Bass-Serre Theory

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Elementary splittings of groups

Definition 1 *Let* A, B, C *be groups and* $\alpha : C \to A, \beta : C \to B$ *injective homomorphisms. The* amalgamated free product *of* A *and* B *over* α *and* β *is the direct limit of this diagram of groups. We abuse notation and denote this*

$$A *_C B.$$

Definition 2 *Now suppose* $A = B$. *Consider* $\{A_i | i \in \mathbb{Z}\}$ *a collection of copies of* A, $\{C_j | j \in \mathbb{Z}\}$ *a collection of copies of* C, *and homomorphisms* $\alpha_i : C_i \to A_i$ *coinciding with* α, $\beta_i : C_i \to A_{i+1}$ *coinciding with* β. *Let* H *be the direct limit of this system, and* $u : H \to H$ *the shift automorphism mapping* $G_i \to G_{i+1}$. *The semidirect product* $H \rtimes_u \mathbb{Z}$ *is called the* HNN-extension *of* A *over* α *and* β, *and abusively denoted*

$$A *_C .$$

A more concrete picture of these constructions is given by their presentations. Let $A = \langle G|R\rangle, B = \langle H|S\rangle$. Then it is easy to write down a presentation for $A *_C B$, namely

$$\langle G, H|R, S, \{\alpha(c)\beta(c^{-1})|c \in C\}\rangle.$$

$A*_C B$ is the freest group into which A, B inject and the images of C are identified.

In the case where $A = B = \langle G|R\rangle$, the HNN-extension $A*_C$ has presentation

$$\langle G, t|R, \{t\alpha(c)t^{-1}\beta(c^{-1})|c \in C\}\rangle.$$

$A*_C$ is the freest group into which G injects and the isomorphism between the images of C is realized as conjugation by an element. The element t is known as the *stable letter*.

Example 3 *The (non-abelian) free group of rank 2, F, can be decomposed in either way. It can be written as the amalgamated free product of two copies of $\mathbb{Z}$ over the trivial group:*

$$F = \mathbb{Z} *_1 \mathbb{Z} = \mathbb{Z} * \mathbb{Z}.$$

It is also the HNN-extension of $\mathbb{Z}$ over the trivial group:

$$F = \mathbb{Z} *_1 .$$

Example 4 *Free abelian groups only decompose in one way, namely as an HNN-extension of a codimension-one subgroup by itself:*

$$\mathbb{Z}^n = \mathbb{Z}^{n-1} *_{\mathbb{Z}^{n-1}} .$$

The homomorphisms $\alpha, \beta : \mathbb{Z}^{n-1} \to \mathbb{Z}^{n-1}$ are both taken to be the identity.

By the end of this talk we will see many more less trivial examples.

The topological perspective

Van Kampen's theorem provides the connection between these decompositions and topological decomposition.

Theorem 5 (Van Kampen) *Let X, Y, Z be path-connected topological spaces and $\alpha : Z \to X$, $\beta : Z \to Y$ π_1-injective continuous maps. Then the fundamental group of the double mapping cylinder of α, β is canonically isomorphic to the amalgamated free product*

$$\pi_1(X) *_{\pi_1(Z)} \pi_1(Y).$$

If $X = Y$ then the fundamental group of the mapping torus of $\alpha \circ \beta^{-1}$ is canonically isomorphic to the HNN-extension

$$\pi_1(X) *_{\pi_1(Z)} .$$

This gives many more examples of elementary splittings. The easiest are for surface groups.

Example 6 *The fundamental group of a surface has many elementary splittings, which can be seen by cutting along simple closed curves. (If the surface has boundary, the curves shouldn't be boundary-parallel.)*

If the curve is separating, the result is an amalgamated free product. If the curve is non-separating, the result is an HNN-extension.

Reduced words

The main technical tool for dealing with elementary splittings is the notion of a reduced word. Consider $G = A *_C B$. A word in the elements of A and B is *reduced* if it is of the form

$$a_1 b_1 a_2 \dots a_n b_n$$

where $a_i \in A, b_i \in B$, and moreover $a_i \notin C$ for $i > 1$ and $b_i \notin C$ for $i < n$.

Theorem 7 *Every element of $G = A *_C B$ can be represented as a reduced word. Moreover, the number n is unique.*

Here is a sketch of the proof

Choose sets of coset representatives for A/C and B/C. Let X be the set of reduced words of the form $ca_1b_1a_2\dots a_nb_n$ where $c \in C$ and a_i, b_i are chosen coset representatives. Then G acts on X by left-multiplication, and the resultant map $G \to X$ defined by

$$g \mapsto g.e$$

provides a left-inverse to the natural map $X \to G$; this gives the surjectivity of $X \to G$ and the uniqueness of the decomposition.

For the HNN-extension $G = A*_C$ a reduced word is of the form

$$a_0t^{\epsilon_1}a_1\dots a_{n-1}t^{\epsilon_n}a_n$$

where $\epsilon_i = \pm 1$; furthermore, if $\epsilon_i = -\epsilon_{i+1}$ then $a_i \notin C$. A similar theorem holds.

Graphs of groups

Definition 8 *Let* Γ *be an oriented connected finite graph. For each vertex* v *let* G_v *be a group. For each edge* e *let* G_e *be a group, and let*

$$\partial_e^{+1} : G_e \to G_{t(e)}$$

and

$$\partial_e^{-1} : G_e \to G_{s(e)}$$

be injective homomorphisms. The data

$$(\Gamma, G, \partial^{+1}, \partial^{-1})$$

defines a graph of groups. *We often abusively denote it just by* Γ.

Example 9 *The simplest non-trivial examples occur when* Γ *has just one edge. There are two cases:* Γ *has one vertex, and* Γ *has two vertices.*

Example 10 *Let T be an oriented graph on which a group G acts simplicially and cocompactly, without edge inversions. Let Γ be the quotient topological space, an oriented finite graph. For each vertex v of Γ choose a lift $\tilde{v} \in T$; set $G_v = \mathrm{Stab}_G(\tilde{v})$. This choice is well defined up to isomorphism, because $\mathrm{Stab}_G(g.\tilde{v}) = g\mathrm{Stab}(\tilde{v})g^{-1}$.*

For each edge e of Γ, choose a lift $\tilde{e}$ in T; set $G_e = \mathrm{Stab}_G(\tilde{e})$; again, this is well defined up to isomorphism. Moreover, the inclusions

$$\mathrm{Stab}_G(\tilde{e}) \hookrightarrow \mathrm{Stab}_G(t(\tilde{e})), \mathrm{Stab}_G(s(\tilde{e}))$$

induce injective homomorphisms

$$G_e \to G_{t(e)}, G_{s(e)}$$

which are well defined up to conjugation by an element of the vertex group; these are taken to define $\partial_e^{+1}, \partial_e^{-1}$, respectively.

The resultant graph of groups Γ is called the quotient *graph of groups; write $\Gamma = T/G$.*

The fundamental group

Let Γ be a graph of groups. Consider the group $F(\Gamma)$ generated by the vertex groups $\{G_v\}$, and the set of edges $\{e\}$ of Γ, subject to the following relations: if e is an edge and $g \in G_e$ then

$$e\partial_e^{+1}(g)e^{-1} = \partial_e^{-1}(g).$$

Let c be a path in Γ, combinatorially represented by the string

$$e_1^{\epsilon_1} \dots e_n^{\epsilon_n}.$$

A *word of type* c is an element of $F(\Gamma)$ of the form

$$g_0 e_1^{\epsilon_1} g_1 \cdots g_{n-1} e_n^{\epsilon_n} g_n$$

where $\epsilon_i = \pm 1$ and, furthermore, if $\epsilon_i = +1$ then $g_i \in G_{t(e)}$ and if $\epsilon_i = -1$ then $g_i \in G_{s(e)}$.

Fix a vertex v_0. Then $\pi_1(\Gamma, v_0)$ is the subgroup of $F(\Gamma)$ consisting of words whose type is a loop based at v_0.

Graphs of groups generalize elementary splittings.

Example 11 *Let* Γ *be a graph of groups with one edge and two vertices; let* A, B *be the vertex groups and* C *the edge group. Then*

$$\pi_1(\Gamma) \cong A *_C B.$$

The isomorphism is given by forgetting the edge elements.

Example 12 *Let* Γ *be a graph of groups with one edge and one vertex; let* A *be the vertex group and* C *the edge group. Then*

$$\pi_1(\Gamma) \cong A *_C .$$

The isomorphism is given by mapping the edge element to the stable letter of the HNN-extension.

An alternative description

It will be useful to be able to think about the fundamental group in a second way. Fix Θ a maximal tree of Γ; let E be the set of edges in $\Gamma - \Theta$. Then the *fundamental group of* Γ *relative to* Θ, denoted $\pi_1(\Gamma, \Theta)$, is the group generated by the vertex groups $\{G_v\}$ and the edges in E, with the following additional relations: if $e \in E$ and $g \in G_e$ then

$$e\partial_e^{+1}(g)e^{-1} = \partial_e^{-1}(g).$$

Note that there is a natural map $F(\Gamma) \to \pi_1(\Gamma, \Theta)$ given by mapping edges in Θ to the identity.

Lemma 13 *Let* Γ *be a graph of groups, let* v_0 *be a vertex, and let* Θ *be a maximal tree. Then the induced map*

$$\pi_1(\Gamma, v_0) \to \pi_1(\Gamma, \Theta).$$

is an isomorphism.

Dévissage

The first aim is to prove a reduced word theorem for graphs of groups. Here is a very useful technical lemmas.

Lemma 14 *Let* Γ *be a graph of groups, and let* Γ' *a subgraph. Let* Δ *be the graph of groups defined by contracting* Γ' *to a single vertex* v *and setting* $G_v = \pi_1(\Gamma')$*. Then the natural map*

$$F(\Gamma) \to F(\Delta)$$

is an isomorphism.

Using dévissage allows results about graphs of groups to be proved by induction.

Reduced words in graphs of groups

A word of type c as above, of the form

$$g_0 e_1^{\epsilon_1} g_1 \cdots g_{n-1} e_n^{\epsilon_n} g_n,$$

is *reduced* if the following two conditions hold:

- if $n = 0$ then $g_0 \neq 1$;
- if $n > 0$, whenever $e_i = e_{i+1}$ and $\epsilon_i = -\epsilon_{i+1}$, $g_i \notin \partial_e^{\epsilon_i}(G_e)$.

Note that this notion of being reduced coincides with the notions of being reduced in the one-edge cases. Note also that every element of $\pi_1(\Gamma)$ can be represented by a reduced word.

Theorem 15 *Consider a reduced word* w *in* $F(\Gamma)$ *of type* c*, where* c *is a path. Then* w *is not the identity.*

Sketch of proof: The theorem follows from the observations that the inclusion map

$$F(\Gamma') \to F(\Gamma)$$

and the dévissage map

$$F(\Gamma) \to F(\Delta)$$

preserve the property of being reduced.

We already know the special case of amalgamated free products; dévissage upgrades this to trees of groups, by induction.

We also already know the special case of HNN-extensions; dévissage upgrades this to roses of groups, by induction.

Using dévissage to contract a maximal tree in an arbitrary graph of groups gives a rose of groups; the theorem follows.

Developing graphs of groups

Theorem 16 *Let* Γ *be a graph of groups, let* Θ *be a maximal tree, and set* $G = \pi_1(\Gamma, \Theta)$. *Then there exists a tree* T *on which* G *acts without edge inversions, and*

$$\Gamma = T/G.$$

It's not hard to see what the vertices and edges of this tree must be. Let $q : T \to \Gamma$ be the quotient map, and suppose $\tilde{x}$ is a vertex or an edge of T. Then

$$\mathrm{Stab}_G(\tilde{x}) = gG_{q(\tilde{x})}g^{-1}$$

for some $g \in G$, by construction. Therefore take as vertices the set

$$\bigsqcup_v G_v \backslash G$$

of left-cosets of vertex groups. Likewise take as edges the set

$$\bigsqcup_e G_e \backslash G$$

of left-cosets of edge groups.

The graph T is defined by attaching the edges to the vertices according to the following formulae:

$$t(gG_e) = geG_{t(e)}$$

and

$$s(gG_e) = gG_{s(e)}.$$

We are thinking of G as the fundamental group relative to a maximal tree Θ; in particular, the element e is the identity if e lies in Θ.

There is an obvious map $T \to \Gamma$ given by

$$gG_x \mapsto x$$

for x a vertex or an edge. This descends to an isomorphism of graphs; the edge groups and vertex groups are isomorphic, and the ∂-maps are equivalent up to conjugation by an element of a vertex group. It remains to see that T is a tree.

First we show T is connected. For every edge e in Θ, there is an edge in T joining the vertices $G_{t(e)}$ and $G_{s(e)}$; therefore Θ lifts to T, and for any pair of vertices u, v of Γ, G_u and G_v are joined in T.

Consider a vertex of the form gG_v of T, for $g \in G_u$. Since G_v is joined to G_u it follows that gG_v is joined to $gG_u = G_u$.

Now consider a vertex of the form eG_v for e an edge. Then eG_v is joined to $eG_{t(e)} = t(G_e)$, and so to $G_{s(e)}$.

But the elements of the vertex groups and the edges generate $\pi_1(\Gamma, \Theta)$; applying this argument inductively gives the connectedness of T.

We now show that T is simply-connected. Consider a path in T, beginning at G_{v_0}. It first crosses an edge $g_0 G_{e_1}$ (for some $g_0 \in G_{v_0}$, $s(e_1) = v_0$) to $g_0 e_1^{\epsilon_1} G_{v_1}$ where $v_1 = t(e_1)$. Repeating this process inductively gives a representation of the final vertex of the loop as the coset

$$g_0 e_1^{\epsilon_1} g_1 \ldots e_n^{\epsilon_n} G_{v_n}.$$

The loop backtracks if and only if, for some i,

$$g_0 e_1^{\epsilon_1} g_1 \ldots e_i^{\epsilon_i} G_{v_i} = g_0 e_1^{\epsilon_1} g_1 \ldots e_i^{\epsilon_i} g_i e_{i+1} G_{v_{i+1}}.$$

This only happens if $e_i = e_{i+1}$, $\epsilon_i = -\epsilon_{i+1}$, and $g_i \in \partial_{e_i}^{\epsilon_i} G_{e_i}$. In other words, backtracking occurs if and only if the word is not reduced.

Now the claim that the path is a loop is equivalent to asserting that

$$g_0 e_1^{\epsilon_1} g_1 \ldots e_n^{\epsilon_n} = g_n \in G_{v_0}$$

or equivalently

$$g_0 e_1^{\epsilon_1} g_1 \ldots e_n^{\epsilon_n} g_n^{-1} = 1.$$

But this is a reduced word, so that can't happen.

Application: $SL_2(\mathbb{Z}) \cong \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$

Recall that $\mathrm{Isom}(\mathbb{H}^2) \cong PSL_2(\mathbb{R})$, so $SL_2(\mathbb{Z})$ acts on $\mathbb{H}^2$ in a natural way, with kernel $\{\pm 1\}$. $G = PSL_2(\mathbb{Z})$ is generated by

$$z \mapsto z + 1$$

and

$$z \mapsto \frac{-1}{z}$$

and a fundamental domain for the action is given by

$$\{|z| \geq 1, |\mathrm{Re} z| \leq \frac{1}{2}\}.$$

The translates of the segment $[i, e^{\frac{i\pi}{3}}]$ form a tree, on which G acts without edge inversions. It is easy to check that $\mathrm{Stab}_G(i)$ is generated by $z \mapsto \frac{-1}{z}$ and is of order 2, while $\mathrm{Stab}_G(e^{\frac{i\pi}{3}})$ is generated by $z \mapsto 1 - \frac{1}{z}$ and is of order 3.

This action on the tree gives a decomposition of G as $\mathbb{Z}/2 * \mathbb{Z}/3$, and of $SL_2(\mathbb{Z})$ as $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$.