Stochastic Analysis with modeled distributions

Josef Teichmann (joint work with David Prömel, Berlin)

ETH Zürich

London, March 2015

Regularity Structures

Regularity structures have been invented by Martin Hairer to provide solution concepts for SPDEs like the KPZ-equation, Φ^4_3 equation, parabolic Anderson model, etc.

The theory is based on an ingenuous algebraic interpretation of (sub-critical) SPDEs. The bridge between the algebraic and analytic world is done by the so called reconstruction operator $\mathcal{R}$. The existence of the reconstruction operator is a beautiful result from wavelet analysis.

Regularity structures come with precise numerical approximation results and are therefore very useful to establish numerical techniques.

KPZ equation

Just to motivate that there is interest from the point of view of mathematical Finance: the Kardar-Parisi-Zhang (KPZ) equation

$$\partial_t u = -\frac{1}{2}(\partial_x u)^2 + \partial_x^2 u + \xi$$

is considered on two dimensional space time. ξ is a space time white noise, i.e. a Gaussian process with vanishing expectation and covariance $\delta(t-s)\delta(x-y)$. The Cole-Hopf transform (under regularity assumptions) $z = \exp(-u)$ leads to a stochastic heat equation with multiplicative noise

$$\partial_t z = \partial_x^2 z + z\xi \,,$$

which is better understood stochastically. From the point of view of mathematical Finance this corresponds to switching between log prices and prices. z could model a continuity of prices, which are driven by independent Brownian motion but still strongly correlated through the drift containing the Laplace.

Stochastic Models for rough phenomena

In several occasions in Finance it might make sense to consider stochastic processes which actually take value in rough objects:

- Forward rates with rough short end (badly defined or singular) short rate.
- Order book dynamics without smoothing the order books.
- Modeling of continua of stocks with drift interaction.

Philosophy of regularity structures

Regularity structures such as rough path theory are deterministic approaches to dealing with rough multi-variate objects, for instance with respect to rough integrators:

- Stochastic Calculus deals with integration against rough paths of semi-martingales by martingale arguments for predictable integrands. Integrals are only measurable with respect to input trajectories.
- Rough path theory deals with integration against rough paths by expansions (up to appropriate orders) for appropriate possibly anticipating integrands. Integrals are continuous with respect to rough paths objects.

It might be interesting to combine both approaches.

The model space

Let $A \subset \mathbb{R}$ be an index set, bounded from below and without accumulation point, and let $T = \bigoplus_{\alpha \in A} T_\alpha$ be a direct sum of Banach spaces T_α graded by A. Let furthermore G be a group of linear operators on T such that, for every $\alpha \in A$, every $\Gamma \in G$, and every $\tau \in T_\alpha$, one has $\Gamma\tau - \tau \in \bigoplus_{\beta<\alpha} T_\beta$.

The triple $\mathcal{T} = (A, T, G)$ is called a *regularity structure* with *model space* T and *structure group* G.

Given $\tau \in T$, we will write $\|\tau\|_\alpha$ for the norm of its T_α-projection.

Meaning: $\mathcal{T}$ represent abstractly expansions of “functions” at some space-time point in terms of “model functions” of some regularity α.

Some notation

Given a test function ϕ on $\mathbb{R}^d$, we write ϕ_x^λ as a shorthand for

$$\phi_x^\lambda(y) = \lambda^{-d}\phi\big(\lambda^{-1}(y-x)\big) \ .$$

Given $r > 0$, we denote by B_r the set of all functions $\phi \colon \mathbb{R}^d \to \mathbb{R}$ with $\phi \in C^r$, its norm $\|\phi\|_{C^r} \leq 1$ and supported in the unit ball around the origin.

Models

Given a regularity structure $\mathcal{T}$ and an integer $d \geq 1$, a *model* for $\mathcal{T}$ on $\mathbb{R}^d$ consists of maps

$$\Pi \colon \mathbb{R}^d \to L\big(T, \mathcal{D}'(\mathbb{R}^d)\big) \qquad\qquad \Gamma \colon \mathbb{R}^d \times \mathbb{R}^d \to G$$
$$x \mapsto \Pi_x \qquad\qquad (x,y) \mapsto \Gamma_{xy}$$

such that $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$ and $\Pi_x \Gamma_{xy} = \Pi_y$. Furthermore, given $r > |\inf A|$, for any compact set $K \subset \mathbb{R}^d$ and constant $\gamma > 0$, there exists a constant C such that the inequalities

$$\big|\big(\Pi_x \tau\big)(\phi_x^\lambda)\big| \leq C \lambda^{|\tau|} \|\tau\|_\alpha \;, \qquad \|\Gamma_{xy}\tau\|_\beta \leq C|x-y|^{\alpha-\beta}\|\tau\|_\alpha \;,$$

hold uniformly over $\phi \in B_r$, $(x,y) \in K$, $\lambda \in]0,1]$, $\tau \in T_\alpha$ with $\alpha \leq \gamma$, and $\beta < \alpha$.

Modeled distributions

The regularity structure allows to speak about abstract expansions at some space time point. We can now introduce spaces of expansion coefficients and under which conditions we can actually associate a generalized function to such coefficient functions.

Given a regularity structure $\mathcal{T}$ equipped with a model (Π, Γ) over $\mathbb{R}^d$, the space $\mathcal{D}^\gamma = \mathcal{D}^\gamma(\mathcal{T}, \Gamma)$ is given by the set of functions $f \colon \mathbb{R}^d \to \bigoplus_{\alpha<\gamma} T_\alpha$ such that, for every compact set K and every $\alpha < \gamma$, the exists a constant C with

$$\|f(x) - \Gamma_{xy} f(y)\|_\alpha \le C|x-y|^{\gamma-\alpha}$$

uniformly over $x, y \in K$.

Reconstruction operator

The most fundamental result in the theory of regularity structures then states that given a coefficient function $f \in \mathcal{D}^\gamma$ with $\gamma > 0$, there exists a *unique* distribution $\mathcal{R}f$ on $\mathbb{R}^d$ such that, for every $x \in \mathbb{R}^d$, $\mathcal{R}f$ equals $\Pi_x f(x)$ near x up to order γ. More precisely, one has the following reconstruction theorem, whose proof relies on deep results from wavelet analysis (multi-resolution analysis, see Martin Hairer's Inventiones article [Hairer(2014)]).

Let $\mathcal{T}$ be a regularity structure and let (Π, Γ) be a model for $\mathcal{T}$ on $\mathbb{R}^d$. Then, there exists a unique linear map $\mathcal{R} \colon \mathcal{D}^\gamma \to \mathcal{D}'(\mathbb{R}^d)$ such that

$$\big|\big(\mathcal{R}f - \Pi_x f(x)\big)(\phi_x^\lambda)\big| \lesssim \lambda^\gamma \,, \tag{1}$$

uniformly over $\phi \in B_r$ and $\lambda \in]0,1]$, and locally uniformly in $x \in \mathbb{R}^d$.

How does it work?

Take the Φ^4_3-equation

$$\partial_t u = \Delta u - u^3 + \xi$$

on $\mathbb{R}_{\geq 0} \times \mathbb{R}^3$ (with periodic boundary conditions in space). ξ is here a space-time white noise, then we obtain the mild formulation through convolution with the Green's function G

$$u = G * (\xi - u^3) + G * u_0 \, .$$

This equation is translated into a fixed point equation on the coefficient space $\mathcal{D}^\gamma$ with respect to taylor-made regularity structure $\mathcal{T}$, i.e.

$$\Phi = \mathcal{I}(\Xi - \Phi^3) + \text{ polynomials}$$

which can be solved for smooth enough Ξ. Re-construction then yields a solution of the Φ^4_3-equation with mollified noise and initial value u_0.

If the mollified noise converges to white noise, one hast to re-normalize the models (which are necessary to define the reconstruction operator) in order to guarantee the convergence of the models. This solutions concept does not depend on the chosen mollification and satisfies all necessary requirements, hence deserves to be called *the* solution of the Φ^4_3-equation.

Another topology on $\mathcal{D}^\gamma$

From a probabilist's point of view it seems much more desirable to work with $\mathcal{L}^p$-norms instead of $\mathcal{L}^\infty$-norms. We introduce analogously to the classical Sobolev-Slobodeckij spaces, but now for a function $f\colon \mathbb{R}^d \to T_\gamma^-$, the norms

$$\|f\|_{\beta,\mathcal{K},p} := \left(\int_{\mathcal{K}} \|f(x)\|_\beta^p \mathrm{d}x \right)^{\frac{1}{p}}$$

and

$$f_{\gamma,p,\mathcal{K}} := \left(\sum_{\alpha<\gamma} \|f\|^p_{\alpha,\overline{\mathcal{K}},p} \right)^{\frac{1}{p}} + \\ + \left(\sum_{\alpha<\gamma} \int_{\overline{\mathcal{K}}} \int_{\mathcal{B}^1(0)} \left(\frac{\|f(x+h) - \Gamma_{x+h,x} f(x)\|_\alpha}{\|h\|^{\gamma-\alpha+d/p}} \right)^p \mathrm{d}h\,\mathrm{d}x \right)^{\frac{1}{p}},$$

for $p \geq 1$, $\gamma > 0$ and a compact set $\mathcal{K} \subset \mathbb{R}^d$. Here $\overline{\mathcal{K}}$ stand for the 1-fattening and we shortened the notation by just writing $\sum_{\alpha<\gamma}$ with the meaning $\sum_{\alpha\in A_\gamma}$.

Definition

Let $\mathcal{K} \subset \mathbb{R}^d$ be a compact set, $p \geq 1$, and $\gamma > 0$. By $\mathcal{D}_p^\gamma$ we denote the space of all function $f : \mathbb{R}^d \to T_\gamma^-$ such that $f_{\gamma,p,\mathcal{K}} < \infty$.

It is easy to check that $\mathcal{D}^{\gamma_1} \subset \mathcal{D}^{\gamma_2}$ and $\mathcal{D}_p^{\gamma_1} \subset \mathcal{D}_p^{\gamma_2}$ for $\gamma_1 \geq \gamma_2$ and $p \in [1, \infty)$.

Let $\gamma > 0$, $p \in [1, \infty)$ and $\mathcal{K} \subset \mathbb{R}^d$ be a compact set. Suppose that $\gamma - \max\{\alpha \in A : \alpha < \gamma\} > d/p$. Then, $\mathcal{D}^{\gamma+d/p}(\mathcal{K})$ embeds continuously into $\mathcal{D}_p^\gamma(\mathcal{K})$.

Reconstruction Theorem

Let (A, T, G) be a regularity structure, (Π, Γ) a model on $\mathbb{R}^d$ and let $r > |\alpha_0| + d/p$ for $\alpha_0 := \inf A$ and $p \geq 1$. Then for every $\gamma > 0$ there exits a linear map $\mathcal{R}\colon \mathcal{D}_p^\gamma \to \mathcal{D}'$ such that for every compact set $\mathcal{K} \subset \mathbb{R}^d$ it holds

$$|\langle \mathcal{R}f - \Pi_x f(x), S_x^\delta \eta \rangle| \lesssim \delta^{\gamma - d/p} \|\Pi\|_{\gamma,\mathcal{K}} f_{\gamma,p,\mathcal{K}} \tag{2}$$

uniformly over all test function $\eta \in \mathcal{B}_0^r$, all $\delta \in (0, 1]$, all $x \in \mathcal{K}$ all $f \in \mathcal{D}_p^\gamma$. Moreover, if $\alpha_0 - d/p \notin \mathbb{Z}$, then $\mathcal{R}f \in \mathcal{C}^{\alpha_0 - d/p}$ for every $f \in \mathcal{D}_p^\gamma$. If $\gamma > d/p$, then the linear map $\mathcal{R}$ is unique.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $I \subset \mathbb{R}$, $(\mathcal{F}_t)_{t\in I}$ be an increasing family of sub-σ-algebra of $\mathcal{F}$ and X be a Banach space with norm $\|\cdot\|_x$. A process $(M_t)_{t\in I}$ is a X*-valued martingale* if and only if $M_t \in L^1(\Omega, \mathcal{F}_t, \mathbb{P}; X)$ for all $t \in I$ and

$$\mathbb{E}[M_t|\mathcal{F}_s] = M_s \quad a.s., \text{ for all } s, t \in I \text{ with } s \leq t.$$

A sequence $(\xi_i)_{i\in\mathbb{N}}$ is called *martingale difference* if $(\sum_{i=0}^{n} \xi_i)_{n\in\mathbb{N}}$ is a X-valued martingale. To rely on a high developed stochastic integration theory on Banach spaces, one needs to require some additional properties on the Banach space X. The definitions are taken from [Brzeźniak(1995)].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

- A Banach space X is of *M-type p* for $p \in [1,2]$ if any X-valued martingale $(M)_{n\in\mathbb{N}}$ satisfies

$$\sup_n \mathbb{E}[\|M_n\|_x^p] \leq C_p(X) \sum_{n\in\mathbb{N}} \mathbb{E}[\|M_{n+1} - M_n\|_X^p]$$

for some constant $C_p(X) > 0$ independent of the martingale $(M_n)_{n\in\mathbb{N}}$.

- A Banach spaces X if of *type p* for $p \in [1,2]$ if any finite sequence $\epsilon_1, ..., \epsilon_n \colon \Omega \to \{-1, 1\}$ of symmetric and i.i.d. random variables and for any finite sequence $x_1, ..., x_n$ of elements of X the inequality

$$\mathbb{E}\left[\left\| \sum_{i=1}^n \epsilon_i x_i \right\|_x^p\right] \leq K_p(X) \sum_{i=1}^n \|x_i\|_x^p$$

holds for some constant $K_p(X) > 0$.

A Banach space X is called a *UMD space* or is said to have the *unconditional martingale property* if for any $p \in (1, \infty)$, for any martingale difference $(\xi_j)_{j \in \mathbb{N}}$ and for any sequence $(\epsilon_i)_{i \in \mathbb{N}} \subset \{-1, 1\}$ the inequality

$$\mathbb{E}\left[\left\| \sum_{i=1}^{n} \epsilon_i \xi_j \right\|_x^p\right] \leq \tilde{K}_p(X) \mathbb{E}\left[\left\| \sum_{i=1}^{n} \xi_i \right\|_x^p\right]$$

holds for all $n \in \mathbb{N}$, where $\tilde{K}_p(X) > 0$ is some constant.

Note that Hilbert spaces and finite dimensional Banach spaces are always UMD spaces. Furthermore, suppose each T_α is a UMD space: Since A is locally finite and the finite product of UMD spaces, the spaces $T_\gamma^- = \bigoplus_{\alpha < \gamma} T_\alpha$ is again a UMD space.

Theorem

Let $\gamma > 0$. Suppose every Banach spaces T_γ^- is a UMD space and of type 2. Then, the space $\mathcal{D}_p^\gamma$ is of M-type 2 for every $p \geq 2$.

[Brzeźniak(1995)] Zdzisław Brzeźniak.

Stochastic partial differential equations in M-type 2 Banach spaces.

Potential Anal., 4(1):1–45, 1995.

ISSN 0926-2601.

[Hairer(2014)] M. Hairer.

A theory of regularity structures.

Invent. Math., 198(2):269–504, 2014.