Lecture on Equivariant Cohomology

Sébastien Racanière

February 20, 2004

I wrote these notes for a 6 hours lecture at Imperial College during January and February 2003. Of course, I tried to track down and remove all mistakes from them. Nevertheless it is very unlikely I succeeded in doing so. If you find any, please let me know.

For a topological space X, we will denote $H^*(X)$ the singular cohomology with coefficients in $\mathbb{R}$. If X is a manifold, it will also sometimes mean the de Rham cohomology. It will either be clear which one is meant, or it will not matter at all.

1 Prerequisite

Theorem 1.1 (Leray-Hirsch Theorem, [5]) *Let $\pi : Y \longrightarrow Z$ be a fibre bundle with fibre X. Assume that for each degree p, the vector space $H^p(X)$ has finite dimension m_p. Assume that, for every p, there exists classes $c_{1,p}, \ldots, c_{m_p,p}$ in $H^p(Y)$ that restricts, on each fibre X, to a basis of the cohomology in degree p. Let $\iota : X \longrightarrow Y$ be an inclusion of a fibre. The map*

$$\begin{array}{ccc} H^*(X) \otimes H^*(Z) & \longrightarrow & H^*(Y) \\ \sum_{i,j,k} \iota^*(c_{i,j}) \otimes b_k & \longmapsto & \sum_{i,j,k} c_{i,j} \wedge b_k \end{array}$$

is an isomorphism of $H^*(Z)$ module.

In the case of Z connected, it is sufficient to check that ι^* is surjective for a particular fibre to apply the theorem.

Theorem 1.2 (Mayer-Vietoris exact sequence, [1]) *Let X be a topological space. Let U and V be two open subsets such that $X = U \cup V$. There exists a long exact sequence*

$$\ldots \longrightarrow H^*(X) \longrightarrow H^*(U) \oplus H^*(V) \longrightarrow H^*(U \cap V) \longrightarrow H^{*+1}(X) \longrightarrow \ldots$$

Theorem 1.3 (Theorem (4.41) in [5]) *If $Y \longrightarrow Z$ is a fibre bundle with fibre X and if Z is path-connected, then there exists a long exact sequence*

$$\ldots \longrightarrow \pi_2(Z) \longrightarrow \pi_1(X) \longrightarrow \pi_1(Y) \longrightarrow \pi_1(Z) \longrightarrow \pi_0(X).$$

The maps $\pi_p(X) \longrightarrow \pi_p(Y)$ and $\pi_p(Y) \longrightarrow \pi_p(Z)$ are induced from the injection of the fibre X in Y and the projection $Y \longrightarrow Z$ respectively.

Theorem 1.4 (Whitehead's theorem, Theorem (4.5) in [5]) *Let X and Y be two connected CW-complex. Let f be a continuous map $X \longrightarrow Y$. Assume that f induces isomorphisms between $\pi_p(X)$ and $\pi_p(Y)$ for all p. Then f is a homotopy equivalence. If f is the inclusion of X in Y, then in addition X is a deformation retract of Y.*

Theorem 1.5 (See [9], Chap. I, §5) *Let $p : Y \longrightarrow X$ be a fibre bundle (in the topological category). Assume that for each x in X, the fibre $p^{-1}(x)$ satisfies $H^*(p^{-1}(x)) = \mathbb{R}$. Then p induces an isomorphism*

$$H^*(Y) \simeq H^*(X).$$

Theorem 1.6 ([3]) *If $p : E \longrightarrow B$ is a bundle with a paracompact base B and if $p^{-1}(b)$ is contractible for each b in B, then p is shrinkable. That is there exists a section s and a homotopy $\Phi : [0,1] \times E \longrightarrow E$ such that*

1. $\Phi(0, u) = u,\ \forall u \in E$,
2. $\Phi(1, u) = s \circ p(u),\ \forall u \in E$,
3. $\Phi(t, u) \in p^{-1}(p(u)),\ \forall (t, u) \in [0,1] \times E$.

Corollary 1.7 *Under the hypothesis of the preceding theorem, E and B have the same homotopy type and all the sections of p are (fibre wise) homotopic.*

Theorem 1.8 (Gleason Lemma, Chap. I, §2 in [6]) *If G is a compact group then for any space E on which G acts freely, the projection $E \longrightarrow E/G$ is locally trivial.*

2 Motivations

Let M be a manifold and G a compact Lie group acting on M. If the action is free then M/G is a 'nice' manifold and has a 'nice' topology. If the action is not free then the quotient may have 'bad' properties. For example, let the circle S^1 act on the sphere S^2 by rotation around the vertical axis. The quotient is a segment. It is contractible and its cohomology is trivial. The problem here is that the two poles of the sphere have non-trivial stabilisers.

The aim of equivariant cohomology is to provide a cohomology that will be the cohomology of the quotient in the 'nice' case and that will keep some trace of the non-trivial stabilisers. One way to do that is to force the action to be free. For example we could replace M by $M \times E$ where E is a space on which the action is free. But then the cohomology of $(M \times E)/G$ depends on the choice of E. We will see that we can get away with this problem by asking E to have all its homotopy groups to vanish.

3 Definition

Let G be a compact Lie group. A topological space X on which G acts is called a G-space.

If X is a G-space, the definition of the equivariant cohomology of X is very simple. It is the usual cohomology of the quotient of $X \times EG$ by the diagonal action of G, where EG is some contractible space on which G acts freely. Such a space is not unique, but because cohomology is a homotopy invariant, we will see that our definition does not depend on the choice of EG.

Let us start by proving the existence of a contractible space with a free action of G.

Proposition 3.1 *There exists a contractible space EG on which G acts freely. The projection $EG \longrightarrow BG$ is a G-principal fibre bundle.*

The proposition remains true if G is simply a topological group (see [8]).

Proof. There exists an injection of G into a $\mathbf{U}(n)$ for n big enough (see [2, Corollary $4.6.5$]). If we find $E\mathbf{U}(n)$ then we can take EG to be $E\mathbf{U}(n)$.

Let $F_n(\mathbb{C}^k)$ be the space of orthonormal families of n vectors in $\mathbb{C}^k$. The group $\mathbf{U}(n)$ acts freely on $F_n(\mathbb{C}^k)$ and the quotient is the Grassmannian $G_n(\mathbb{C}^k)$

of n-dimensional subvector spaces of $\mathbb{C}^k$. The map

$$\begin{array}{ccc} F_n(\mathbb{C}^k) & \longrightarrow & S^{2k-1} \\ (e_1, \dots, e_n) & \longmapsto & e_n \end{array}$$

is a fibre bundle of fibre $F_{n-1}(\mathbb{C}^{k-1})$. Thus because $\pi_p(S^{2k-1})$ is trivial and because of the long exact sequence of Theorem 1.3, we have

$$\pi_p(F_n(\mathbb{C}^k)) = \pi_p(F_{n-1}(\mathbb{C}^{k-1}))$$

whenever $p \leq 2k-2$. By taking k big enough, precisely for $k > \frac{1}{2}p + n - 1$, we can repeat the process and get

$$\pi_p(F_n(\mathbb{C}^k)) = \pi_p(F_{n-1}(\mathbb{C}^{k-1})) = \ldots = \pi_p(F_1(\mathbb{C}^{k+1-n})) = \pi_p(S^{k-n}).$$

This last group is trivial for $k > n + p$. Let

$$E\mathbf{U}(n) = \lim_{\rightarrow}{}_{k\to\infty} F_n(\mathbb{C}^k)$$

be the direct limit of all the $F_n(\mathbb{C}^k)$ (with the induced topology), which we will also denote by $F_n(\mathbb{C}^\infty)$.

Lemma 3.2 *The group $\pi_p(F_n(\mathbb{C}^\infty))$ is trivial for all p.*

Proof. Let γ be a map from the sphere S^p to $F_n(\mathbb{C}^\infty)$. As S^p is compact, there exists k such that $\gamma(S^p)$ is included in $F_n(\mathbb{C}^k)$. By taking k big enough, we see that γ is homotopic, with respect to the base point, to the constant map. $\square$

In addition, $\mathbf{U}(n)$ acts freely on $F_n(\mathbb{C}^\infty)$. The spaces $F_n(\mathbb{C}^k)$ and $G_n(\mathbb{C}^k)$ are CW-complexes. One can find a decomposition of these spaces into CW-complexes such that the decomposition of $F_n(\mathbb{C}^k)$, resp. $G_n(\mathbb{C}^k)$, is induced by restriction of the one for $F_n(\mathbb{C}^{k+1})$, resp. $G_n(\mathbb{C}^{k+1})$ (the details are left to the reader). Thus $F_n(\mathbb{C}^\infty)$ (and also $G_n(\mathbb{C}^\infty)$) is a CW-complexe. By Theorem 1.4 and Lemma 3.2, $F_n(\mathbb{C}^\infty)$ is contractible. $\square$

Remark 3.3 • *The space EG is not unique. Indeed we can replace EG by $EG \times E$ where E is any contractible space on which G acts in whatever way.*

- *To prove Proposition 3.1 for $G = \mathbf{GL}(n, \mathbb{C})$, one just has to replace $F_n(\mathbb{C}^k)$ with the set of free families of n vectors in $\mathbb{C}^k$.*

Definition 3.4 *We call the quotient $BG := EG/G$ the classifying space of G.*

Proposition 3.5 *Let X be a paracompact G-space. The space $X_G = (X \times EG)/G$, is well defined up to homotopy.*

Proof. Let E_1 and E_2 be two contractible spaces on which G acts freely. Let E be equal to $E_1 \times E_2$. We make G act on E by the diagonal action. The natural map $(X \times E)/G \longrightarrow (X \times E_1)/G$ is locally trivial with fibre E_1. Because Corollary 1.7 applies, this map is a homotopy equivalence. The proposition follows since we can do the same with $(X \times E)/G$ and $(X \times E_2)/G$. $\square$

Corollary 3.6 *The space BG is well defined up to homotopy.*

The following Remark justifies the name 'classifying space' for BG.

Remark 3.7 *If M is a paracompact manifold and $P \longrightarrow M$ is a principal G-bundle, then there exists a map $f : M \longrightarrow BG$, well defined up to homotopy, such that P is isomorphic to $f^*(EG)$, the pull-back of the G-bundle $EG \longrightarrow BG$ by f.*

Proof. On one hand, the pull-back of the bundle $\pi : EG \longrightarrow BG$ by the natural projection $P \times_G EG \longrightarrow BG$ is the the bundle $P \times G$. On the other hand, the pull-back of the principal G-bundle $P \longrightarrow M$ by the projection $p : P \times_G EG \longrightarrow M$ is also $P \times EG$

$$\begin{tikzcd} P \arrow[d] & P \times EG \arrow[l] \arrow[r] \arrow[d] & EG \arrow[d, "\pi"] \\ M & P \times_G EG \arrow[l, "p"'] \arrow[r] & BG. \end{tikzcd}$$

Since p is a fibration with contractible fibre EG, Corollary 1.7 applies. Sections of p exist. To such a section s we associate the composition with the projection $P \times_G EG \longrightarrow BG$. The map we get is the f we were looking for.
For the uniqueness up to homotopy, notice that there exists a one to one correspondence between maps $f : M \longrightarrow BG$ such that $f^*EG \longrightarrow M$ is isomorphic

to $P \longrightarrow M$ and sections of p. We have just seen how to associate a f to a section. Inversely, assume that f is given. Let Φ be an isomorphism between f^*EG and P

$$\Phi : \{(x, u) \in M \times EG \mid f(x) = \pi(u)\} \longrightarrow P.$$

Now, simply define a section by

$$\begin{array}{ccc} M & \longrightarrow & P \times_G EG \\ x & \longrightarrow & [\Phi(x, u), u]. \end{array}$$

Because all sections of p are homotopic, the homotopy class of f is unique. $\square$

Proposition 3.8 *For all p, the groups $\pi_p(BG)$ and $\pi_{p-1}(G)$ are isomorphic.*

Proof. Study the long exact sequence arising from the fibre bundle $EG \longrightarrow BG$ of fiber G. $\square$

Definition 3.9 *Let X be a G space. We define the G-equivariant cohomology of X to be*

$$H^*_G(X) = H^*(X_G).$$

When X is paracompact, because of Proposition 3.5, the equivariant cohomology of X is well defined. In general one can use Theorem 1.5. Indeed, let E_1 and E_2 be two contractible spaces on which G acts freely. By Theorem 1.5, the maps $X \times_G (E_1 \times E_2) \longrightarrow X \times_G E_i$, $i = 1, 2$, induce isomorphisms

$$H^*(X \times_G E_i) \longrightarrow H^*(X \times_G (E_1 \times E_2)),\ i = 1, 2.$$

Hence, the equivariant cohomoloyg is well-defined.

The equivariant cohomology is a ring and the natural projection $X_G \longrightarrow BG$ makes it into a module over $H^*(BG)$. This cohomology, as we will see, is a 'nice' one but it lacks certain properties of the usual cohomology of a manifold. For example, Poincaré duality does not work since there is usually no top cohomology class.
It is worth mentioning that equivariant cohomology is also well-defined when the group is not a compact Lie group.

4 Examples and properties

Recall that G is a compact Lie group.

When we motivated the definition of equivariant cohomology, we said that it should be isomorphic to the cohomology of the quotient when the action is free. Let us prove it is indeed the case.

Theorem 4.1 *Let X be a G-space. Assume that the action is free. Then the equivariant cohomology of X is the cohomology of the quotient X/G.*

In fact, the Theorem remains true for a locally free action.

Proof. The map $p : X_G \longrightarrow X/G$ is a fibre bundle with contractible fibre EG. By Theorem 1.5

$$H^*_G(X) \simeq H^*(X/G).$$

□

Lemma 4.2 *If X is a contractible G-space, its cohomology is the cohomology of the classifying space BG.*

Proof. Indeed, $X \times EG$ is a contractible space on which G acts freely. □

The following Proposition tells us what kind of object the equivariant cohomology of a space is. Its proof is easy.

Proposition 4.3 *Let X be a G-space. Its equivariant cohomology is a ring. Also, the projection*

$$X \times_G EG \longrightarrow BG$$

is well-defined up to homotopy. It makes $H^*_G(X)$ into a $H^*(BG)$-module.

Let us now see what happens in the example we gave in Section 2: the circle action on the 2-sphere.

Definition 4.4 *A G-space X is said to be equivariantly formal (for the action of G) if*

$$H^*_G(X) \simeq H^*(BG) \otimes H^*(X)$$

as $H^*(BG)$ modules.

If $H^*(X)$ is finitely generated, then by the Leray-Hirsch Theorem 1.1 it is enough to check that $H^*_G(X) \simeq H^*(BG) \otimes H^*(X)$ as vector spaces to prove that X is equivariantly formal.

A space on which a group G acts trivially is equivariantly formal. The reverse is not true as the next Proposition 4.5 will show.

Proposition 4.5 *Let S^2 be the unit sphere in $\mathbb{R}^3$. The action, by rotation around the vertical axis, of the circle S^1 on the sphere S^2 is equivariantly formal.*

Proof. The proof requires the following result: $H^*(BS^1)$ is a polynomial algebra in one variable, c_1, of degree 2

$$H^*(BS^1) = \mathbb{R}[c_1], \text{ with } \deg(c_1) = 2.$$

This will be proved in Proposition 4.13. In particular, $H^{2k}(BS^1)$ is of dimension one and $H^{2k-1}(BS^1)$ is of dimension zero.
Let N, resp. S, be the open consisting of points above the tropic of Capricorn, resp. below the tropic of Cancer. Both N and S are contractible and S^1 invariant. Thus, by Lemma 4.2, $H^*_{S^1}(N) \simeq H^*_{S^1}(S) \simeq H^*(BS^1)$. The circle acts freely on the intersection of N and S. The quotient is a segment, hence $H^*(N \cap S) \simeq H^*(\{\text{point}\})$. Using the Mayer-Vietoris sequence for the decomposition of $S^2 \times_{S^1} ES^1$ in $N \times_{S^1} ES^1$ and $S \times_{S^1} ES^1$, we deduce the long exact sequence

$$\ldots \longrightarrow H^{*-1}(\{\text{pt}\}) \longrightarrow H^*_{S^1}(S^2) \longrightarrow H^*(BS^1) \oplus H^*(BS^1) \longrightarrow H^*(\{\text{pt}\}) \longrightarrow \ldots$$

By comparing dimensions, we deduce that S^2 is equivariantly formal for the action of S^1. $\square$

Notice that Frances Kirwan proved ([7]) that if a compact Lie group G acts in Hamiltonian way on a compact symplectic manifold M then M is equivariantly formal for the action of G. In particular, since our action of S^1 on the sphere S^2 is Hamiltonian (the symplectic form is the volume form induced by the Euclidean metric of $\mathbb{R}^3$ and the moment map is the height function), Proposition 4.5 follows from Kirwan's result.

Let us say a few words about equivariant cohomologies with non compact groups.

Example 4.6 *If G is $\mathbb{Z}$, then $EG = \mathbb{R}$ and $BG = S^1$. If G is $\mathbb{R}$, then EG is $\mathbb{R}$ as well and BG is just a point.*

This last example can be generalised.

Proposition 4.7 *Let H be a topological group and K a subgroup. Assume that H/K is contractible. If X is a H-space then the cohomologies $H_H^*(X)$ and $H_K^*(X)$ are naturally isomorphic.*

In particular, by taking K a connected compact Lie group and H its complexification, we see that H and K equivariant cohomologies are isomorphic. Also, if $\mathbb{R}^n$ acts on X, then $H^*_{\mathbb{R}^n}(X) = H^*(X)$. We see this example is not going to be very exciting.

Proof. Let $EH \longrightarrow BH$ be a H-principal bundle with a contractible total space. The restricted action of K on EH is also free. Consider $X \times_K EH \longrightarrow X \times_H EH$. It is a fibre bundle with contractible fibre H/K. The proposition follows from Theorem 1.5. $\square$

Let us get back to more serious things and see some examples of classifying spaces. We showed in the proof of Proposition 3.1 that

$$S^\infty = \varinjlim_{k \longrightarrow \infty} S^{2k-1}$$

is contractible. But S^∞ is also the direct limit of spheres of either all dimensions or of dimensions $4k-1$. This remark leads to the following examples.

Example 4.8

1. *The group $\mathbf{U}(1)$ acts freely on the odd dimensional spheres, thus*

$$B\mathbf{U}(1) = \mathbb{C}\mathrm{P}^\infty = \varinjlim_{k \longrightarrow \infty} \mathbb{C}\mathrm{P}^k.$$

2. *The group $\mathbb{Z}_2$ acts freely on all spheres, thus*

$$B\mathbb{Z}_2 = \mathbb{R}\mathrm{P}^\infty = \varinjlim_{k \longrightarrow \infty} \mathbb{R}\mathrm{P}^k.$$

3. *The group $\mathbf{SU}(2)$ acts freely on the S^{4k-1}, the spheres of quaternionic vector spaces, thus*

$$B\mathbf{SU}(2) = \mathbb{H}\mathrm{P}^\infty = \varinjlim_{k \longrightarrow \infty} \mathbb{H}\mathrm{P}^k.$$

We wish now to compute the cohomology of the $B\mathbf{U}(n)$'s. To do so we need to develop some tools.

From now on, G is a compact connected Lie group. Let us recall some basic facts about this kind of Lie groups.

Proposition 4.9 *There exists a maximal torus T in G. All such maximal tori are conjugated. Let $N(T)$ be the normaliser of T. The group $N(T)/T$ is called the Weyl group W. It is a finite group, let $|W|$ be its cardinal. It acts on G/T and $W \longrightarrow G/T \longrightarrow G/N(T)$ is a finite covering.*
Also, the Bruhat decomposition of G/T is a cell decomposition with $|W|$ cells of even dimensions.

The first part of the Proposition can be found in almost any book on compact Lie group, see for example [2]. I have to admit I didn't find any reference for the Bruhat decomposition (suggestions are welcome).

In the case of $\mathbf{U}(n)$, we have

Proposition 4.10 *A maximal torus in $\mathbf{U}(n)$ is given by the subgroup T of diagonal matrices. The Weyl group is the symmetric group of a set of n elements. It acts on T by permuting the diagonal entries.*

We will now investigate the relation between equivariant cohomology of a space with respect to a compact Lie group G and with respect to a maximal torus T.

Proposition 4.11 *Let T be a maximal torus in G, then*

$$H^*(G/N(T)) \simeq H^*(G/T)^W \simeq H^*(\{pt\}).$$

Proof. The existence of the finite covering $W \longrightarrow G/T \longrightarrow G/N(T)$ implies

$$H^*(G/N(T)) \simeq H^*(G/T)^W.$$

Also, noting χ the Euler number of a manifold

$$\chi(G/N(T)) = \frac{1}{|W|}\chi(G/T).$$

Because of the Bruhat decomposition of G/T, the odd degree cohomology of G/T vanishes. The odd degree cohomology of $G/N(T)$ also then vanishes and

$$\dim(H^*(G/N(T))) = \chi(G/N(T)) = \frac{1}{|W|}\chi(G/T) = 1.$$

□

We will now use the previous proposition to prove

Theorem 4.12 *Let X be a G-space. Let T be a maximal torus for G. The Weyl group acts on X_T and*

$$H_G^*(X) \simeq H_{N(T)}^*(X) \simeq H_T^*(X)^W.$$

Proof. The fibre $G/N(T)$ of the bundle $X_{N(T)} \longrightarrow X_G$ has trivial (real) cohomology. We deduce from Theorem 1.5 that $H_G^*(X) \simeq H_{N(T)}^*(X)$. Next consider the covering

$$(EG \times X) \times_G G/T \longrightarrow (EG \times X) \times_G G/N(T)$$

with covering group W. We have

$$(EG \times X) \times_G G/T = ((EG \times X) \times_G G)/T = EG \times_T X, \tag{1}$$

in the same fashion

$$(EG \times X) \times_G G/N(T) = EG \times_{N(T)} X. \tag{2}$$

Beware, in Equality (1), the action of T on $(EG \times X) \times_G G$ is induced only by the multiplication on the right of T on G, the action of T on X or EG is not involved. The theorem follows from Equality (1) and Equality (2). $\square$

Proposition 4.13 *The cohomology of the classifying space $H^*(B\mathbf{U}(n))$ is a ring of polynomials in n variables $c_1, \ldots, c_n$ where c_p is of degree $2p$.*

A similar theorem is true for any compact connected Lie group.
This Proposition shows in particular that the $\mathbf{U}(n)$-equivariant cohomology of a point does not have any top cohomology.

Proof. Let us first consider the case $n = 1$. In this case, $U(1)$ is the circle S^1 and the universal bundle is $S^\infty \longrightarrow \mathbb{C}\mathrm{P}^\infty$. It is well known (see [1] for example) that the cohomology of $\mathbb{C}\mathrm{P}^k$ is isomorphic to $\mathbb{R}[c_1]/c_1^{k+1}$, where c_1 is the Euler class of the $U(1)$-bundle $S^{2k+1} \longrightarrow \mathbb{C}\mathrm{P}^k$ (if you are more of an algebraic geometer than a differential geometer, you may prefer to take c_1 to be the Euler class tautological bundle over $\mathbb{C}\mathrm{P}^k$, the two different approaches will give the same result), and that the injections $\mathbb{C}\mathrm{P}^k \longrightarrow \mathbb{C}\mathrm{P}^{k+1}$, for $k \in \mathbb{N}^*$, are compatible with these presentations of the cohomology of the projective spaces. The Proposition is proven for $n = 1$.

In the general case, let T be the subgroup of diagonal matrices. It is a maximal torus in $\mathbf{U}(n)$. Its classifying space is $(\mathbb{C}\mathrm{P}^\infty)^n$ and its cohomology is $\mathbb{R}[x_1, \ldots, x_n]$, where x_i is the 'Euler class' of the tautological bundle over the i-th $\mathbb{C}\mathrm{P}^\infty$. The Weyl group acts on T by permuting the diagonal entries, hence it acts on $(\mathbb{C}\mathrm{P}^\infty)^n$ by permutation of the factors. The induce action on its cohomology is the permutation of the x_i's. We deduce

$$H^*(B\mathbf{U}(n)) = \mathbb{R}[c_1, \ldots, c_n],$$

where the c_i's are the symmetric polynomials in the x_i's. □

Let M be a paracompact manifold with a complex vector bundle $V \longrightarrow M$. Choose a hermitian metric on V. Let $P \longrightarrow M$ be the associated $\mathbf{U}(n)$ bundle. Then V is isomorphic to $P \times_{\mathbf{U}(n)} \mathbb{C}^n$. Let $f : M \longrightarrow B\mathbf{U}(n)$ be such that the pull-back bundle $f^*(E\mathbf{U}(n))$ is isomorphic to P. Hence V is isomorphic to $f^*(E\mathbf{U}(n) \times_{\mathbf{U}(n)} \mathbb{C}^n)$. Let us define the Chern classes of the complex vector bundle V.

Definition 4.14 *The $k-th$ Chern class $c_k(V)$ of V is $f^*(c_k)$.*

A different choice of a hermitian metric would give an isomorphic $\mathbf{U}(n)$ bundle. Thus $c_k(V)$ is independent of the choice of the metric.

More generally:

Definition 4.15 *Let X be a paracompact space. Let $P \longrightarrow X$ be a $\mathbf{U}(n)$-principal bundle. We define the k-th Chern class of P to be*

$$c_k(P) = f^*(c_k)$$

in $H^{2k}(X)$, where f is a map $X \longrightarrow B\mathbf{U}(n)$ such that $f^*(E\mathbf{U}(n)) \simeq P$.

There exist many ways of defining the Chern classes for a vector bundle over a manifold (using a connection or by studying the cohomology of the projectivised bundle for example). There exists an axiomatic definition of these classes. To check that our definition coincides with other definitions, we just have to check that $c_1, \ldots, c_n$ satisfy these axioms:

1. For the tautological line bundle over $\mathbb{C}\mathrm{P}^k$, c_1 is the Euler class.

2. If $E \longrightarrow M$ is a sum of complex line bundles $L_1 \oplus \ldots \oplus L_n$, then the total class $c(E) = 1 + c_1(E) + \ldots + c_n(E)$ satisfies

$$c(E) = \prod_{i=1}^{n} (1 + c_1(L_i)).$$

3. If $E \longrightarrow M$ is a complex vector bundle and $h : N \longrightarrow M$ a continuous map, then
$$c(h^*E) = h^*c(E).$$

The first point follows from the proof of Proposition 4.13. The third point is clear by definition. To prove the second point, we just have to notice that decomposing a vector bundle into a sum of line bundles corresponds to reducing the structure group from $\mathbf{U}(n)$ to a maximal torus. This, with the proof of Proposition 4.13, proves the second axiom is verified.

Let us take a small break. Recall the Whitehead's Theorem states that if a map between CW-complexes induces isomorphisms between all the homotopy groups, then it is a homotopy equivalence. Compare this result to the following Lemma.

Lemma 4.16 *The* 2 *sphere* S^2 *and* $\mathbb{C}\mathrm{P}^\infty \times S^3$ *are both CW-complexes whose homotopy groups are isomorphic. Nevertheless, they have not the same homotopy type.*

The problem here is that the isomorphisms between the homotopy groups are not given by a map.

Proof. We already know that S^2 and $\mathbb{C}\mathrm{P}^\infty \times S^3$ are CW complexes. The homotopy groups of S^2 and S^3 are the same except for the second one, indeed the Hopf fibration

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \\ & & S^2 \end{array}$$

induces a long exact sequence

$$\ldots \pi_p(S^1) \longrightarrow \pi_p(S^3) \longrightarrow \pi_p(S^2) \longrightarrow \pi_{p-1}(S^1) \longrightarrow \ldots.$$

All the homotopy groups of S^1 vanish, except its fundamental which is isomorphic to $\mathbb{Z}$. Also, the first and second homotopy groups of S^3 vanish. We deduce that

S^2 and S^3 have isomorphic homotopy groups, apart from the second one which is trivial for S^3 and $\mathbb{Z}$ for S^2.

In addition, because of the fibration

$$\begin{tikzcd} S^1 \arrow[r] & S^\infty \arrow[d] \\ & \mathbb{C}\mathrm{P}^\infty \end{tikzcd}$$

and because S^∞ is contractible, the second homotopy group of $\mathbb{C}\mathrm{P}^\infty$ is isomorphic to $\mathbb{Z}$ whereas all the others are trivial.

It follows that S^2 and $S^3 \times \mathbb{C}\mathrm{P}^\infty$ have isomorphic homotopy groups. But, because of Proposition 4.13, they clearly have very different cohomologies. This proves they don't have the same homotopy type. $\square$

Assume M is a manifold and F is a closed submanifold. The restriction map $H^*(M) \longrightarrow H^*(F)$ is of course never injective. This is why I found the next Theorem (due to Borel and Hsiang) very surprising the first time I read it.

Theorem 4.17 (Theo. 11.4.5 in [4]) *Assume M is a manifold acted on smoothly by a compact torus T. Suppose also that $H^*(M)$ is finitely generated and that the action of T is equivariantly formal. Let F be the set of fixed points. The restriction map*

$$H_T^*(M) \longrightarrow H_T^*(F)$$

is injective.

Proof. We will prove the Theorem by induction on the dimension of T. Assume first that $T = S^1$. The set of fixed point F is a closed submanifold of M. Let U be a S^1 invariant neighbourhood of F that retracts onto F. Let $V = M - F$. Because every proper subgroup of S^1 is finite, the action of S^1 on V and $U \cap V$ is locally free. By Theorem 4.1, we have

$$\begin{aligned} H_{S^1}^*(V) &\simeq H^*(V/S^1), \\ H_{S^1}^*(U \cap V) &\simeq H^*(U \cap V/S^1). \end{aligned}$$

Because U contracts to F, we also have

$$H_{S^1}^*(U) \simeq H_{S^1}^*(F) \simeq H^*(F) \otimes H^*(BS^1).$$

This cohomology groups fit into the Mayer-Vietoris exact sequence

$$\ldots \longrightarrow H^*_{S^1}(M) \longrightarrow H^*(F)\otimes H^*(BS^1)\oplus H^*(V/S^1) \longrightarrow H^*(U\cap V/S^1) \longrightarrow \ldots$$

For $*$ big,

$$H^*(V/S^1) \simeq H^*(U \cap V/S^1) \simeq \{0\}$$

hence the restriction map $r : H^*_{S^1}(M) \longrightarrow H^*_{S^1}(F)$ is an isomorphism for $*$ big. Assume $\alpha \in H^*_{S^1}(M)$ satisfies $r(\alpha) = 0$. On one hand, multiplication by $c_1 \in H^*(BS^1)$ is injective in $H^*_{S^1}(M)$. On the other hand, for all p in $\mathbb{N}$, $r(c^p \wedge \alpha) = 0$. By choosing p big enough, we deduce that $c^p \wedge \alpha = 0$ and that $\alpha = 0$. We have proved the Theorem for $T = S^1$.

Assume the Theorem is true for tori of dimension 1 and $n-1$. Assume $\dim(T) = n$. Write

$$T = S^1 \times \ldots \times S^1.$$

Let A denote $S^1 \times \ldots S^1 \times \{1\}$ the first $n-1$ factors in T and B be the last factor. We first have to prove that the action of A is equivariantly formal on M. The T equivariant cohomology of M is the cohomology of

$$M \times_T (ES^1 \times \ldots \times ES^1) = (M \times_A (ES^1)^{n-1}) \times_B ES^1.$$

The inclusion of M in $M \times_T (ES^1 \times \ldots \times ES^1)$ factorises through $M \times_A (ES^1)^{n-1}$ and gives rise to a commutative diagram

$$\begin{tikzcd} H^*(M) & H^*_T(M) \arrow[l] \arrow[d, leftrightarrow] \\ H^*_A(M) \arrow[u] & H^*_B(M \times_A ES^1). \arrow[l, "\pi"'] \end{tikzcd}$$

The vertical double arrow is an isomorphism. The top horizontal arrow is surjective. It follows that the left vertical arrow is also surjective. By the Leray-Hirsch Theorem 1.1, this means that the action of A on M is equivariantly formal. We can give an explicit isomorphism. Choose elements, denoted $b_1, \ldots, b_m$, in $H^*_T(M)$ which restrict on M to a basis of $H^*(M)$. Still denote $b_1, \ldots, b_m$ their restrictions to $H^*_A(M)$ and to $H^*(M)$. The maps

$$\begin{array}{ccc} H^*(BT) \otimes H^*(M) & \longrightarrow & H^*_T(M) \\ P(x_1, \ldots, x_n) \otimes b_i & \longmapsto & P \wedge b_i \end{array}$$

and

$$\begin{array}{ccc} H^*((BS^1)^{n-1}) \otimes H^*(M) & \longrightarrow & H^*_A(M) \\ P(x_1, \ldots, x_{n-1}) \otimes b_i & \longmapsto & P \wedge b_i \end{array}$$

are isomorphisms. Let $F(A)$ be the set of fixed points in M for the action of A. The group B acts on $F(A)$ and its set of fixed points is F. There is a canonical isomorphism

$$F(A) \times_A (ES^1)^{n-1} \simeq F(A) \times (BS^1)^{n-1}.$$

Consider the commutative diagram

$$\begin{tikzcd}
F(A) \times (BS^1)^{n-1} \arrow[r] \arrow[d] & M \times_A (ES^1)^{n-1} \arrow[d] \\
(F(A) \times (BS^1)^{n-1}) \times_B ES^1 \arrow[r] & (M \times_A (ES^1)^{n-1}) \times_B ES^1.
\end{tikzcd}$$

It induces another commutative diagram

$$\begin{tikzcd}
H^*(F(A) \times (BS^1)^{n-1}) & H^*_A(M) \arrow[l] \\
H^*_B(F(A) \times (BS^1)^{n-1}) \arrow[u] & H^*_T(M). \arrow[l] \arrow[u, "\pi"']
\end{tikzcd}$$

Assume $\alpha \in H^*_T(M)$ satisfies $r(\alpha) = 0$. Because the top arrow in the previous diagram is injective (this is our theorem for tori of dimension $n-1$), we also have $\pi(\alpha) = 0$. Write

$$\alpha = \sum_i P_i(x_1, \ldots, x_n) \otimes b_i.$$

The image of α by π is

$$\pi(\alpha) = \sum_i P_i(x_1, x_2, \ldots, x_{n-1}, 0) \otimes b_i = 0.$$

It follows that there exist polynomials P_i' such that for each i

$$P_i(x_1, \ldots, x_n) = x_n P_i'(x_1, \ldots, x_n).$$

Let

$$\alpha' = \sum_i P_i'(x_1, \ldots, x_n) \otimes b_i,$$

be the ‘division’ of α by x_n. We have

$$r(\alpha) = r(x_n \wedge \alpha') = x_n \wedge r(\alpha') = 0.$$

Because multiplication by x_n is an injective map in $H_T^*(F)$, the image of α' by r must be zero. We can repeat with α' what was done with α, and so on and so on. We deduce by induction that the P_i's are infinitely divisible by x_n. Therefore they are equal to the zero polynomial and $\alpha = 0$.

We have proved that

$$H_T^*(M) \longrightarrow H_B^*(F(A) \times (BS^1)^{n-1}) \tag{3}$$

is an injective homomorphism. The space on the right hand side is canonically isomorphic to $H_B^*(F(A)) \otimes H^*((BS^1)^{n-1})$. If we apply the induction hypothesis for the action of the 1-dimensionnal torus B on $F(A)$, we find an injection

$$H_B^*(F(A)) \longrightarrow H_B^*(F). \tag{4}$$

In the above injection, the tensor product of the left hand side by $H^*((BS^1)^{n-1})$ is naturally isomorphic to $H_B^*(F(A) \times (BS^1)^{n-1})$, whereas the tensor product of the right hand side by the same space is naturally isomorphic to $H_T^*(F)$. The composition of the injection (3) with the tensor product of the injection (4) with $H^*((BS^1)^{n-1})$ proves the theorem is true for T of dimension n. By induction it is true for any torus. $\square$

The assumption that the group is a torus in Theorem 4.17 is necessary. Indeed, let $\mathbf{SU}(2)$ act on $\mathbb{C}\mathrm{P}^1$

$$\begin{array}{ccc} \mathbf{SU}(2) \times \mathbb{C}\mathrm{P}^1 & \longrightarrow & \mathbb{C}\mathrm{P}^1 \\ \left(\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, [z_0, z_1] \right) & \longmapsto & [az_0 + bz_1, -\bar{b}z_0 + \bar{a}z_1]. \end{array}$$

This action is equivariantly formal (Lemma 4.19) but it doesn't have any fixed point so that F is empty and $H^*_{\mathbf{SU}(2)}(F) = \{0\}$. In particular the restriction map

$$H^*_{\mathbf{SU}(2)}(\mathbb{C}\mathrm{P}^1) \longrightarrow H^*_{\mathbf{SU}(2)}(F)$$

is not injective.

We will see that the above action of $\mathbf{SU}(2)$ is equivariantly formal. But before doing so we need to know what is the cohomology of $B\mathbf{SU}(2)$.

Lemma 4.18 *The cohomology of $B\mathbf{SU}(2)$ is a polynomial ring in one variable c_2 of degree 4*

$$H^*(B\mathbf{SU}(2)) = \mathbb{R}[c_2].$$

Proof. The proof is similar to that of Theorem 4.13. A maximal torus in $\mathbf{SU}(2)$ is given by the set of diagonal matrices

$$T = \left\{ \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \lambda \in S^1 \right\}.$$

The Weyl group is the symetric group on a set of 2 elements. It acts on T by replacing λ by $\bar{\lambda} = \lambda^{-1}$. In particular, it replaces the tautological bundle over $\mathbb{C}\mathrm{P}^k$ by its inverse bundle and multiplies x_1 in $H^*(BT) = \mathbb{R}[x_1]$ by -1. This means

$$H^*(B\mathbf{SU}(2)) = \mathbb{R}[c_2]$$

where c_2 is x_1^2, a class of degree 4. □

Notice that the c_2 of $B\mathbf{SU}(2)$ is the 'same' as the c_2 of $B\mathbf{U}(2)$. By this I mean that whenever a complex vector bundle of rank 2 is given, the choice of a hermitian metric reduces the structure group to $\mathbf{U}(2)$. The structure group can be further reduce to $\mathbf{SU}(2)$ if and only if the determinant bundle is trivial. In this case, the c_2's we defined in $H^*(B\mathbf{SU}(2))$ and $H^*(B\mathbf{U}(2))$ both induce the second Chern class on the vector bundle. Also, whenever the determinant bundle is trivial, the first Chern class vanishes. That is why there is no 'c_1' in $H^*(B\mathbf{SU}(2))$.

More generally, one can prove the cohomology of $B\mathbf{SU}(n)$ is a polynomial ring in $n-1$ variables $c_2, \ldots, c_n$ where $\deg c_i = 2i$.

Lemma 4.19 *The above action of* $\mathbf{SU}(2)$ *on* $\mathbb{C}\mathrm{P}^1$ *is equivariantly formal.*

Proof. The graded vector space $H^*(\mathbb{C}\mathrm{P}^1) \otimes H^*(B\mathbf{SU}(2))$ has dimension 1 in even degree and vanish in odd degree.

The computation of $H^*_{\mathbf{SU}(2)}(\mathbb{C}\mathrm{P}^1)$ is a special case of the next Lemma.

Lemma 4.20 *Let G be a compact Lie group. Let K be a subgroup. The group G acts on G/K and*

$$H^*_G(G/K) \simeq H^*(BK).$$

So far, all the actions we considered were either locally free or equivariantly formal. This of course is far from being the general case, as the above Lemma 4.20 is showing.

Proof. Define a map

$$\begin{array}{ccc} G/K \times_G EG & \longrightarrow & EG/K \\ [gK, u] & \longmapsto & [g^{-1}u]. \end{array}$$

It is well-defined because if $[gK, u] = [g'K, u']$, there exists h in G and k in K such that

$$u' = gu \text{ and } g' = hgk.$$

Therefore $g'^{-1}u' = k^{-1}g^{-1}h^{-1}hu = k^{-1}g^{-1}u \equiv g^{-1}u \bmod K$. Moreover, the previous map is a homeomorphism. Indeed, we can explicitly give an inverse

$$\begin{array}{ccc} EG/K & \longrightarrow & G/K \times_G EG \\ [u] & \longmapsto & [K, u]. \end{array}$$

This proves the lemma. □

The action of $\mathbf{SU}(2)$ on $\mathbb{C}\mathrm{P}^1$ is transitive. The stabiliser of $[1, 0]$ is the subgroup of diagonal matrices in $\mathbf{SU}(2)$. According to Lemma 4.20, we have

$$H^*_{\mathbf{SU}(2)}(\mathbb{C}\mathrm{P}^1) \simeq H^*(BS^1).$$

This graded vector space has indeed dimension 1 in every even degree and vanish in odd degrees. □

Remark 4.21 *The restriction of the preceding action of $\mathbf{SU}(2)$ to the maximal torus of diagonal matrices is the same as the action of S^1 on S^2 in Proposition 4.5, once we have identified S^2 with $\mathbb{C}\mathrm{P}^1$ through the stereographic projection. Hence, one may have proved Lemma 4.19 using Theorem 4.12.*

Let us now have a look at the functorial properties of equivariant cohomology.

Proposition 4.22 *Let $\rho : K \longrightarrow G$ be a homomorphism of groups and X a space on which G acts. The homomorphism ρ induces an action of K on X and there exists a natural homomorphism*

$$H^*_K(X) \longrightarrow H^*_G(X).$$

Proof. Let K act on $EK \times EG$ by

$$h \cdot (e_1, e_2) = (h \cdot e_1, h \cdot e_2), \text{ for } h \in K,\ (e_1, e_2) \in EK \times EG.$$

The map

$$\begin{array}{ccc} X \times_K (EK \times EG) & \longrightarrow & X \times_G EG \\ [x, e_1, e_2]_H & \longmapsto & [x, e_2]_G \end{array}$$

is well-defined and induces an homomorphism as in the proposition. □

Proposition 4.23 *Let X and Y be G spaces. Let $f : X \longrightarrow Y$ an equivariant map (that is $f(g \cdot x) = g \cdot f(x)$). There exists a natural homomorphism of $H^*(BG)$-modules*

$$H^*_G(Y) \longrightarrow H^*_G(X).$$

Proof. Look at the map $X \times_G EG \longrightarrow Y \times_G EG$ induced by f. □

There is not in general an equivalent of the Künneth formula. This is illustrated by the two following examples.

Example 4.24 *1. Let $G = S^1$ act on $X = Y = S^1$ by multiplication. We have*

$$H^*_G(X \times Y) = H^*(S^1)$$

but

$$H^p_G(X) \otimes H^q_G(Y) = \begin{cases} \mathbb{R}, & \text{if } p = q = 0 \\ \{0\}, & \text{otherwise.} \end{cases}$$

2. Let X be a G space. Let Y be a single point. Then

$$H^*_G(X \times Y) = H^*_G(X)$$

and

$$H^*_G(X) \otimes H^*_G(Y) = H^*_G(X) \otimes H^*(BG).$$

In the last example, we nevertheless have

$$H^*_G(X \times Y) = H^*_G(X) \otimes_{H^*(BG)} H^*_G(Y).$$

This is a special case of the following proposition.

Proposition 4.25 *Let X and Y be G-spaces. Assume that Y is equivariantly formal for the action of G. Assume also that the usual cohomology of Y is finitely generated as a ring. Then*

$$H_G^*(X \times Y) = H_G^*(X) \otimes_{H^*(BG)} H_G^*(Y).$$

Proof. By the Leray-Hirsh Theorem 1.1, to say that Y is equivariantly formal is to say that the inclusion of Y into $Y \times_G EG$ induces a surjective homomorphism from $H_G^*(Y)$ onto $H^*(Y)$. The injection of Y into $Y \times_G EG$ can be factorised through $(X \times Y) \times_G EG$, hence the homomorphism $H_G^*(X \times Y) \longrightarrow H^*(Y)$ is surjective. Because the map $(X \times Y) \times_G EG \longrightarrow X \times_G EG$ is a fibration with fibre Y, by the Leray-Hirsh Theorem we have

$$\begin{array}{rcl} H_G^*(X \times Y) & \simeq & H_G^*(X) \otimes H^*(Y) \\ & \simeq & H_G^*(X) \otimes_{H^*(BG)} H_G^*(Y). \end{array}$$

□

Corollary 4.26 *Let X be an equivariantly trivial G space. Then X^r, for $r \in \mathbb{N}^*$, with the diagonal action of G is equivariantly trivial.*

Proof. The proof is straightforward by induction. The Corollary is true for $r = 1$. If it is true for $r - 1$, then

$$\begin{array}{rcl} H_G^*(X^r) & \simeq & H_G^*(X) \otimes_{H^*(BG)} H_G^*(X^{r-1}) \\ & \simeq & H^*(X) \otimes H^*(X^{r-1}) \otimes H^*(BG) \\ & \simeq & H^*(X^r) \otimes H^*(BG). \end{array}$$

□

References

[1] R. BOTT, L. W. TU – *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics 82, Springer.

[2] J. J. DUISTERMAAT and J. A. Kolk, – *Lie Groups*, Universitext, Springer.

[3] A. DOLD – *Partitions of Unity in the Theory of Fibrations*,Annals of Math., vol. 78, No 2 (1963).

[4] V. W. GUILLEMIN and S. Sternberg, – *Supersymmetry and Equivariant de Rham Theory*, Mathematics Past and Present, Springer.

[5] A. HATCHER – *Algebraic Topology*, Cambridge University Press.

[6] W. Y. HSIANG – *Cohomology Theory of Topological Transformation Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Brand 85, Springer-Verlag.

[7] F. C. KIRWAN –*Cohomology of Quotients in Algebraic and Symplectic Geometry*, Princeton University Press.

[8] J. MILNOR – *Construction of Universal Bundles, II*, Annals of Math., vol. 63, No. 3 (1956), 430–436.

[9] J. P. SERRE – *Homologie Singulière des espaces fibrés*, Ann. of Math., vol. 54 (1951), 425–505.