# Rate Regions for the Separated Two-Way Relay Channel

Deniz Gündüz[*†], Ertem Tuncel[‡], Jayanth Nayak[‡]

[*]Dept. of Electrical Engineering, Princeton University, Princeton, NJ
[†]Dept. of Electrical Engineering, Stanford University, Stanford, CA
[‡]Dept. of Electrical Engineering, University of California, Riverside, CA

***Abstract*—A two-way relay channel in which two users communicate with each other over a relay terminal is considered. In particular, a "separated" two-way relay channel, in which the users do not receive each other's signals is studied. Various achievable schemes are proposed and corresponding achievable rate regions are characterized. Specifically, a combination of partial decode-and-forward and compress-and-forward schemes is proposed. In addition, compress-and-forward relaying with two layered quantization, in which one of the users receive a better description of the relay received signal is studied. Extension of these achievable schemes to the Gaussian separated two-way relay channel is presented. It is shown that the compress-and-forward scheme achieves rates within half bit of the capacity region in the Gaussian setting. Numerical results are also presented for comparison of the proposed achievable schemes in the Gaussian case.**

## I. Introduction

We consider a two-way relay channel (TRC) [1], [2] in which two users exchange independent messages with the help of a dedicated relay terminal. TRC models scenarios such as ad-hoc networks, or two mobile terminals communicating with each other over a base station or a satellite. In its most general form, this multi-user channel model can be considered as a combination of various other well-studied models such as the relay, the multiple access, the broadcast and the two-way channels.

If we let one of the messages to be constant and ignore the channel output at the other user, the system model reduces the classical relay channel studied by Cover and El Gamal [3]. On the other hand, if the relay terminal has no channel input, then the model reduces to the two-way channel model studied by Shannon [4]. Neither of these two special cases of the TRC model has been fully understood, in the sense that, we do not have the corresponding finite letter capacity expressions.

In this paper, we consider a special TRC, which we call the *separated two-way relay channel* (sTRC), in which the two terminals receive signals only from the relay terminal. This corresponds to a scenario where the two terminals are physically separated, and can only communicate through the relay terminal which is located in between the users and is connected to both of them.

We first propose an achievable rate region for the sTRC based on a combination of partial decode and forward (pDF) and compress-and-forward (CF) schemes proposed in [3] for the classical relay channel. However, different from the usual relay channel, in the sTRC model, the relay helps both users simultaneously, hence both the decoded parts of the messages and the compressed version of the relay's received signal need to be broadcast to the two users. On the other hand, this is also different from the usual broadcast channel in the sense that, both users already know their own messages, and their own channel input, which serves as correlated side information for what the relay is broadcasting. Hence, broadcasting of the decoded parts and the compressed relay signal can be considered under the framework of Slepian-Wolf over broadcast channel [5] and Wyner-Ziv over broadcast channel [6], [7], respectively.

When broadcasting the relay received signal to the users in a lossy fashion in the CF scenario, due to the differences between the channel qualities from the relay to the users as well as the qualities of the available side information at the users, a single quantization may not be simultaneously optimal for both users. It is shown in [6], [7] that, we can improve the performance by transmitting refinement information to the "better" user. While the performance measure in those works is the average distortion of the reconstruction at the users, here we are interested in the achieved rates. However, similar arguments apply in the CF scenario, since the achieved rates increase as the users receive higher quality descriptions of the relay's received signal.

We also provide extension of the achievable rate regions to the Gaussian TRC setting. We present a comparison of the rate regions achieved by CF with single-layer and two-layer quantization schemes as well as the combined scheme of pDF and the CF. In addition, we show that, for Gaussian channels with symmetric noise variances at the users single-layer CF suffices to achieve within half bit of the capacity region.

The TRC model has been popular recently, both because it models many practical communication scenarios, and because it represents a very simple yet theoretically challenging model in which we can observe benefits of network coding in the physical domain. The network coding aspects of this model has been illustrated in [8] where the capacity region of the binary additive sTRC is shown to be achievable with binary linear block codes. The relay only decodes and forward the binary sum of the messages, which suffices for each user as they already know their own messages. For this binary setting,

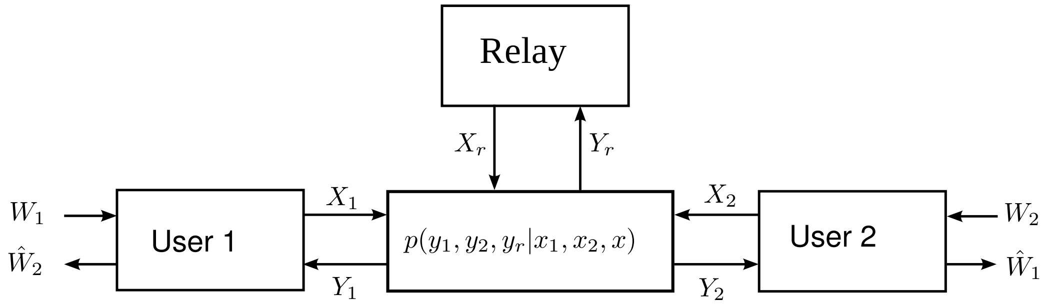


Fig. 1. Two-way relay channel model. User 1 and User 2 exchange information over the relay terminal.

all the known random coding schemes fall short of achieving the rates on the boundary of the capacity region. Extension of this structured coding approach to the Gaussian channel setup through lattice codes is given in [9], [10].

The rest of the paper is organized as follows. We present the system model in Section II. Several achievable rate regions are provided in Section III together with an outer bound. In Section IV, we focus on the Gaussian channel setting and extend our achievability results and the outer bound to the Gaussian setting. We also provide numerical results providing a comparison between the achievable rate regions and the outer bound. Finally we conclude the paper in Section V followed by the Appendices.

## II. System Model

The discrete memoryless two-way relay channel is denoted by $(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_r, p(y_1, y_2, y_r|x_1, x_2, x_r), \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y})$, where $\mathcal{X}_1, \mathcal{X}_2$ and $\mathcal{X}_r$ are the finite input alphabets, $\mathcal{Y}_1, \mathcal{Y}_2$ and $\mathcal{Y}$ are the finite output alphabets while $p(y_1, y_2, y_r|x_1, x_2, x_r)$ is the probability transition function from the inputs to the outputs. $x_i$ and $y_i$ model the input and output signals, respectively, of user $i$, while $x_r$ and $y_r$ are the input and output signals, respectively, of the relay terminal.

An $(2^{nR_1}, 2^{nR_2}, n)$ code for the TRC consists of two sets of integers $\mathcal{W}_1 = \{1, 2, \ldots, 2^{nR_1}\}, \mathcal{W}_2 = \{1, 2, \ldots, 2^{nR_2}\}$ as the message sets, two sets of encoding functions $\{f_{i,j}\}_{j=1}^n$ at the users such that

$$x_{i,j} = f_{i,j}(W_i, Y_{i,1}, \ldots, Y_{i,j-1}), \quad i = 1, 2, \quad 1 \le j \le n,$$

a set of encoding functions $\{f_{r,j}\}_{j=1}^n$ at the relay such that

$$x_{r,j} = f_{r,j}(Y_{r,1}, \ldots, Y_{r,j-1}), \quad 1 \le j \le n,$$

and two decoding functions $g_1 : \mathcal{Y}_1^n \times \mathcal{W}_1 \to \mathcal{W}_2, g_2 : \mathcal{Y}_2^n \times \mathcal{W}_2 \to \mathcal{W}_1$.

The average probability of error for this system is defined as

$$P_e^n = \Pr\{g_1(W_1, Y_1^n) \ne W_2 \text{ or } g_2(W_2, Y_2^n) \ne W_1\}.$$

Note that, $P_e^n \to 0$ implies that individual average error probabilities also go to zero. We assume that the messages $W_i$, $i = 1, 2$, are chosen independently and uniformly over the message sets $\mathcal{W}_i$.

*Definition 1:* A rate pair $(R_1, R_2)$ is said to be *achievable* for TRC, if there exists a sequence of $(2^{nR_1}, 2^{nR_2}, n)$ codes such that $P_e^n \to 0$ as $n \to \infty$. The *capacity region* of the TRC is the convex closure of all achievable rate pairs.

We focus on separated TRC in which $Y_i$ depends only on $X_r$ and $X_i$, $i = 1, 2$, that is, $X_1 - (X_r, X_2) - Y_2$ and $X_2 - (X_r, X_1) - Y_1$ form Markov chains at each channel realization. For this class of TRC's, no positive rate can be achieved without the help of the relay.

## III. Achievable Rate Regions and an Outer Bound

In this section, we provide achievable rate regions for the sTRC. We only consider coding functions at the users which depend on the message at the corresponding user, that is, the encoding functions are independent of the previously received messages. This is similar to the "restricted" two-way channel model of Shannon [4]. However, while it is possible to characterize the capacity region of the two-way channel under the restricted encoder assumption, this does not directly extend to the restricted sTRC model, and the capacity of the general sTRC remains to be open.

In the first achievability scheme, each user splits its message into two. While the first part of each message is decoded by the relay, the second parts are only decoded by the other user. The relay terminal broadcasts decoded parts of the messages to the users by exploiting the fact that each user already knows its own part. Simultaneously, the relay also compresses its own received signal and broadcasts the quantized version to the users by exploiting the fact that each user already knows the decoded parts as well as their own transmitted signals, which are correlated with the relay received signal. While the coding strategy for broadcasting the decoded parts is as in [5], [11], the strategy for broadcasting the relay's received signal follows [6]. The following theorem can be considered as an extension of the Theorem 7 in [3] where partial DF and CF schemes are combined.

*Theorem 1:* In a separated TRC $(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_r, p(y_1, y_2, y_r|x_1, x_2, x_r), \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_r)$, the convex hull of all rate pairs $(R_1, R_2)$ that satisfy the equations at the bottom of the next page subject to

$$\begin{aligned} I(Y_r; \hat{Y}_r|X_r, X_1, U_2, Q) &\le I(X_r; Y_1|V, X_1, Q) \\ I(Y_r; \hat{Y}_r|X_r, X_2, U_1, Q) &\le I(X_r; Y_2|V, X_2, Q) \end{aligned}$$

for some joint probability distribution $p(q,v,u_1,u_2,x_1,x_2,x_r,y_1,y_2,y_r,\hat{y}_r) = p(q)p(u_1|q)p(u_2|q)$ $p(x_1|u_1)p(x_2|u_2)p(v|q)p(x_r|v)p(y_1,y_2,y_r|x_1,x_2,x_r)$ $\cdot p(\hat{y}_r|x_r,u_1,u_2,y_r)$ is achievable, where $q \in \mathcal{Q}$, $u_i \in \mathcal{U}_i$, $v \in \mathcal{V}$ and $\hat{y}_r \in \hat{\mathcal{Y}}_r$ and $\mathcal{Q}$, $\mathcal{U}_1$, $\mathcal{U}_2$ , $\mathcal{V}$ and $\hat{\mathcal{Y}}_r$ all have bounded cardinalities.

*Proof:* A sketch of the proof can be found in Appendix A. ■

*Remark 1:* If we constrain the relay to pDF, that is $V = X_r$, $U_1 = X_1$, $U_2 = X_2$ and $\hat{\mathcal{Y}}_r = \emptyset$, the transmission scheme is equivalent to DF since there is no direct link between the users to help them decode the rest of the messages. Then the convex hull of the set of all rate pairs $(R_1,R_2)$ that satisfy

$$R_1 \leq \min\{I(X_1;Y_r|X_r,Q), I(X_r;Y_2|X_2,Q)\} \quad (1)$$
$$R_2 \leq \min\{I(X_2;Y_r|X_r,Q), I(X_r;Y_1|X_1,Q)\} \quad (2)$$
$$R_1+R_2 \leq I(X_1,X_2;Y_r|X_r,Q) \quad (3)$$

for some joint probability distribution

$$p(q,x_1,x_2,x_r,y_1,y_2,y_r) = p(q)p(x_1|q)p(x_2|q)p(x_r|q) \cdot p(y_1,y_2,y_r|x_1,x_2,x_r)$$

is achievable.

This DF based scheme was studied in [12] for half-duplex relay channels. While the first terms in the constraints in the right hand side (RHS) of (1)-(3) are due to the multiple access channel from the users to the relay, the second terms in the RHS of (1)-(2) are due to the broadcasting stage from the relay to the users. These latter rate constraints can also be obtained as a special case of the joint source-channel coding scheme in [13], [5]. Next, we consider pure CF relaying at the relay. The following lemma gives the corresponding achievable rate region which can be found in [14] and [15] for half-duplex relays.

*Lemma 1:* In a separated TRC, by compress-and-forward relaying, that is $\mathcal{U}_1 = \mathcal{U}_2 = \mathcal{V} = \emptyset$, convex hull of the set of all rate pairs $(R_1,R_2)$ that satisfy

$$R_1 \leq I(X_1;\hat{Y}_r|X_r,X_2,Q), \quad (4)$$
$$R_2 \leq I(X_2;\hat{Y}_r|X_r,X_1,Q) \quad (5)$$

subject to

$$I(Y_r;\hat{Y}_r|X_r,X_1,Q) \leq I(X_r;Y_1|X_1,Q), \quad (6)$$
$$I(Y_r;\hat{Y}_r|X_r,X_2,Q) \leq I(X_r;Y_2|X_2,Q) \quad (7)$$

for some joint probability distribution

$$p(q,x_1,x_2,x_r,y_1,y_2,y_r,\hat{y}_r) = p(q)p(x_1|q)p(x_2|q)p(x_r|q) p(y_1,y_2,y_r|x_1,x_2,x_r)p(\hat{y}_r|x_r,y_r)$$

is achievable.

In the achievability of Lemma 1, relay terminal quantizes its received signal and broadcasts this quantized version to both users exploiting their correlated side information. Each user decodes other user's message from this quantized version of the relay's received signal. However, since neither the channels from the relay to the users, nor their side information qualities are equivalent, this single layer broadcast scheme is limited by the quantization that can be transmitted to the worst user. However, as in the case of broadcasting a common source to two receivers with different channel and side information qualities, studied in [6], in general it is possible to broadcast a common layer to both users while transmitting a refinement layer to the better user. Naturally, this can also be combined with the partial decoding scheme as in Theorem 1, but here we will only present the CF with two layer broadcasting for the sake of brevity. We give some definitions first.

We denote by $\mathcal{R}_1^{CF}$ the set of all rate pairs $(R_1,R_2)$ that satisfy

$$R_1 \leq I(X_1;\hat{Y}_r^2|X_r,X_2,Q)$$
$$R_2 \leq I(X_2;\hat{Y}_r^1|X_r,X_1,Q)$$

subject to

$$I(Y_r;\hat{Y}_r^1|X_r,X_1,Q) \leq I(V;Y_1|X_1,Q)$$
$$I(Y_r;\hat{Y}_r^1|X_r,X_2,Q) \leq I(V;Y_2|X_2,Q)$$
$$I(Y_r;\hat{Y}_r^2|X_r,X_2,\hat{Y}_r^1,Q) \leq I(X_r;Y_1|V,X_1,Q)$$

for some joint probability distribution

$$p(q,x_1,x_2,v,x_r,y_1,y_2,y_r,\hat{y}_r^1,\hat{y}_r^2) = p(q)p(x_1|q)p(x_2|q) \cdot p(v|q)p(x_r|v)p(y_1,y_2,y_r|x_1,x_2,x_r)p(\hat{y}_r^2|x_r,y_r)p(\hat{y}_r^1|x_r,\hat{y}_r^2) \quad (8)$$

such that $V - X_r - (Y_1,Y_2)$ form a Markov chain.

Similarly, we denote by $\mathcal{R}_2^{CF}$ the set of all rate pairs $(R_1,R_2)$ that satisfy

$$R_1 \leq I(X_1;\hat{Y}_r^1|X_r,X_2,Q)$$
$$R_2 \leq I(X_2;\hat{Y}_r^2|X_r,X_1,Q)$$

subject to

$$I(Y_r;\hat{Y}_r^1|X_r,X_1,Q) \leq I(V;Y_1|X_1,Q)$$
$$I(Y_r;\hat{Y}_r^1|X_r,X_2,Q) \leq I(V;Y_2|X_2,Q)$$
$$I(Y_r;\hat{Y}_r^2|X_r,X_1,\hat{Y}_r^1,Q) \leq I(X_r;Y_2|V,X_2,Q)$$

for some joint probability distribution of the form (8), such that $V - X_r - (Y_1,Y_2)$ form a Markov chain.

---

$$R_1 \leq \min\left\{I(U_1;Y_r|U_2,X_r,V,Q)+I(X_1;\hat{Y}_r|X_r,U_1,X_2,Q), I(X_r;Y_2|X_2,Q)-I(Y_r;\hat{Y}_r|X_r,U_1,X_1,X_2,Q)\right\}$$
$$R_2 \leq \min\left\{I(U_2;Y_r|U_1,X_r,V,Q)+I(X_2;\hat{Y}_r|X_r,U_2,X_1,Q), I(X_r;Y_1|X_1,Q)-I(Y_r;\hat{Y}_r|X_r,U_2,X_2,X_1,Q)\right\}$$
$$R_1+R_2 \leq I(U_1,U_2;Y_r|X_r,V,Q)+I(X_1;\hat{Y}_r|X_r,U_1,X_2,Q)+I(X_2;\hat{Y}_r|X_r,U_2,X_1,Q)$$

Note that, the regions $\mathcal{R}_1^{CF}$ and $\mathcal{R}_2^{CF}$ are based on which user will receive only a single quantization $\hat{Y}_r^1$ of the relay received signal, and which one will also receive the refinement $\hat{Y}_r^2$.

*Theorem 2:* In a separated TRC, by compress-and-forward relaying in two layers, convex hull of the region $\mathcal{R}_1^{CF} \bigcup \mathcal{R}_2^{CF}$ is achievable.

*Proof:* The proof of the theorem is skipped due to space limitations. ■

The achievability scheme is based on quantizing the relay received signal into two layers: the base layer $\hat{Y}_r^1$, and an enhancement layer $\hat{Y}_r^2$. Then the relay broadcasts the base layer to both users, while transmitting the enhancement layer to only one of the users. However, this is not the usual broadcast channel due to the availability of correlated side information at the receivers. As described in detail in [6], there can be different transmission schemes by using successive decoding or dirty paper encoding to mitigate the interference among the transmission of these two layers. Here, for brevity, we consider only one of the schemes, in which the base layer codeword is broadcast while the refinement layer codeword is treated as noise. In [6], it is shown that this scheme dominates the other possibilities in terms of the achieved distortion performance in the case of Gaussian sources and Gaussian channels. We have not performed such comparison in the current setting, hence it is possible that the set of achievable rate pairs can be enlarged by using one of the alternative transmission schemes in [6].

We provide an outer bound for the capacity region of the separated TRC in the following proposition.

*Proposition 1:* Any rate pair achievable in the separated TRC must satisfy

$$R_1 \leq \min\{I(X_1;Y_r|X,X_2), I(X;Y_2|X_2)\}, \quad (9)$$
$$R_2 \leq \min\{I(X_2;Y_r|X,X_1), I(X;Y_1|X_1)\}, \quad (10)$$

for some joint distribution

$$p(x_1,x_2,x,y_1,y_2,y_r) = p(x,x_1,x_2)p(y_1,y_2,y_r|x_1,x_2,x).$$

*Proof:* Proof follows from the usual cut-set bound arguments [16]. ■

A tighter outer bound can be obtained if we constrain the model to the restricted sTRC setting in which the encoders at the users only depend on their own messages, and ignore the channel outputs. Note that, all the achievable schemes considered in this paper fall into this category.

*Proposition 2:* Any achievable rate pair $(R_1, R_2)$ for the separated TRC with restricted encoders must satisfy

$$R_1 \leq \min\{I(X_1;Y_r|X_r,X_2,Q), I(X_r;Y_2|X_2,U_2,Q)\}$$
$$R_2 \leq \min\{I(X_2;Y_r|X_r,X_1,Q), I(X_r;Y_1|X_1,U_1,Q)\}$$

for some joint distribution of the form $p(q)p(x_1,u_1|q)p(x_2,u_2|q)p(x_r|u_1,u_2,q)p(y_1,y_2,y_r|x_1,x_2,x_r)$.

*Proof:* See Appendix C for a proof of the proposition. ■

## IV. Gaussian Two-way Relay Channel

In this section, we focus on the Gaussian separated TRC. Since self interference can simply be subtracted at each node, the additive white Gaussian noise channels can be modeled as below.

$$Y_r = X_1 + X_2 + Z_r \quad (11)$$
$$Y_i = X_i + Z_i, \text{ for } i = 1,2, \quad (12)$$

where $Z_r$ is the zero-mean Gaussian noise at the relay with variance $N_r$, while $Z_i$, $i = 1,2$, is the zero-mean Gaussian noise term at user $i$ with variance $N_i$. These noise terms are independent of each other and the channel inputs. Average power constraints on the transmitted signals apply:

$$\frac{1}{n}E\left[\sum_{j=1}^{n} x_{r,j}^2\right] \leq P_r \text{ and } \frac{1}{n}E\left[\sum_{j=1}^{n} x_{i,j}^2\right] \leq P_i, \quad i = 1,2.$$

We first consider the CF scheme with single layer for the Gaussian sTRC where the user $i$ transmits at power $\bar{P}_i \leq P_i$, while the relay transmits at its highest possible power $P_r$. Let the forward test channel for the relay received signal be given by $\hat{Y}_r = Y_r + Q$ with $Q \perp Y_r$, where $Q$ is the quantization error. Let $Q$ be zero mean Gaussian with variance $N_Q$. We have

$$I(Y_r;\hat{Y}_r|X_1) = \frac{1}{2}\log\left(1 + \frac{\bar{P}_2 + N_r}{N_Q}\right). \quad (13)$$

For successful transmission of the quantized version to both users, we need

$$N_Q \geq \max_{(i,j)=(1,2),(2,1)} \frac{(\bar{P}_i + N_r)N_j}{P_r}. \quad (14)$$

Then the achievable rate pair $(R_1, R_2)$ for this scheme can be found as

$$\begin{aligned} R_i &= I(X_i;\hat{Y}_r|X_j) \\ &= I(X_i;X_i + X_j + Z + Q|X_j) \\ &= \frac{1}{2}\log\left(1 + \frac{\bar{P}_i}{N + N_Q}\right), \end{aligned} \quad (15)$$

with $(i,j) \in \{(1,2),(2,1)\}$.

Without loss of generality, we assume $\bar{P}_1 \geq \bar{P}_2$ and $N_r = 1$. We also fix $N_1 = N_2 = 1$, which represents the cases $N_1 = N_2$ in general. Then we achieve the following rate pair with single layer CF.

$$\begin{aligned} R_1 &= \frac{1}{2}\log\left(\frac{1 + \bar{P}_1 + P_r + \bar{P}_1 P_r}{1 + \bar{P}_1 + P_r}\right) \\ R_2 &= \frac{1}{2}\log\left(\frac{1 + \bar{P}_1 + P_r + \bar{P}_2 P_r}{1 + \bar{P}_1 + P_r}\right) \end{aligned} \quad (16)$$

The whole set of achievable rates with the CF scheme can be found by taking the union of achievable pairs over all power allocations satisfying $\bar{P}_i \leq P_i$, $i = 1,2$.

The outer bound on rate $R_i$ in this setup is found as below for $i = 1,2$.

$$R_i \leq \min\left\{\frac{1}{2}\log(1 + P_1), \frac{1}{2}\log(1 + P_r)\right\}$$

The following lemma shows that, for any given triplet of $(P_1, P_2, P_r)$, CF scheme achieves rates within $1/2$ bit of the capacity outer bound. That is, the rate loss of the CF scheme with respect to the optimal cannot be larger than $1/2$ bit.

*Proposition 3:* The achievable rate rate pair $(R_1, R_2)$ of CF scheme given in (16)-(17) is within half bit of the capacity region of the Gaussian separate two-way relay channel, i.e., $(R_1 + \frac{1}{2}, R_2 + \frac{1}{2})$ is not achievable.

*Proof:* See Appendix B for a proof of this proposition. ■

Next, we consider the achievable rate region with the more advanced schemes for Gaussian sTRC. However, note that, in the symmetric case of $N_1 = N_2$, since the single layer CF already achieves rates within half bit of the capacity, none of these schemes can improve the rate region beyond half bit. The following rate region can be achieved by pDF combined with single layer CF scheme considered in Theorem 1.

Let $0 \leq \alpha_i \leq 1$ and $\alpha_i \leq \beta_i \leq 1$, $i = 1, 2$, and $0 \leq \alpha_r \leq 1$. Then the rate pairs $(R_1, R_2)$, as defined below, are achievable.

$$R_i = R_{df,i} + R_{cf,i}$$

for $i = 1, 2$, where

$$R_{df,1} \leq \min\left\{\frac{1}{2}\log\left(1 + \frac{\alpha_1 P_1}{(\beta_1 - \alpha_1)P_1 + \beta P_2 + N_r}\right), \frac{1}{2}\log\left(1 + \frac{\alpha_r P_r}{(1-\alpha_r)P_r + N_2}\right)\right\}$$

$$R_{df,2} \leq \min\left\{\frac{1}{2}\log\left(1 + \frac{\alpha_2 P_2}{(\beta_2 - \alpha_2)P_2 + \beta P_1 + N_r}\right), \frac{1}{2}\log\left(1 + \frac{\alpha_r P_r}{(1-\alpha_r)P_r + N_1}\right)\right\}$$

$$R_{df,1} + R_{df,2} \leq \frac{1}{2}\log\left(1 + \frac{\alpha_1 P_1 + \alpha_2 P_2}{(\beta_1 - \alpha_1)P_1 + (\beta_2 - \alpha_2)P_2 + N_r}\right)$$

while, for $i = 1, 2$,

$$R_{cf,i} = \frac{1}{2}\log\left(1 + \frac{(\beta_i - \alpha_i)P_i}{N_r + N_Q}\right)$$

and

$$N_Q = \max_{(i,j)=(1,2),(2,1)} \left\{\frac{((\beta_i - \alpha_i)P_i + N_r)N_j}{(1-\alpha_r)P_r}\right\}.$$

In Fig. 2, on the left, we plot the portion of the rate regions achieved by DF, single layer CF and the above partial DF schemes for the case $P_1 = P_r = 50$ dB and $P_2 = 20$ dB while $N_2 = 4$, $N_1 = N_r = 1$. We should note that the rate region of partial DF scheme is not just the convex combination of the pure DF and pure CF regions. In a concurrent work [17], it is shown that the combined scheme of pDF and single-layer CF achieves within $3/2$ bits of the capacity region for all Gaussian TRCs.

Next, we consider the two layered transmission with CF. We have two level quantization of the relay received signal $Y_r$ as follows. Let $\hat{Y}_r^1 = Y_r + Q_1$ with $Q_1 \perp Y_r$, where $Q_1 \sim \mathcal{N}(0, N_Q^1)$ and $\hat{Y}_r^2 = \hat{Y}_r^1 + Q_2$ with $Q_2 \perp (Y_r, Q_1)$, where $Q_2 \sim \mathcal{N}(0, N_Q^2)$. The relay broadcasts the coarse quantization $\hat{Y}_r^2$ to both users, using $\alpha P_r$ of its power. On the other hand, it transmits the fine quantization to only one user using the remaining power. The codeword corresponding to the fine quantization acts as noise for the coarse quantization. We have

$$N_Q^1 + N_Q^2 = \max_{(i,j)=(1,2),(2,1)} \frac{(\alpha\bar{P}_i + N_r)(N_j + (1-\alpha)P_r)}{\alpha P_r},$$

and

$$N_Q^1 = \frac{(\bar{P}_2 + N_r)}{(1 + (1-\alpha)P_r)(1 + \frac{\bar{P}_2 + N_r}{N_Q^1 + N_Q^2}) - 1}.$$

In Fig. 2, on the right, we have the achievable rate regions for DF, partial DF, CF with single layer and two layer transmissions. While in general partial DF enlarges the rate region compared to both DF and CF with single layer, we see that CF with two layers achieves rate pairs that cannot be achieved by partial DF.

We note here that, in the Gaussian case, in addition to the two-layer quantization, further gains might be achieved by considering hybrid digital-analog techniques as in [7]. This, in a sense, combines amplify-and-forward relaying with the other techniques.

## V. Conclusions

We have considered sTRC, in which two users, which can not overhear each other's signal, exchange information over a relay terminal. We have focused on the the restricted coding model in which the encoder of each user only depends on its own message, and ignores the previously received channel outputs. Our main result is an achievable rate region based on a combination of partial decode-and-forward scheme together with compress-and-forward in which the relay decodes only parts of the messages and forwards quantized versions of its received signal to the users in addition to the decoded parts. We have shown that the rate region can be enhanced by considered two levels of quantization at the relay where one of the users receives a better description of relay's received signal, and hence, a higher rate. We have also considered the Gaussian setting, and shown that the proposed coding scheme achieves within half bit of the capacity region.

## Appendix A
## Proof of Theorem 1

In the following proof we will use the notion of strong typicality. Given a random variable $X \sim p(x)$, $x \in \mathcal{X}$, the strong typical set for block length $n$ is denoted by $T_\delta^n(X)$.

For the brevity of the presentation, we consider $|\mathcal{Q}| = 1$ in the proof here. Generalization to arbitrary finite cardinalities follows from the usual techniques. We use the classical block Markov encoding scheme [3], in which $B - 1$ messages from each user is transmitted over $B$ blocks of $n$ symbols.

1) *Code generation:*
   - Generate $2^{nR_{11}}$ i.i.d. sequences $u_1^n$ with probability distribution $p(u_1^n) = \prod_{i=1}^n p(u_{1,i})$, and label these sequences as $u_1^n(w_1')$, $w_1' \in [1, 2^{nR_{11}}]$. Similarly, generate $2^{nR_{21}}$ i.i.d. sequences $u_2^n$ with probability

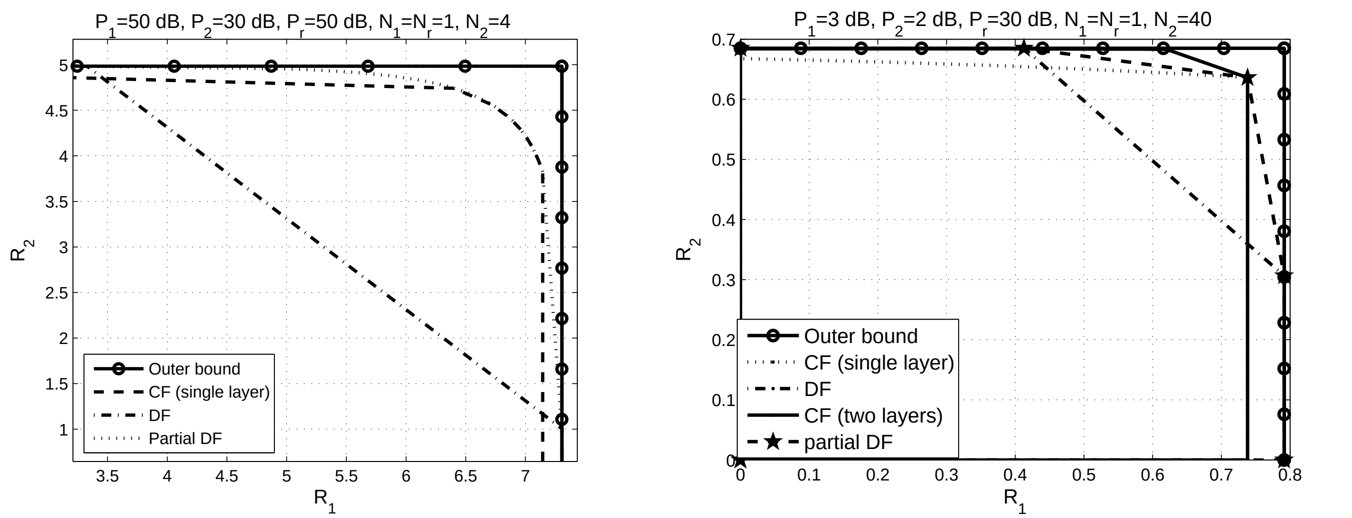


Fig. 2. Achievable rate regions and the outer bound for the Gaussian separated two-way relay channel. On the left we plot the portion of the rate region illustrating the gain of partial DF compared to DF and single layer CF schemes. The figure on the right illustrates a setup in which two layer CF improves upon partial DF scheme.

distribution $p(u_2^n) = \prod_{i=1}^n p(u_{2,i})$, and label these sequences as $u_2^n(w_2')$, $w_2' \in [1, 2^{nR_{21}}]$.

- Generate $2^{n(R_{11}+R_{21})}$ i.i.d. sequences $v^n$ with probability $p(v^n) = \prod_{i=1}^n p(v_i)$. Label these sequences as $v^n(w_1', w_2')$ where $w_1' \in [1, 2^{nR_{11}}]$ and $w_2' \in [1, 2^{nR_{21}}]$.
- For every $u_1^n(w_1')$ generate $2^{nR_{12}}$ i.i.d. sequences $x_1^n$ with probability
$$p(x_1^n|u_1^n(w_1')) = \prod_{i=1}^n p(x_{1,i}|u_{1,i}(w_1')).$$
Label these sequences as $x_1^n(w_1''|w_1')$, $w_1'' \in [1, 2^{nR_{12}}]$. Also, for every $u_2^n(w_2')$ generate $2^{nR_{22}}$ i.i.d. sequences $x_2^n$ with probability $p(x_2^n|u_2^n(w_2')) = \prod_{i=1}^n p(x_{2,i}|u_{2,i}(w_2'))$. Label these sequences as $x_2^n(w_2''|w_2')$, $w_2'' \in [1, 2^{nR_{22}}]$.
- For every $v^n(w_1', w_2')$ generate $2^{n(I(Y_r;\hat{Y}_r|X_r,U_1,U_2)+\epsilon)}$ i.i.d. sequences $x_r^n$ with probability
$$p(x_r^n|v^n(w_1', w_2')) = \prod_{i=1}^n p(x_{r,i}|v_i(w_1', w_2')).$$
Label these sequences as $x_r^n(z|w_1', w_2')$, $z \in [1, 2^{n(I(Y_r;\hat{Y}_r|X_r,U_1,U_2)+\epsilon)}]$.
- For every $(x_r^n(s|w_1', w_2'), u_1^n(w_1'), u_2^n(w_2'))$, generate $2^{n(I(Y_r;\hat{Y}_r|X_r,U_1,U_2)+\epsilon)}$ i.i.d. sequences $\hat{y}_r^n$ with probability
$$p(\hat{y}_r^n|x_r^n(s|w_1', w_2'), u_1^n(w_1'), u_2^n(w_2')) = \prod_{i=1}^n p(\hat{y}_{r,i}|x_{r,i}(s|w_1', w_2'), u_{1,i}(w_1'), u_{2,i}(w_2'))$$
for every $x_r \in \mathcal{X}_r$, $u_1 \in \mathcal{U}_1$ and $u_2 \in \mathcal{U}_2$. Label these sequences as $\hat{y}_r^n(z|w_1', w_2', s)$ where $z \in [1, 2^{n(I(Y;\hat{Y}_r|X,U_1,U_2)+\epsilon)}]$.

2) *Encoding:*
Let $w_{1,i} = (w_{1,i}', w_{1,i}'')$ and $w_{2,i} = (w_{2,i}', w_{2,i}'')$ be the messages that each user wants to transmit to the other at block $i$ for $i = 1, \ldots, B-1$. We have $w_{k,i}' \in [1, 2^{nR_{k1}}]$ and $w_{k,i}'' \in [1, 2^{nR_{k2}}]$ for $k = 1, 2$ and $i = 1, \ldots, B-1$. As the number of blocks goes to infinity, we achieve a rate pair of $\frac{B-1}{B}(R_1, R_2) \to (R_1, R_2)$ where $R_1 = R_{11} + R_{12}$ and $R_2 = R_{21} + R_{22}$.
Assume that
$$\big(\hat{y}_r^n(w_{1,i-1}', w_{2,i-1}', s_{i-1}), y_r(i-1), u_1^n(w_{1,i-1}'), u_2^n(w_{2,i-1}'), x_r^n(z_{i-2}|w_{1,i-1}', w_{2,i-1}')\big) \in T_\delta^n(\hat{Y}_r, Y_r, U_1, U_2, X_r).$$
Then the following codeword triplet is transmitted: $(x_1^n(w_{1,i}''|w_{1,i}'), x_2^n(w_{2,i}''|w_{2,i}'), x_r^n(z_{i-1}|w_{1,i-1}', w_{2,i-1}'))$.

3) *Decoding:*
We explain the decoding strategy at the end of block $i$.
- After receiving $y_r(i)$, the relay declares that $(\hat{w}_1', \hat{w}_2')$ was transmitted if
$$(u_1^n(\hat{w}_1'), u_2^n(\hat{w}_2'), y_r^n(i), x_r^n(z_{i-1}|w_{1,i-1}', w_{2,i-1}')) \in T_\delta^n(U_1, U_2, X_r, Y_r).$$
For sufficiently large $n$, we have $(\hat{w}_1', \hat{w}_2') = (w_{1,i}', w_{2,i}')$ with high probability if
$$\begin{aligned} R_{11} &< I(U_1; Y_r|U_2, X_r, V) \\ R_{21} &< I(U_2; Y_r|U_1, X_r, V) \\ R_{11} + R_{21} &< I(U_1, U_2; Y_r|X_r, V) \end{aligned}$$
- The relay also estimates $z_i$ such that
$$(\hat{y}_r^n(z_i|w_{1,i}', w_{2,i}', z_i), y_r^n(i), x_r^n(z_i|w_{1,i}', w_{2,i}')) \in T_\delta^n(\hat{Y}_r, Y_r, X_r).$$

Using the properties of typical sets, we can show that such a $z_i$ exist with high probability for sufficiently large $n$. Hence, the relay knows $z_i$.

- The User 1 declares $\hat{w}'_{2,i-1}$ was transmitted in block $i-1$ by User 2 if

$$(v^n(w'_{1,i-1}, w'_{2,i-1}), y_2^n(i), x_1^n(i)) \in T_\delta^n(V, Y_1, X_1).$$

Since User 1 knows $w'_{1,i-1}$, it can correctly find $\hat{w}'_{2,i-1} = w'_{2,i-1}$ with high probability if

$$R_{11} < I(V; Y_2|X_2),$$

and $n$ is sufficiently large due to the Markov chain $X_2 - (X_1, X_r) - Y_1$.
Similarly, User 2 can find the correct $\hat{w}'_{1,i-1} = w'_{1,i-1}$ with high probability if

$$R_{21} < I(V; Y_1|X_1),$$

and $n$ is sufficiently large due to the Markov chain $X_1 - (X_2, X_r) - Y_2$.

- Then the User 1 declares $\hat{z}_{i-1}$ for the $z$ index if

$$(x_r^n(z_{i-1}|w'_{1,i-1}, \hat{w}'_{2,i-1}), y_1^n(i), x_1^n(i), v^n(i)) \in T_\delta^n(X_r, Y_1, X_1, V).$$

and

$$\big(x_1^n(w''_{1,i-1}|w'_{1,i-1}), \hat{y}_r^n(z_{i-1}|w'_{1,i-1}, \hat{w}'_{2,i-1}, z_{i-2}), u_1^n(w'_{1,i}), u_2^n(w'_{2,i}), x_r^n(z_{i-2}|w'_{1,i-2}, w'_{2,i-2})\big) \in T_\delta^n(X_1, \hat{Y}_r, U_1, U_2, X_r)$$

are satisfied simultaneously for some index $z_{i-1}$. We can show that $\hat{z}_{i-1} = z_{i-1}$ with high probability for sufficiently large $n$ if

$$I(Y_r; \hat{Y}_r|X_r, U_1, U_2) + \epsilon < I(X_r; Y_1|X_1, V) + I(X_1; \hat{Y}_r|U_1, U_2, X_r).$$

due to the Markov chain $X_2 - (X_1, X_r) - Y_1$. Equivalently, if

$$I(Y_r; \hat{Y}_r|X_r, X_1, U_2) + \epsilon < I(X_r; Y_1|X_1, V).$$

Similarly, User 2 estimates $\hat{z}_{i-1} = z_{i-1}$ correctly with high probability if, $n$ is large enough and

$$I(Y_r; \hat{Y}_r|X_r, X_2, U_1) + \epsilon < I(X_r; Y_2|X_2, V).$$

due to the Markov chain $X_1 - (X_2, X_r) - Y_2$.

- Using $\hat{y}_r^n(\hat{z}_{i-1}|w'_{1,i-1}, \hat{w}'_{2,i-1}, \hat{z}_{i-1})$ and $y_1^n(i)$, User 1 declares $\hat{w}''_{i-1}$ was sent by User 2 in block $i-1$ if

$$\big(x_2^n(w''_{2,i-1}|w'_{2,i-1}), \hat{y}_r^n(z_{i-1}|w'_{1,i-1}, \hat{w}'_{2,i-1}, z_{i-2}), x_1^n(i-1), x_r^n(i-1), y_1^n(i)\big) \in T_\delta^n(X_2, \hat{Y}_r, X_1, X_r, Y_1).$$

We have $\hat{w}''_{2,i-1} = w''_{2,i-1}$ with high probability for sufficiently large $n$ if

$$R_{22} < I(X_2; \hat{Y}_r|X_1, X_r, U_2) - \epsilon.$$

Similarly, User 1 can find the correct $\hat{w}''_{1,i-1} = w''_{1,i-1}$ with high probability for sufficiently large $n$ if

$$R_{12} < I(X_1; \hat{Y}_r|X_2, X_r, U_1) - \epsilon.$$

Combining the bounds for $R_{11}, R_{12}, R_{21}$ and $R_{22}$, we can obtain the bounds given in the theorem.

## APPENDIX B
## PROOF OF PROPOSITION 3

Under the assumption of $P_1 \geq P_2$, we consider three cases separately depending on the relay power.

1) $P_r \geq P_1 \geq P_2$ : For the upper bound we have $R_i^{UB} = \frac{1}{2}\log(1+P_i)$. For the achievable rate, let $\bar{P}_i = P_i$. We have

$$\begin{aligned} R_1 &= \frac{1}{2}\log(1+P_1) + \frac{1}{2}\log\left(\frac{1+P_r}{1+P_1+P_r}\right) \\ &\geq R_1^{UB} - \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} R_2 &= \frac{1}{2}\log\left(\frac{1+P_2+P_r+P_2P_r}{1+P_1+P_r}\right) \\ &\geq R_2^{UB} - \frac{1}{2}. \end{aligned}$$

2) $P_1 \geq P_r \geq P_2$ : We have $R_1^{UB} = \frac{1}{2}\log(1+P_r)$ and $R_2^{UB} = \frac{1}{2}\log(1+P_2)$. Let $\bar{P}_1 = P_1$ and $\bar{P}_2 = P_2$. We have

$$\begin{aligned} R_1 &= \frac{1}{2}\log(1+P_r) + \frac{1}{2}\log(\frac{1+P_r}{1+2P_r}) \\ &\geq R_1^{UB} - \frac{1}{2}. \end{aligned}$$

$$\begin{aligned} R_2 &= \frac{1}{2}\log\left(\frac{1+P_2+P_r+P_2P_r}{1+P_1+P_r}\right) \\ &\geq R_2^{UB} - \frac{1}{2}. \end{aligned}$$

3) $P_1 \geq P_2 \geq P_r$ : We have $R_i^{UB} = \frac{1}{2}\log(1+P_r)$. Let $\bar{P}_i = P_2$. We have

$$\begin{aligned} R_i &= \frac{1}{2}\log(1+P_r) + \frac{1}{2}\log(\frac{1+P_2}{1+P_2+P_r}) \\ &\geq R_i^{UB} - \frac{1}{2}. \end{aligned}$$

## APPENDIX C
## PROOF OF PROPOSITION 2

From Fano's inequality, we have, for $i = 1, 2$,

$$H(W_i|Y_i^n) \leq n\delta_n,$$

where $\delta_n \to 0$ for $n \to \infty$. We can also write

$$\begin{aligned} H(W_1|W_2, Y_r^n) &= H(W_1|W_2, X_2^n, Y_r^n, X_r^n) && (17) \\ &\leq H(W_1|X_2^n, X_r^n) && (18) \\ &= H(W_1|X_2^n, X_r^n, Y_2^n) && (19) \\ &\leq H(W_1|Y_2^n) && (20) \\ &\leq n\delta_n, && (21) \end{aligned}$$

where (17) follows since, from the restricted coding constraint, $X_2^n$ is a function of $W_2$ and $X_r^n$ is a function of $Y_r^n$; (19) follows as $Y_2^n - (X_2^n, X_r^n) - W_1$ form a Markov chain based on the separated channel assumption; (20) follows since conditioning reduces entropy; and finally (21) follows from Fano's inequality. Similarly, we can also show

$$H(W_2|W_1, Y_r^n) \leq n\delta_n.$$

It follows that

$$\begin{aligned} nR_1 &= H(W_1) = H(W_1|W_2) && (22)\\ &\leq I(W_1; Y_r^n|W_2) + n\delta_n && (23)\\ &= \sum_{i=1}^n I(W_1; Y_{r,i}|W_2, Y_r^{i-1}) + n\delta_n \\ &= \sum_{i=1}^n H(Y_{r,i}|Y_r^{i-1}, W_2) - H(Y_{r,i}|W_1, W_2, Y_r^{i-1}) + \delta_n \\ &= \sum_{i=1}^n H(Y_{r,i}|Y_r^{i-1}, W_2, X_{2i}, X_{r,i}) \\ &\quad - H(Y_{r,i}|W_1, W_2, Y_r^{i-1}, X_{1i}, X_{2i}, X_{r,i}) + \delta_n && (24)\\ &\leq \sum_{i=1}^n H(Y_{r,i}|X_{2i}, X_{r,i}) - H(Y_{r,i}|X_{1i}, X_{2i}, X_{r,i}) + \delta_n && (25)\\ &= \sum_{i=1}^n I(X_{1i}; Y_{r,i}|X_{2i}, X_{r,i}) + \delta_n, && (26) \end{aligned}$$

where (23) follows form (21); (24) follows as $X_{1i}$ and $X_{2i}$ are functions of $W_1$ and $W_2$, respectively, and $X_{r,i}$ is a function of $Y_r^{i-1}$; (25) follows from the fact that conditioning reduces entropy and also the fact that $Y_{r,i} - (X_{1i}, X_{2i}, X_{r,i}) - (W_1, W_2, Y_r^{i-1})$. Similarly, we can show that

$$nR_2 \leq \sum_{i=1}^n I(X_{2i}; Y_{r,i}|X_{1i}, X_{r,i}). \qquad (27)$$

Furthermore, we have

$$\begin{aligned} nR_1 &\leq I(W_1; Y_2^n|W_2) + n\delta_n H(W_1) \\ &= \sum_{i=1}^n H(Y_{2i}|W_2, Y_2^{i-1}) - H(Y_{2i}|W_1, W_2, Y_2^{i-1}) + n\delta_n \\ &= \sum_{i=1}^n H(Y_{2i}|W_2, Y_2^{i-1}, X_{2i}) - \\ &\quad H(Y_{2i}|W_1, W_2, X_{1i}, X_{2i}, X_{r,i}, Y_2^{i-1}) + n\delta_n && (28)\\ &\leq \sum_{i=1}^n H(Y_{1i}|W_2, X_{2i}) - H(Y_{2i}|X_{1i}, X_{2i}, X_{r,i}) + n\delta_n && (29)\\ &= \sum_{i=1}^n H(Y_{2i}|W_2, X_{2i}) - H(Y_{2i}|X_{2i}, X_{r,i}) + n\delta_n && (30)\\ &\leq \sum_{i=1}^n H(Y_{2i}|U_{2i}, X_{2i}) - H(Y_{2i}|X_{2i}, X_{r,i}, U_{2i}) + n\delta_n && (31)\\ &= \sum_{i=1}^n I(X_{r,i}; Y_{2i}|U_{2i}, X_{2i}) + n\delta_n. && (32) \end{aligned}$$

We can similarly obtain

$$nR_2 \leq \sum_{i=1}^n I(X_{r,i}; Y_{1i}|U_{1i}, X_{1i}) + n\delta_n.$$

Finally, we introduce the time-sharing random variable $Q$ uniformly distributed over the set $\{1, 2, .., n\}$ and defining $X_j \triangleq X_{jQ}$, $Y_j = Y_{jQ}$ and $U_j = U_{jQ}$ for $j = 1, 2, r$, we complete the proof of the outer bound.

## REFERENCES


[1] B. Rankov and A. Wittneben, "Spectral efficient signaling for half-duplex relay channels," in *Proc. 39th Asilomar Conference on Signals, Systems, and Computers*, Pacific Grove, CA, November 2005.

[2] P. Popovski and H. Yomo, "The anti-packets can increase the achievable throughput of a wireless multi-hop network," in *Proc. IEEE Int'l Conf. on Communication (ICC*.

[3] T. M. Cover and A. El Gamal, "Capacity theorems for the relay channel," *IEEE Trans. on Information Theory*, vol. 25, no. 5, pp. 572 – 584, September 1979.

[4] C. E. Shannon, "Two-way communication channels," in *Proc. 4th Berkeley Symp. Math. Stat. Prob.*, vol. 1, Berkeley, CA, 1961, pp. 611–644.

[5] E. Tuncel, "Slepian-Wolf coding over broadcast channels," *IEEE Trans. on Information Theory*, vol. 52, no. 4, pp. 1469–1482, April 2006.

[6] J. Nayak, E. Tuncel, and D. Gündüz, "Wyner-Ziv coding over broadcast channels: Digital schemes," *submitted to IEEE Trans. on Information Theory*, 2008.

[7] D. Gündüz, J. Nayak, and E. Tuncel, "Wyner-Ziv coding over broadcast channels using hybrid digital/analog transmission," in *Proc. IEEE International Symposium on Information Theory*, Toronto, ON, Canada, July 2008.

[8] R. Knopp, "Two-way wireless communication via a relay station," in *The 3 Ms of future mobile communications: Multi-user, Multi-antenna, Multi-hop systems*, Paris, France, March 2007.

[9] I. J. Baik and S. Y. Chung, "Network coding for two-way relay channels using lattices," in *Proc. IEEE Int'l Conference on Communications (ICC)*, Beijing, China, May 2008.

[10] K. Narayanan, M. P. Wilson, and A. Sprintson, "Joint physical layer coding and network coding for bi-directional relaying," in *Proc. 45th Annual Allerton Conference on Communication, Control, and Computing*, Monticello, IL, September 2007.

[11] D. Gündüz and E. Erkip, "Reliable cooperative source transmission with side information," in *Proc. IEEE Information Theory Workshop (ITW)*, Bergen, Norway, July 2007.

[12] R. Knopp, "Two-way radio networks with a star topology," in *Proc. IEEE International Zurich Seminar on Communications*, Zurich, Switzerland, Feb. 2006, pp. 154–157.

[13] E. Tuncel, "Lossless joint source-channel coding across broadcast channels with decoder side information," in *IEEE International Symposium on Information Theory*, Adelaide, Australia, September 2005.

[14] C. Schnurr, T. J. Oechtering, and S. Stanczak, "Achievable rates for the restricted half-duplex two-way relay channel," in *Proc. Asilomar Conf. on Signals, Systems and Computers*, Monterey, CA, Nov. 2007.

[15] S. J. Kim, N. Devroye, P. Mitran, and V. Tarokh, "Comparison of bi-directional relaying protocols," in *Proc. IEEE Sarnoff Symposium*, Princeton, NJ, Apr. 2008.

[16] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. Wiley-Interscience, 1991.

[17] S. Avestimehr, A. Sezgin, and D. Tse, "Approximate capacity of the two-way relay channel: A deterministic approach," in *46th Annual Allerton Conference on Communication, Control, and Computing*, Monticello, IL, September 2008.