ by Proposition 3.6. On $\overline{Z}_-$ the map $u \mapsto \frac{1}{4u}\left(\frac{\Psi'(z)^2}{\Psi(z)} - 2\Psi''(z)\right) + \frac{\Psi'(z)^2}{16}$ is clearly also non-decreasing on $\mathbb{R}_+^*$. Therefore $(Z_+(u))_{u>0}$ is a non-decreasing family of sets and $Z_+(0) = \overline{Z}_+$. The inequality

$$(u\varphi(u))^2 \leq \inf_{z\in\overline{Z}_-\bigcap Z_+(u)} \frac{\left(1-\frac{z\Psi'(z)}{2\Psi(z)}\right)^2}{\frac{1}{4u}\left(\frac{\Psi'(z)^2}{\Psi(z)} - 2\Psi''(z)\right) + \frac{\Psi'(z)^2}{16}}, \quad \text{for all } u \in (0, \theta_\infty)$$

is equivalent to

$$M_\infty^2 \leq \inf_{z\in\overline{Z}_-\bigcap Z_+(\theta_\infty)} \frac{\left(1-\frac{z\Psi'(z)}{2\Psi(z)}\right)^2}{\frac{1}{4\theta_\infty}\left(\frac{\Psi'(z)^2}{\Psi(z)} - 2\Psi''(z)\right) + \frac{\Psi'(z)^2}{16}}.$$

Note that in case $\theta_\infty = \infty$ this simplifies to:

$$M_\infty \leq \inf_{z\in\overline{Z}_-} \left|\frac{4}{\Psi'(z)} - \frac{2z}{\Psi(z)}\right|.$$

Statement (ii) in the proposition is then an immediate consequence of Proposition 4.2 and the computation above. Let us remark also that in this case $\overline{Z}_+$ is not empty. If it is then we can then choose $a > 0$ such that $\Psi'(z) > 0$ for all $z > a$. This implies that $\Psi(z)^{-1} \leq 2\Psi''(z)/\Psi'(z)^2$ for all $z > a$, which in turn yields $\int_a^z \Psi(b)^{-1}\mathrm{d}b \leq 2\left(\Psi'(\alpha)^{-1} - \Psi'(z)^{-1}\right)$. The integral diverges to infinity as z tends to infinity since $\Psi(z) \sim \alpha_+ z$ whereas the right-hand side is bounded by Assumption 2.2. The same argument shows that $\overline{Z}_+$ is not bounded from above. □

Coupling condition (ii) with $\theta_\infty = \infty$, we can also take u to ∞, and we obtain the necessary condition

$$M_\infty \leq \inf_{z\in\mathbb{R}} \left|\frac{4}{\Psi'(z)} - \frac{2z}{\Psi(z)}\right|;$$

in particular $M_\infty \leq 2/\sup\{|\alpha_+|, |\alpha_-|\}$.

4.3. **Application to symmetric SVI.** As in Section 3.2 above, we show that in the symmetric SVI case, all our expressions above are easily computed and give rise to simple formulations. Let us define the functions Y, A and and A^* by

$$A(y,u) := \frac{16uy(y+1)}{8(y-2) + uy(y-1)}, \tag{4.6}$$

$$Y(u) := \frac{2}{1-u/4} + \sqrt{\left(\frac{2}{1-u/4}\right)^2 + \frac{2}{1-u/4}}, \tag{4.7}$$

$$A^*(u) := A(Y(u), u). \tag{4.8}$$

Of course we only define these functions on their effective domains, the forms of which we omit for clarity. The following proposition makes the conditions of Proposition 4.3 explicit in the uncorrelated SVI case.

Proposition 4.4. *In the symmetric SVI* (2.1) *case $\rho = 0$, there is no butterfly arbitrage if and only if*

$$u\varphi(u) \leq A^*(u)\mathbb{1}_{\{u<4\}} + 16\,\mathbb{1}_{\{u\geq 4\}}, \quad \text{for all } u \in (0, \theta_\infty).$$

Proof. For any $z \in \mathbb{R}$, using (3.3) and letting $\xi_z := z^2$ and $y_z := \sqrt{1+\xi_z}$, we can compute

$$\begin{aligned} \left(\Psi'(\xi_z)\right)^2 - 2\Psi(\xi_z)\Psi''(\xi_z) &= \frac{1}{4}\frac{(y_z-2)(y_z+1)^2}{y_z^3}, \\ 1 - \frac{z\Psi'(\xi_z)}{2\Psi(\xi_z)} &= \frac{1}{2}\left(1+\frac{1}{y_z}\right), \\ \Psi'(\xi_z)^2 &= \frac{1}{4}\left(1-\frac{1}{y_z^2}\right), \end{aligned}$$

where by $\Psi'(\xi_z)$ we really mean $\partial_u \Psi(u)|_{u=\xi_z}$. From the first equation, the set $\overline{Z}_+$ defined in (4.4) is exactly $\{z \in \mathbb{R} : y_z > 2\} = \mathbb{R} \setminus [-\sqrt{3}, \sqrt{3}]$. For any fixed u, the function appearing on the right-hand side of Proposition 4.3 (ii) can be computed explicitly and is equal (after simplifications) to $A(y_z, u)$ given in (4.3). In particular $A(2,u) = 48$ and $\lim_{y\nearrow\infty} A(y,u) = 16$. For any $u \geq 0$, we have

$$\partial_y A(y,u) = \frac{128 u B_u(y)}{\left(8y - 16 + y^2 u - yu\right)^2},$$

where $B_u(y) := \left(1 - \frac{u}{4}\right) y^2 - 4y - 2$. When $u \geq 4$, B_u is concave on $(2,\infty)$ with $B_u(2) = -(6+u) < 0$, and hence the map $y \mapsto A(y,u)$ is decreasing on $(2,\infty)$ and its infimum is equal to $\lim_{y\nearrow\infty} A(y,u) = 16$. For $u \in [0,4)$, the strict convexity of B_u and the inequality $B_u(2) = -(6+u) < 0$ implies that the equation $B_u(y) = 0$ has a unique solution in $(2,\infty)$, which in fact is equal to $Y(u)$ given in (4.7). Then the map $y \mapsto A(y,u)$ is decreasing on $(2, Y(u))$ and increasing on $(Y(u),\infty)$. Its infimum is attained at $Y(u)$ and is equal to $A^*(u)$ defined in (4.8). □

Remark 4.5. In [11], the authors prove that the two conditions (altogether) $u\varphi(u) < 4$ and $u\varphi(u)^2 < 4$ (for all $u \geq 0$) are sufficient to prevent butterfly arbitrage in the uncorrelated $\rho = 0$ case. These two conditions can be combined to obtain $(u\varphi(u))^2 < 16\min(1, \varphi(u)^{-1})$. A tedious yet straightforward computation shows that A^* is increasing on $[0,4)$ and maps this interval to $[0,16)$. Notwithstanding the fact that our condition is necessary and sufficient, it is then clear that

(i) for $u \geq 4$, it is also stronger than the one in [11] whenever $\varphi(u) < 1$;
(ii) for $u < 4$ (which corresponds to most practically relevant cases) it is also stronger whenever $16/\varphi(u) < A^*(u)$.

In particular, item (ii) could be used as a (moving) sufficient and necessary lower bound condition for the function φ on $[0,4)$.

5. Generalised Gatheral-Jacquier surfaces and absence of arbitrage

5.1. **Roper Large-Moneyness Behaviour condition.** As recalled in Definitions 3.1 and 4.1, Gatheral and Jacquier [11] define the absence of calendar spread arbitrage and of butterfly arbitrage by the non-negativity of the numerator and denominator of Dupire's equation in implied volatility. These properties are necessary, yet not sufficient, as shown by Roper [18]. More precisely, the additional condition from [18, Theorem 2.9], namely the Large-Moneyness Behaviour (LMB), reads

$$\lim_{k\nearrow+\infty} d_+(k, w(k,t)) = -\infty, \tag{5.1}$$

with the usual notation from the Black-Scholes formula $d_\pm(k,w) := -k/\sqrt{w} \pm \sqrt{w}/2$. Note that this condition is equivalent to call option prices tending to zero as the strike tends to (positive) infinity, as proved in [19, Theorem 5.3]. As mentioned in the footnote on page 5, this condition has just been added to Definition 4.1 in a revised version of [11]. The following lemma however shows that some other asymptotic behaviours of d_+ and d_- hold in great generality. This was proved by Rogers and Tehranchi [19] in a general framework, and we include here a short self-contained proof.

Lemma 5.1. *Let w be any positive real function. Then*

(i) $\lim_{k\nearrow+\infty} d_-(k, w(k)) = -\infty$*;*

(ii) $\lim_{k\searrow-\infty} d_+(k, w(k)) = +\infty$*.*

Proof. The arithmetic mean-geometric mean inequality implies

$$\begin{aligned} -d_-(k, w(k)) &= \frac{k}{\sqrt{w(k)}} + \frac{\sqrt{w(k)}}{2} \geq \sqrt{2k}, \qquad \text{for } k > 0, \\ d_+(k, w(k)) &= \frac{-k}{\sqrt{w(k)}} + \frac{\sqrt{w(k)}}{2} \geq \sqrt{-2k}, \qquad \text{for } k < 0. \end{aligned}$$

In the first line, the term on the right-hand side tends to $+\infty$ as k tends to $+\infty$, proving (i). Statement (ii) follows by the same argument. □

The following proposition bridges the gap between our framework and [18, Theorem 2.9]:

Proposition 5.2. *Assume in the setting of Proposition 4.3 that the inequality in Condition (ii) is strict. Then the corresponding implied volatility surface is free of static arbitrage.*

Proof. In our setting (with Ψ being strictly asymptotically linear) $\lim_{k\nearrow+\infty}(w(k,t)/k) = \theta_t\varphi(\theta_t)\alpha_+$, so that all we need to prove is that $\theta_t\varphi(\theta_t) < 2/\alpha_+$. For any $z \in \overline{Z}_+$ (defined in (4.4)), note that

$$\frac{\left(1 - \frac{z\Psi'(z)}{2\Psi(z)}\right)^2}{\frac{1}{4\theta_t}\left(\frac{\Psi'(z)^2}{\Psi(z)} - 2\Psi''(z)\right) + \frac{\Psi'(z)^2}{16}} \leq \frac{\left(1 - \frac{z\Psi'(z)}{2\Psi(z)}\right)^2}{\frac{\Psi'(z)^2}{16}}.$$

Choosing a sequence in $\overline{Z}_+$ diverging to infinity, we have $(\theta_t\varphi(\theta_t))^2 < 4/\alpha_+^2$ and the proposition follows. □

5.2. **Small-moneyness behaviour, no mass at zero, and symmetry.** The missing statements in Lemma 5.1 are the LMB Condition (5.1) and the Small-Moneyness Behaviour (SMB):

$$\lim_{k\searrow-\infty} d_-(k, v(k)) = +\infty.$$

To investigate further, let us remark that the framework developed in [18] encompasses situations where the underlying stock price can be null with positive probability. This can indeed be useful to model the probability of default of the underlyer. Computations similar in spirit to [18] show that the marginal law of the stock price at some fixed time $t > 0$ has no mass at zero if and only if

$$\lim_{K\searrow 0} \partial_K \mathrm{C}(K, t) = -1,$$

which is a statement about a 'small-moneyness' behaviour. This can be fully recast in terms of implied volatility, and the above missing conditions then come naturally into play:

Proposition 5.3. *(Symmetry under small-moneyness behaviour)*
Let a $\mathcal{C}^2$ real function v satisfy

(I) $v(k) > 0$ *and* $\mathcal{L}v(k) \geq 0$ *for all* $k \in \mathbb{R}$*;*

(II) $\lim_{k \searrow -\infty} d_-(k, v(k)) = +\infty$ *(SMB Condition);*

(III) $\lim_{k \nearrow +\infty} d_+(k, v(k)) = -\infty$ *(LMB Condition).*

Define two functions p_- *and* p_+ *by* $k \mapsto p_\pm(k) := (2\pi v(k))^{-1/2} \exp\left(-\frac{1}{2} d_\pm^2(k, v(k))\right) \mathcal{L}v(k)$*. Then*

(1) p_+ *and* p_- *define two densities of probability measures on* $\mathbb{R}$ *with respect to the Lebesgue measure, i.e.* $\int_{\mathbb{R}} p_-(k)\mathrm{d}k = \int_{\mathbb{R}} p_+(k)\mathrm{d}k = 1$*.*

(2) $p_+(k) = \mathrm{e}^k p_-(k)$*, so that* $\int_{-\infty}^{\infty} \mathrm{e}^k p_-(k)\mathrm{d}k = \int_{-\infty}^{\infty} \mathrm{e}^{-k} p_+(k)\mathrm{d}k = 1$*.*

(3) p_- *is the density of probability associated to Call option prices with implied volatility* v*, in the sense that* $p_-(k) \equiv \partial^2_{KK} \mathrm{C}_{\mathrm{BS}}(K, v(\log(K)))\big|_{K=\mathrm{e}^k}$*, and* $k \mapsto p_+(-k)$ *is the density of probability associated to Call option prices with implied volatility* $k \mapsto w(k) := v(-k)$*.*

Remark 5.4. Note that the strict positivity of the function v in Assumption (I) ensures that the support of the underlying distribution is the whole real line. One could bypass this assumption by considering finite support as in [19]. In the latter case—slightly more general case—the statements and proofs would be very analogous but much more notationally inconvenient to write down.

Proof. The functions p_- and p_+ are clearly well-defined and non-negative. Consider first p_-. It is readily seen that the function $D(k) \equiv \partial_K \mathrm{C}_{\mathrm{BS}}(K, v(\log(K)))\big|_{K=\mathrm{e}^k}$ is a primitive of p_-. We now proceed to prove that p_- is indeed a density. Let $\mathcal{N}$ denote the cumulative distribution function of the standard Gaussian distribution. An explicit computation yields (the reverse one can be found in [11, Lemma 4.1])

$$\begin{aligned}
\partial_K \mathrm{C}_{\mathrm{BS}}(K, v(\log(K))) &= \partial_K d_+ \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-d_+^2/2} - \mathcal{N}(d_-) - K\partial_K d_- \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-d_-^2/2} \\
&= \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-d_+^2/2} \left(\partial_K d_+ - K\partial_K d_- \mathrm{e}^{d_+\sqrt{v} - v/2}\right) - \mathcal{N}(d_-) \\
&= \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-d_+^2/2} \left(\partial_K d_+ - \partial_K d_-\right) - \mathcal{N}(d_-),
\end{aligned}$$

where all the functions are evaluated at $(K, \log(K))$, and where we have used the equality $d_- = d_+ - \sqrt{v}$. Evaluating the right-hand side at $K = \mathrm{e}^k$, and using the fact that $-k - \frac{1}{2}d_+^2 = -\frac{1}{2}d_-^2$, we obtain

$$D(k) = \frac{v'(k)}{2\sqrt{2\pi v(k)}} \exp\left(-\frac{d_-(k, v(k))^2}{2}\right) - \mathcal{N}(d_-).$$

Therefore if

$$\lim_{k \to \pm\infty} \frac{v'(k)}{2\sqrt{2\pi v(k)}} \exp\left(-\frac{d_-(k, v(k))^2}{2}\right) = 0, \tag{5.2}$$

then

$$\int_{\mathbb{R}} p_-(k)\mathrm{d}k = \lim_{k \searrow -\infty} \mathcal{N}(d_-(k, v(k))) - \lim_{k \nearrow +\infty} \mathcal{N}(d_-(k, v(k))) = 1,$$

where we have used Assumption (II): $\lim_{k \searrow -\infty} d_-(k, v(k)) = +\infty$. Let us first deal with the case where k tends to (positive) infinity. From Lemma 5.1, $\exp\left(-\frac{1}{2}d_-(k, v(k))^2\right)$ tends to zero. The key point is to observe that D is the primitive of a non-negative function, therefore it is non-decreasing and has a (generalised) limit $L \in [-\infty, \infty]$ as $k \nearrow +\infty$. Since $\mathcal{N}(d_-(k, v(k)))$ converges to zero by Lemma 5.1 (ii), we deduce that $\frac{v'(k)}{2\sqrt{2\pi v(k)}} \exp\left(-\frac{1}{2}d_-(k, v(k))^2\right)$ tends to L. From [19, Proof of Theorem 5.3], the inequality $v' < \sqrt{2v/k}$ holds for any $k > 0$ so that L is necessarily non-positive. Assume that L is negative (possibly infinite). Since $\frac{v'(k)}{2\sqrt{v(k)}} \equiv \partial_k \sqrt{v(k)}$, if L is negative then $\sqrt{v}$ is eventually decreasing. Since it is bounded from below by 0, there is a sequence $(k_n)_{n\geq 0}$ going to infinity such that $\partial_k \sqrt{v(k_n)}$ converges to zero by the mean value theorem. Therefore L must be 0.

Let us now consider the case where k tends to negative infinity. Using similar arguments, the quantity $\frac{v'(k)}{2\sqrt{2\pi v(k)}} \exp\left(-\frac{1}{2}d_-(k, v(k))^2\right)$ tends to $M \in [-\infty, \infty]$. Assume that $M < 0$. Then v is decreasing for k small enough. Since v is positive, this implies that $v(k) > \varepsilon$ for some $\varepsilon > 0$ and k small enough. In particular $1/v(k)$ is bounded. Since $v'(k) > -4$ for all $k \in \mathbb{R}$ by [19, Theorem 5.1], then for k small enough, the inequalities $-4 < v'(k) \leq 0$ hold, and the term outside the exponential in (5.2) is bounded. Since the exponential converges to zero by Lemma 5.1 (ii), we obtain $M = 0$. If $M > 0$, then v is increasing for k small enough. We conclude as above by the mean value theorem since $\sqrt{v}$ is increasing and bounded from below. Therefore $M = 0$ and the limit (5.2) holds.

So far we have proved that p_- is the density of probability associated to Call option prices with implied volatility $k \mapsto v(k)$. Consider now the function $w(k) \equiv (-k)$. Then for all $k \in \mathbb{R}$, $\partial_k w(k) = -\partial_k v(-k)$, and it follows by inspection that $\mathcal{L}w(k) = \mathcal{L}v(-k) \geq 0$. Consider the function $\widehat{p}_-$ associated to w, i.e.

$$\widehat{p}_-(k) := (2\pi w(k))^{-1/2} \exp\left(-\frac{1}{2}d_-^2(k, w(k))\right)\mathcal{L}w(k), \quad \text{for all } k \in \mathbb{R}.$$

Now $d_-(k, w(k)) = -\frac{k}{\sqrt{v(-k)}} - \frac{\sqrt{v(-k)}}{2} = -d_+(-k, v(-k))$, so that $\widehat{p}_-(k) = p_+(-k)$. In order for $\widehat{p}_-$ to be a genuine density, we need to check conditions symmetric to those ensuring that p_- is a density. The condition symmetric to the SMB assumption (II) is precisely Condition (i) in Lemma 5.1, and the condition symmetric to the Lemma 5.1 (ii) is precisely the LMB assumption (III). Therefore $k \mapsto p_+(-k)$ is also a density, associated to Call option prices with implied volatility w. Finally the identity $p_+(k) = \mathrm{e}^k p_-(k)$ follows immediately from the equality $-k - \frac{1}{2}d_+^2 = -\frac{1}{2}d_-^2$. □

The previous proposition has been intentionally stated in a maturity-free way: it is indeed a purely 'marginal' statement. A natural question is then to determine whether such a function v, satisfying the assumptions of Proposition 5.3, can represent the total implied variance smile at time 1 associated to some martingale (issued from 1 at time 0)? The answer is indeed positive and this can be proved as follows. Consider the natural filtration $\mathbb{B}$ of a standard (one-dimensional) Brownian motion, $(B_t)_{t\geq 0}$. Let P be the cumulated distribution function associated to p_- characterised in Proposition 5.3, and let $\mathcal{N}$ be the Gaussian cumulative distribution function. Then the random variable $X := P^{-1}(\mathcal{N}(B_1))$ has law P, and $E(X) = 1$. Set now $M_s := E(X|\mathbb{B}_s)$, then M is a martingale issued from 1. Note that M is even a Brownian martingale and therefore a continuous martingale. The associated Call option prices $E[(M_s - K)_+]$ uniquely determine a total implied variance surface $(t, k) \mapsto w(k, t)$ such that $v = w(1, \cdot)$.

5.3. A simplified version of Lee moment formula in the strictly asymptotically linear case.

So far we have stressed that, following Roper or the variant in Proposition 5.3 (no mass at the origin), the positivity of the operator $\mathcal{L}$ in (4.1) guarantees the existence of a martingale explaining market prices. As a consequence, the celebrated moment formula [17] holds. We show in this subsection that, at least in the strictly asymptotically linear case, it can be derived in a purely analytic fashion.

Proposition 5.5. *Consider a $\mathcal{C}^2$ function v satisfying the following conditions:*

(1) $v(k) > 0$ and $\mathcal{L}v(k) \geq 0$ for all $k \in \mathbb{R}$;

(2) $\lim_{k\nearrow+\infty} v'(k) = \alpha \in [0,2)$;

(3) $\lim_{k\nearrow+\infty} v''(k) = 0$.

Let X be a random variable with density p, where p is the probability density associated to v by Proposition 5.3. Then X satisfies $\mathbb{E}(X) = 1$ and $\sup\left\{m \geq 0 : \mathbb{E}\left(X^{1+m}\right) < \infty\right\} = \frac{1}{2}\left(\frac{\alpha}{4} - 1 + \alpha\right)$.

Proof. Condition (1) clearly implies Condition (I) in Proposition 5.3, and Conditions (2) and (3) imply the SMB and LMB limits (II) and (III) in Proposition 5.3. Therefore by Proposition 5.3, the centred probability density p is well defined on $\mathbb{R}$. For $m \in \mathbb{R}$, we therefore have $\mathrm{e}^{(1+m)k}p(k) = f(k)\mathrm{e}^{-g(k)}$, where

$$f(k) \equiv (2\pi v(k))^{-1/2}\,\mathcal{L}v(k) \qquad \text{and} \qquad g(k) \equiv \frac{1}{2}\left(\frac{k^2}{v(k)} + \frac{v(k)}{4} + k\right) - (1+m)k.$$

As k tends to infinity, straightforward computations show that

$$f(k) \sim \frac{4-\alpha^2}{16\sqrt{2\pi\alpha k}} \qquad \text{and} \qquad \lim_{k\nearrow+\infty} \frac{g(k)}{k} = \frac{(\alpha-2)^2 - 8m\alpha}{8\alpha} := \frac{P_m(\alpha)}{\alpha}.$$

Since P_m is a second-order strictly convex polynomial with $P_m(0) > 0$, the function $k \mapsto f(k)\mathrm{e}^{(1+m)k}p(k)$ is integrable as long as $P_m(\alpha) > 0$, i.e. $\alpha < 2 - 4\left(\sqrt{m^2+m} - m\right)$, or

$$m < \frac{\alpha}{8} - \frac{1}{2} + \frac{1}{2\alpha}.$$

In other words, we have proved that:

$$\sup\left\{m > 0 : \mathbb{E}\left(X^{1+m}\right) < \infty\right\} = \frac{1}{2}\left(\frac{\alpha}{4} - 1 + \alpha\right).$$

□

5.4. **A variant of Roper's theorem for non-smooth implied volatilities.** Let us observe that the price formulation of arbitrage freeness in [18, Theorem 2.1] is minimal in the sense that the regularity conditions on the Call option prices are necessary and sufficient: to be convex in the strike direction and non-decreasing in the maturity direction. The implied volatility formulation [18, Theorem 2.9, condition IV.1] however, assumes that the total variance is twice differentiable in the strike direction. This regularity is certainly not required; in fact, the author [18, Theorem 2.9] proves the latter by checking the necessary assumptions of Theorem 2.1 on the properties of the Call option prices defined by the formula

$$\mathrm{C}(\mathrm{e}^k, t) = \mathrm{C}_{\mathrm{BS}}(\mathrm{e}^k, w(k,t)), \quad \text{for all } k \in \mathbb{R}, t \geq 0, \tag{5.3}$$

where C_{BS} represents the Black-Scholes call price. More precisely, Roper uses the regularity assumption in k of w in order to define pointwise the second derivative of C with respect to the strike. He then proves that the latter is positive, henceforth obtaining the convexity of the price with respect to the strike (Assumption A.1 of Theorem 2.1). It turns out that the same result can be obtained without this regularity assumption: assume that for any t, the function $k \mapsto w(k,t)$ is continuous and almost everywhere differentiable. Since $\partial_k w$ is defined almost everywhere, and since the term in $\partial_{kk} w$ is linear, $\mathcal{L}w$ can be defined as a distribution and we replace the assumption $\mathcal{L}w(k,t) \geq 0$ everywhere by $\mathcal{L}w(k,t) \geq 0$ in the distribution sense. In order to do so on $\mathbb{R}^*$—saving us from dealing with the boundary behaviour at the origin—we can assume additionally the SMB Condition (Proposition 5.3 (II)) and work on $\mathbb{R}_+^*$. To conclude, we need to prove that the Call options C defined in (5.3) are convex function in $K = \mathrm{e}^k$. Since a continuous function is convex if and only if its second derivative as a distribution is a a positive distribution, the only remaining point to check is that the second derivative as a distribution of C is $\mathcal{L}w$.

But this the same computation already carried out in [18], and the result hence follows. Let us finally note that our assumptions on w are indeed minimal: conversely, if we start from option prices convex in K, their first derivative are defined almost everywhere, and so are those of w (in K or k) since the Black-Scholes mapping in total variance is smooth.

6. The quest of Ψ alternative to SVI

In order to find examples of pairs φ, Ψ, with Ψ different from the SVI parameterisation (2.1), observe first that the second coupling condition (Proposition 4.2) is more geared towards finding out φ given Ψ than the other way round. We first start with a partial result in the other direction.

6.1. An effective bound for Ψ.

We consider the quasi-linear case $\lim_{k\nearrow\infty} w'(k) = \alpha_+ > 0$, when $\theta_\infty = \infty$.

Proposition 6.1. *Let w be the generalised SVI surface* (2.2). *If the surface is free of static arbitrage, then there exists $z_+ \geq 0$ and $\kappa \geq 0$ such that for all $z \geq z_+$ the upper bound*

$$\Psi(z) \leq \kappa^2 + \frac{2z}{M_\infty} - \kappa\sqrt{\kappa^2 + \frac{2z}{M_\infty}},$$

holds, where M_∞ is defined in (4.5).

Proof. In the generalised SVI case (2.2), the function Ψ is strictly asymptotically linear (see Definition 3.5) with $\lim_{z\nearrow\infty} \Psi'(z) = \alpha_+ > 0$, and $\theta_\infty = \infty$. From subsection 4.2 the condition $M_\infty \leq \left|\frac{4}{\Psi'(z)} - \frac{2z}{\Psi(z)}\right|$ holds for all $z \in Z_+(\theta_\infty) = \mathbb{R}$. Since $\lim_{z\nearrow\infty}\left(\frac{4}{\Psi'(z)} - \frac{2z}{\Psi(z)}\right) = \frac{2}{\alpha_+}$, we can define

$$z_+ := \inf\left\{z \in \mathbb{R}_+ : \inf_{y\geq z}\left(\frac{4}{\Psi'(y)} - \frac{2y}{\Psi(y)}\right) > 0\right\} < \infty,$$

and therefore

$$\frac{4}{\Psi'(z)} - \frac{2z}{\Psi(z)} \geq M_\infty, \quad \text{for all } z \geq z_+. \tag{6.1}$$

Note in passing that $M_\infty \leq 2/\alpha_+$. Define $u_+ := \Psi(z_+)$. Since the (continuous) function Ψ is increasing on $[z_+, +\infty)$, we can define its inverse $g : [u_+, +\infty) \to [z_+, +\infty)$. Using the equality $\exp\left(-\int_{u_+}^u \frac{\mathrm{d}v}{2v}\right) = \sqrt{\frac{u_+}{u}}$, we can rewrite (6.1) as

$$\begin{aligned}
\frac{4}{\Psi'(z)} - \frac{2z}{\Psi(z)} \geq M_\infty &\iff g'(u) - \frac{g(u)}{2u} \geq \frac{M_\infty}{4} \\
&\iff \partial_u\left(g(u)\exp\left(-\int_{u_+}^u \frac{dv}{2v}\right)\right) \geq \frac{M_\infty}{4}\sqrt{\frac{u_+}{u}} \\
&\iff g(u)\sqrt{\frac{u_+}{u}} - g(u_+) \geq \frac{M_\infty}{2}\sqrt{u_+}\left(\sqrt{u} - \sqrt{u_+}\right) \\
&\iff g(u)\sqrt{\frac{u_+}{u}} - g(u_+) \geq \frac{M_\infty}{2}\sqrt{u_+}\left(\sqrt{u} - \sqrt{u_+}\right) \\
&\iff g(u) \geq g(u_+)\sqrt{\frac{u}{u+}} + \frac{M_\infty}{2}\sqrt{u}\left(\sqrt{u} - \sqrt{u_+}\right).
\end{aligned}$$

where all the inequalities on the right-hand side are considered for $u \geq u_+$. The third line is obtained by integration between u_+ and u on both sides of second line. Let $K_l := M_\infty/2$ and $K_s := u_+^{-1/2} g(u_+) -$

$\frac{1}{2}M_\infty\sqrt{u_+}$. We then obtain the condition

$$g(u) \geq K_s\sqrt{u} + K_l u, \qquad \text{for all } u \geq u_+. \tag{6.2}$$

Note that K_s remains non-negative if we increase z_+ or decrease M_∞; indeed $\lim_{z\nearrow\infty}(2z/\Psi(z)) = 2/\alpha_+$, so that the condition $M_\infty \leq 2/\alpha_+$ is equivalent to $M_\infty \leq 2z/\Psi(z) = 2g(u)/u$. Finally let us translate condition (6.2) into conditions on Ψ. Fix $u \geq u_+$ and denote $z := g(u)$, then $(z - K_l u)^2 \geq K_s^2 u$, which is equivalent to $K_l^2 u^2 - (K_s^2 + 2K_l z)u + z^2 \geq 0$. The discriminant is equal to $K_s^2(K_s^2 + 4zK_l)$ and is clearly non-negative. Condition (6.2) is therefore equivalent to

$$\Psi(z) \notin \left[\frac{K_s^2 + 2K_l z - K_s\sqrt{K_s^2 + 2K_l z}}{2K_l^2}, \frac{K_s^2 + 2K_l z + K_s\sqrt{K_s^2 + 2K_l z}}{2K_l^2}\right].$$

Given $z - K_l u \geq 0$ (equivalently $\Psi(z) \leq z/K_l$) we obtain $\Psi(z) \leq \kappa^2 + \lambda z - \kappa\sqrt{\kappa^2 + \lambda z}$, where $\kappa := K_s/\left(\sqrt{2}K_l\right)$ and $\lambda := K_l^{-1}$. □

6.2. **An example of non-SVI function.** We now provide a triplet (θ, φ, Ψ), different from the SVI form (2.1), which characterises an arbitrage-free volatility surface via (2.2). Define the function θ by $\theta_t \equiv t$ and the functions φ and Ψ by

$$\varphi(u) := \begin{cases} \dfrac{1 - \mathrm{e}^{-u}}{u}, & \text{if } u > 0 \\ 1, & \text{if } u = 0 \end{cases} \qquad \text{and} \qquad \Psi(z) := |z| + \frac{1}{2}\left(1 + \sqrt{1 + |z|}\right), \quad \text{for all } z \in \mathbb{R}.$$

A few remarks are in order:

- the function φ is continuous on $\mathbb{R}_+$;
- $\theta_\infty = \infty$;
- the map $u \mapsto u\varphi(u)$ is increasing and its limit is $M_\infty = 1$;
- the function Ψ—directly inspired from the computations in Section 6.1—is symmetric and continuous on $\mathbb{R}$. It is also $\mathcal{C}^\infty$ on $\mathbb{R}^*$, and strictly asymptotically linear. Its derivative has a discontinuity at the origin, but the results in Section 5.1 allow it.

With these functions, the total implied variance (2.2) reads

$$w(k,t) = k\left(1 - \mathrm{e}^{-t}\right) + \frac{\sqrt{t}}{2}\left(\sqrt{t} + \sqrt{k\left(1 - \mathrm{e}^{-t}\right) + t}\right), \quad \text{for all } k \in \mathbb{R}.$$

Proposition 6.2. *The surface w is free of static arbitrage.*

Proof. The function f defined in (3.1) therefore reads $f(u) = \frac{(u+1)\mathrm{e}^{-u} - 1}{1 - \mathrm{e}^{-u}}$, and is decreasing from 0 to -1. Regarding the function F, it is clearly continuous, increasing from 0 to 1 and

$$F(z) = \frac{|z|\left(4\sqrt{1 + |z|} + 1\right)}{2\sqrt{1 + |z|}\left(2|z| + 1 + \sqrt{1 + |z|}\right)}, \quad \text{for all } z \in \mathbb{R}^*,$$

with $F(0) = 0$. By Proposition 3.2, straightforward computations then show that the volatility surface is free of calendar-spread arbitrage. Now, for any $z \geq 0$, we have

$$\frac{\Psi'(z)^2}{\Psi(z)} - 2\Psi''(z) = \frac{(16z + 19)\sqrt{1 + z} + 12z + 10}{16(1 + z)^{3/2}\Psi(z)},$$

which is a decreasing function of z with limit equal to zero. Therefore $\overline{Z}_+ = \mathbb{R}$, and for any $u \in \mathbb{R}_+^*$, $Z_+(u) = \mathbb{R}$. Let us check that the generalised SVI surface w parameterised by the previous triplet (θ, φ, Ψ) satisfies $\mathcal{L}w \geq 0$ as a distribution. Indeed we only checked that $\{\mathcal{L}w\} \geq 0$ as a function defined everywhere except at the origin (where as usual in distribution notations $\{\mathcal{L}w\}$ is a function defined where w'' is defined). Here $\Psi'' = \{\Psi''\} + \frac{5}{2}\delta_0$ (where δ_0 stands for the Dirac mass at the origin), so that $\mathcal{L}w = \{\mathcal{L}w\} + 5(\theta\phi(\theta))^2\delta_0$, which is positive since $\{\mathcal{L}w\} \geq 0$. Finally,

$$\frac{4}{\Psi'(z)} - \frac{2z}{\Psi(z)} = 4\frac{4(z+1)^{3/2} + 3z + 4}{\left(4\sqrt{z+1}+1\right)\left(2z+1+\sqrt{z+1}\right)}$$

decreases to $2 \geq M_\infty$. Since $\Psi'(z)^2 \geq 1$, the condition $(u\varphi(u))^2 \leq 4$ is sufficient for absence of butterfly arbitrage. □

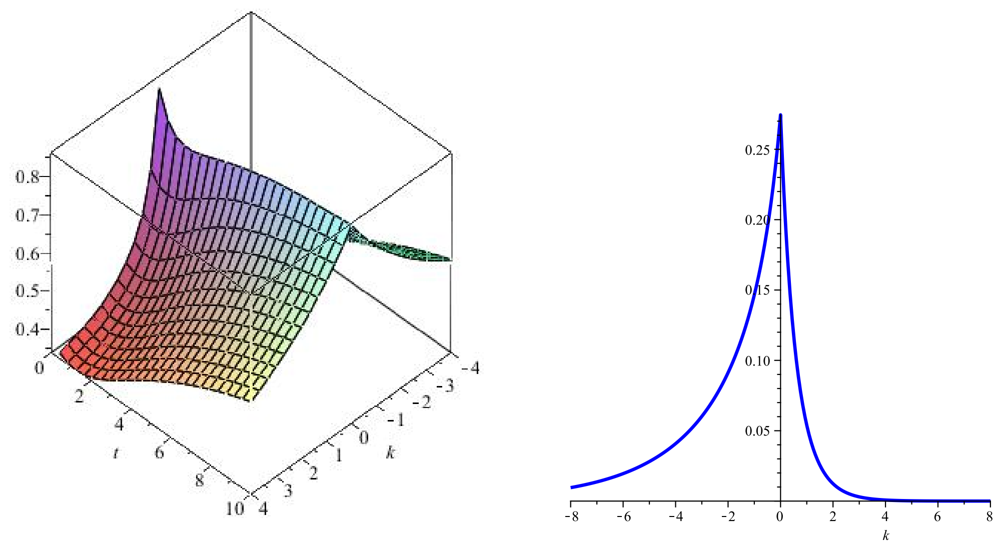

FIGURE 1. Plot of the differential operator $\mathcal{L}$ (left) defined in (4.1) in the non-SVI case of Section 6.2, and of the density at time $t = 1$ (right). Note that $\mathcal{L}w(k, t)$ remains positive for all $k \in \mathbb{R}$ and $t \in \mathbb{R}_+$ as expected.

6.3. **An example of non-SVI parametric family.** We propose a new triplet (θ, φ, Ψ) characterising an arbitrage-free volatility surface via (2.2). Define the function θ by $\theta_t \equiv t$ and

$$\varphi(u) := \begin{cases} \alpha\dfrac{1-\mathrm{e}^{-u}}{u}, & \text{if } u > 0 \\ \alpha, & \text{if } u = 0 \end{cases} \qquad \text{and} \qquad \Psi_\nu(z) := \left(1 + |z|^\nu\right)^{1/\nu}, \quad \text{for } z \in \mathbb{R},$$

where ν is a real number in $(1, \infty)$ and $\alpha \in (0, \overline{\alpha})$ with $\overline{\alpha} \approx 1.33$. Note that when $\nu = 2$, modulo a constant, the function Ψ_2 corresponds to SVI.

Remark 6.3. We could in principle let α depend on ν. The reason for the construction above is that we want to show that the corresponding implied volatility surface is free of static arbitrage for all $\nu > 1$.

The same remarks as in the example in Section 6.2 hold: φ is continuous on $\mathbb{R}_+$, $\theta_\infty = \infty$, $u \mapsto u\varphi(u)$ is increasing to $M_\infty = \alpha$ and Ψ_ν is symmetric and continuous on $\mathbb{R}$. It is also $\mathcal{C}^\infty$ on $\mathbb{R}^*$, $\mathcal{C}^1$ on $\mathbb{R}$, and strictly asymptotically linear. With these functions, the total implied variance (2.2) reads

$$w(k,t) = \theta_t \left(1 + \frac{(1-\mathrm{e}^{-\theta_t})^\nu}{\theta_t^\nu}\alpha^\nu |k|^\nu\right)^{1/\nu}, \quad \text{for all } k \in \mathbb{R}, t > 0.$$

Proposition 6.4. *The surface w is free of static arbitrage.*

Proof. The function f defined in (3.1) therefore reads $f(u) = \frac{(u+1)\mathrm{e}^{-u}-1}{1-\mathrm{e}^{-u}}$, with $f(0) = 0$ and is strictly decreasing from 0 to -1. Regarding the function F, it is clearly continuous, increasing from 0 to 1 and $F(z) = |z|^\nu/(1+|z|^\nu)$ for all $z \in \mathbb{R}^*$, with $F(0) = 0$. Since $\theta_\cdot$ is increasing and $1 + f(u)F(z) \geq 0$ for all $(u,z) \in \mathbb{R}_+^* \times \mathbb{R}$, the first coupling conditions in Proposition 3.2 are satisfied, and the volatility surface is free of calendar spread arbitrage. We now need to check the second coupling condition, namely Proposition 4.2. For $\nu \geq 2$, Ψ is $\mathcal{C}^2$, and we can indeed apply Proposition 4.2. Since Ψ is asymptotically linear, we can alternatively check Proposition 4.3. The equality

$$\Phi_\nu(z) \equiv \frac{\Psi_\nu'(z)^2}{\Psi_\nu(z)} - 2\Psi_\nu''(z) = (1+|z|^\nu)^{1/\nu-2}\,|z|^{\nu-2}\left(|z|^\nu - 2(\nu-1)\right) \tag{6.3}$$

holds for all $z \in \mathbb{R}^*$ and hence the sets $\overline{Z}_+$ and $\overline{Z}_-$ defined in (4.4) are equal to $\overline{Z}_- = [z_-^*, z_+^*]$ and $\overline{Z}_+ = \mathbb{R} \setminus [z_-^*, z_+^*]$, where $z_\pm^* := \pm[2(\nu-1)]^{1/\nu}$. The two conditions in Proposition 4.3 read

$$M_\infty^2 \leq \inf_{z \in \overline{Z}_- \bigcap Z_+(\theta_\infty)} \frac{\left(1 - \frac{z\Psi_\nu'(z)}{2\Psi_\nu(z)}\right)^2}{\frac{1}{4\theta_\infty}\Phi_n(z) + \frac{\Psi_\nu'(z)^2}{16}};$$

$$(u\varphi(u))^2 \leq \inf_{z \in \overline{Z}_+} \frac{\left(1 - \frac{z\Psi_\nu'(z)}{2\Psi_\nu(z)}\right)^2}{\frac{1}{4u}\Phi_n(z) + \frac{\Psi_\nu'(z)^2}{16}}, \quad \text{for any } u \in (0,\infty). \tag{6.4}$$

From the proof of Proposition 4.3, we know that when $\theta_\infty = \infty$, the first condition simplifies to

$$M_\infty \leq \inf_{z \in [z_-^*, z_+^*]} \left|\frac{4}{\Psi_\nu'(z)} - \frac{2z}{\Psi_\nu(z)}\right|. \tag{6.5}$$

Now, immediate computations yield

$$\left|\frac{4}{\Psi_\nu'(z)} - \frac{2z}{\Psi_\nu(z)}\right| = \left|\frac{2z}{|z|^\nu}\frac{2+|z|^\nu}{(1+|z|^\nu)^{1/\nu}}\right|,$$

which, as a function of z is defined on $\mathbb{R}^*$, is strictly increasing on $\mathbb{R}_-^*$ and strictly decreasing on $\mathbb{R}_+^*$. Therefore, is infimum z_ν over the interval $[z_-^*, z_+^*]$ is precisely attained at $z_\pm^*$ (by symmetry) and is equal to $4\nu(2\nu-2)^{(1-\nu)/\nu}(2\nu-1)^{-1/\nu}$. Since by construction $M_\infty = \alpha$, Inequality (6.5) is therefore equivalent to $\alpha \leq z_\nu$. This inequality is clearly not true for any $\nu > 1$ and $\alpha > 0$; however straightforward considerations show that there exists a unique $\nu^* > 1$ such that the map $(1,\infty) \ni \nu \mapsto z_\nu$ is strictly increasing on $(1,\nu^*)$ and strictly decreasing on (ν^*,∞) with $z_1 = 4$ and $\lim_{z\nearrow+\infty} z_\nu = 2$. Therefore the inequality $\alpha \leq z_\nu$ is satisfied for all $\nu > 1$ if and only if $\alpha \leq 2$.

We now check the second inequality (6.4) above. Straightforward computations show that

$$\left(1 - \frac{z\Psi_\nu'(z)}{2\Psi_\nu(z)}\right)^2 = \frac{1}{4}\left(\frac{2+|z|^\nu}{1+|z|^\nu}\right)^2,$$

which increases on $\mathbb{R}_-$ from $1/4$ to 1 and decreases on $\mathbb{R}_+$ from 1 to $1/4$. The map $z \mapsto \Psi'_\nu(z)^2$ is decreasing on $\mathbb{R}_-$, increasing on $\mathbb{R}_+$ and maps the real line to the interval $(0,1)$. Therefore, for any $u > 0$, and any $z \in \overline{Z}_+$, we have

$$\frac{\left(1 - \frac{z\Psi'_\nu(z)}{2\Psi_\nu(z)}\right)^2}{\frac{1}{4u}\Phi_n(z) + \frac{\Psi'_\nu(z)^2}{16}} \geq \frac{1}{4}\frac{1}{\frac{1}{4u}\Phi_\nu(z) + \frac{1}{16}} = \left(\frac{\Phi_\nu(z)}{u} + \frac{1}{4}\right)^{-1},$$

with Φ_ν defined in (6.3). Now a quick look at the function Φ_ν shows that it is bounded above by $\Phi_\nu(z^*_\nu) \in (0,1)$, with $z^*_\nu := [\nu(\nu-1) - 2 + \sqrt{\nu(\nu-1)(\nu^2+3\nu-2)}]^{1/\nu}$. Define the function g_α by $g_\alpha(u) \equiv (u\varphi(u))^2\left(\frac{1}{u} + \frac{1}{4}\right)$. There exists a unique $u^* \approx 1.87$ such that g_α is strictly increasing on $(0, u^*)$ and strictly decreasing on (u^*, ∞) with $g_\alpha(u^*) = g_1(u^*)\alpha^2$. Setting $\overline{\alpha} := g_1(u^*)^{-1/2} \approx 1.33$, the inequality $g_\alpha(u) \leq 1$ is clearly satisfied for any $u > 0$ and all $\alpha \in (0, \overline{\alpha})$. To conclude, note that for $1 < \nu < 2$, the second derivative has a mass at the origin, but Ψ_ν is convex which implies that this mass is positive and that $\mathcal{L}w(k,t) \geq 0$ in the distributional sense following Section 5.4. Therefore the implied volatility surface is free of static arbitrage and the proposition follows. □

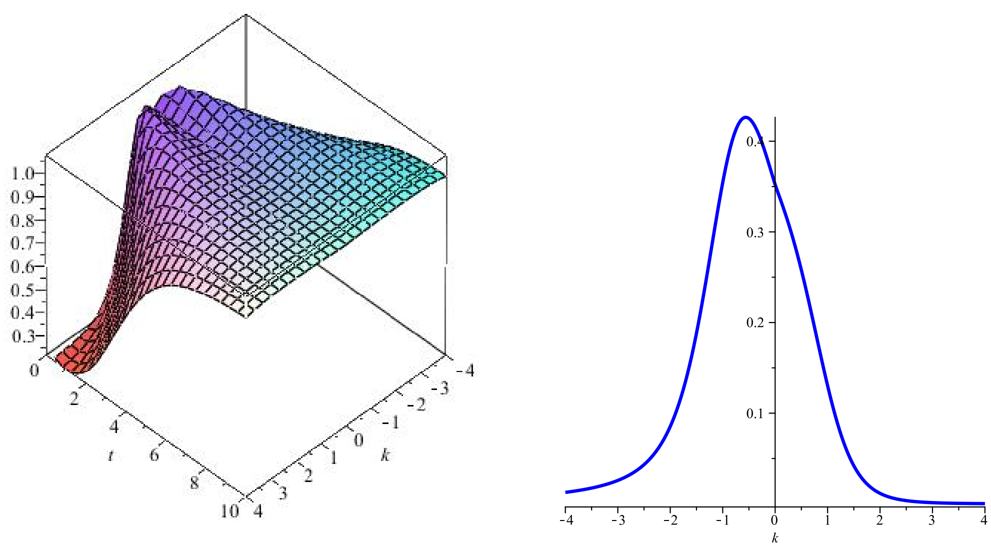

FIGURE 2. Plot of the differential operator $\mathcal{L}$ (left) defined in (4.1) in the non-SVI case of Section 6.3, and of the density at time $t = 1$ (right), with $\nu = 3.5$ and $\alpha = 1$. Note that $\mathcal{L}w(k,t)$ remains positive for all $k \in \mathbb{R}$ and $t \in \mathbb{R}_+$ as expected. Note further that in this example, the density does not have a spike at the origin.

7. Conclusion

Adopting a slightly more abstract point of view than [11], we formulate necessary and sufficient conditions for a wide family of implied volatility surfaces to be free of arbitrage. This allows us to settle some questions raised in [11]. In particular we characterise the exact set of functions φ for which butterfly arbitrage cannot occur in the symmetric SVI model. We precise the notion of arbitrage freeness use in [11] in relation with Roper conditions in [18], and complement those conditions in order to deal with marginal probabilities not weighing zero and non-smooth volatility functions. As an illustration, we provide two examples of smooth non-SVI arbitrage-free volatility surfaces.

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Ecole Polytechnique & Zeliade Systems, Paris

Department of Mathematics, Imperial College London
E-mail address: ajacquie@imperial.ac.uk

Zeliade Systems, Paris
E-mail address: cmartini@zeliade.com

Ecole Polytechnique & Zeliade Systems, Paris