LIE ALGEBROIDS, LIE GROUPOIDS AND POISSON GEOMETRY

by

Sébastien Racanière

Abstract. — I wrote these notes for a series of lectures at Imperial College during the Summer Term 2004. The aim was to introduce symplectic groupoids, Hamiltonian actions of these groupoids and how this generalises other moment map theories.

The study of these subjects in these notes is by no mean thorough. My hope is that these notes will instead consist in a nice introduction to the subject of symplectic groupoids and Poisson geometry. For a more detailed introduction, the reader might read Vaisman [**8**] for example.

Contents

I will denote $\langle\, ,\rangle$ the natural pairing between a space and its dual. If Q is a manifold then TQ is its tangent space. Let f be a smooth map between Q and a vector space V. If X belongs to the tangent space of Q at a point p, then X acts on f and give an element $X \cdot f$ in V.

Real numbers are $\mathbb{R}$, complex numbers are $\mathbb{C}$, and so on ....

1. Lie Algebroids

A Lie algebroid is a real vector bundle with a Lie bracket on its space of sections which satisfies the Leibniz identity. More precisely,

Definition 1.1. — *Let Q be a smooth manifold. A (real) Lie algebroid over Q is a vector bundle $A \longrightarrow Q$ with an antisymmetric $\mathbb{R}$-bilinear map on the space of smooth sections of A*

$$\begin{array}{rcl}\Gamma(A)\times\Gamma(A) & \longrightarrow & \Gamma(A)\\ (\alpha,\beta) & \longmapsto & [\alpha,\beta],\end{array}$$

such that

$$[[\alpha,\beta],\gamma]+[[\beta,\gamma],\alpha]+[[\gamma,\alpha],\beta]=0, \quad \textit{for } \alpha,\beta,\gamma\in\Gamma(A) \qquad \textit{(Jacobi)},$$

and an endomorphism of vector bundles (called the anchor map)

$$\rho : A \longrightarrow TQ$$

which induces a homomorphism of Lie algebra between $\Gamma(A)$ and $\mathfrak{X} = \Gamma(TQ)$; moreover it should satisfy

$$[\alpha, f\beta] = f[\alpha,\beta] + (\rho(\alpha)\cdot f)\beta \qquad \textit{(Leibniz)}.$$

Remark 1.2. — *It follows form the Leibniz identity that if β and β' are two sections of A that agree on a neighbourhood of $p \in Q$, then*

$$[\alpha,\beta](p) = [\alpha,\beta'](p),\ \forall\alpha\in\Gamma(A).$$

This means that $[\alpha,\beta](p)$ can be computed locally and in local coordinates, it depends only on $\alpha(p)$, $\beta(p)$ and the first derivatives of α and β at p.

Examples of Lie algebroids are numerous (not to say manifold). Later, we will see that Poisson manifolds can be defined in terms of Lie algebroids.

Example 1.3. — *Let Q be a manifold. Sections of $TQ \longrightarrow Q$ are vector fields. The tangent bundle $TQ \longrightarrow Q$ with bracket of sections the usual bracket of vector fields and anchor map the identity $TQ \longrightarrow TQ$ is a Lie algebroid called a* **pair algebroid**.

Example 1.4. — *A Lie algebra is a Lie algebroid over a point.*

Example 1.5. — *Let Q be a manifold and $\mathfrak{k}$ a real Lie algebra acting on Q; in other words we have a morphism of Lie algebras*

$$\begin{array}{rcl}\mathfrak{k} & \longrightarrow & \Gamma(TQ)\\ \xi & \longmapsto & v_\xi.\end{array}$$

Consider $A = \mathfrak{k}\times Q$ a trivial vector bundle over Q. Identify sections of A with maps $Q \longrightarrow \mathfrak{k}$. Define a bracket on sections by

$$[\alpha,\beta](p) = [\alpha(p),\beta(p)] + v_{\beta(p)}\cdot\alpha - v_{\alpha(p)}\cdot\beta,$$

and a morphism of vector bundles

$$\begin{array}{rcl} \rho : A & \longrightarrow & TQ \\ (\xi, p) & \longmapsto & v_\xi(p). \end{array}$$

These data define a Lie algebroid called an **action algebroid**.

2. Poisson manifolds

Definition 2.1. — *A Poisson structure on a manifold Q is a Lie algebroid structure on $T^*Q \longrightarrow Q$ such that if $[\,,]$ is the bracket on 1-forms and $\rho : T^*Q \longrightarrow TQ$ is the anchor map, then for any functions f, g on Q and $p \in Q$*

$$[df, dg](p) = d_p\langle dg, \rho(df)\rangle.$$

Notice that ρ is automatically antisymmetric. Indeed $[\mathrm{d}f, \mathrm{d}f] = 0$ for all functions f implies that $\mathrm{d}\langle \mathrm{d}f, \rho(\mathrm{d}f)\rangle$ is constant for all f. Let $c(f)$ be this value. Assume there exists f such that $c(f) \neq 0$, then

$$c(f^2) = 4f^2c(f),$$

and f^2 is constant and equal to $\frac{c(f^2)}{4c(f)}$ which implies that f is constant and $c(f) = 0$ . . .

Proposition 2.2. — *Let Q be a Poisson manifold. There exists a Lie bracket on its algebra of smooth functions that is a $\mathbb{R}$-bilinear antisymmetric map*

$$\begin{array}{rcl} C^\infty(Q) \times C^\infty(Q) & \longrightarrow & C^\infty(Q) \\ (f, g) & \longmapsto & \{f, g\} \end{array}$$

which satisfies for all functions f, g and h

$$\{fg, h\} = f\{g, h\} + g\{f, h\} \quad \textit{(Leibniz)},$$

and

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \quad \textit{(Jacobi)}.$$

The converse is also true, that is given a bracket on the algebra of smooth functions that satisfies the Leibniz and Jacobi identities, then Q is a Poisson manifold. This result is left as an exercise for those interested (alternatively, you can read [**4**]).

Proof. — Define the bracket in the following way: let f and g be functions on Q and set

$$\begin{array}{rcl} \{f, g\}(p) & = & \langle \mathrm{d}_p g, \rho(\mathrm{d}_p f)\rangle \\ & = & \rho(d_p f) \cdot g. \end{array}$$

This bracket is clearly $\mathbb{R}$-bilinear. It is antisymmetric because

$$\begin{aligned} \{f,g\}(p) &= \langle \mathrm{d}_p g, \rho(\mathrm{d}_p f)\rangle \\ &= \langle -\rho(\mathrm{d}_p g), \mathrm{d}_p f\rangle \\ &= -\{g,f\}(p). \end{aligned}$$

It satisfies the Leibniz identity because

$$\begin{aligned} \{fg,h\}(p) &= \langle \mathrm{d}_p h, \rho(\mathrm{d}_p(fg))\rangle \\ &= \langle \mathrm{d}_p h, \rho(f(p)\mathrm{d}_p g + g(p)\mathrm{d}_p f)\rangle \\ &= f(p)\{g,h\}(p) + g(p)\{f,h\}(p). \end{aligned}$$

Finally, it satisfies the Jacobi identity because

$$\{\{f,g\},h\} = \langle \mathrm{d}h, \rho([\mathrm{d}f,\mathrm{d}g])\rangle$$

and

$$\begin{aligned} \{\{g,h\},f\} + \{\{h,f\},g\} &= \langle \mathrm{d}f, \rho(\mathrm{d}(\rho(\mathrm{d}g)\cdot h))\rangle + \langle \mathrm{d}g, -\rho(\mathrm{d}(\rho(\mathrm{d}f)\cdot h))\rangle \\ &= -\rho(\mathrm{d}f)\cdot(\rho(\mathrm{d}g)\cdot h) + \rho(\mathrm{d}g)\cdot(\rho(\mathrm{d}f)\cdot h) \\ &= -\langle \mathrm{d}h, [\rho(\mathrm{d}f),\rho(\mathrm{d}g)]\rangle, \end{aligned}$$

so that

$$\begin{aligned} \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} &= \langle \mathrm{d}h, \rho([\mathrm{d}f,\mathrm{d}g]) - [\rho(\mathrm{d}f),\rho(\mathrm{d}g)] \\ &= 0, \end{aligned}$$

because ρ is a homomorphism of Lie algebra. □

Definition 2.3. — *If Q and P are Poisson manifolds, a smooth map $\varphi : P \longrightarrow Q$ is called Poisson if for all functions f, g in $C^\infty(Q)$*

$$\varphi^*\{f,g\} = \{\varphi^* f, \varphi^* g\}.$$

It is called anti-Poisson if

$$\varphi^*\{f,g\} = -\{\varphi^* f, \varphi^* g\}.$$

Particular examples of Poisson manifolds are the symplectic ones.

Definition 2.4. — *A symplectic manifold (Q,ω) is a manifold Q with a non-degenerate closed 2-form, that is $d\omega = 0$ and the map*

$$\begin{aligned} TQ &\longrightarrow T^*Q \\ \xi &\longmapsto \omega(\xi,\cdot) \end{aligned}$$

is an isomorphism.

Proposition 2.5. — *A symplectic manifold (Q,ω) is naturally a Poisson manifold.*

Proof. — Use the isomorphism between TQ and T^*Q to define a Lie algebroid structure on T^*Q. □

3. Lie Groupoids

In the language of categories, a groupoid is a small category in which all morphisms are invertible. A Lie groupoid is then a groupoid with a nice smooth structure.

Definition 3.1. — *A Lie groupoid is given by*

- *two smooth manifolds G (the morphims or arrows) and Q (the objects or points),*
- *two smooth maps $s : G \longrightarrow Q$ the source map and $t : G \longrightarrow Q$ the target map,*
- *a smooth embedding $\iota : Q \longrightarrow G$ (the identities or constant arrows),*
- *a smooth involution $I : G \longrightarrow G$, also denoted $x \longmapsto x^{-1}$,*
- *a multiplication*

$$\begin{array}{rcl} m : G_2 & \longrightarrow & G \\ (x, y) & \longmapsto & x \cdot y, \end{array}$$

where $G_2 = G_s \times_t G = \{(x, y) \in G \times G | s(x) = t(y)\}$,

such that the source map and target map are surjective submersions (hence G_2 is a smooth manifold because t and s are submersions), the multiplication is smooth and

1. $s(x \cdot y) = s(y)$, $t(x \cdot y) = t(x)$,
2. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$,
3. *ι is a section of both s and t,*
4. $\iota(t(x)) \cdot x = x = x \cdot \iota(s(x))$,
5. $s(x^{-1}) = t(x)$, $t(x^{-1}) = s(x)$,
6. $x \cdot x^{-1} = \iota(t(x))$, $x^{-1} \cdot x = \iota(s(x))$,

whenever (x, y) and (y, z) are in G_2.

I will identify Q with its image in G using ι; so that if $p \in Q$ then also $p \in G$.

Lie groupoids are almost as numerous as Lie algebroids. Here are few examples.

Example 3.2. — *Let Q be a manifold. Let $G = Q \times Q$ and define*

$$\begin{array}{rcl} s(p, q) & = & q \\ t(p, q) & = & p \\ (p, q) \cdot (q, r) & = & (p, r) \\ I(p, q) & = & (q, p) \\ \iota(p) & = & (p, p). \end{array}$$

These data define a Lie groupoid called a **pair groupoid**.

Example 3.3. — *A Lie group is a Lie groupoid with Q a singleton.*

Example 3.4. — *Let Q be a manifold and K a Lie group acting on the left on this manifold. Let $G = K \times Q$ and define*

$$\begin{array}{rcl} s(k,p) & = & k^{-1}\cdot p \\ t(k,p) & = & p \\ (k,p)\cdot(h,k^{-1}\cdot p) & = & (kh,p) \\ I(k,p) & = & (k^{-1},k^{-1}\cdot p) \\ \iota(p) & = & (e,p) \quad \text{where } e \text{ is the unit of } K. \end{array}$$

These data form a Lie groupoid called an **action groupoid**.

Example 3.5. — *A particular and important case of the last example is when Q is $\mathfrak{k}^*$ the dual of a Lie algebra $\mathfrak{k}$ and K, a Lie group integrating $\mathfrak{k}$, acts on $\mathfrak{k}^*$ by the co-adjoint action. The total space of the action groupoid is $K \times \mathfrak{k}^*$. It can be identified with T^*K by the map*

$$\begin{array}{rcl} K\times\mathfrak{k}^* & \longrightarrow & T^*K \\ (k,\alpha) & \longmapsto & \alpha\circ R_{k^{-1}}, \end{array}$$

where R_k is the tangent map $\mathfrak{k} \simeq T_eK \longrightarrow T_kK$ to the map induced by multiplication by k on the right.

As for Lie groups, an important property of Lie groupoids is that they can act on spaces.

Definition 3.6. — *Let G be a Lie groupoid over a manifold Q. Let M be a manifold. A left action of G on M consists in the following data*

- *a smooth map $J : M \longrightarrow Q$,*
- *a smooth map*

$$\begin{array}{rcl} G_s\times_J M & \longrightarrow & M \\ (x,m) & \longmapsto & x\cdot m, \end{array}$$

such that

1. *$J(x\cdot m) = t(x)$, when $(x,m) \in G_s \times_J M$,*
2. *$y\cdot(x\cdot m) = (y\cdot x)\cdot m$ when $s(y) = t(x)$ and $(x,m) \in G_s \times_J M$.*

Notice that $G_s \times_J M$ is a manifold because s is a submersion.

Example 3.7. — *If K acts on the left on Q, then the action groupoid $K \times Q$ acts on the left on Q with $J =$ the identity map of Q and*

$$(k,k^{-1}\cdot p)\cdot p = k\cdot p.$$

4. The Lie Algebroid of a Lie Groupoid

Let G be a Lie groupoid over Q. Since the target map t is a submersion, its kernel is a vector bundle over G. Call it $T^tG \longrightarrow G$. Let x be in G. Multiplication by x on the left induces a diffeomorphism

$$t^{-1}(s(x)) \longrightarrow t^{-1}(t(x)),$$

and its tangent map at $y \in t^{-1}(s(x))$ induces an isomorphism

$$L_x : T^t_yG \longrightarrow T^t_{x \cdot y}G.$$

A vector field X on G is called left invariant if X takes its value in T^tG and for x in G and y in $t^{-1}(s(x))$

$$L_xX(y) = X(x \cdot y).$$

Let $A \longrightarrow Q$ be the restriction of T^tG to Q (in other words, $A = \iota^*T^tG$).

Let α be a section of A.

Lemma 4.1. — *There exists a unique extension of α to a left invariant vector field X_α on G.*

Proof. — One just has to put

$$X_\alpha(x) = L_x\alpha(s(x)).$$

□

We have identified left invariant vector fields on G with sections of A.

Lemma 4.2. — *The bracket of two left invariant vector fields is a left invariant vector field; hence, the bracket of left invariant vector fields induces a bracket, denoted $[\,,]$, on $\Gamma(A)$.*

Proof. — Let X and Y be two left invariant vector fields on G. Let y be in G and $p = t(y)$. Both X and Y are tangent to the submanifold $t^{-1}(p)$. We deduce that $[X, Y](y)$ is tangent to $t^{-1}(p)$ and

$$[X, Y]_G(y) = [X, Y]_{t^{-1}(p)}(y),$$

moreover, let $x \in G$ such that $s(x) = t(y)$,

$$\begin{aligned} L_x[X, Y]_{t^{-1}(p)}(y) &= [L_xX, L_xY]_{t^{-1}(p)}(x \cdot y) \\ &= [X, Y]_{t^{-1}(p)}(x \cdot y). \end{aligned}$$

□

Proposition 4.3. — *Let α and β be sections of $A \longrightarrow Q$. Then*

$$ds([\alpha, \beta]) = [ds(\alpha), ds(\beta)],$$

where on the right hand side, $[\,,]$ is the bracket on vector fields over Q.

Proof. — Extend α and β to left invariant vector fields X and Y, respectively. Let φ_u and ψ_r be their respective flows

$$X_y = \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=0} \varphi_u(y),\ Y_y = \left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=0} \psi_r(y).$$

Lemma 4.4. — *We have*

$$\varphi_u(x \cdot y) = x \cdot \varphi_u(y),$$

and

$$\psi_r(x \cdot y) = x \cdot \psi_r(y)$$

whenever $x \cdot y$ is defined.

Proof. — For $x \in G$ with $s(x) = t(y)$, let

$$\begin{array}{rcl} \varphi'_u : t^{-1}(t(x)) & \longrightarrow & t^{-1}(t(x)) \\ z & \longmapsto & x \cdot \varphi_u(x^{-1} \cdot z). \end{array}$$

We have

$$\varphi'_0(z) = z$$

and

$$\begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{d}u}\varphi'_u(z) & = & L_x \frac{\mathrm{d}}{\mathrm{d}u}\varphi_u(x^{-1} \cdot z) \\ & = & L_x X_{x^{-1} \cdot z} \\ & = & X_z. \end{array}$$

The result follows for φ. A similar proof holds for ψ. □

Lemma 4.5. — *We have*

$$s \circ \varphi_u = s \circ \varphi_u \circ s.$$

Proof. — It is a matter of a simple calculation

$$\begin{array}{rcl} s \circ \varphi_u(y) & = & s \circ \varphi_u(y \cdot s(y)) \\ & = & s \circ L_y \circ \varphi_u(s(y)) \\ & = & s \circ \varphi_u(s(y)). \end{array}$$

□

We can now prove the Proposition. We have

$$\begin{array}{rcl} (s \circ \varphi_u) \circ (s \circ \varphi_{u'}) & = & s \circ \varphi_u \varphi_{u'} \\ & = & s \circ \varphi_{u+u'}. \end{array}$$

which proves that $p \longmapsto s \circ \varphi_u(p)$ is the flow of the vector field $\mathrm{d}s(X)$ on Q. Similarly, $p \longmapsto s \circ \psi_r(p)$ is the flow of the vector field $\mathrm{d}s(Y)$ on Q. The Proposition follows because by definition of the bracket of vector fields

$$[X,Y]_p = \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=0} \left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=0} \varphi_{-u} \circ \psi_r \circ \varphi_u(p).$$

Hence

$$\begin{aligned}
\mathrm{d}_p s[X,Y] &= \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=0} \left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=0} s \circ \varphi_{-u} \circ \psi_r \circ \varphi_u(p) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=0} \left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=0} (s \circ \varphi_{-u}) \circ (s \circ \psi_r) \circ (s \circ \varphi_u)(p) \\
&= [d_p s(X_p), \mathrm{d}_p s(Y_p)].
\end{aligned}$$

□

We conclude

Theorem 4.6. — *The bundle $A \longrightarrow Q$ is a Lie algebroid with the above bracket on its space of sections and anchor given by the differential $\mathrm{d}s$ of the source map s. It is called the Lie algebroid of $G \rightrightarrows Q$.*

Proof. — We already know that the bracket satisfies the Jacobi identity and that the anchor map is a morphism of Lie algebra. There only remain to check the Leibniz identity: it is satisfied because for f a function on Q, the extension of $f\beta$ to a left invariant vector field on G is $(f \circ s)Y$. □

The construction of a Lie algebroid from a Lie groupoid resembles the construction of a Lie algebra from a Lie group; and indeed

Example 4.7. — *A Lie group is a Lie groupoid over a point and its Lie algebroid is the Lie algebra of the group seen as a Lie algebroid over a point.*

Example 4.8. — *The Lie algebroid of a pair groupoid $Q \times Q \rightrightarrows Q$ is the pair algebroid $TQ \longrightarrow Q$.*
The Lie algebroid of an action groupoid $K \times Q \rightrightarrows Q$ is the action algebroid $\mathfrak{k} \times Q \longrightarrow Q$.

A particular case of the last example if when Q is the dual $\mathfrak{k}^*$ of the Lie algebra $\mathfrak{k}$ of a Lie group K and K acts on $\mathfrak{k}^*$ by the co-adjoint action. Its Lie algebroid is the action Lie algebroid $\mathfrak{k} \times \mathfrak{k}^* \longrightarrow \mathfrak{k}^*$. Notice also that this Lie algebroid together with the natural identification of $\mathfrak{k} \times \mathfrak{k}^*$ with $T^*\mathfrak{k}^*$ makes $\mathfrak{k}^*$ a Poisson manifold (it is called a Lie-Poisson manifold). Its bracket is given by

$$\{f,g\}(\theta) = -\theta([\mathrm{d}_\theta f, \mathrm{d}_\theta g]).$$

We will see that this example is of particular importance is the theory of Hamiltonian spaces with equivariant moment maps.

5. Lie's Third Theorem

We have seen that from any Lie groupoid one can derive a Lie algebroid. It is natural to ask if the converse is also true, that is can any Lie algebroid be integrated to a Lie groupoid. This is a difficult question which found a complete satisfactory answer only recently, see Crainic and Fernandes [**4**].

In short, the answer is that a Lie algebroid can NOT in general be integrated to a Lie groupoid, moreover, to get a satisfactory theory one needs to allow the total space of a Lie groupoid to be a non Hausdorff manifold.

However, I will quote the following Theorem without proving it.

Theorem 5.1. — *Let $A \longrightarrow Q$ be Lie algebroid. Assume there exists a Lie groupoid $G \rightrightarrows Q$ integrating $A \longrightarrow Q$. Then, there exists a unique, up to isomorphism, source simply connected Lie groupoid (with non necessarily Hausdorff total space G) integrating $A \longrightarrow Q$.*

A Lie groupoid is source simply connected if the fibres of the source map are simply connected. In this case, the fibres of the target map are also simply connected.

6. Symplectic Lie groupoids

We will see in this section that when the Lie algebroid of a Poisson manifold can be integrated to a Lie groupoid, this groupoid carries a symplectic form compatible with the structure of groupoid.

Let $G \rightrightarrows Q$ a Lie groupoid with multiplication $m : G_2 \longrightarrow G$. Let ω be a 2-form on G and call pr_1, respectively pr_2, the projection from G_2 on the first, respectively second, factor of $G \times G$.

Lemma 6.1. — *The following properties are equivalent*

1. $$m^*\omega = pr_1^*\omega + pr_2^*\omega,$$
2. *the 2-form $\omega \oplus \omega \oplus -\omega$ on $G \times G \times G$ vanishes on the submanifold $\Lambda = \{(x, y, x \cdot y) \in G \times G \times G | t(y) = s(x)\}$.*

A form which satisfies these properties is called multiplicative.

Proof. — The map

$$\begin{array}{ccc} G_2 & \longrightarrow & \Lambda \\ (x,y) & \longmapsto & (x,y,x\cdot y) \end{array}$$

is a diffeomorphism. It pulls back $\omega \oplus \omega \oplus -\omega$ to $pr_1^*\omega + pr_2^*\omega - m^*\omega$. □

Definition 6.2. — *A symplectic groupoid is a Lie groupoid $G \rightrightarrows Q$ with a multiplicative symplectic form ω.*

An important property of symplectic groupoids is that their space of objects is naturally a Poisson manifold.

Theorem 6.3. — *Let $(G \rightrightarrows Q, \omega)$ be a symplectic groupoid. Then Q has a unique Poisson structure such that s is Poisson and t is anti-Poisson.*

Proof. — Let $A \longrightarrow Q$ be the algebroid of $G \rightrightarrows Q$ and denote by $\rho : A \longrightarrow TQ$ its anchor map. Because $t = s \circ I$ (I is the inversion map), the following Lemma shows that if s if Poisson then t is anti-Poisson.

Lemma 6.4. — *We have*

$$\iota^*\omega = 0$$

and

$$I^*\omega = -\omega.$$

Proof. — Consider the embedding

$$\begin{array}{rccc} \iota' : & Q & \longrightarrow & G \times G \times G \\ & p & \longmapsto & (p, p, p). \end{array}$$

Its image is in Λ and $\iota'^*(\omega \oplus \omega \oplus -\omega) = \iota^*\omega$, thus $\iota^*\omega = 0$.

The second part of the Lemma is equivalent to: the 2-form $\omega \oplus \omega$ vanishes on $\{(x, x^{-1}),\ x \in G\}$. But $\iota^*\omega = 0$ implies $m^*\omega$ vanishes on this manifold; hence $\omega \oplus \omega$ on $\{(x, x^{-1}),\ x \in G\}$ is the restriction of $pr_1^*\omega + pr_2^*\omega - m^*\omega$ defined on G_2. □

Let $x \in G$ and $p = s(x)$. Let ξ be in A_p. Define $\overrightarrow{\xi}_x = L_x\xi$, this is a vector in $T_x^t G$. Similarly, define $\overleftarrow{\xi}_x = R_x I_* \xi$, a vector in $T_x^s G$ the kernel of $\mathrm{d}_x s$. Sometimes, the subscript x will be omitted when it is clear from the context at which point we are working.

Lemma 6.5. — *Let ζ and ξ be in A_p, then*

$$\omega_x(\overleftarrow{\zeta}, \overrightarrow{\xi}) = 0.$$

Proof. — Indeed, both vectors $(\overleftarrow{\zeta}_x, 0_{t(x)}, \overleftarrow{\zeta}_x)$ and $(0_x, \overrightarrow{\xi}_{(t(x)}, \overrightarrow{\xi}_x)$ are in $T_{(x,t(x),x)}\Lambda$ and $\omega \oplus \omega \oplus -\omega$ vanishes on Λ. □

The following Lemma will also be needed to prove the Theorem.

Lemma 6.6. — *Let p be a point in Q, then*

$$T_pG = T_pQ \oplus \overrightarrow{A}_p = T_pQ \oplus \overleftarrow{A}_p, \tag{1}$$

and the map

$$\phi : T_pQ \longrightarrow A_p^*$$

defined by: for v in T_pQ and ξ in A_p

$$\phi(v)(\xi) = \omega_p(v, \overrightarrow{\xi})$$

is an isomorphism. In particular, the dimension of G as a manifold is equal to twice the dimension of Q

$$\mathit{dim}\, G = 2\mathit{dim}\, Q.$$

Proof. — We have $\overrightarrow{A}_p = \operatorname{Ker} \mathrm{d}_p t$ and $\overleftarrow{A}_p = \operatorname{Ker} \mathrm{d}_p s$; since $s \circ \iota = \mathrm{id} = t \circ \iota$, the equality (1) follows.

Assume $\phi(v) = 0$. For all ξ in A_p

$$\omega_p(v, \overrightarrow{\xi}) = 0,$$

but

$$\omega(v, u) = 0$$

for all u in T_pQ as well, hence $v = 0$ by (1). Assume that $\xi \in A_p$ is such that $\phi(v)(\xi) = 0$ for all v in T_pQ. We also have

$$\omega_p(\overrightarrow{\xi}, \overleftarrow{\zeta}) = 0$$

for all ζ in A_p. Again, this proves that $\xi = 0$, thus ϕ is surjective. □

The symplectic form ω on G defines a Poisson structure with anchor map $\omega^{-1} : T^*G \longrightarrow TG$. Assume $\eta : T^*Q \longrightarrow TQ$ is the anchor map of a Poisson structure on Q. For $x \in G$, let

$$s_{*,x} : T_xG \longrightarrow T_{s(x)}Q$$

be the tangent map of s; it induces, by pull-back, a map

$$s^*_x : T^*_{s(x)}Q \longrightarrow T^*_xG.$$

The source map s is Poisson if and only if for all functions f, g on Q

$$\begin{aligned} s^*\{f, g\} &= \{s^*f, s^*g\} \\ s^*(\mathrm{d}g(\eta(\mathrm{d}f))) &= (s^*\mathrm{d}g)(\omega^{-1}(s^*\mathrm{d}f)) \\ \langle \mathrm{d}_{s(x)}g, \eta_{s(x)}\mathrm{d}_{s(x)}f\rangle &= \langle \mathrm{d}_{s(x)}g, s_{*,x}\omega_x^{-1}s^*_x\mathrm{d}_{s(x)}f\rangle, \quad \text{for all } x \in G. \end{aligned}$$

Hence, s is Poisson iff

$$\eta_{s(x)} = s_{*,x} \circ \omega_x^{-1} \circ s^*_x, \quad \text{for all } x \in G. \tag{2}$$

Since s is surjective, this proves the uniqueness of η if it exists. To prove its existence, we must prove that the above formula for η depends only on $s(x)$ and not on x. This will follow from the next Lemma.

Lemma 6.7. — *Let $p = s(x)$. Consider*

$$\phi_p^{-1} \circ \rho^*_p : T^*_pQ \longrightarrow T_pQ.$$

We have $s_{*,x} \circ \omega_x^{-1} \circ s^*_x = \phi_p^{-1} \circ \rho^*_p$.

Proof. — Indeed,

$$\begin{array}{rrcl} & s_{*,x} \circ \omega_x^{-1} \circ s_x^* & = & \phi_p^{-1} \circ \rho_p^* \\ \iff & s_{*,x} \circ \omega_x^{-1} \circ s_x^* & = & \phi_p^{-1} \circ L_x^* \circ s_x^* \quad (1) \\ \Longleftarrow & s_{*,x} \circ \omega_x^{-1} & = & \phi_p^{-1} \circ L_x^* \\ \iff & \phi_p \circ s_{*,x} & = & L_x^* \circ \omega_x \\ \iff & L_p^* \circ \omega_p \circ \iota_{*,p} \circ s_{*,x} & = & L_x^* \circ \omega_x, \end{array}$$

this last line is true iff for all $X_x \in T_xG$ and $\xi \in A_p$

(3) $$\omega_x(X_x, \overrightarrow{\xi}_x) = \omega_p(\iota_{*,p} \circ s_{*,x} X_x, \xi_p).$$

Let $X \in T_xG$ and $\xi \in A_p$. If $y \in G$ is such that $(x, y) \in G_2$ then

$$T_{(x,y)}G_2 = \{(Y_1, Y_2) \in T_xG \oplus T_yG \mid s_*(Y_1) = t_*(Y_2)\}.$$

We have $s_*(X) = t_* \circ \iota_* \circ s_*(X)$, thus $(X, \iota_* \circ s_*(X)) \in T_{(x,\iota(p))}G_2$. Moreover $m_*(X, \iota_* \circ s_*(X)) = X$; indeed, if γ is a path satisfying $\gamma(0) = x$ and $\dot\gamma(0) = X$, then

$$m_*(X, \iota_* \circ s_*(X)) = \overbrace{m(\gamma, \iota \circ s \circ \gamma)}^{\cdot}(0) = \dot\gamma(0) = X.$$

In a similar way (that is by choosing appropriate paths), one can prove that

$$(\overrightarrow{\xi}_x, \iota_{*,p} \circ \rho(\xi) + \xi) \in T_{(x,p)}G_2$$

and

$$m_*(\overrightarrow{\xi}_x, \iota_{*,p} \circ \rho(\xi) + \xi) = 2\overrightarrow{\xi}_x.$$

Because of Property (1) in Lemma 6.1

$$\begin{array}{l} \omega_x(m_*(X, \iota_{*,p} \circ s_{*,x}(X)), m_*(\overrightarrow{\xi}_x, \iota_{*,p} \circ \rho(\xi) + \xi)) = \\ \omega_x(X, \overrightarrow{\xi}_x) + \omega_p(\iota_{*,p} \circ s_{*,x}(X), \iota_{*,p} \circ \rho(\xi) + \xi), \end{array}$$

but the left hand side of this equation is equal to $2\omega_x(X, \overrightarrow{\xi}_x)$ and the right hand side is equal to $\omega_x(X, \overrightarrow{\xi}_x) + \omega_p(\iota_{*,p} \circ s_{*,x}(X), \xi)$. □

Thus η is well defined. It defines a bracket on the space of sections of T^*Q by the formula of Definition 2.1. That this bracket satisfies the Jacobi and the Leibniz identity automatically follows from those same identities for the bracket of sections of $T^*G \longrightarrow G$. □

Because not every algebroid can be integrated to a groupoid, the converse of Theorem 6.3 is not true. We need to assume that the Poisson structure comes from an integrable Lie algebroid.

---

(1) The fibre A_p equals T_p^tG. On this space, the anchor map is $s_{*,p}$. It is then easy to deduce that $\rho_p = s_{*,x} \circ L_x$.

Theorem 6.8. — *Let Q be a Poisson manifold. Assume that the Lie algebroid $T^*Q \to Q$ with anchor map $\eta : T^*Q \longrightarrow TQ$ integrates to a source connected Lie groupoid $G \rightrightarrows Q$. Then there exists at most one symplectic form ω on G such that $(G \rightrightarrows Q, \omega)$ is a symplectic groupoid and such that the source map is Poisson.*

In fact it can be proved (see [**2**] in a much more general context), that whenever T^*Q can be integrated to a source simply connected groupoid $G \rightrightarrows Q$, there always exists such a symplectic form on G. I will not prove this fact here because it necessitates to know the construction of G and I did not give such a construction in these notes.

Proof. — Assume that ω is such a form. Let x be an arrow in G and $p = s(x)$. Then it will satisfy

$$\iota^*\omega = 0,$$

and formula (3)

$$\omega_x(X, \overrightarrow{\xi}_x) = \omega_p(\iota_{*,p} \circ s_{*,x}(X), \xi),$$

for all X in T_xG and ξ in $A_p = T_p^*Q \simeq T_p^tG$. Moreover, since

$$T_pG = T_pQ \oplus \overrightarrow{A}_p,$$

and according to Equation (2)

$$\eta_{s(x)} = s_{*,x} \circ \omega_x^{-1} \circ s_x^*, \quad \text{for all } x \in G,$$

the form ω is entirely defined along Q by: for X, Y in T_pQ and ξ, ζ in A_p

$$\omega_p(X \oplus \xi, Y \oplus \zeta) = \zeta(s_{*,p}(X)) - \xi(s_{*,p}(Y)) + \xi(\eta(\zeta)).$$

Assume that ω' is another symplectic form satisfying the properties of the Theorem. Then for every v parallel to a fibre of s, the interior product of $\omega - \omega'$ with v vanishes. Also, because $\omega - \omega'$ is closed, the Lie derivative $L_v(\omega - \omega')$ vanishes. Hence there exists a 2-form σ on Q such that $\omega - \omega' = s^*\sigma$. Since $\omega - \omega'$ vanishes along M, we must have $\eta = 0$ and $\omega = \omega'$. □

Let us see some examples.

Example 6.9. — *Assume that Q has the zero Poisson structure (the anchor map $T^*Q \longrightarrow TQ$ is the zero map). The Lie algebroid $T^*Q \longrightarrow Q$ can be integrated to a Lie groupoid $T^*Q \rightrightarrows Q$ where the source map and target map are equal to the natural projection and where multiplication is given by the addition in the fibres. The symplectic form on T^*Q is then the usual symplectic form on the cotangent bundle of a manifold (that is $\omega = d\theta$ where θ is the 1-form on T^*Q characterised by $\alpha^*\theta = \alpha$ for every 1-form α on Q).*

Example 6.10. — *Assume that the Poisson structure on Q comes from a symplectic form σ. Then the symplectic form induces an isomorphism of Lie algebroids between $T^*Q \longrightarrow Q$ and $TQ \longrightarrow Q$. This algebroid integrates to the pair groupoid $G \times G$. It becomes a symplectic groupoid with the symplectic form $\sigma \oplus -\sigma$.*

Example 6.11. — *Take $Q = \mathfrak{k}^*$, the dual of the Lie algebra $\mathfrak{k}$ of a Lie group K. The Lie algebroid $T^*\mathfrak{k}^*$ integrates to the action groupoid $K \times \mathfrak{k}^* \rightrightarrows \mathfrak{k}^*$ for the co-adjoint action. It becomes a symplectic Lie groupoid with the usual symplectic form on the cotangent bundle $T^*K \longrightarrow K$ (see Example 6.9 for the definition of this symplectic form).*

7. Hamiltonian actions of symplectic groupoids

In this section I will define a moment map theory for actions of symplectic groupoids. I will also show this reduces to the classical theory of equivariant moment maps in the case of the symplectic groupoid $K \times \mathfrak{k}^* \rightrightarrows \mathfrak{k}^*$. To finish, I will introduce Lu's moment map theory for the action of a Poisson group.

Definition 7.1. — *Let $(G \rightrightarrows Q, \omega)$ be a symplectic groupoid. Assume that (M, σ) is a symplectic manifold. A Hamiltonian action of $(G \rightrightarrows Q, \omega)$ on (M, σ) with moment map $J : M \longrightarrow Q$ is an action of $G \rightrightarrows Q$ on M via $J : M \longrightarrow Q$ such that $\Lambda_J = \{(x, m, x \cdot m) \mid (x, m) \in G \times M,\ s(x) = J(m)\}$ is an isotropic submanifold (that is a submanifold on which a symplectic form vanishes) of $(G \times M \times M, \omega \oplus \sigma \oplus -\sigma)$.*

This definition generalises the classical definition of a Hamiltonian action of a Lie group as we will see in the next subsection.

It is a good exercise to check what are the spaces acted on in a Hamiltonian way by the symplectic groupoid of a symplectic manifold (seen as a Poisson manifold).

7.1. Hamiltonian action of a Lie group. —

Definition 7.2. — *Let (M, σ) be a symplectic manifold. Let K be a Lie group acting smoothly on M. This action is called Hamiltonian if*

- *the form σ is invariant,*
- *there is an equivariant map $J : M \longrightarrow \mathfrak{k}^*$ (equivariant for the co-adjoint action of K on $\mathfrak{k}^*$) called the moment map such that for any $X \in \mathfrak{k} = (\mathfrak{k}^*)^* \subset \Omega^1(\mathfrak{k}^*)$*

$$\imath_{v_\xi}\sigma = J^* X, \tag{4}$$
[2]

where v_ξ is the fundamental vector field on M generated by ξ and $\imath_{v_\xi}$ is the inner product with this fundamental vector field.

[2] The vector X defines a function on $\mathfrak{k}^*$, in this formula I consider the differential of this function, still denoted X.

Theorem 7.3. — *The symplectic groupoid $(K \times \mathfrak{k}^*, \omega)$ of Example 6.11 acts in a Hamiltonian way on (M, σ) with moment map $J : M \longrightarrow \mathfrak{k}^*$ if and only if K acts in a Hamiltonian way on (M, σ) with moment map $J : M \longrightarrow \mathfrak{k}^*$.*

Proof. — Identify $K \times \mathfrak{k}^*$ with T^*K by

$$\begin{array}{ccc} K \times \mathfrak{k}^* & \longrightarrow & T^*K \\ (k, \alpha) & \longmapsto & \alpha \circ R_{k^{-1}}. \end{array}$$

Define a symplectic form $\omega = -\mathrm{d}\theta$ on T^*K as in Example 6.11. Identify $\mathfrak{k}$ with T_kK by $X \longmapsto R_k(X) = X \cdot k$. This allows us to identify $T_{(k,\alpha)}K \times \mathfrak{k}^*$ with $\mathfrak{k} \times \mathfrak{k}^*$. In this identification the multiplicative symplectic form[(3)] ω is given by

$$\omega_{(k,\alpha)}\big((X, \beta), (Y, \gamma)\big) = \gamma(X) - \beta(Y) - \alpha([X, Y])$$

and the induced Poisson structure on $\mathfrak{k}^*$ is given by

$$\begin{array}{ccc} T^*_\alpha \mathfrak{k}^* & \stackrel{\eta}{\longrightarrow} & T_\alpha \mathfrak{k}^* \\ \zeta & \longmapsto & \eta_\alpha(\zeta) := -\alpha \circ \mathrm{ad}_\zeta, \end{array}$$

that is on functions f, g in $C^\infty(\mathfrak{k}^*)$

$$\{f, g\}(\alpha) = -\alpha([\mathrm{d}_\alpha f, \mathrm{d}_\alpha g]).$$

The source map on $K \times \mathfrak{k}^* \rightrightarrows \mathfrak{k}^*$ is $s(k, \alpha) = \alpha \circ \mathrm{Ad}_k$ and the target map is $t(k, \alpha) = \alpha$.

Assume $K \times \mathfrak{k}^* \rightrightarrows \mathfrak{k}^*$ acts on M with moment map $J : M \longrightarrow \mathfrak{k}^*$. In particular we have a map

$$\begin{array}{ccc} \mathcal{A} : (K \times \mathfrak{k}^*)_s \times_J M & \longrightarrow & M \\ (k, J(m) \circ \mathrm{Ad}_{k^{-1}}, m) & \longmapsto & \mathcal{A}(k, J(m) \circ \mathrm{Ad}_{k^{-1}}, m). \end{array}$$

This is equivalent to having an action

$$\begin{array}{ccc} K \times M & \longrightarrow & M \\ (k, m) & \longmapsto & k \cdot m \end{array}$$

and an equivariant map $J : M \longrightarrow \mathfrak{k}^*$ [(4)]. That the action of the symplectic groupoid is Hamiltonian is equivalent to: the manifold $\{\Lambda_J = (k, J(m) \circ \mathrm{Ad}_{k^{-1}}, m, k \cdot m) \mid k \in K, m \in M\}$ is isotropic in $(K \times \mathfrak{k}^* \times M \times M, \omega \oplus \sigma \oplus -\sigma)$. A vector tangent to Λ_J at $(k, J(m) \circ \mathrm{Ad}_{k^{-1}}, m, k \cdot m)$ is of the form

$$Z = (X, J_{*,m}(Y) \circ \mathrm{Ad}_{k^{-1}} - J(m) \circ \mathrm{Ad}_{k^{-1}} \circ \mathrm{ad}_X, Y, v_X(k \cdot m) + k \cdot Y)$$

where $X \in \mathfrak{k}$ and $Y \in T_mM$. Let

$$Z' = (X', J_{*,m}(Y') \circ \mathrm{Ad}_{k^{-1}} - J(m) \circ \mathrm{Ad}_{k^{-1}} \circ \mathrm{ad}_{X'}, Y', v_{X'}(k \cdot m) + k \cdot Y')$$

---

(3) The verification that ω is multiplicative is left to the reader.

(4) The verification of this fact is straightforward. The actions of the group and the groupoid are linked by $k \cdot m = \mathcal{A}(k, J(m) \circ \mathrm{Ad}_{k^{-1}}, m)$.

be another such vector. The isotropy of Λ_J is equivalent to

$$(\omega \oplus \sigma \oplus -\sigma)(Z, Z') = 0,$$

that is

$$J_*(Y') \circ \mathrm{Ad}_{k^{-1}}(X) - J(m) \circ \mathrm{Ad}_{k^{-1}}[X', X] - J_*(Y) \circ \mathrm{Ad}_{k^{-1}}(X') + J(m) \circ \mathrm{Ad}_{k^{-1}}[X, X'] + \sigma_m(Y, Y') - \sigma_{k\cdot m}(k \cdot v_{\mathrm{Ad}_{k^{-1}}X}(m) + k \cdot Y, k \cdot v_{\mathrm{Ad}_{k^{-1}}X'}(m) + k \cdot Y')) - J(m) \circ \mathrm{Ad}_{k^{-1}}[X, X'] = 0.$$

Taking $X = 0$ and $X' = 0$ in the above expression shows that σ is K-invariant. So that after some simplifications we obtain

$$(5) \qquad J_*(Y') \circ \mathrm{Ad}_{k^{-1}}(X) - J_*(Y) \circ \mathrm{Ad}_{k^{-1}}(X') - J(m) \circ \mathrm{Ad}_{k^{-1}}[X', X]$$
$$-\sigma_m(v_{\mathrm{Ad}_{k^{-1}}X}, v_{\mathrm{Ad}_{k^{-1}}X'}) - \sigma_m(Y, v_{\mathrm{Ad}_{k^{-1}}X'}) - \sigma_m(v_{\mathrm{Ad}_{k^{-1}}X}, Y') = 0,$$

By taking $X' = 0$ in the above expression, we get

$$J_*(Y') \circ \mathrm{Ad}_{k^{-1}}(X) = \sigma_m(v_{\mathrm{Ad}_{k^{-1}}X}, Y').$$

This last equation is true for all X, Y if and only if

$$\imath_{v_X}\sigma = J^* X,$$

that is if the action of K is Hamiltonian. Conversely, if the action is Hamiltonian, we know that Equation (5) is true whenever $X' = 0$. We need to check that it is also true when $Y' = 0$. In this case it reduces to

$$J(m) \circ \mathrm{Ad}_{k^{-1}}[X, X'] \underline{- J_*(Y) \circ \mathrm{Ad}_{k^{-1}}(X')} = \sigma_m(v_{\mathrm{Ad}_{k^{-1}}X}, v_{\mathrm{Ad}_{k^{-1}}X'}) + \underline{\sigma_m(Y, v_{\mathrm{Ad}_{k^{-1}}X'})}.$$

The two above underlined terms are equal by Equation (2) and the two others are equal by the equivariance of the moment map and Equation (2). □

7.2. Lu's moment map for Poisson Lie groups actions. —

Definition 7.4. — *A Poisson Lie group is a Lie group K with a Poisson structure $\eta : T^*K \longrightarrow TK$ such that the multiplication*

$$K \times K \longrightarrow K$$

is Poisson (for the product Poisson structure on $K \times K$).

Let K be a Poisson Lie group. The anchor η can be seen as a section of $\Lambda^2 TK \longrightarrow K$. This bundle can be trivialised using left translations, hence η is equivalent to a map $K \longrightarrow \Lambda^2\mathfrak{k}$. It can be proved that this map necessarily vanishes at the identity and that its derivative at the identity, a linear morphism $\mathfrak{k} \longrightarrow \Lambda^2\mathfrak{k}$ defines by duality a Lie bracket $\Lambda^2\mathfrak{k}^* \longrightarrow \mathfrak{k}^*$ on $\mathfrak{k}^*$. Thus $\mathfrak{k}^*$ is naturally a Lie algebra. Denote K^* the simply connected Lie group integrating $\mathfrak{k}^*$.

It is proved in [7] using an Iwasawa decomposition that any semi-simple compact connected Lie group has a non-trivial Poisson Lie structure. From now on, I will assume that K is a compact connected and simply connected (hence semi-simple) Lie group with such a Poisson Lie structure. In this case, K and K^* are both subgroups of the complexified $K^{\mathbb{C}}$ and $K^{\mathbb{C}} \simeq K^*K$. Because $K^* \simeq K^{\mathbb{C}}/K$, the group $K^{\mathbb{C}}$ acts on K^* and this action restrict to an action of K on K^* called the (left) dressing action.

Definition 7.5. — *Let (M, σ) be a symplectic manifold. An action of K on M is called Poisson if the action map*

$$K \times M \longrightarrow M$$

is Poisson. An equivariant map $J : M \longrightarrow K^*$ is called a Poisson Lie moment map if for every X in $\mathfrak{k}$

$$\imath_{v_X}\omega = J^*\langle \overline{\theta}_{K^*}, X\rangle,$$

where $\overline{\theta}_{K^*}$ is the right invariant Maurer Cartan form on K^*.

Just as for usual $\mathfrak{k}^*$-valued moment maps, it turns out that the theory of Poisson Lie moment maps can be described as a theory of Hamiltonian actions for a symplectic groupoid (see [**7**] and [**9**])(5). The groupoid can be constructed in the following way: let $k \longmapsto \overline{k}$ and $a \longmapsto \overline{a}$ be the injections of K and, respectively, K^* in $K^{\mathbb{C}}$; consider

$$G = \{(k, a, b, l) \in K \times K^* \times K^* \times K \mid \overline{k}\overline{a} = \overline{b}\overline{l},$$

this is a groupoid over K with source and target maps

$$s(k, a, b, l) = l,\ t(k, a, b, l) = k,$$

multiplication

$$(k, a, b, l) \cdot (l, c, d, h) = (k, ac, bd, h)$$

and inversion map

$$I(k, a, b, l) = (l, a^{-1}, b^{-1}, k).$$

The manifold G is diffeomorphic to $K^{\mathbb{C}}$ by

$$\begin{array}{ccc} G & \longrightarrow & K^{\mathbb{C}} \\ (k, a, b, l) & \longrightarrow & \overline{k}\overline{a}. \end{array}$$

The symplectic structure on $G \simeq K^{\mathbb{C}}$ is constructed from the Poisson structure on K and K^* using the fact that $\mathfrak{k}$ and $\mathfrak{k}^*$ are subalgebras of the Lie algebra of $K^{\mathbb{C}}$ and $\mathfrak{k}^{\mathbb{C}} = \mathfrak{k}^* \oplus \mathfrak{k}$ as vector spaces (for more details, see [**6**]).

(5) In fact the theory of Lie Poisson moment maps and the theory of k^* valued moment maps are equivalent. See [**1**] and [**9**].

Appendix: Poisson Lie structure on a simply connected compact Lie group

In this Appendix, I will explain how Lu and Weinstein [7] constructed a non-trivial Poisson Lie structure on a simply connected compact Lie group.

Let K be such a group. Let

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \oplus i\mathfrak{k}.$$

Let T be a choice of a maximal torus in K and $\mathfrak{t}$ its Lie algebra. Let $\mathfrak{a} = i\mathfrak{t}$ and $\mathfrak{n}$ be the sum of positive roots spaces. These are both subalgebras of $\mathfrak{k}^{\mathbb{C}}$ and

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Let $K^{\mathbb{C}}$ be the simply connected group integrating $\mathfrak{k}^{\mathbb{C}}$ (the complexified group of K). Let $A = \mathrm{Exp}(\mathfrak{a})$ and $N = \mathrm{Exp}(\mathfrak{n})$. Then

$$K^{\mathbb{C}} = KAN = ANK,$$

this is the Iwasawa decomposition of $K^{\mathbb{C}}$. The Killing form on K extends to an hermitian form on $K^{\mathbb{C}}$ whose imaginary part can be used to identify $\mathfrak{k}^*$ and $\mathfrak{a} \oplus \mathfrak{n}$. Hence K and $K^* = AN$ are Poisson groups dual to each others.

Let us see an example. Take $K = SU(2)$, then $K^{\mathbb{C}} = SL(2, \mathbb{C})$. Also

$$A = \left\{ \begin{bmatrix} r & 0 \\ 0 & r^{-1} \end{bmatrix}, r \in \mathbb{R}^*_+ \right\}$$

and

$$N = \left\{ \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}, z \in \mathbb{C} \right\},$$

so that

Proposition 7.6. — *When $K = SU(2)$, the Lie group K^* consists in the set of upper triangular complex 2×2 matrices of determinant 1 with positive reals on the diagonal.*

This can be generalised to $SU(n)$ for any n.

References

[1] A. ALEKSEEV, –*On Poisson actions of compact Lie groups on symplectic manifolds*, J. Diff. Geom. 45 (1997), 241–256.

[2] H. BURSZTYN, M. CRAINIC, A. WEINSTEIN and C. ZHU, –*Integration of twisted Dirac brackets*, math.DG/0303180.

[3] H. BURSZTYN and M. CRAINIC, –*Dirac structures, moment maps and quasi-Poisson manifolds*, math.DG/0310445.

[4] M. CRAINIC and R. L. FERNANDES, –*Integrability of Lie bracket*, math.DG/0105033.

[5] N. P. LANDSMAN, – *Mathematical topics between classical and quantum mechanics*, Springer.

[6] J.-H. LU and A. WEINSTEIN, –*Groupoïdes symplectiques doubles des groupes de Lie-Poisson*, C. R. Acad. Sci. Paris, t. 309, Série I, p. 951-954 (1989).

[7] J.-H. LU and A. WEINSTEIN, –*Poisson Lie Groups, Dressing Transformations, and Bruhat Decompositions*, J. Differential Geometry, 31 (1990) 501–526.

[8] I. VAISMAN, –*Lectures on the Geometry on Poisson manifolds*, Progress in Mathematics, Birhä user.

[9] P. XU, –*Moment Maps and Morita Equivalence*, math.SG/0307319.

June 1, 2004

SÉBASTIEN RACANIÈRE