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Low Reynolds number hydrodynamics: Stokes' flow

Let us start with the familiar Navier-Stokes equation for an incompressible fluid

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u} + \frac{1}{\rho}\nabla p - \frac{\mathbf{f}}{\rho} = \frac{\mu}{\rho}\nabla^2\mathbf{u}, \tag{1}$$

in which $\mathbf{f}$ is a general external body force per unit volume (such as gravity $-\rho g\hat{\mathbf{k}}$). We are particularly interested in the relative importance of the second term on the left-hand side and the term on the right hand side. If L is a typically length scale over which the flow varies, and U is a typical scale of the velocity,

$$\frac{\mathbf{u}.\nabla\mathbf{u}}{\frac{\mu}{\rho}\nabla^2\mathbf{u}} \sim \frac{UL\rho}{\mu} = \mathrm{Re}, \tag{2}$$

Where 'Re' is the Reynolds number. It is this reasonable that if $\mathrm{Re} \ll 1$, we might neglect the $\mathbf{u}.\nabla\mathbf{u}$ term in Equation 1. This limit holds for small. slowly moving, viscous flows. Most discussions I have seen simply assume $\frac{\partial \mathbf{u}}{\partial t} = 0$ as well, which then leave you with Stokes' flow

$$\nabla p - \mathbf{f} = \mu\nabla^2\mathbf{u}. \tag{3}$$

But why should we take $\frac{\partial \mathbf{u}}{\partial t} = 0$? Is it immediately obvious that it follows from $\mathrm{Re} \ll 1$? Personally, I don't think so. Let's put it back in to Equation 3

$$\rho\frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \mathbf{f} + \mu\nabla^2\mathbf{u}. \tag{4}$$

What happens if the RHS is not zero, and there is an imbalance between the driving forces and the dissipation? The flow will respond until this imbalance disappears, by changing $\mathbf{u}$. And the lower the Reynolds number, the faster the response. So in low Reynolds number flows, $\mathbf{u}$ quickly reaches the state in which viscous and other forces balance each other, the condition embodied by Equation 3.

When we have fixed boundary conditions for a system, this is then the whole story. $\mathbf{u}$ rapidly relaxes until Equation 3 holds, in such a way that $\mathbf{u}$ satisfies said boundary conditions. We have some linear partial differential equations to solve, a procedure that should be extremely familiar by now. But often we are interested in boundary conditions that are not fixed in time − for instance, when we are trying to pull something away from a sticky surface. In this case, we assume that the fluid responds rapidly to our changing boundary conditions, so that at any instance in time $\nabla p - \mathbf{f} = \mu\nabla^2\mathbf{u}$ holds. This assumption is reasonable, because we know that low Reynolds number flows quickly relax to such a state. In other words, with time-varying boundary conditions we proceed in the following way. We solve $\nabla p - \mathbf{f} = \mu\nabla^2\mathbf{u}$ given our externally imposed boundary conditions at each point in time. The fluid itself has no memory of what the boundary conditions where in the past - the current flow is entirely determined by the current boundary conditions.

The above property of Stokes flow is know as *instantaneity.* From instantaneity, *time-reversibility* immediately follows. This means that if I apply some set of time-varying boundary conditions and generate a flow $\mathbf{u}(t)$, then applying the opposite boundary conditions in reverse will generate exactly the opposite flow, and we will get back to where we started. As a result, if I'm trying to swim in the low Reynolds number limit, there is no point in using a time-reversible swimming stroke like the opening and closing of a scallop shell (which is like implying time-reversible boundary conditions). If I tried, the net water flow during opening would exactly balance that during closing and I wouldn't move anywhere. Note that this still holds even if I open and close at different rates (provided I remain in the low Reynolds number limit), as the aggregate water flow when opening or closing doesn't depend on the speed with which I change my boundary conditions.