Useful theorems in complex geometry

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Abstract

This is a list of main theorems in complex geometry that I will use throughout the course on Calabi-Yau manifolds and Mirror Symmetry. In practice it is a very brief summary of results from Chapters 0 and 1 of Griffiths-Harris. I will only remind their statements as I need them during the course. The main purpose of this list is to make the course accessible also to those people who only have a vague familiarity with these theorems. Hopefully, during the course, people will see some of their nice applications which, due to physical exhaustion before reaching the end of Chapters 0 and 1, they were not able to see before (this was at least my experience for a while).

1 Dolbeaut's Theorem

Theorem 1 *Let X be a complex n-dimensional compact manifold and $E \to X$ a holomorphic vector bundle. If we denote by $\mathcal{E}$ the sheaf of holomorphic sections of E and by Ω^q_X the sheaf of holomorphic q-forms, then we have:*

$$H^p_{\check{C}ech}(X, \Omega^q_X \otimes \mathcal{E}) \cong H^{q,p}_{\overline{\partial}}(X, E).$$

This theorem is analogous to a classical theorem of de Rham saying that the Čech cohomology over $\mathbb{R}$ of a smooth manifold is isomorphic to its de Rham cohomology. Here we have two types of cohomology groups. To form the one on the lefthand side we do the usual Čech cohomology of a (pre)sheaf on a sufficiently fine cover of X. On the righthand side are the Dolbeaut cohomology groups of a vector bundle. These are just like ordinary Dolbeaut cohomology groups except that $\overline{\partial}$ acts on the space $\Omega^{p,q}(E)$ of (p, q)-forms with coefficients in E. Notice that $\overline{\partial}$ is a connection on both $\Lambda^{p,q}T^*X$ and E and therefore it makes sense to consider $\overline{\partial}$ on their tensor product. For more information on sheaves, bundles, Dolbeaut cohomology and a proof of the above theorem consult [1, Sections 0.3 and 0.5].

2 Hodge decomposition

Theorem 2 *Let X be a compact n-dimensional Kähler manifold. Then*

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C}). \tag{1}$$

Moreover:

$$H^{p,q}(X,\mathbb{C}) = \overline{H^{q,p}(X,\mathbb{C})} \tag{2}$$

This very important theorem is a consequence of the Hodge theorem and of some identities which hold on Kähler manifolds. Let $\overline{\partial}^* : \Omega^{p,q}(X) \to \Omega^{p,q-1}(M)$ be the formal adjoint of $\overline{\partial}$ with respect to some L^2-norm on $\Omega^k(X)$ coming from an hermitian inner product on TX. Then we can define the Laplacian $\Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$, a second order selfadjoint operator on $\Omega^{p,q}(X)$. The space $\mathcal{H}^{p,q}(X) := \ker \Delta_{\overline{\partial}}$ is called the space of $\overline{\partial}$-harmonic forms and consists of (p,q)-forms which are both $\overline{\partial}$ and $\overline{\partial}^*$ closed. Hodge's theorem states that $\mathcal{H}^{p,q}(X)$ is finite dimensional and that in each cohomology class $[\phi] \in H^{p,q}(X,\mathbb{C})$ there exists a unique $\overline{\partial}$-harmonic representative, and therefore that

$$\mathcal{H}^{p,q}(X) \cong H^{p,q}(X,\mathbb{C}).$$

Hodge's theorem holds on any compact complex manifold (even non Kähaler ones). On a Kähler manifold it turns out that when $\overline{\partial}^*$ is calculated using the Kähler metric, then

$$2\Delta_{\overline{\partial}} = \Delta_d,$$

where on the righthand side we have the analogous Laplacian for the operator d, (it takes quite a bit of effort to check this equality!). This implies that $\Delta_{\overline{\partial}}$ is a real operator and that d preserves the (p,q)-type of a form, i.e. that $\overline{\partial}$-harmonic forms are also d-harmonic (in particular d-closed) and that all the components of pure type of a d-harmonic form are $\overline{\partial}$-harmonic. From such considerations we obtain both ismorphisms (1) and (2).

For details consult [1, Section 0.6 and 0.7], although maybe for a proof of Hodge's theorem [3] is better.

3 Kodaira-Serre duality

Theorem 3 *On any compact n-dimensional complex manifold X we have a non degenerate pairing*

$$H^{p,q}(X,\mathbb{C}) \otimes H^{n-p,n-q}(X,\mathbb{C}) \to \mathbb{C},$$

in particular

$$H^{p,q}(X,\mathbb{C}) \cong H^{n-p,n-q}(X,\mathbb{C}).$$

At the level of forms the pairing is simply given by wedging the two forms ϕ and ψ of type (p,q) and $(n-p,n-q)$ respectively and integrating over X. The fact that it descends to a well defined and non-degenerate pairing follows from Hodge's theorem (cfr. previous section) and the fact that

$$\Delta_{\overline{\partial}}\star = \star\Delta_{\overline{\partial}},$$

where $\star$ is Hodge's star operator $\star : \Omega^{p,q}(X) \to \Omega^{n-p,n-q}(X)$ (for a definition see [1, pag.82], but suffices to know that it is an isometry and that $\star^2 = \pm\,\mathrm{Id}$). In particular $\star$ sends harmonic forms to harmonic ones and being an isometry it is an isomorphism (this shows at least the last part of the statement). For the Kodaira-Serre duality we do not require X to be Kähler!

4 Line bundles and the Picard group

Let $\mathcal{O}$ denote the sheaf of holomorphic functions and $\mathcal{O}^*$ the multiplicative sheaf of nowhere zero holomorphic functions. We have

Theorem 4 *The group of holomorphic line bundles, modulo isomorphism, is naturally isomorphic to $H^1(X,\mathcal{O}^*)$, also called the Picard group.*

This theorem is quite simple. Let L be a holomorphic line bundle. Take $\{U_\alpha, s_\alpha\}$ to be a sufficiently fine covering of X with choices of trivializing local holomorphic sections s_α of L. Then on overlaps $U_\alpha \cap U_\beta$ we have

$$s_\alpha = g_{\alpha\beta}s_\beta,$$

where $g_{\alpha\beta} \in \mathcal{O}^*$. One can see that $\{U_\alpha \cap U_\beta, g_{\alpha\beta}\}$ is a Čech cocycle (i.e. an element of $H^1(X,\mathcal{O}^*)$) and that it is exact if and only if L is the trivial bundle.

5 Exponential sequence and first Chern class

Notice that we have the natural exponential exact sequence:

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0,$$

where the second arrow is just inclusion and the third is $f(z) \mapsto e^{2\pi i f(z)}$. This sequence translates in cohomology to the long exact sequence

$$\ldots \to H^k(X,\mathbb{Z}) \to H^k(X,\mathcal{O}) \to H^k(X,\mathcal{O}^*) \xrightarrow{\delta} H^{k+1}(X,\mathbb{Z}) \ldots$$

In particular one notices that the first non-trivial part of the sequence gives

$$0 \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}) \to H^1(X,\mathcal{O}^*) \xrightarrow{\delta} H^2(X,\mathbb{Z}).$$

The homomorphism δ maps the Picard group of X into $H^2(X,\mathbb{Z})$, giving us an invariant of holomorphic line bundles. Its computability comes from the important observation that δ is almost the first Chern class. To be more precise, let us denote by $\tilde{H}^2(X,\mathbb{Z}) \subset H^2(X,\mathbb{R})$ the image of $H^2(X,\mathbb{Z})$ in $H^2(X,\mathbb{R})$ induced by the inclusion $\mathbb{Z} \to \mathbb{R}$ and by $c_1(L)$ the first Chern class of a line bundle (computed for example using a connection) then we have the commuting diagram

$$\begin{array}{ccc} H^1(X,\mathcal{O}^*) & \xrightarrow{\delta} & H^2(X,\mathbb{Z}) \\ c_1 & \searrow & \downarrow \\ & & \tilde{H}^2(X,\mathbb{Z}) \end{array}$$

6 Divisors and first Chern class

Theorem 5 *Let $L \to X$ be a holomorphic line bundle over the complex n-dimensional manifold X. Suppose L has a meromorphic section s having zeroes along a variety Z and poles along a variety P. Suppose $Z = \cup_k Z_k$ and $P = \cup_k P_k$ are the decompositions of Z and P into their irreducible components and suppose also that s has zero of order r_k and pole of order q_k along Z_k and P_k respectively. Then we may form the divisor*

$$D = \sum_l r_l Z_l - \sum_k q_k P_k,$$

reppresenting a homology class in $H_{n-2}(X,\mathbb{Z})$. We have

$$c_1(L) = PD(D),$$

where $PD(\cdot)$ denotes the Poincaré dual.

I would just like to remind you some definitions here. Given a small open set U, after trivializing L over U, the section s can be assumed to be a meromorphic function (which changes by multiplication by a function in $\mathcal{O}^*$ when we change trivialization). Now, a theorem in complex analysis tells us that s has a unique decomposition in irreducible factors:

$$s = hg_1^{r_1} \dots g_m^{r_m} f_1^{-q_1} \dots f_l^{-q_l},$$

where $h \in \mathcal{O}^*$ and the g_k's and f_k's vanish in some non-empty subset of U (defining $Z_k \cap U$ and $P_k \cap U$ respectively). The above decomposition also defines the notion of the orders r_k and q_k of zeroes and poles respectively. The proof of the above theorem and a discussion of the above definitions is in [1, Section 1.1].

7 Lefschetz's hyperplane section theorem

Theorem 6 *Let M be a smooth $n+1$-dimensional complex submanifold of $\mathbb{P}^N$ for some N and let $V = H \cap M$, where H is a hyperplane. For sufficiently general H, V will be a smooth manifold, called a hyperplane section of M. Then the homomorphism*

$$H^q(M, \mathbb{Q}) \to H^q(V, \mathbb{Q})$$

given by restriction, is an isomorphism for $q \leq n-1$ and injective for $q = n$.

A nice proof of this useful theorem using Morse theory is in [2].

References

[1] P. Griffiths and J. Harris. *Principles of Algebraic Geometry.* Wiley Interscience, 1978.

[2] J. Milnor. *Morse Theory.* Princeton Univ. Press, Princeton, NJ, 1963.

[3] R. O. Wells. *Differential Analysis on Complex Manifolds.* GTM. Springer-Verlag, 1980.