Energy-Distortion Exponents in Lossy Transmission of Gaussian Sources Over Gaussian Channels

Erman Köken, *Student Member, IEEE*, Deniz Gündüz, *Senior Member, IEEE*, Ertem Tuncel, *Member, IEEE*

***Abstract*—Lossy transmission of Gaussian sources over energy-limited Gaussian point-to-point and broadcast channels is studied under the infinite bandwidth regime, i.e., when the number of channel uses is unlimited. Using previously known asymptotic achievability and converse results, the *energy-distortion exponent*, defined as the rate of decay of the square-error distortion as the available energy-to-noise ratio increases without bound, is completely characterized for both the point-to-point and broadcast channel cases. Turning then to the scenario of zero-delay transmission, where outage events with arbitrarily small probability are allowed, it is shown that the same energy-distortion exponent as in the infinite-delay case can be achieved in all the studied scenarios.**

***Index Terms*—Broadcast channels, energy-distortion exponent, energy-distortion tradeoff, energy-limited transmission, joint source-channel coding, zero-delay.**

I. Introduction

In information theory, performance of a communication system is typically analyzed under the average power constraint per unit bandwidth (i.e., Joules/second/Hertz), which automatically translates into infinite energy consumption per source sample when the bandwidth is unlimited. This does not correspond to a meaningful setting for sensor networks which are limited by the total energy available in finite-size batteries, while the relative channel bandwidth per source sample is abundant when the source signal changes slowly over time and each source sample can be transmitted over many uses of the channel. A more appropriate performance measure for the sensor network scenario is the *energy-distortion tradeoff* [8], [9], which characterizes the minimum average reconstruction distortion that can be achieved under a total energy constraint (per source sample) without any limitation on the channel bandwidth.

In this paper, we introduce the *energy-distortion exponent* as the exponential rate of decay of the square-error distortion as the energy-to-noise ratio (ENR) approaches infinity. Our motivation for defining this measure is the same as in typical high signal-to-noise ratio (SNR) analyses that appear in the literature: in the absence of a completely characterized energy-distortion tradeoff, energy-distortion exponent will provide us with a rough benchmark to strive for when designing practical systems.

Two prominent examples where the energy-distortion characterization is not fully known are (i) the transmission of a single Gaussian source over a Gaussian broadcast channel where each receiver reconstructs its own estimate of the source (see [15] and [17] for inner and outer bounds derived for average power constraint and finite bandwidth), and (ii) the transmission of a pair of correlated Gaussian sources over a Gaussian broadcast channel, where each receiver is interested in reconstructing only one of the sources (similarly, see [2], [3], [5], [10], [18], [19], [20] for existing results). In both cases, the tradeoff is between two distortion levels achieved at each receiver for a given energy budget. Similarly, there will be a tradeoff between energy-distortion exponents at each receiver.

Our first result is a closed-form characterization of the achievable pairs of energy-distortion exponents in the first scenario. More specifically, we show that the achievability and converse results in [17] coincide in the energy-distortion regime for very high ENR.

For the second scenario we prove a similar result. Namely, we show that the converse result in [2], when translated into the energy-distortion tradeoff, yields a pair of energy-distortion exponents that can be achieved using a simple energy splitting scheme.

For both of the broadcast scenarios, as well as the point-to-point channel, we then investigate the energy-distortion exponents in the extreme case of *zero-delay*[1]. Zero-delay transmission is relevant in applications where delay could not be tolerated, such as smart-grid systems where smart-meter measurements are used for monitoring the grid for energy outages. In a typical smart meter scenario, one measurement is taken every 15 minutes and must be transmitted as soon as possible to the central control unit [12]. With the same motivation, energy-distortion tradeoff for zero-delay transmission over a Gaussian broadcast channel *with perfect channel output feedback* was studied in [13].

Our last result is that, in the zero-delay regime, if we allow for a small *outage* event whose probability is vanishingly small, the same energy-distortion exponent(s) can be achieved as in the aforementioned infinite delay scenarios.

The rest of the paper is organized as follows. Section II is dedicated to preliminaries and notation. In Section III, achievable energy-distortion exponents are derived for both of the broadcast scenarios under the infinite-delay regime. Then,

This work was in part presented in IEEE Information Theory Workshop, Jerusalem, Israel, April 2015.

This work was supported in part by the National Science Foundation Grant CCF-1423570, and by the European Research Council (ERC) through Starting Grant BEACON (agreement #677854).

Erman Köken and Ertem Tuncel are with the Department of Electrical and Computer Engineering, University of California, Riverside, CA 92521. E-mail: ekoken@ee.ucr.edu, ertem.tuncel@ucr.edu.

Deniz Gündüz is with the Department of Electrical and Electronic Engineering, Imperial College London, London, UK. E-mail: d.gunduz@imperial.ac.uk.

[1]To clarify, in our terminology zero-delay refers to transmission being complete before the next source sample is generated. In other words, zero *source* delay is incurred.

in Section IV, we focus on zero-delay transmission for both point-to-point and broadcast channels, and show that the same energy-distortion exponents as in the infinite delay regime can be achieved with distortion outages.

II. Preliminaries and Notation

A. Point-To-Point Transmission

Let $X^M \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_M)$ be an independent and identically distributed (i.i.d.) Gaussian source sequence to be transmitted over the channel

$$V^N = U^N + W^N \ , \tag{1}$$

where U^N and V^N are the channel input and output, respectively, and the channel noise $W^N \sim \mathcal{N}\left(\mathbf{0}, \sigma_W^2 \mathbf{I}_N\right)$ is independent of U^N. The encoder

$$\phi_{M,N} : \mathbb{R}^M \longrightarrow \mathbb{R}^N \tag{2}$$

maps X^M into U^N, and the receiver

$$\psi_{M,N} : \mathbb{R}^N \longrightarrow \mathbb{R}^M \tag{3}$$

estimates X^M as $\hat{X}^M$. The ratio $\kappa = \frac{N}{M}$ is usually referred to as the bandwidth expansion factor, and it is measured in channel uses per source symbol.

Definition 1. A pair (D, E) is achievable for point-to-point transmission if for any $\epsilon > 0$, there exists large enough M, N, and a corresponding encoder-decoder pair $(\phi_{M,N}, \psi_{M,N})$, such that

$$\begin{aligned} \frac{1}{M}\mathbb{E}\left[||U^N||^2\right] &\leq E + \epsilon \\ \frac{1}{M}\mathbb{E}\left[||X^M - \hat{X}^M||^2\right] &\leq D + \epsilon \ . \end{aligned}$$

As usual, we denote by $D(E)$ the minimum possible distortion such that (D, E) is achievable.

Note that in the above definition, the expended energy is measured *per source symbol*. This is in contrast with power-limited transmission where the channel power is measured *per channel symbol*. However, by expressing the energy constraint alternatively as

$$\frac{1}{N}\mathbb{E}\left[||U^N||^2\right] \leq \frac{E}{\kappa} + \epsilon' \ ,$$

one can utilize existing power-constrained channel transmission results. For instance, it directly follows from the separation theorem that (D, E) is achievable if and only if

$$R(D) \leq \sup_{\kappa > 0} \ \kappa C\left(\frac{E}{\kappa}\right) \tag{4}$$

where $C(P)$ is the capacity with power constraint

$$C(P) = \frac{1}{2}\log_2\left(1 + \frac{P}{\sigma_W^2}\right) \ ,$$

and $R(D)$ is the rate-distortion function given by

$$R(D) = \frac{1}{2}\log_2 \frac{1}{D}.$$

Translating (4) then yields

$$\begin{aligned} D(E) &= \inf_{\kappa > 0}\left(1 + \frac{E}{\kappa\sigma_W^2}\right)^{-\kappa} \\ &= \lim_{\kappa \to \infty}\left(1 + \frac{E}{\kappa\sigma_W^2}\right)^{-\kappa} \\ &= e^{-\frac{E}{\sigma_W^2}} \ . \end{aligned} \tag{5}$$

To emphasize the fact that the minimum achievable distortion $D(E)$ depends only on the energy-to-noise ratio (ENR), defined as

$$\gamma \triangleq \frac{E}{\sigma_W^2} \ ,$$

we write (5) in the sequel as

$$D(E) = e^{-\gamma} \ . \tag{6}$$

In all the scenarios we consider in the sequel, we will observe similar energy-distortion behaviors as $E \to \infty$. That motivates us to define

$$\lim_{E \to \infty} -\frac{1}{\gamma}\ln D(E)$$

as the *energy-distortion exponent* for each scenario. Therefore, when we say that a receiver achieves an energy-distortion exponent of β, it is equivalent to stating that the average distortion at that receiver decays to zero as $e^{-\beta\gamma}$ in the high-energy high-bandwidth regime. Thus, we observe that the energy-distortion tradeoff $D(E)$ in (6) achieves an exponent of 1.

B. Transmission of a Single Source Over a Broadcast Channel

Let the i.i.d. Gaussian source X^M be transmitted over the Gaussian broadcast channel

$$V_i^N = U^N + W_i^N \tag{7}$$

for $i = 1, 2$, where U^N and V_i^N are the channel input and output at the ith receiver, respectively, and the channel noise sequences $W_i^N \sim \mathcal{N}\left(\mathbf{0}, \sigma_{W_i}^2 \mathbf{I}_N\right)$ are independent of U^N and each other.

Let the encoder be the same as given in (2). At the ith receiver, the decoder

$$\psi_{M,N}^{(i)} : \mathbb{R}^N \to \mathbb{R}^M \tag{8}$$

maps the observation V_i^N into the estimation $\hat{X}_i^M = \psi_{M,N}^{(i)}\left(V_i^N\right)$. We refer the reader to Fig. 1 for the block diagram of the system.

Definition 2. An energy-distortion triplet (D_1, D_2, E) is achievable if for any $\epsilon > 0$, there exists large enough M, N and $(\phi_{M,N}, \psi_{M,N}^1, \psi_{M,N}^2)$ such that

$$\begin{aligned} \frac{1}{M}\mathbb{E}\left[||U^N||^2\right] &\leq E + \epsilon \\ \frac{1}{M}\mathbb{E}\left[||X^M - \hat{X}_i^M||^2\right] &\leq D_i + \epsilon \end{aligned}$$

for $i = 1, 2$.

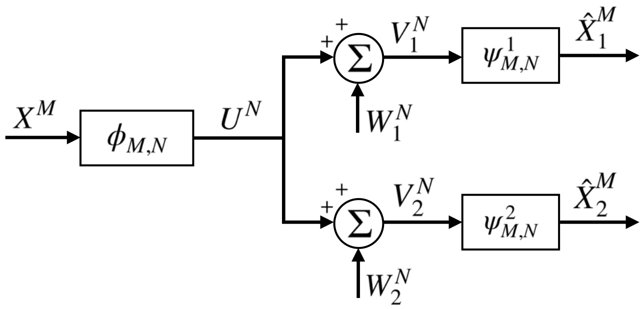

Figure 1. The block diagram for transmission of a single Gaussian source sequence X^M over the Gaussian broadcast channel $V_i^N = U^N + W_i^N$, $i = 1, 2$. Each receiver estimates its version $\hat{X}_i^M$ of the source.

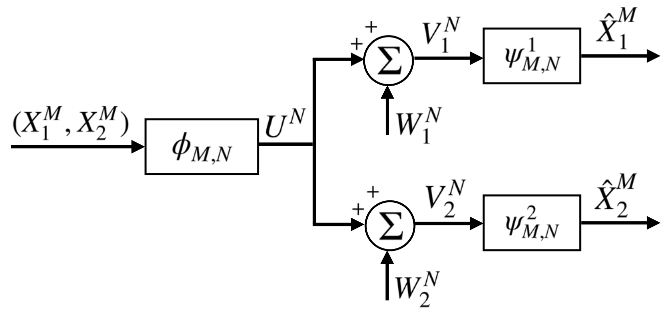

Figure 2. The block diagram for transmission of a bivariate Gaussian source sequence (X_1^M, X_2^M) over the Gaussian broadcast channel $V_i^N = U^N + W_i^N$, $i = 1, 2$. Each receiver estimates only one source.

As in the point-to-point case, for a fixed energy budget E, the tradeoff between D_1 and D_2 will depend only on the ENR values $\gamma_i = \frac{E}{\sigma^2_{W_i}}$. Without loss of generality, we assume that the second receiver is the "better" one, i.e., $\sigma^2_{W_1} = g\sigma^2_{W_2}$ for some $g > 1$. This also implies $\gamma_2 = g\gamma_1$ for all E.

Definition 3. An *energy-distortion exponent pair* (β_1, β_2) is achievable if there exist functions $D_1(E)$ and $D_2(E)$ such that $(D_1(E), D_2(E), E)$ is achievable for all $E > 0$ and

$$\lim_{E\to\infty} -\frac{1}{\gamma_i} \ln D_i\left(E\right) = \beta_i$$

for $i = 1, 2$.

C. Transmission of a Bivariate Source Over a Broadcast Channel

Consider the transmission of an i.i.d. bivariate zero-mean Gaussian source $\left(X_1^M, X_2^M\right)$ over the same channel given in (7), where $(X_{1,m}X_{2,m}) \sim \mathcal{N}\left(0, \Sigma\right)$ with covariance matrix

$$\Sigma = \left[\begin{array}{cc} 1 & \rho \\ \rho & 1 \end{array}\right]$$

and $|\rho| < 1$. The encoder (2) is modified as

$$\phi_{M,N} : \mathbb{R}^M \times \mathbb{R}^M \longrightarrow \mathbb{R}^N \quad (9)$$

which maps $\left(X_1^M, X_2^M\right)$ into U^N, and at the ith receiver the decoder

$$\psi^{(i)}_{M,N} : \mathbb{R}^N \to \mathbb{R}^M \quad (10)$$

estimates the ith source as $\hat{X}_i^M = \psi^{(i)}_{M,N}\left(V_i^N\right)$. See Fig. 2 for the pictorial description of the system.

Definition 4. An energy-distortion triplet (D_1, D_2, E) is achievable if for any $\epsilon > 0$, there exists large enough M, N and $(\phi_{M,N}, \psi^1_{M,N}, \psi^2_{M,N})$ such that

$$\begin{aligned} \frac{1}{M}\mathbb{E}\left[||U^N||^2\right] &\leq E + \epsilon \\ \frac{1}{M}\mathbb{E}\left[||X_i^M - \hat{X}_i^M||^2\right] &\leq D_i + \epsilon \end{aligned}$$

for $i = 1, 2$.

Definition of achievable energy-distortion exponent pairs for this scenario is exactly as given in Definition 3.

III. Achievable Energy-Distortion Exponents

We begin by characterizing the energy-distortion exponent pair for the transmission of a single Gaussian source over a Gaussian broadcast channel. To that end, we utilize the inner and outer bounds on the achievable distortion region given in [17] for a fixed bandwidth expansion factor κ and average channel input power P. In particular, the bounds coincide when translated into the energy-distortion exponent regime. We note in passing that even though the inner bound of [17] was improved in [15], the former suffices for our purposes.

Theorem 5. *The set of achievable energy-distortion exponent pairs for the transmission of a single Gaussian source over a Gaussian broadcast channel is given as*

$$\mathcal{B}_{\text{single}} = \left\{(\beta_1, \beta_2) \middle| 0 \leq \beta_1 \leq 1,\ 0 \leq \beta_2 \leq \frac{\beta_1}{g} + (1 - \beta_1)\right\}$$

as shown in Fig. 3.

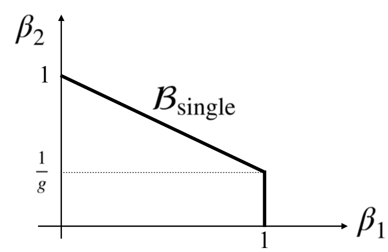

Figure 3. The region of all achievable energy-distortion exponent pairs (β_1, β_2) for transmission of a single Gaussian source over a Gaussian broadcast channel.

Proof: We first show that all energy-distortion exponent pairs in $\mathcal{B}_{\text{single}}$ are achievable. For a fixed bandwidth expansion factor κ and an average power budget P per channel use, the scheme in [17] achieves

$$D_1 = \frac{1}{1 + \frac{P}{\sigma^2_{W_1}}} \left(1 + \frac{\alpha P}{\sigma^2_{W_1} + (1 - \alpha)\, P}\right)^{1-\kappa} \quad (11)$$

$$\begin{aligned} D_2 = {} & \frac{1}{1 + \frac{P}{\sigma^2_{W_2}}} \left(1 + \frac{\alpha P}{\sigma^2_{W_1} + (1 - \alpha)\, P}\right)^{1-\kappa} \\ & \cdot \left(1 + \frac{(1 - \alpha) P}{\sigma^2_{W_2}}\right)^{1-\kappa} \quad (12) \end{aligned}$$

$$D_2 \geq \sup_{m>0} \frac{m}{\left(\frac{\sigma_{W_1}^2}{\sigma_{W_2}^2}\left[\delta + m\left(\frac{P}{\sigma_{W_1}^2}+1\right)^{\kappa}\right]^{1/\kappa} - \left(\frac{\sigma_{W_1}^2}{\sigma_{W_2}^2}-1\right)(1+m)^{1/\kappa}\right)^{\kappa} - 1} . \tag{13}$$

for all $0 \leq \alpha \leq 1$.

Substituting in (11) and (12) the identity $P = \frac{E}{\kappa}$, and letting $\kappa \to \infty$, we obtain

$$\begin{aligned} D_1 &= \lim_{\kappa\to\infty} \frac{1}{1+\frac{\gamma_1}{\kappa}} \left(1 + \frac{\alpha\gamma_1}{\kappa + (1-\alpha)\gamma_1}\right)^{1-\kappa} \\ &= \lim_{\kappa\to\infty} \left(1 + \frac{\alpha\gamma_1}{\kappa + (1-\alpha)\gamma_1}\right)^{1-\kappa} \\ &= \lim_{\kappa'\to\infty} \left(1+\frac{1}{\kappa'}\right)^{1+(1-\alpha)\gamma_1 - \alpha\gamma_1\kappa'} \\ &= \lim_{\kappa'\to\infty} \left(1+\frac{1}{\kappa'}\right)^{1+(1-\alpha)\gamma_1} \lim_{\kappa'\to\infty} \left(1+\frac{1}{\kappa'}\right)^{-\alpha\gamma_1\kappa'} \\ &= \lim_{\kappa'\to\infty} \left(1+\frac{1}{\kappa'}\right)^{-\alpha\gamma_1\kappa'} \\ &= e^{-\alpha\gamma_1} \end{aligned} \tag{14}$$

where $\kappa' = \frac{\kappa+(1-\alpha)\gamma_1}{\alpha\gamma_1}$, and

$$\begin{aligned} D_2 &= \lim_{\kappa\to\infty} \left[\frac{1}{1+\frac{\gamma_2}{\kappa}} \left(1 + \frac{\alpha\gamma_1}{\kappa + (1-\alpha)\gamma_1}\right)^{1-\kappa} \cdot \left(1 + (1-\alpha)\frac{\gamma_2}{\kappa}\right)^{1-\kappa}\right] \\ &= \lim_{\kappa\to\infty} \left(1 + \frac{\alpha\gamma_1}{\kappa + (1-\alpha)\gamma_1}\right)^{1-\kappa} \cdot \lim_{\kappa\to\infty} \left(1 + (1-\alpha)\frac{\gamma_2}{\kappa}\right)^{1-\kappa} \\ &= e^{-\alpha\gamma_1} e^{-(1-\alpha)\gamma_2} \\ &= e^{-\gamma_2\left[\frac{\alpha}{g}+1-\alpha\right]} . \end{aligned} \tag{15}$$

It follows from (14) and (15) that all energy-distortion pairs in the set $\mathcal{B}_{\text{single}}$ are indeed achievable.

For the converse, we use the outer bound for the finite bandwidth power-constrained problem given in [17] as follows. For any achievable (D_1, D_2) such that

$$D_1 = \delta \left(1 + \frac{P}{\sigma_{W_1}^2}\right)^{-\kappa} ,$$

with $\delta > 1$, D_2 must satisfy (13) shown at the top of this page.

In the energy-distortion regime, this bound translates to

$$\begin{aligned} D_1 &= \lim_{\kappa\to\infty} \delta \left(1 + \frac{\gamma_1}{\kappa}\right)^{-\kappa} \\ &= \delta e^{-\gamma_1} \end{aligned} \tag{16}$$

and

$$D_2 \geq \sup_{m>0} \frac{m}{(1+m)L(m) - 1} \tag{17}$$

where

$$\begin{aligned} L(m) &= \frac{1}{1+m} \lim_{\kappa\to\infty} \left(g\left[\delta + m\left(1+\frac{\gamma_1}{\kappa}\right)^{\kappa}\right]^{1/\kappa} - (g-1)(1+m)^{1/\kappa}\right)^{\kappa} \\ &= \lim_{\kappa\to\infty} \left(1 + g\left[\left(\frac{\delta + m\left(1+\frac{\gamma_1}{\kappa}\right)^{\kappa}}{1+m}\right)^{\frac{1}{\kappa}} - 1\right]\right)^{\kappa} . \end{aligned}$$

Now, for $f(\kappa)$ non-decreasing in κ, and $h(\theta,\kappa)$ non-decreasing in θ, we can write

$$h(f(\kappa_-),\kappa) \leq h(f(\kappa),\kappa) \leq h(f(\kappa_+),\kappa) \tag{18}$$

for any $\kappa_- \leq \kappa \leq \kappa_+$. Moreover, the inequality chain in (18) remains intact if we first let $\kappa_+ \to \infty$, then let $\kappa \to \infty$, and finally $\kappa_- \to \infty$, to obtain

$$\begin{aligned} &\lim_{\kappa_-\to\infty} \lim_{\kappa\to\infty} h(f(\kappa_-),\kappa) \\ &\quad \leq \lim_{\kappa\to\infty} h(f(\kappa),\kappa) \\ &\quad\quad \leq \lim_{\kappa\to\infty} h\big(\lim_{\kappa_+\to\infty} f(\kappa_+),\kappa\big) \end{aligned} \tag{19}$$

whenever the above limits exist. Setting

$$f(\kappa) = \frac{\delta + m\left(1+\frac{\gamma_1}{\kappa}\right)^{\kappa}}{1+m}$$

and

$$h(\theta,\kappa) = \left(1 + g\left[\theta^{\frac{1}{\kappa}} - 1\right]\right)^{\kappa} ,$$

together with the observation that

$$\lim_{\kappa\to\infty} f(\kappa) = \frac{\delta + me^{\gamma_1}}{1+m}$$

and

$$\lim_{\kappa\to\infty} h(\theta,\kappa) = \theta^g$$

for any fixed θ, we notice that the upper and lower bounds in (19) collapse and yield

$$L(m) = \left(\frac{\delta + me^{\gamma_1}}{1+m}\right)^g .$$

Therefore (17) is the same as

$$D_2 \geq \sup_{m>0} \frac{m}{(1+m)\left(\frac{\delta+me^{\gamma_1}}{1+m}\right)^g - 1} .$$

The supremum above is difficult to compute. However, substitution of any $m > 0$ obviously results in a (looser) lower bound on the achievable D_2. In particular, it is easy to show after some algebra that the choice

$$m = \frac{\delta}{(g-1)e^{\gamma_1} - \delta g}.$$

results in

$$D_2 \geq \frac{\delta}{(g-1)(e^{\gamma_1}-\delta)\left(\frac{\delta g}{g-1}\right)^g - (g-1)e^{\gamma_1} + \delta g} \quad (20)$$

for any fixed $D_1 = \delta e^{-\gamma_1}$.

Note that we are interested in the asymptotic behavior of achievable (D_1, D_2) as $E \to \infty$. To that end, let $D_1(E) = \delta(E)e^{-\gamma_1}$ for some arbitrary $\delta(E) > 1$ such that

$$\lim_{E\to\infty} -\frac{1}{\gamma_1}\ln D_1(E) = \beta_1 \quad (21)$$

for some $0 \leq \beta_1 \leq 1$. This implies that for any such $\delta(E)$,

$$\lim_{E\to\infty} -\frac{1}{\gamma_1}\ln \delta(E) = \beta_1 - 1 \ ,$$

or in other words, $\delta(E)$ must grow as $e^{\gamma_1(1-\beta_1)}$. For the second energy-distortion exponent, (20) then translates to the upper bound

$$\begin{aligned}
&\lim_{E\to\infty} -\frac{1}{\gamma_2}\ln D_2(E) \\
&\leq \lim_{E\to\infty}\Bigg\{-\frac{1}{\gamma_2}\ln\delta(E) \\
&\quad + \frac{1}{\gamma_2}\ln\Bigg[(g-1)(e^{\gamma_1}-\delta(E))\left(\frac{\delta(E)g}{g-1}\right)^g \\
&\qquad - (g-1)e^{\gamma_1} + g\delta(E)\Bigg]\Bigg\} \\
&= \frac{\beta_1 - 1}{g} \\
&\quad + \lim_{E\to\infty}\frac{1}{\gamma_2}\ln\Bigg[\frac{g^g}{(g-1)^{g-1}}[e^{\gamma_1}-\delta(E)]\delta(E)^g \\
&\qquad - (g-1)e^{\gamma_1} + g\delta(E)\Bigg] \\
&\stackrel{(a)}{=} \frac{\beta_1 - 1}{g} + \frac{1}{g} + 1 - \beta_1 \\
&= \frac{\beta_1}{g} + 1 - \beta_1 \quad (22)
\end{aligned}$$

where (a) follows from the fact that $[e^{\gamma_1} - \delta(E)]\delta(E)^g$ grows as $e^{\gamma_1+\gamma_2(1-\beta_1)}$, which is faster than the other terms e^{γ_1} and $\delta(E)$. The proof is therefore complete because (21) and (22) implies that $\mathcal{B}_{\text{single}}$ is indeed an outer bound to achievable energy-distortion exponents. ■

In the next theorem, we characterize the achievable energy-distortion exponent pairs for the transmission of bivariate Gaussian sources over the Gaussian broadcast channel. As in the single source case, we utilize an existing outer bound introduced in [2] on achievable (D_1, D_2) pairs for a given channel input power P and bandwidth expansion factor κ. Interestingly, a very simple coding scheme achieves the same energy-distortion exponents as the outer bound.

Theorem 6. *The set of achievable energy-distortion exponent pairs for the transmission of a bivariate Gaussian source over a Gaussian broadcast channel is given as*

$$\mathcal{B}_{\text{bivariate}} = \{(\beta_1, \beta_2)\,|0 \leq \beta_1 \leq 1,\, 0 \leq \beta_2 \leq 1 - \beta_1\}$$

as shown in Fig. 4.

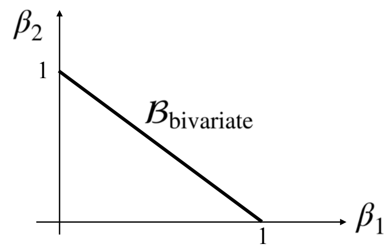

Figure 4. The region of all achievable energy-distortion exponent pairs (β_1, β_2) for transmission of a bivariate Gaussian source over a Gaussian broadcast channel.

Proof: We start with the converse. It follows from the outer bound derived in [2] that for fixed channel input power P and bandwidth expansion factor κ, (D_1, D_2) is achievable only if there exists $0 \leq \alpha \leq 1$ such that

$$D_1 \geq \left(1 + \frac{\alpha P}{(1-\alpha)P + \sigma_{W_1}^2}\right)^{-\kappa} \quad (23)$$

$$D_2 \geq \left(1-\rho^2\right)\left(1 + \frac{(1-\alpha)P}{\sigma_{W_2}^2}\right)^{-\kappa} . \quad (24)$$

In the energy-distortion framework, this implies that for any fixed energy budget E per source symbol, there must exist $0 \leq \alpha \leq 1$ such that

$$\begin{aligned}
D_1 &\geq \lim_{\kappa\to\infty}\left(1 + \frac{\alpha\gamma_1}{(1-\alpha)\gamma_1 + \kappa}\right)^{-\kappa} \\
&= e^{-\alpha\gamma_1} \\
D_2 &\geq \lim_{\kappa\to\infty}\left(1-\rho^2\right)\left(1 + \frac{(1-\alpha)\gamma_2}{\kappa}\right)^{-\kappa} \\
&= \left(1-\rho^2\right)e^{-(1-\alpha)\gamma_2} .
\end{aligned}$$

Thus, for any $D_1(E)$ and $D_2(E)$ such that $(D_1(E), D_2(E), E)$ is achievable, we must have

$$\begin{aligned}
\lim_{E\to\infty} -\frac{1}{\gamma_1}\ln D_1(E) &\leq \alpha \\
\lim_{E\to\infty} -\frac{1}{\gamma_2}\ln D_2(E) &\leq 1-\alpha
\end{aligned}$$

for some $0 \leq \alpha \leq 1$, proving that $\mathcal{B}_{\text{bivariate}}$ is indeed an outer bound for all achievable exponent pairs (β_1, β_2).

To prove achievability of any $(\beta_1, \beta_2) \in \mathcal{B}_{\text{bivariate}}$, it suffices to simply send the source pair with two rounds of transmission, where in each round $i = 1, 2$, we transmit X_i^M with energy $\beta_i E$ and bandwidth expansion factor κ_i. Note that (i) $\beta_1 E + \beta_2 E \leq E$, and therefore this is a feasible choice, and (ii) the individual κ_i can be arbitrarily taken to infinity, resulting in

$$\begin{aligned}
D_1(E) &= e^{-\beta_1\gamma_1} \\
D_2(E) &= e^{-\beta_2\gamma_2}
\end{aligned}$$

which follows from (6). The proof is therefore complete. ■

IV. Zero-Delay Communication with Distortion Outage

In this section we focus on the extreme case of zero source delay, i.e., $M = 1$. In other words, a single random variable X is mapped into the channel input U^N where the channel, the encoder, and the decoder(s) are in the same form in each aforementioned scenario. We have $\kappa = N$, and once again, we are interested in the energy-distortion tradeoff when the bandwidth is not limited, i.e., $N \to \infty$. However, we slightly change the achievability definition for distortion by allowing a vanishingly small probability of *distortion outage*, and evaluating the expected distortion conditioned on no distortion outages.

The motivation behind this change is as follows. While one should ultimately search for an analog mapping between X and U^N, that proves a difficult task for even moderate values of N [1], [4], [7], [21], let alone $N \to \infty$. That leaves the alternative of either *digital* coding or hybrid digital/analog coding. On the other hand, any coding scheme that transmits some digital information through the channel is prone to error in decoding of that information. Regardless of how small the probability of incorrect decoding is, the overall expected distortion might still be very adversely affected.

We generalize this "error event" as the *outage region* in the product space of (X, W^N), and formally define the energy-distortion-outage tradeoff. We then show that in each scenario we consider, zero-delay communication with distortion outage achieves the same energy-distortion exponent as in the infinite-delay case discussed in the previous section[2].

A. Point-to-Point Transmission

Definition 7. A triple (D, E, δ) is achievable for zero-delay point-to-point transmission with distortion outage if for any $\epsilon > 0$, there exist a large enough N, an encoder-decoder pair $(\phi_{1,N}, \psi_{1,N})$, and an outage region $\mathcal{O} \in \mathbb{R} \times \mathbb{R}^N$ such that

$$\begin{aligned} \mathbb{E}\left[||U^N||^2\right] &\leq E + \epsilon \\ \Pr\left[\left(X, W^N\right) \in \mathcal{O}\right] &\leq \delta \\ \mathbb{E}\left[(X - \hat{X})^2 \middle| \mathcal{O}^c\right] &\leq D + \epsilon\ . \end{aligned}$$

Also denote by $D(E, \delta)$ the minimum possible distortion such that (D, E, δ) is achievable.

It should be clear that the region of all achievable $(D, E, 0)$ coincides with the set of achievable (D, E) as in Definition 1, and therefore this is a more general achievability concept.

We modify the definition of energy-distortion exponents accordingly as follows.

Definition 8. An *energy-distortion exponent* β is achievable for zero-delay point-to-point transmission with distortion outage if

$$\beta = \lim_{\delta \to 0} \lim_{E \to \infty} -\frac{1}{\gamma} \ln D(E, \delta)$$

where, as before, $\gamma = \frac{E}{\sigma_W^2}$.

In what follows we show that we can achieve $\beta = 1$ just as in the infinite-delay case (6).

Theorem 9. *$\beta = 1$ is an achievable energy-distortion exponent for zero-delay point-to-point transmission of a Gaussian source with distortion outage.*

Proof: We quantize the single random variable X with $N \gg 1$ levels and use orthogonal signaling to transmit the quantization index. At the receiver, we use maximum likelihood decoding, and classify incorrect decoding as the distortion outage event.

It is well-known [6] that the optimal high-resolution quantizer has the point density function $\lambda(x)$ given by

$$\lambda(x) = \frac{f_X(x)^{\frac{1}{3}}}{\int_{-\infty}^{\infty} f_X(x')^{\frac{1}{3}}\, dx'} \tag{25}$$

which, for $X \sim \mathcal{N}(0, 1)$, boils down to a Gaussian with zero mean and variance 3. The resultant distortion can be approximated using the Bennett integral [6] as

$$\begin{aligned} D &\approx \frac{1}{12N^2} \int_{-\infty}^{\infty} \frac{f_X(x)}{\lambda(x)^2} dx \\ &= \frac{1}{12N^2} \left(\int_{-\infty}^{\infty} f_X(x')^{\frac{1}{3}}\, dx' \right)^3 \\ &= \frac{\sqrt{3}\pi}{2N^2}\ . \end{aligned} \tag{26}$$

One can formalize this approximation by

$$D \leq \frac{\sqrt{3}\pi}{2N^2} + \epsilon \tag{27}$$

for arbitrarily small $\epsilon > 0$ and large enough N.

The quantized indices are mapped into orthogonal channel input vectors U^N such that

$$U_t = \begin{cases} \sqrt{E} & t = k(X) \\ 0 & t \neq k(X) \end{cases}$$

where $1 \leq k(X) \leq N$ is the integer quantization index. Note that $\left\|U^N\right\|^2 = E$ always. At the receiver end, upon receiving $V^N = U^N + W^N$, the decoder simply selects

$$\hat{K} = \arg\max_{1 \leq i \leq N} V_i$$

and then outputs

$$\hat{X} = r_{\hat{K}}$$

where r_k is the kth reconstruction level of the quantizer. Thus, the distortion outage event is given by

$$\mathcal{O} = \left\{ k(X) \neq \hat{K} \right\}\ .$$

[2]For point-to-point transmission, one can alternatively consider the more popular criterion of vanishingly small *excess distortion probability* as in [11], and observe the same energy-distortion exponent for the case of $M = 1$ as in the asymptotics of $M \to \infty$. However, for the broadcast scenarios we consider, the asymptotic results depend heavily on *expected distortion*. In particular, it is not immediately clear how the converses in [17] and [2] can be adapted to excess distortion probability.

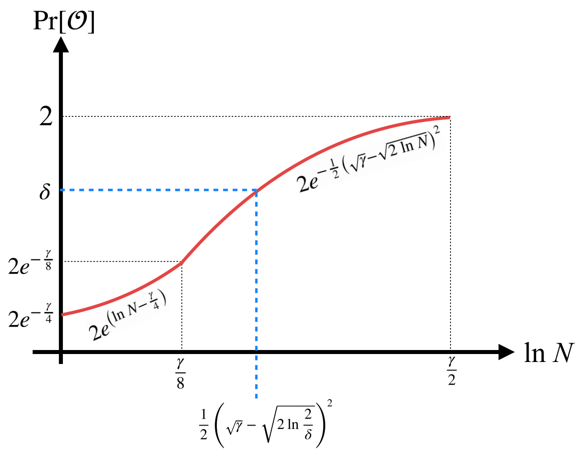

Figure 5. The upper bound on the probability of distortion outage as a function of $\ln N$ for a fixed ENR γ. The maximum allowed $\ln N$ to guarantee $\Pr[\mathcal{O}] \le \delta$ when $\delta \ge 2e^{-\frac{\gamma}{8}}$ is also shown.

Since the analysis of the probability of decoding error for orthogonal signaling can be found in the literature (for example, see [16, Section 6.6]), we include it only for convenience and defer it to Appendix A[3]. The analysis yields

$$\Pr[\mathcal{O}] \le \begin{cases} 2e^{\left(\ln N-\frac{\gamma}{4}\right)} & \ln N < \frac{\gamma}{8} \\ 2e^{-\frac{1}{2}\left(\sqrt{\gamma}-\sqrt{2\ln N}\right)^2} & \frac{\gamma}{8} \le \ln N \le \frac{\gamma}{2} \end{cases} . \quad (28)$$

This upper bound is depicted in Fig. 5, for a given ENR γ.

Now, for a given $\delta > 0$ and ENR γ, we need to use the maximum possible number of quantization levels $N_{\max}(\gamma,\delta)$ such that $\Pr[\mathcal{O}] \le \delta$ to minimize the distortion (see Fig. 5). It follows from (28) that

$$N_{\max}(\gamma,\delta) = \begin{cases} \frac{\delta}{2}e^{\frac{\gamma}{4}} & 4\ln\frac{2}{\delta} < \gamma < 8\ln\frac{2}{\delta} \\ e^{\frac{1}{2}\left(\sqrt{\gamma}-\sqrt{2\ln\frac{2}{\delta}}\right)^2} & \gamma \ge 8\ln\frac{2}{\delta} \end{cases} .$$

It then follows by choosing $N = N_{\max}(\gamma,\delta)$ in (27) that for any $\delta > 0$,

$$\mathbb{E}\left[||X-\hat{X}||^2\middle|\mathcal{O}^c\right] \le \frac{\sqrt{3}\pi}{2}e^{-\left(\sqrt{\gamma}-\sqrt{2\ln\frac{2}{\delta}}\right)^2} + \epsilon .$$

for arbitrarily small $\epsilon > 0$ and large enough[4] γ. Thus,

$$D(E,\delta) \le \frac{\sqrt{3}\pi}{2}e^{-\left(\sqrt{\gamma}-\sqrt{2\ln\frac{2}{\delta}}\right)^2}$$

for any $\delta > 0$ and large enough E, and therefore

$$\lim_{\delta\to 0}\lim_{E\to\infty} -\frac{1}{\gamma}\ln D(E,\delta) \ge 1$$

finishing the proof. ■

[3]We also refer the reader to [14] for a similar analysis.

[4]Large enough γ is necessary because (i) we need $\gamma \ge 8\ln\frac{2}{\delta}$, and (ii) $N_{\max}(\gamma,\delta)$ must be large enough for the Bennett approximation (27) to be valid.

B. Broadcasting of a Single Gaussian Source

Definition 10. A quadruple (D_1, D_2, E, δ) is achievable for zero-delay broadcasting of a single source with distortion outage if for any $\epsilon > 0$, there exist a large enough N, an encoder $\phi_{1,N}$, decoders $\psi_{1,N}^{(i)}$, and outage regions $\mathcal{O}_i \in \mathbb{R}\times\mathbb{R}^N$ for $i = 1,2$ such that

$$\begin{aligned} \mathbb{E}\left[||U^N||^2\right] &\le E+\epsilon \\ \Pr\left[\left(X, W_i^N\right)\in\mathcal{O}_i\right] &\le \delta \\ \mathbb{E}\left[(X-\hat{X}_i)^2\middle|\mathcal{O}_i^c\right] &\le D_i+\epsilon . \end{aligned}$$

Definition 11. An *energy-distortion exponent pair* (β_1,β_2) is achievable for zero-delay broadcasting of a single source with distortion outage if there exist functions $D_1(E,\delta)$ and $D_2(E,\delta)$ such that $(D_1(E,\delta), D_2(E,\delta), E, \delta)$ is achievable for all $E > 0, \delta > 0$ and

$$\lim_{\delta\to 0}\lim_{E\to\infty} -\frac{1}{\gamma_i}\ln D_i(E,\delta) = \beta_i$$

for $i = 1,2$.

We are now ready to state and prove the following theorem.

Theorem 12. *Any pair $(\beta_1,\beta_2) \in \mathcal{B}_{\text{single}}$ is achievable for zero-delay broadcasting of a single Gaussian source with distortion outage.*

Proof: We quantize X with successive refinement with $N_1 \gg 1$ levels in the base layer and $N_2 \gg 1$ levels in the refinement layer. We then use orthogonal signaling and maximum likelihood decoding as in point-to-point transmission, with the modification that the transmission is done in two rounds: In the ith round, $i = 1,2$, the channel is used N_i times to transmit the ith layer quantization index. Although both receivers have access to both rounds, only the second receiver attempts to decode the refinement layer. We define $\mathcal{O}_1$ as the event that receiver 1 decodes the base layer index incorrectly, as in point-to-point transmission. On the other hand, we let $\mathcal{O}_2$ indicate that the second receiver incorrectly decodes either of the quantization indices.

It is clear from (27) that if the point density function $\lambda(x)$ for the base layer is chosen as in (25), there exist large enough N_1 such that

$$\mathbb{E}\left[(X-\hat{X}_1)^2\middle|\mathcal{O}_1^c\right] \le \frac{\sqrt{3}\pi}{N_1^2}+\epsilon \quad (29)$$

for any $\epsilon > 0$. We claim that for large enough N_1 and N_2, one can simultaneously achieve (29) and

$$\mathbb{E}\left[(X-\hat{X}_2)^2\middle|\mathcal{O}_2^c\right] = \frac{\sqrt{3}\pi}{N_1^2N_2^2}+\epsilon \quad (30)$$

for any $\epsilon > 0$. To that end, it suffices to recall that high-resolution quantization is equivalent to mapping the sample X onto the interval $[0,1]$ using $G(x) = \int_{-\infty}^{x}\lambda(z)dz$ followed by uniform quantization. Thus, not only does dividing the interval $[0,1]$ into N_1 equal-width intervals (followed by the inverse mapping G^{-1}) yield the optimal quantizer for the base layer, but further dividing each subinterval into N_2 equal-width intervals yield the optimal quantizer for the refinement layer. In

$$N_{2,\max}(g,\alpha,\gamma_1,\delta)=\begin{cases}\frac{\delta'}{2}e^{\frac{g(1-\alpha)\gamma_1}{4}} & 4\ln\frac{2}{\delta'}<g(1-\alpha)\gamma_1<8\ln\frac{2}{\delta'}\\ e^{\frac{1}{2}\left(\sqrt{g(1-\alpha)\gamma_1}-\sqrt{2\ln\frac{2}{\delta'}}\right)^2} & g(1-\alpha)\gamma_1\geq 8\ln\frac{2}{\delta'}\end{cases}. \tag{31}$$

other words, X is successively refineable in the high-resolution regime[5].

Let the two rounds of transmission expend energies αE and $(1-\alpha)E$, respectively, for some $0<\alpha<1$. Then, as in the proof of Theorem 9, one can upper bound $\Pr[\mathcal{O}_1]$ as

$$\Pr[\mathcal{O}_1]\leq\begin{cases}2e^{\left(\ln N_1-\frac{\alpha\gamma_1}{4}\right)} & \ln N_1<\frac{\alpha\gamma_1}{8}\\ 2e^{-\frac{1}{2}\left(\sqrt{\alpha\gamma_1}-\sqrt{2\ln N_1}\right)^2} & \frac{\alpha\gamma_1}{8}\leq\ln N_1\leq\frac{\alpha\gamma_1}{2}\end{cases}. \tag{32}$$

Then, also as in the proof of Theorem 9, to guarantee $\Pr[\mathcal{O}_1]\leq\delta$, we need to satisfy $N_1\leq N_{1,\max}(\alpha,\gamma_1,\delta)$, where

$$N_{1,\max}(\alpha,\gamma_1,\delta)=\begin{cases}\frac{\delta}{2}e^{\frac{\alpha\gamma_1}{4}} & 4\ln\frac{2}{\delta}<\alpha\gamma_1<8\ln\frac{2}{\delta}\\ e^{\frac{1}{2}\left(\sqrt{\alpha\gamma_1}-\sqrt{2\ln\frac{2}{\delta}}\right)^2} & \alpha\gamma_1\geq 8\ln\frac{2}{\delta}\end{cases}. \tag{33}$$

Combining (29) and (33) then yields for any $\epsilon>0$, $\delta>0$, and large enough E that

$$\mathbb{E}\left[(X-\hat{X}_1)^2\middle|\mathcal{O}_1^c\right]\leq D_1(E,\delta)+\epsilon$$

with

$$D_1(E,\delta)=\frac{\sqrt{3}\pi}{2}e^{-\left(\sqrt{\alpha\gamma_1}-\sqrt{2\ln\frac{2}{\delta}}\right)^2}. \tag{34}$$

At the second receiver, one can use the union bound to write

$$\Pr[\mathcal{O}_2]\leq P_{e,1}+P_{e_2}$$

where $P_{e,1}$ and $P_{e,2}$ are the probabilities of the second receiver incorrectly decoding the base and refinement layer index incorrectly, respectively. Since the second channel is less noisy, it is clear that

$$P_{e,1}<\Pr[\mathcal{O}_1]\leq\delta\ .$$

We can in fact tighten this upper bound by first translating (32) for the second receiver as

$$P_{e,1}\leq\begin{cases}2e^{\left(\ln N_1-\frac{\alpha\gamma_2}{4}\right)} & \ln N_1<\frac{\alpha\gamma_2}{8}\\ 2e^{-\frac{1}{2}\left(\sqrt{\alpha\gamma_2}-\sqrt{2\ln N_1}\right)^2} & \frac{\alpha\gamma_2}{8}\leq\ln N_1\leq\frac{\alpha\gamma_2}{2}\end{cases} \tag{35}$$

and then assuming $\alpha\gamma_1\geq 8\ln\frac{2}{\delta}$ without loss of generality (because we will eventually let $E\to\infty$), using (33) to show both

$$\begin{aligned}2e^{\left(\ln N_1-\frac{\alpha\gamma_2}{4}\right)} &\leq 2e^{\left(\frac{1}{2}\left(\sqrt{\alpha\gamma_1}-\sqrt{2\ln\frac{2}{\delta}}\right)^2-\frac{g\alpha\gamma_1}{4}\right)}\\ &\leq 2e^{\left(\frac{1}{2}\left(\sqrt{\alpha\gamma_1}-\sqrt{2\ln\frac{2}{\delta}}\right)^2-2g\ln\frac{2}{\delta}\right)}\\ &\leq 2e^{-2g\ln\frac{2}{\delta}}\\ &= 2\left(\frac{\delta}{2}\right)^{2g}\end{aligned} \tag{36}$$

[5]This argument is independent of the PDF of X. The notion of successive refineability here is not to be confused with the notion that appears in the literature. The latter deals with finite rates but infinite blocklengths, whereas we are interested in infinite rates and scalar coding.

and

$$\begin{aligned}2e^{-\frac{1}{2}\left(\sqrt{\alpha\gamma_2}-\sqrt{2\ln N_1}\right)^2} &\leq 2e^{-\frac{1}{2}\left((\sqrt{g}-1)\sqrt{\alpha\gamma_1}+\sqrt{2\ln\frac{2}{\delta}}\right)^2}\\ &\leq 2e^{-\frac{1}{2}\left(2(\sqrt{g}-1)\sqrt{2\ln\frac{2}{\delta}}+\sqrt{2\ln\frac{2}{\delta}}\right)^2}\\ &\leq 2e^{-\left(2\sqrt{g}-1\right)^2\ln\frac{2}{\delta}}\\ &\leq 2\left(\frac{\delta}{2}\right)^{g}.\end{aligned} \tag{37}$$

Bringing together (35)-(37), we obtain

$$P_{e,1}\leq 2\left(\frac{\delta}{2}\right)^{g}. \tag{38}$$

Letting $\delta'=\delta-2\left(\frac{\delta}{2}\right)^g$, we can once again use the analysis in Theorem 9 to conclude that

$$P_{e,2}\leq 2e^{\left(\ln N_2-\frac{(1-\alpha)\gamma_2}{4}\right)}$$

when $\ln N_2<\frac{(1-\alpha)\gamma_2}{8}$, and

$$P_{e,2}\leq 2e^{-\frac{1}{2}\left(\sqrt{(1-\alpha)\gamma_2}-\sqrt{2\ln N_2}\right)^2}$$

when $\frac{(1-\alpha)\gamma_2}{8}\leq\ln N_2\leq\frac{(1-\alpha)\gamma_2}{2}$. Thus, to guarantee $\Pr[\mathcal{O}_2]\leq\delta$, it suffices to choose $N_2\leq N_{2,\max}(g,\alpha,\gamma_1,\delta)$, where $N_{2,\max}(g,\alpha,\gamma_1,\delta)$ is as given in (31) at the top of this page.

Combining (30), (33), and (31) then yields for any $\epsilon>0$, $\delta>0$, and large enough E that

$$\mathbb{E}\left[(X-\hat{X})^2\middle|\mathcal{O}_2^c\right]\leq D_2(E,\delta)+\epsilon$$

with

$$D_2(E,\delta)=\frac{\sqrt{3}\pi}{2}e^{-\left(\sqrt{\alpha\gamma_1}-\sqrt{2\ln\frac{2}{\delta}}\right)^2}e^{-\left(\sqrt{g(1-\alpha)\gamma_1}-\sqrt{2\ln\frac{2}{\delta'}}\right)^2}. \tag{39}$$

The proof is complete by observing that for any $\delta>0$,

$$\lim_{E\to\infty}-\frac{1}{\gamma_1}\ln D_1(E,\delta)=\alpha$$

and

$$\lim_{E\to\infty}-\frac{1}{\gamma_2}\ln D_2(E,\delta)=\frac{\alpha}{g}+1-\alpha\ .$$

■

C. Broadcasting of a Bivariate Gaussian Source

Definition 13. A quadruple (D_1,D_2,E,δ) is achievable for zero-delay broadcasting of bivariate sources with distortion outage if for any $\epsilon>0$, there exist a large enough N, an encoder $\phi_{1,N}$, decoders $\psi_{1,N}^{(i)}$, and outage regions $\mathcal{O}_i\in\mathbb{R}\times\mathbb{R}^N$ for $i=1,2$ such that

$$\begin{aligned}\mathbb{E}\left[||U^N||^2\right] &\leq E+\epsilon\\ \Pr\left[\left(X_1,X_2,W_i^N\right)\in\mathcal{O}_i\right] &\leq \delta\\ \mathbb{E}\left[(X_i-\hat{X}_i)^2\middle|\mathcal{O}_i^c\right] &\leq D_i+\epsilon\ .\end{aligned}$$

The definition of achievable energy-distortion exponent pairs is exactly as in Definiton 11.

Just as in the proof of Theorem 6, separately encoding the two sources and splitting the available energy E into αE and $(1-\alpha)E$ to transmit the quantization indices using orthogonal signaling is a sufficient strategy to achieve the same energy-distortion exponents in $\mathcal{B}_{\text{bivariate}}$ as stated in the next theorem.

Theorem 14. *Any pair $(\beta_1, \beta_2) \in \mathcal{B}_{\text{bivariate}}$ is achievable for zero-delay broadcasting of bivariate Gaussian sources with distortion outage.*

Proof: We provide only a sketch of the proof as it is straightforward. Using the same technique as in the proof of Theorem 12, it is possible to show that for any $\epsilon > 0$, $\delta > 0$, and large enough E,

$$\begin{aligned} \mathbb{E}\left[||U^N||^2\right] &\leq E + \epsilon \\ \Pr\left[\left(X_1, X_2, W_i^N\right) \in \mathcal{O}_i\right] &\leq \delta \\ \mathbb{E}\left[(X_i - \hat{X}_i)^2 \middle| \mathcal{O}_i^c\right] &\leq D_i(E, \delta) + \epsilon \end{aligned}$$

for $i = 1, 2$, can be simultaneously satisfied, where

$$\begin{aligned} D_1(E, \delta) &= \frac{\sqrt{3}\pi}{2} e^{-\left(\sqrt{\alpha\gamma_1} - \sqrt{2\ln\frac{2}{\delta}}\right)^2} \\ D_2(E, \delta) &= \frac{\sqrt{3}\pi}{2} e^{-\left(\sqrt{(1-\alpha)\gamma_2} - \sqrt{2\ln\frac{2}{\delta}}\right)^2} . \end{aligned}$$

Therefore for any $\delta > 0$,

$$\lim_{E\to\infty} -\frac{1}{\gamma_1} \ln D_1(E, \delta) = \alpha$$

and

$$\lim_{E\to\infty} -\frac{1}{\gamma_2} \ln D_2(E, \delta) = 1 - \alpha$$

and the proof is complete. ■

Appendix A
Analysis of the Probability of Decoding Error

Without loss of generality, we assume that the first codeword is sent. Using the normalized noise vector $Z^N = \frac{W^N}{\sigma_W}$, the probability of erroneous decoding for fixed N and E becomes

$$\begin{aligned} &\Pr[\mathcal{O}] \\ &= 1 - \int_{-\infty}^{\infty} f_Z(z_1) \Pr\left[\max_{2\leq i\leq N}\{Z_i\} < \sqrt{\gamma} + z_1\right] dz_1 \\ &= 1 - \int_{-\infty}^{\infty} f_Z(z_1) \prod_{i=2}^{N} \Pr\left[Z_i < \sqrt{\gamma} + z_1\right] dz_1 \\ &= \int_{-\infty}^{\infty} f_Z(z_1) \left\{1 - \left(1 - Q\left(\sqrt{\gamma} + z_1\right)\right)^{N-1}\right\} dz_1 . \end{aligned}$$

Here, we use the standard definition of the Q-function as

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{s^2}{2}} ds.$$

It is well-known that the Chernoff bound on the Q-function is given by

$$Q(x) \leq e^{-\frac{x^2}{2}} \tag{40}$$

for all $x \geq 0$. Although there are other established bounds that are tighter than (40), the Chernoff bound will suffice for our analysis.

From this point on, we will assume that $\gamma \geq 2\ln N$. Then defining

$$\alpha = \sqrt{2\ln N} - \sqrt{\gamma} \tag{41}$$

which is always non-positive, we write

$$\Pr[\mathcal{O}] = P_{\mathcal{O},1} + P_{\mathcal{O},2}$$

with

$$\begin{aligned} P_{\mathcal{O},1} &= \int_{-\infty}^{\alpha} f_Z(z_1) \left\{1 - \left(1 - Q\left(\sqrt{\gamma} + z_1\right)\right)^{N-1}\right\} dz_1 \\ P_{\mathcal{O},2} &= \int_{\alpha}^{\infty} f_Z(z_1) \left\{1 - \left(1 - Q\left(\sqrt{\gamma} + z_1\right)\right)^{N-1}\right\} dz_1 . \end{aligned}$$

We then bound $P_{\mathcal{O},1}$ as

$$\begin{aligned} P_{\mathcal{O},1} &\leq \int_{-\infty}^{\alpha} f_Z(z_1) dz_1 \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{z_1^2}{2}} dz_1 \\ &= 1 - Q(\alpha) \\ &= Q(-\alpha) \\ &\leq e^{-\frac{\alpha^2}{2}} \end{aligned} \tag{42}$$

where the last inequality follows from (40) and the fact that $\alpha \leq 0$. Also, since it follows from (41) that $\alpha > -\sqrt{\gamma}$, we have for all $z_1 \geq \alpha$ that

$$\begin{aligned} 1 - \left(1 - Q\left(\sqrt{\gamma} + z_1\right)\right)^{N-1} &\leq (N-1) Q\left(\sqrt{\gamma} + z_1\right) \\ &\leq N e^{-\frac{(\sqrt{\gamma}+z_1)^2}{2}} \end{aligned}$$

again using (40). Therefore,

$$\begin{aligned} P_{\mathcal{O},2} &\leq N \int_{\alpha}^{\infty} f_Z(z_1) e^{-\frac{(\sqrt{\gamma}+z_1)^2}{2}} dz \\ &= \frac{N}{\sqrt{2\pi}} e^{-\frac{\gamma}{4}} \int_{\alpha+\sqrt{\frac{\gamma}{4}}}^{\infty} e^{-s^2} ds \\ &= \frac{N}{2\sqrt{\pi}} e^{-\frac{\gamma}{4}} Q\left(\sqrt{2}\left(\alpha + \sqrt{\frac{\gamma}{4}}\right)\right) \\ &= \begin{cases} \frac{N}{2\sqrt{\pi}} e^{-\frac{\gamma}{4} - \left(\alpha + \sqrt{\frac{\gamma}{4}}\right)^2} & \alpha \geq -\sqrt{\frac{\gamma}{4}} \\ \frac{N}{2\sqrt{\pi}} e^{-\frac{\gamma}{4}} & \alpha < -\sqrt{\frac{\gamma}{4}} \end{cases} \end{aligned} \tag{43}$$

After some algebraic manipulations, it can be shown using (41) that

$$N e^{-\frac{\gamma}{4} - \left(\alpha + \sqrt{\frac{\gamma}{4}}\right)^2} = e^{-\frac{\alpha^2}{2}}. \tag{44}$$

Finally, bringing (41)-(44) together, we find for all $e^{\frac{\gamma}{8}} \leq N \leq e^{\frac{\gamma}{2}}$ that

$$\begin{aligned} \Pr[\mathcal{O}] &\leq \left(1 + \frac{1}{2\sqrt{\pi}}\right) e^{-\frac{\alpha^2}{2}} \\ &\leq 2e^{-\frac{1}{2}\left[\sqrt{\gamma} - \sqrt{2\ln N}\right]^2} \end{aligned} \tag{45}$$

and similarly for all $N < e^{\frac{\gamma}{8}}$ that

$$\begin{aligned} \Pr[\mathcal{O}] &\leq N e^{-\frac{\gamma}{4}} \left(\frac{1}{2\sqrt{\pi}} + e^{-\left(\sqrt{2\ln N} - \sqrt{\frac{\gamma}{4}}\right)^2}\right) \\ &\leq 2N e^{-\frac{\gamma}{4}} . \end{aligned} \tag{46}$$

References

[1] E. Akyol, K. Rose, K. Viswanatha, and T. Ramstad, "On zero-delay source-channel coding, *IEEE Transactions on Information Theory*, vol. 60, no. 12, pp. 7473–7489, Dec. 2014

[2] H. Behroozi, F. Alajaji, and T. Linder, "On the performance of hybrid digital/analog coding for broadcasting correlated Gaussian sources," *IEEE Transactions on Communications*, vol. 59, no. 12, pp. 3335–3343, Dec. 2011.

[3] S. I. Bross, A. Lapidoth, and S. Tinguely, "Broadcasting correlated Gaussians," *IEEE Transactions on Information Theory*, vol. 56, no. 7, pp. 3057–3068, Jul. 2010.

[4] X. Chen and E. Tuncel, "Zero-delay joint source-channel coding using hybrid digital-analog schemes in the Wyner-Ziv setting," *IEEE Transactions on Communications*, vol. 62, no. 2, pp. 726–735, Feb. 2014.

[5] Y. Gao and E. Tuncel, "Separate source-channel coding for transmitting correlated Gaussian sources over degraded broadcast channels," *IEEE Transactions on Information Theory*, vol. 59, no. 6, pp. 3619–3634, Jun. 2013.

[6] A.Gersho and R.M.Gray, *Vector Quantization and Signal Compression.* Kluwer Academic Publishers, Boston, MA, 1992.

[7] F. Hekland, P. A. Floor, and T. A. Ramstad, "Shannon-Kotel'nikov mappings in joint source-channel coding," *IEEE Transactions on Communications,* vol. 57, no. 1, pp. 94–105, Jan. 2009.

[8] A. Jain, D. Gündüz, S. R. Kulkarni, H. V. Poor, and S. Verdú, "Energy-distortion tradeoffs in Gaussian joint source-channel coding problems," *IEEE Transactions on Information Theory*, vol. 58, no. 5, pp. 3153–3168, May 2012.

[9] N. Jiang, Y. Yang, A. Host-Madsen, and Z. Xiong, "On the minimum energy of sending correlated sources over the Gaussian MAC," *IEEE Transactions on Information Theory*, vol. 60, no. 10, pp. 6254–6275, Aug 2014.

[10] K. Khezeli and J. Chen, "A source-channel separation theorem with application to the source broadcast problem," *IEEE Transactions on Information Theory*, vol. 62, no. 4, pp. 1764–1781, Apr. 2016.

[11] V. Kostina and S. Verdú, "Lossy joint source-channel coding in the finite block length regime," *IEEE Transactions on Information Theory*, vol. 59, no. 5, pp. 2545–2575, May 2013.

[12] R. .R. Mohassel, A. S. Fung, F. Mohammadi, and K. Raahemifar, "A survey on advanced metering infrastructure and its application in smart grids," *IEEE 27th Canadian Conference on Electrical and Computer Engineering*, Toronto, ON, May 2014, pp. 1–8.

[13] Y. Murin, Y. Kaspi, R. Dabora, and D. Gunduz, "On the energy-distortion tradeoff of Gaussian broadcast channels with feedback", submitted. Available at https://arxiv.org/abs/1605.00204.

[14] Y. Polyansky, V. Poor, and S. Verdu, "Minimum energy to send k bits through the Gaussian channel with and without feedback," *IEEE Transactions on Information Theory*, vol. 51, no. 8, pp. 4880–4922, Aug. 2011.

[15] V. M. Prabhakaran, R. Puri, and K. Ramchandran, "Hybrid digital-analog codes for source-channel broadcast of Gaussian sources over Gaussian channels," *IEEE Transactions on Information Theory*, vol. 57, no. 1, pp. 60–64, Jan. 2011.

[16] J. G. Proakis, M. Salehi, *Digital communications Wiley 2008*

[17] Z. Reznic, M. Feder, and R. Zamir, "Distortion bounds for broadcasting with bandwidth expansion," *IEEE Transactions on Information Theory*, vol. 52, no. 8, pp. 3778–3788, Aug. 2006.

[18] L. Song, J. Chen, and C. Tian, "Broadcasting correlated vector Gaussians," *IEEE Transactions on Information Theory*, vol. 61, no. 5, pp. 2465–2477, May 2015.

[19] R. Soundararajan and S. Vishwanath, "Hybrid coding for Gaussian broadcast channels with Gaussian sources," *Proc. IEEE Int. Symp. Inf. Theory (ISIT 2009)*, Seoul, Korea, June 2009.

[20] C. Tian, S. Diggavi, and S. Shamai, "The achievable distortion region of sending a bivariate Gaussian source on the Gaussian broadcast channel," *IEEE Transactions on Information Theory*, vol. 57, no. 10, pp. 6419–6427, October 2010.

[21] N. Wernersson, M. Skoglund, and T. Ramstad, "Polynomial Based Analog Source-Channel Codes," *IEEE Transactions on Communications*, vol. 57, no. 9, pp. 2600–2606, Sep. 2009.

Erman Köken [S'16] received the B.S. and M.S. degrees in electrical and electronics engineering at Middle East Technical University, Turkey, in 2008 and 2011, respectively. He is currently pursuing his PhD degree in electrical and computer engineering at the University of California, Riverside. He is an IEEE Student Member and a founding member of IEEE HKN Sigma Lambda Chapter. His research interests include joint source-channel coding, zero-delay transmission scenarios, and wireless communications.

Deniz Gündüz [S'03-M'08-SM'13] received the B.S. degree in electrical and electronics engineering from METU, Turkey in 2002, and the M.S. and Ph.D. degrees in electrical engineering from NYU Polytechnic School of Engineering in 2004 and 2007, respectively. After his PhD, he served as a postdoctoral research associate at Princeton University, and as a consulting assistant professor at Stanford University. He was a research associate at CTTC in Barcelona, Spain until September 2012, when he joined the Electrical and Electronic Engineering Department of Imperial College London, UK, as a Lecturer. Currently he is a Reader in the same department.

Dr. Gündüz is an Editor of the IEEE Transactions on Communications, and IEEE Transactions on Green Communications and Networking. He is the recipient of a Starting Grant of the European Research Council (ERC), the 2014 IEEE Communications Society Best Young Researcher Award for the Europe, Middle East, and Africa Region, Best Paper Award at the 2016 IEEE Wireless Communications and Networking Conference (WCNC), and the Best Student Paper Award at the 2007 IEEE International Symposium on Information Theory (ISIT). He served as the General Co-chair of the 2016 IEEE Information Theory Workshop, and the 2012 IEEE European School of Information Theory (ESIT). His research interests lie in the areas of communication theory and information theory with special emphasis on joint source-channel coding, multi-user networks, energy efficient communications and privacy in cyber-physical systems.

Ertem Tuncel [S'99-M'04] received the B.S. degree in electrical and electronics engineering from the Middle East Technical University, Ankara, Turkey in 1995, the M.S. degree in electrical and electronics engineering from Bilkent University, Ankara, Turkey in 1997, and the Ph.D. degree in electrical and computer engineering from University of California, Santa Barbara, in 2002. In 2003, he joined the Department of Electrical and Computer Engineering, University of California, Riverside, where he is now a Professor.

Dr. Tuncel received the National Science Foundation CAREER Award in 2007. He is an Associate Editor of the IEEE TRANSACTIONS ON INFORMATION THEORY.

His main research interests are multi-user networks, joint source-channel coding, zero-error information theory, zero-delay communication, and content-based retrieval in high-dimensional databases.