

On the Distortion-Energy Tradeoff for Zero-Delay Transmission of a Gaussian Source Over the AWGN Channel

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Abstract—An achievable scheme for zero-delay transmission of an i.i.d. Gaussian source over an additive white Gaussian channel with no bandwidth limitation is introduced, and its energy-distortion performance is analyzed. By the nature of the problem, one must transmit each source sample separately but can use the channel infinitely many times. We introduce an outage concept, and analyze the expected distortion conditioned on no outage. We show that the proposed scheme can approach the asymptotical decay for large enough energy for arbitrary outage probability. The proposed scheme builds on separation of source and channel coding, whereby the source is quantized with a high-resolution optimal quantizer. In the high energy-to-noise ratio (ENR) regime, the minimum energy required to obtain a given distortion level in the proposed scheme can approach arbitrarily close the Shannon bound, which can only be achieved using infinite delay.

Index Terms—Energy-distortion exponents, Energy-distortion tradeoff, high-resolution quantization, zero-delay.

I. INTRODUCTION

We consider the zero-delay lossy transmission of a Gaussian random variable $X \sim \mathcal{N}(0, 1)$ over an energy-limited additive white Gaussian noise (AWGN) channel with N channel uses where N can be arbitrarily large. This setting is motivated by the communication scenarios where the source sampling rate is really low, e.g., the source might consist of sensor measurements taken every hour, or even every minute¹. In those cases, even if the channel bandwidth is actually narrow, a large bandwidth expansion factor becomes feasible. We study the idealized (and mathematically tractable) case where the bandwidth is infinite.

The encoder

$$\phi_N : \mathbb{R} \rightarrow \mathbb{R}^N$$

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¹To clarify, in our terminology zero-delay refers to transmission being complete before the next source sample is generated.

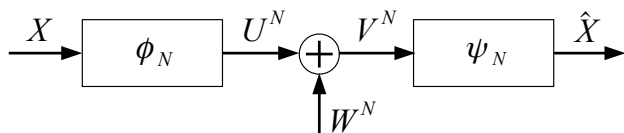


Figure 1. The system model.

maps the source X into the channel input $U^N = \phi(X)$ where U^N is energy-limited as

$$\sum_{t=1}^N \mathbb{E}\{U_t^2\} \leq E. \quad (1)$$

At the receiver, the function

$$\psi_N : \mathbb{R}^N \rightarrow \mathbb{R}$$

maps the observation $V^N = U^N + W^N$ into the estimation $\hat{X} = \psi_N(V^N)$, where $W^N \sim \mathcal{N}(0, \sigma_W^2 \mathbf{I}_N)$ is the additive channel noise. Without loss of generality, we take $\sigma_W^2 = 1$ so that E is also the energy-to-noise ratio (ENR), which is an important parameter in the sequel. The scenario is illustrated in Fig. 1.

Definition 1: A triplet (E, D, ϵ) of energy, distortion, and outage probability is achievable if for some N , there exist an outage region $\mathcal{O} \subset \mathbb{R} \times \mathbb{R}^N$ in the source-channel space such that

$$\Pr[(X, W^N) \in \mathcal{O}] \leq \epsilon \quad (2)$$

and an encoder-decoder pair (ϕ_N, ψ_N) satisfying (1) and

$$\mathbb{E}[(X - \hat{X})^2 | \mathcal{O}^c] \leq D. \quad (3)$$

Remark 1: It should be clear that when the probability density functions (pdf) satisfy $f_X(x) > 0$ and $f_W(w) > 0$ for all x and w , as is the case for Gaussian sources and channels, then $(E, D, 0)$ is achievable if and only if $\mathcal{O} = \emptyset$, thereby reducing (3) to

$$\mathbb{E}[(X - \hat{X})^2] \leq D. \quad (4)$$

Therefore, our achievability definition is more general than the classical energy-distortion tradeoff requiring (1) and (4).

The motivation behind considering distortion outages is that any coding scheme that transmits some *digital* information is prone to error in decoding that information. Regardless of how small the probability of incorrect decoding is, the overall expected distortion might still be very adversely affected. We essentially allow for catastrophic reconstruction provided that it occurs with a small enough probability.

One simple coding strategy is linear (a.k.a. uncoded) transmission: Ignoring all the available bandwidth, simply set $N = 1$ and use the encoder-decoder pair

$$\phi_1(x) = \sqrt{E}x \quad \psi_1(v) = \frac{\sqrt{E}}{1+E}v,$$

together with the outage region $\mathcal{O} = \emptyset$ to achieve the triplet $(E, D, 0)$ with

$$D = \frac{1}{1+E}. \quad (5)$$

It might initially be thought that this is a very poor utilization of the available bandwidth, and therefore the performance can be improved. However, it is well-established (see, for example, [4]) that even if the total energy E was spread across channel uses, i.e., if $U_t = \alpha_t X$ for $t = 1, \dots, N$ with

$$\sum_{t=1}^N \alpha_t^2 = E,$$

the resultant minimum expected distortion would still be given as in (5). Therefore, for large ENR, the distortion decays as $\frac{1}{\text{ENR}}$ when linear transmission is used.

If we allowed for infinite delay, then $(E, D, 0)$ would be achievable if²

$$\begin{aligned} D &\geq \lim_{N \rightarrow \infty} \left(1 + \frac{E}{N}\right)^{-N} \\ &= \exp(-E) \end{aligned} \quad (6)$$

as was discussed in [2]. Motivated by this exponential decay, we define the energy-distortion exponent.

Definition 2: An *energy-distortion exponent* $\beta > 0$ is achievable if for any $\epsilon > 0$, there exist an achievable triplet (E, D, ϵ) with large enough E , satisfying

$$-\frac{1}{E} \log D > \beta - \epsilon.$$

The authors of this work recently addressed the same problem in [3], where they proposed a scheme based on quantization of the source random variable into equiprobable cells and transmission of the quantization index through

²Throughout, \log and \exp refer to natural logarithm and its inverse.

the channel using capacity-achieving codes. It was shown in [3] that for any $\epsilon > 0$, it is possible to achieve

$$D = \frac{13}{6E} \exp\left(-\frac{E}{2}\right) \quad (7)$$

for large enough ENR E . Therefore, implicit in that result was that, according to the above definition of energy-distortion exponents, the same approach achieves $\beta = 0.5$.

In what follows, we propose another separable coding scheme, and subsequently show that it achieves $\beta = 1$. We need to emphasize that ENR is assumed to be large to allow for high-resolution quantizers.

II. THE CODING SCHEME

The single random variable X is first quantized with a high-resolution optimal quantizer with N levels. It is well-known [1] that the optimal quantizer has a point density function $\lambda(x)$ given by

$$\lambda(x) = \frac{f_X(x)^{\frac{1}{3}}}{\int_{-\infty}^{\infty} f_X(x')^{\frac{1}{3}} dx'}$$

which is a Gaussian with variance 3. The resultant distortion can be approximated as

$$\begin{aligned} D &\approx \frac{1}{12N^2} \int_{-\infty}^{\infty} \frac{f_X(x)}{\lambda(x)^2} dx \\ &= \frac{1}{12N^2} \left(\int_{-\infty}^{\infty} f_X(x)^{\frac{1}{3}} dx \right)^3 \\ &= \frac{\sqrt{3}\pi}{2N^2}. \end{aligned} \quad (8)$$

The quantized indices $k(X)$ are mapped into orthogonal channel input vectors U^N such that

$$U_t = \begin{cases} \sqrt{E} & t = k(X) \\ 0 & t \neq k(X) \end{cases}$$

and therefore $\|U^N\|^2 = E$. Note that we use the channel only N times (instead of infinitely many) but we will eventually let N grow without bound. At the receiver end, upon receiving $V^N = U^N + W^N$, the decoder simply selects

$$\hat{K} = \arg \max_{1 \leq i \leq N} V_i$$

and then outputs

$$\hat{X} = r_{\hat{K}}$$

where r_k is the k th reconstruction level.

Remark 2: Although simplex encoding can improve over orthogonal signaling, the ENR gain is $\frac{N}{N-1}$ which approaches 1 as the number of codewords N goes to infinity.

III. PERFORMANCE ANALYSIS

We regard the codeword decoding error as the outage event. We use the probability of error analysis in [5, Section 6.6], which we adapt to our notation and include below for convenience³. Using the bounds therein, we can solve for the maximum allowed number of codewords N for a given energy and maximum allowable probability of error pair (E, ϵ) , and translate that N to expected distortion D .

A. Bounds on the Probability of Error

Without loss of generality, we assume that the first codeword is sent. The probability of erroneous decoding for fixed N and E is then

$$\begin{aligned} P_e &= 1 - \int_{-\infty}^{\infty} f_W(w_1) \Pr \left[\max_{2 \leq i \leq N} \{W_i\} < \sqrt{E} + w_1 \right] dw_1 \\ &= 1 - \int_{-\infty}^{\infty} f_W(w_1) \prod_{i=2}^N \Pr [W_i < \sqrt{E} + w_1] dw_1 \\ &= \int_{-\infty}^{\infty} f_W(w_1) \left\{ 1 - \left(1 - Q(\sqrt{E} + w_1) \right)^{N-1} \right\} dw_1 \\ &= P_{e,1} + P_{e,2} \end{aligned}$$

where the integral is divided into two as

$$\begin{aligned} P_{e,1} &= \int_{-\infty}^{\alpha} f_W(w_1) \left\{ 1 - \left(1 - Q(\sqrt{E} + w_1) \right)^{N-1} \right\} dw_1 \\ P_{e,2} &= \int_{\alpha}^{\infty} f_W(w_1) \left\{ 1 - \left(1 - Q(\sqrt{E} + w_1) \right)^{N-1} \right\} dw_1 \end{aligned}$$

with

$$\alpha = \sqrt{2 \log N} - \sqrt{E}.$$

Here, we use the standard definition of the Q-function as

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left(-\frac{x^2}{2}\right) dx.$$

It is well-known that the Chernoff bound on the Q-function is given by

$$Q(x) \leq \exp\left(-\frac{x^2}{2}\right) \quad (9)$$

for all $x \geq 0$. Although there are other established bounds that are tighter than (9), the Chernoff bound will suffice for our analysis.

Now, it follows from

$$1 - \left(1 - Q(\sqrt{E} + w_1) \right)^{N-1} \leq 1$$

³We also refer the reader to [6] for a similar analysis.

that

$$\begin{aligned} P_{e,1} &\leq \int_{-\infty}^{\alpha} f_W(w_1) dw_1 \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} \exp\left(-\frac{w_1^2}{2}\right) dw_1 \\ &= 1 - Q(\alpha) \\ &\stackrel{(a)}{=} Q(-\alpha) \\ &\leq \exp\left(-\frac{\alpha^2}{2}\right) \end{aligned} \quad (10)$$

where (a) follows from (9) and holds if $\alpha \leq 0$, or equivalently, $E \geq 2 \log N$.

Also, since $\alpha > -\sqrt{E}$, we have for all $w_1 \geq \alpha$ that

$$\begin{aligned} 1 - \left(1 - Q(\sqrt{E} + w_1) \right)^{N-1} &\leq (N-1) Q(\sqrt{E} + w_1) \\ &\leq N \exp\left(-\frac{(\sqrt{E} + w_1)^2}{2}\right) \end{aligned}$$

again using (9). Therefore,

$$\begin{aligned} P_{e,2} &\leq N \int_{\alpha}^{\infty} f_W(w_1) \exp\left(-\frac{(\sqrt{E} + w_1)^2}{2}\right) dw_1 \\ &= \frac{N}{\sqrt{2\pi}} \exp\left(-\frac{E}{4}\right) \int_{\alpha + \sqrt{\frac{E}{4}}}^{\infty} \exp(-u^2) du \\ &= \frac{N}{\sqrt{2}} \exp\left(-\frac{E}{4}\right) Q\left(\sqrt{2}\left(\alpha + \sqrt{\frac{E}{4}}\right)\right) \\ &= \begin{cases} \frac{N}{\sqrt{2}} \exp\left(-\frac{E}{4} - \left(\alpha + \sqrt{\frac{E}{4}}\right)^2\right) & \alpha \geq -\sqrt{\frac{E}{4}} \\ \frac{N}{\sqrt{2}} \exp\left(-\frac{E}{4}\right) & \alpha < -\sqrt{\frac{E}{4}} \end{cases} \end{aligned} \quad (11)$$

After some algebraic manipulation, it can be shown that

$$N \exp\left(-\frac{E}{4} - \left(\alpha + \sqrt{\frac{E}{4}}\right)^2\right) = \exp\left(-\frac{\alpha^2}{2}\right)$$

Since $\alpha \geq -\sqrt{\frac{E}{4}}$ is the same as $E \leq 8 \log N$, (11) simplifies to

$$P_{e,2} \leq \begin{cases} \frac{1}{\sqrt{2}} \exp\left(-\frac{[\sqrt{E} - \sqrt{2 \log N}]^2}{2}\right) & E \leq 8 \log N \\ \frac{N}{\sqrt{2}} \exp\left(-\frac{E}{4}\right) & E > 8 \log N \end{cases} \quad (12)$$

Bringing (10) and (12) together, we observe that if

$$2 \log N \leq E \leq 8 \log N, \quad (13)$$

then the error probability can be upper bounded as

$$P_e \leq 2 \exp \left(-\frac{1}{2} \left[\sqrt{E} - \sqrt{2 \log N} \right]^2 \right). \quad (14)$$

On the other hand, if

$$E > 8 \log N, \quad (15)$$

then we have

$$\begin{aligned} P_e &\leq N \exp \left(-\frac{E}{4} \right) \left(1 + \exp \left[-\left(\sqrt{2 \log N} - \sqrt{\frac{E}{4}} \right)^2 \right] \right) \\ &\leq 2N \exp \left(-\frac{E}{4} \right) \end{aligned} \quad (16)$$

B. An Achievable Region

Since the distortion decreases with increasing N , we need to choose for fixed (E, ϵ) the maximum possible N . Once the maximum allowed codeword number N for any given (E, ϵ) is found, we can calculate D according to (8).

Theorem 1: Let $\eta(E, \epsilon)$ and $N_{\max}(E, \epsilon)$ be defined as

$$\eta(E, \epsilon) = \begin{cases} \frac{E}{2} - 2 \log \frac{2}{\epsilon} & 4 \log \frac{2}{\epsilon} < E < 8 \log \frac{2}{\epsilon} \\ \left(\sqrt{E} - \sqrt{2 \log \frac{2}{\epsilon}} \right)^2 & E \geq 8 \log \frac{2}{\epsilon} \end{cases}$$

and

$$N_{\max}(E, \epsilon) = \begin{cases} \frac{\epsilon}{2} \exp \left(\frac{E}{4} \right) & 4 \log \frac{2}{\epsilon} < E < 8 \log \frac{2}{\epsilon} \\ \exp \left(\frac{\eta(E, \epsilon)}{2} \right) & E \geq 8 \log \frac{2}{\epsilon} \end{cases}.$$

Then all triplets (E, D, ϵ) such that

$$D \approx \frac{\sqrt{3\pi}}{2} \exp(-\eta(E, \epsilon)) \quad (17)$$

are achievable provided the high-resolution approximation (8) is accurate for $N = N_{\max}(E, \epsilon)$.

Remark 3: When E is larger than (or at least close to) $8 \log \frac{2}{\epsilon}$, $N_{\max}(E, \epsilon)$ is larger than (or, respectively, close to) $\frac{\epsilon}{2}$. Therefore, in the intended (i.e., low outage) regime $\epsilon \rightarrow 0$, N_{\max} is indeed very large, thereby making the high resolution distortion approximation accurate. On the other hand, if E is close to $4 \log \frac{2}{\epsilon}$, N_{\max} becomes very small, and the theorem would not be applicable. Fortunately, we will utilize this theorem mainly in the high energy regime $E \geq 8 \log \frac{2}{\epsilon}$.

Proof: Manipulating (14), we observe that for $P_e \leq \epsilon$ we need

$$N \leq \exp \left(\frac{\eta(E, \epsilon)}{2} \right).$$

With the choice $N = \exp \left(\frac{\eta(E, \epsilon)}{2} \right)$, the high-resolution distortion (8) translates to (17). On the other hand, for this

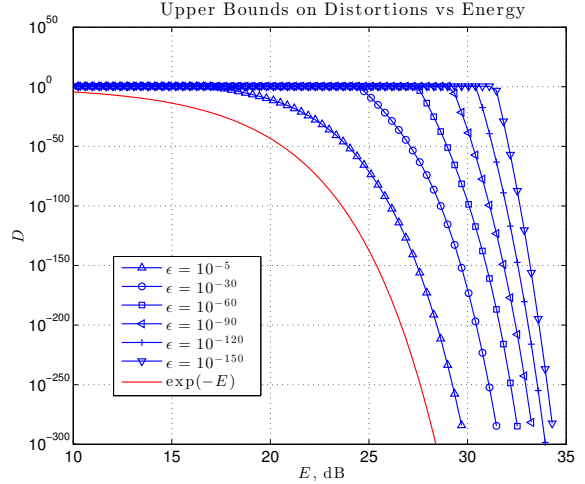


Figure 2. The blue curves represent the achievable energy-distortion tradeoffs for the given different ϵ values. The red curve represents the asymptotical distortion expression (6).

choice to be valid, we need from (13) that

$$\frac{E}{8} \leq \left(\sqrt{\frac{E}{2}} - \sqrt{\log \frac{2}{\epsilon}} \right)^2 \leq \frac{E}{2}$$

which yields

$$E \geq 8 \log \frac{2}{\epsilon}. \quad (18)$$

Similarly manipulating (15) and (16), we find that for $P_e \leq \epsilon$, we need

$$N \leq \frac{\epsilon}{2} \exp \left(\frac{E}{4} \right)$$

for fixed (E, ϵ) satisfying

$$4 \log \left(\frac{2}{\epsilon} \right) < E < 8 \log \left(\frac{2}{\epsilon} \right). \quad (19)$$

Note that the left inequality in (19) stems from the fact that we need, at a minimum, $N > 1$. With the choice $N = \frac{\epsilon}{2} \exp \left(\frac{E}{4} \right)$, once again (8) reduces to (17). ■

In Fig. 2, we exhibit the behavior of the (E, D, ϵ) achievability region for fixed and small values of ϵ . As expected, with decreasing ϵ , the minimum energy E needed to obtain a certain D increases. Another observation is that the large energy behavior seems to suggest that the asymptotic (i.e., infinite-delay) performance might be eventually attained, were we not limited by our computational tools. Our main theorem in the next subsection states exactly that.

C. Optimal Zero-Delay Distortion-Energy Exponent

Theorem 2: The proposed scheme achieves an energy-distortion exponent of $\beta = 1$.

Proof: For arbitrarily small $\epsilon > 0$, we operate in the high ENR regime $E \geq 8 \log \frac{2}{\epsilon}$. Therefore, it follows from Theorem 1 that

$$D = \frac{\sqrt{3\pi}}{2} \exp \left(- \left(\sqrt{E} - \sqrt{2 \log \frac{2}{\epsilon}} \right)^2 \right)$$

can be achieved by the proposed scheme. Hence,

$$\begin{aligned} -\frac{1}{E} \log D &= \frac{\left(\sqrt{E} - \sqrt{2 \log \frac{2}{\epsilon}} \right)^2 + \log \left(\frac{2}{\sqrt{3\pi}} \right)}{E} \\ &\geq 1 - \epsilon \end{aligned}$$

for large enough E . ■

IV. CONCLUSION

We considered a scenario where a Gaussian random variable is transmitted over a bandwidth-unlimited but energy-limited AWGN channel in a zero-delay fashion. We showed in this paper that the same exponential decay that can be achieved with an infinite-delay-infinite-bandwidth coding scheme can also be achieved with a separable zero-delay coding scheme with infinite-bandwidth if we allow for arbitrarily small outage probability.

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