A class of approximate Greek weights

Imperial-ETH Workshop on Mathematical Finance,
Imperial College London

Ivo Mihaylov

5 March 2015

Content

1. Overview
2. Weights H
3. Weak Taylor schemes
4. Δ
5. Heston Δ
6. Γ

Asset price dynamics

- Process $X = (X_t)_{t\geq 0}$ take values in $\mathbb{R}$, with dynamics described by the SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \ , \quad X_0 = x \in \mathbb{R} \ , \tag{1}$$

where $W = (W_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}$.

Asset price dynamics

- Process $X = (X_t)_{t\geq 0}$ take values in $\mathbb{R}$, with dynamics described by the SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \ , \quad X_0 = x \in \mathbb{R} \ , \qquad (1)$$

where $W = (W_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}$.

- Fix number of time steps $n \in \mathbb{N}^+$ and a time horizon $T > 0$.

Asset price dynamics

- Process $X = (X_t)_{t \geq 0}$ take values in $\mathbb{R}$, with dynamics described by the SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \ , \quad X_0 = x \in \mathbb{R} \ , \tag{1}$$

where $W = (W_t)_{t \geq 0}$ is a Brownian motion in $\mathbb{R}$.

- Fix number of time steps $n \in \mathbb{N}^+$ and a time horizon $T > 0$.
- Define a partition on the interval $[0, T]$ by

$$\pi := \{0 = t_0 < t_1 < \ldots < t_n = T\}.$$

Option Price and Greeks

- Let g be a function of process X at terminal time T.

Option Price and Greeks

- Let g be a function of process X at terminal time T.
- **Option price:** $V(x)$, given the initial condition $X_0 = x$:

$$V(x) := \mathbb{E}\left[g(X_{t_n})|X_0 = x\right].$$

Option Price and Greeks

- Let g be a function of process X at terminal time T.
- **Option price:** $V(x)$, given the initial condition $X_0 = x$:

$$V(x) := \mathbb{E}\left[g(X_{t_n})|X_0 = x\right].$$

- **Greeks:** sensitivities of option price.

Option Price and Greeks

- Let g be a function of process X at terminal time T.
- **Option price:** $V(x)$, given the initial condition $X_0 = x$:

$$V(x) := \mathbb{E}\left[g(X_{t_n})|X_0 = x\right].$$

- **Greeks:** sensitivities of option price.
- Δ: sensitivity w.r.t. to x using a central-difference

$$\Delta_{C,h} := \frac{V(x+h) - V(x-h)}{2h}.$$

Setting

- Recall SDE (1). Value function $u : [0, T] \times \mathbb{R} \to \mathbb{R}$ is such that

$$\begin{aligned} L^{(0)} u(t, X_t) &= 0 \qquad \text{for } t \in [0, T), \\ u(T, \cdot) &= g(\cdot), \end{aligned} \tag{2}$$

where the operators are defined as

$$\begin{aligned} L^{(0)} &:= \partial_t + \mu(x)\partial_x + \frac{1}{2}\sigma(x)^2 \partial_x^2 \\ L^{(1)} &:= \sigma(x)\partial_x . \end{aligned}$$

Setting

- Recall SDE (1). Value function $u : [0, T] \times \mathbb{R} \to \mathbb{R}$ is such that

$$\begin{aligned} L^{(0)} u(t, X_t) &= 0 \qquad \text{for } t \in [0, T), \\ u(T, \cdot) &= g(\cdot), \end{aligned} \tag{2}$$

where the operators are defined as

$$\begin{aligned} L^{(0)} &:= \partial_t + \mu(x)\partial_x + \frac{1}{2}\sigma(x)^2\partial_x^2 \\ L^{(1)} &:= \sigma(x)\partial_x. \end{aligned}$$

- Assumption on the smoothness of the value function u imposed.

Aim

Work with approximations $\hat{X} = (\hat{X}_t)_{t\in[0,T]}$ using grid π, where $h := |\pi| := T/n$.

Aim

Work with approximations $\hat{X} = (\hat{X}_t)_{t\in[0,T]}$ using grid π, where $h := |\pi| := T/n$.

1. Find weights H such that for a general diffusion X:

$$\text{Greek} = \mathbb{E}[Hg(\hat{X}_T)] + \mathcal{O}(h^l),$$

where H is some $\mathcal{F}_h$-measurable weight.

Aim

Work with approximations $\hat{X} = (\hat{X}_t)_{t\in[0,T]}$ using grid π, where $h := |\pi| := T/n$.

1. Find weights H such that for a general diffusion X:

$$\text{Greek} = \mathbb{E}[Hg(\hat{X}_T)] + \mathcal{O}(h^l),$$

   where H is some $\mathcal{F}_h$-measurable weight.

2. Control MSE for convergence results of the Greek approximations.

Aim

Work with approximations $\hat{X} = (\hat{X}_t)_{t\in[0,T]}$ using grid π, where $h := |\pi| := T/n$.

1. Find weights H such that for a general diffusion X:

$$\text{Greek} = \mathbb{E}[Hg(\hat{X}_T)] + \mathcal{O}(h^l),$$

   where H is some $\mathcal{F}_h$-measurable weight.
2. Control MSE for convergence results of the Greek approximations.
3. Higher order schemes and extrapolation techniques.

Theoretical Coefficients H^{ψ} [CC14]

- Fix $l \in \mathbb{N}$. Define $\mathcal{B}^{l}_{[0,1]}$ as the set of bounded, measurable functions $\psi : [0,1] \to \mathbb{R}$ such that

$$\int_0^1 \psi(s) ds = 1,$$

$$\int_0^1 \psi(s) s^k ds = 0, \text{if } l \in \mathbb{N}^+, \forall 1 \leq k \leq l.$$

Theoretical Coefficients H^{ψ} [CC14]

- Fix $l \in \mathbb{N}$. Define $\mathcal{B}^{l}_{[0,1]}$ as the set of bounded, measurable functions $\psi : [0,1] \to \mathbb{R}$ such that

$$\int_0^1 \psi(s)\mathrm{d}s = 1,$$

$$\int_0^1 \psi(s)s^k \mathrm{d}s = 0, \text{if } l \in \mathbb{N}^+, \forall 1 \leq k \leq l.$$

- Define weights $H^{\psi}_{\cdot}$ to approximate the Δ:

Definition 1 (H^{ψ}_h-functionals)

Let $\psi \in \mathcal{B}^{l}_{[0,1]}$, and for $0 < h \leq T$, define H^{ψ}_h as

$$H^{\psi}_h := \frac{1}{h} \int_{s=0}^{h} \psi\left(\frac{s}{h}\right) \mathrm{d}W_s.$$

Examples of $\psi \in \mathcal{B}^l_{[0,1]}$ and H^ψ_h

0. $l = 0$: $\psi \equiv 1 \in \mathcal{B}^0_{[0,1]}$, and weight $H^\psi_h := W_h/h$.

Examples of $\psi \in \mathcal{B}^l_{[0,1]}$ and H^ψ_h

0. $l = 0$: $\psi \equiv 1 \in \mathcal{B}^0_{[0,1]}$, and weight $H^\psi_h := W_h/h$.
1. $l = 1$: $\mathcal{B}^1_{[0,1]}$
   (a) Linear equation $\psi_{p,1}(u) \equiv 4 - 6u$.

$$H_h^{\psi_{p,1}} = \frac{4}{h} W_h - \frac{6}{h^2} \int_0^h s \mathrm{d} W_s.$$

Examples of $\psi \in \mathcal{B}^l_{[0,1]}$ and H^ψ_h

0. $l = 0$: $\psi \equiv 1 \in \mathcal{B}^0_{[0,1]}$, and weight $H^\psi_h := W_h/h$.
1. $l = 1$: $\mathcal{B}^1_{[0,1]}$
   (a) Linear equation $\psi_{p,1}(u) \equiv 4 - 6u$.

$$H^{\psi_{p,1}}_h = \frac{4}{h} W_h - \frac{6}{h^2} \int_0^h s \mathrm{d} W_s.$$

   (b) Fix $c \in (0,1)$, the function $\psi_{s,1}(u) \equiv \frac{1}{c(c-1)} \mathbb{1}_{[1-c,1]}(u) + \frac{c-2}{c-1}$.

$$H^{\psi_{s,1}}_h = \frac{c-1}{c} \frac{W_h}{h} + \frac{1}{c} \frac{W_{h(1-c)}}{h(1-c)}.$$

Examples of $\psi \in \mathcal{B}^l_{[0,1]}$ and H^ψ_h

0. $l = 0$: $\psi \equiv 1 \in \mathcal{B}^0_{[0,1]}$, and weight $H^\psi_h := W_h/h$.
1. $l = 1$: $\mathcal{B}^1_{[0,1]}$
   (a) Linear equation $\psi_{p,1}(u) \equiv 4 - 6u$.

$$H^{\psi_{p,1}}_h = \frac{4}{h} W_h - \frac{6}{h^2} \int_0^h s \mathrm{d} W_s.$$

   (b) Fix $c \in (0,1)$, the function $\psi_{s,1}(u) \equiv \frac{1}{c(c-1)} \mathbb{1}_{[1-c,1]}(u) + \frac{c-2}{c-1}$.

$$H^{\psi_{s,1}}_h = \frac{c-1}{c} \frac{W_h}{h} + \frac{1}{c} \frac{W_{h(1-c)}}{h(1-c)}.$$

2. $l = 2$: the unique quadratic belonging to $\mathcal{B}^2_{[0,1]}$ is $\psi_{p,2}(u) \equiv 9 - 36u + 30u^2$.

- The variance of the weights $H_h^{\psi_\cdot,l}$ grows with the order l.

- The variance of the weights $H_h^{\psi_{\cdot},l}$ grows with the order l.
- Polynomial weights have slightly lower variance than step functions.

- The variance of the weights $H_h^{\psi_{\cdot},l}$ grows with the order l.
- Polynomial weights have slightly lower variance than step functions.
- If X can be perfectly simulated, $\psi \equiv 1 \in \mathcal{B}^0_{[0,1]}$ (i.e. order $l = 0$) recovers the Malliavin Δ weight $H_T = W_T/(xT\sigma)$.

- The variance of the weights $H_h^{\psi_\cdot, l}$ grows with the order l.
- Polynomial weights have slightly lower variance than step functions.
- If X can be perfectly simulated, $\psi \equiv 1 \in \mathcal{B}^0_{[0,1]}$ (i.e. order $l = 0$) recovers the Malliavin Δ weight $H_T = W_T/(xT\sigma)$.
- Family of weights used in the BSDE literature to approximate the Z process (i.e. $Z_t = \sigma(X_t)\partial_x u(t, X_t)$), which contains the Δ [CC14].

- The variance of the weights $H_h^{\psi_{\cdot},l}$ grows with the order l.
- Polynomial weights have slightly lower variance than step functions.
- If X can be perfectly simulated, $\psi \equiv 1 \in \mathcal{B}^0_{[0,1]}$ (i.e. order $l = 0$) recovers the Malliavin Δ weight $H_T = W_T/(xT\sigma)$.
- Family of weights used in the BSDE literature to approximate the Z process (i.e. $Z_t = \sigma(X_t)\partial_x u(t, X_t)$), which contains the Δ [CC14].

Lemma 2 ([CC14, Proposition 2.4])

For $\psi \in \mathcal{B}^l_{[0,1]}$ and value function sufficiently smooth,

$$\mathbb{E}\left[H_h^{\psi} g(X_T)\right] = L^{(1)}u(0,x) + \mathcal{O}(h^{l+1}),$$

where $L^{(1)}u(0,x) = \sigma(x)\Delta$ (i.e. expression containing the Δ).

Weak Taylor schemes

$\hat{X}_{t_0} = x$. For $i = 1, \ldots, n-1$, define

$$h_{i+1} := t_{i+1} - t_i, \qquad \Delta W_{i+1} := \int_{t_i}^{t_{i+1}} \mathrm{d}W_s, \qquad \Delta Z_{i+1} := \int_{t_i}^{t_{i+1}} W_s \mathrm{d}s.$$

Weak Taylor schemes

$\hat{X}_{t_0} = x$. For $i = 1, \ldots, n-1$, define

$$h_{i+1} := t_{i+1} - t_i, \qquad \Delta W_{i+1} := \int_{t_i}^{t_{i+1}} \mathrm{d}W_s, \qquad \Delta Z_{i+1} := \int_{t_i}^{t_{i+1}} W_s \mathrm{d}s.$$

1. Euler scheme (weak Taylor scheme order 1).

$$\hat{X}_{t_{i+1}} := \hat{X}_{t_i} + \mu(\hat{X}_{t_i}) h_{i+1} + \sigma(\hat{X}_{t_i}) \Delta W_{i+1}.$$

Weak Taylor schemes

$\hat{X}_{t_0} = x$. For $i = 1, \ldots, n-1$, define

$$h_{i+1} := t_{i+1} - t_i, \qquad \Delta W_{i+1} := \int_{t_i}^{t_{i+1}} \mathrm{d}W_s, \qquad \Delta Z_{i+1} := \int_{t_i}^{t_{i+1}} W_s \mathrm{d}s.$$

1. Euler scheme (weak Taylor scheme order 1).

$$\hat{X}_{t_{i+1}} := \hat{X}_{t_i} + \mu(\hat{X}_{t_i}) h_{i+1} + \sigma(\hat{X}_{t_i}) \Delta W_{i+1}.$$

2. Weak Taylor scheme of order 2

$$\begin{aligned}
\hat{X}_{t_{i+1}} := \text{Euler} &+ \frac{1}{2}\sigma(\hat{X}_{t_i})\sigma'(\hat{X}_{t_i})\left((\Delta W_{i+1})^2 - h_{i+1}\right) \\
&+ \mu'(\hat{X}_{t_i})\sigma(\hat{X}_{t_i})\Delta Z_{i+1} + \frac{1}{2}\left(\mu(\hat{X}_{t_i})\mu'(\hat{X}_{t_i}) + \frac{1}{2}\mu''(\hat{X}_{t_i})\sigma^2(\hat{X}_{t_i})\right) h_{i+1}^2 \\
&+ \left(\mu(\hat{X}_{t_i})\sigma'(\hat{X}_{t_i}) + \frac{1}{2}\sigma''(\hat{X}_{t_i})\sigma^2(\hat{X}_{t_i})\right)\left(\Delta W_{i+1} h_{i+1} - \Delta Z_{i+1}\right).
\end{aligned}$$

Euler scheme

- On $[0, h]$, the Euler scheme is a BM with drift $f(y)$ diffusion $\sigma(y)$ if the process X starts at y at time $t = 0$, i.e.

$$\hat{X}_h = y + \mu(y)h + \sigma(y)\sqrt{h}Z,$$

for some $Z \sim \mathrm{N}(0, 1)$.

- Define the operators $\hat{L}_y^{(j)}$, $j = 0, 1$ associated to this process:

Definition 3 (Fixed space operators)

For function $\varphi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and some $y \in \mathbb{R}$, define the operators $\hat{L}_y^{(j)}$ on φ by

$$\{\hat{L}_y^{(1)}\varphi\}(t, x) := \sigma(y)\partial_x\varphi(t, x),$$
$$\{\hat{L}_y^{(0)}\varphi\}(t, x) := \left(\partial_t + \mu(y)\partial_x + \frac{1}{2}\hat{L}_y^{(1)} \circ \hat{L}_y^{(1)}\right)\varphi(t, x),$$

where ∂_t and ∂_x are partial derivatives w.r.t. time and space.

Remark 1

Considering the explicit Euler scheme and fixing $y = \hat{X}_{t_i}$*, then* $\hat{L}_y^{(0)}$ *is the operator associated to the diffusion process* $(\hat{X}_t)_{t\in[t_i,t_{i+1}]}$*. Recall the operators defined in* (2)*; note that*

$$L^{(0)}\varphi(t,X_t) = \hat{L}_{X_t}^{(0)}\varphi(t,X_t), \qquad L^{(1)}\varphi(t,X_t) = \hat{L}_{X_t}^{(1)}\varphi(t,X_t).$$

Choosing the appropriate weight and weak Taylor scheme and sufficient smoothness of the value function:

Lemma 4

Fix $l \in \mathbb{N}$. Suppose u is sufficiently smooth, and $L^{(0)}u. = 0$, $\psi \in \mathcal{B}^{l}_{[0,1]}$, weak Taylor scheme order $l+1$. Then,

$$\begin{aligned} \mathbb{E}\left[H_h^{\psi} u(h, \hat{X}_h)\right] &= L^{(1)} u(0,x) + \mathcal{O}(h^{l+1}) \\ &= \sigma(x)\Delta + \mathcal{O}(h^{l+1}). \end{aligned}$$

Idea of proof

Consider one time step of SDE: $\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t$, with $X_0 = x$.

1. Euler scheme on $[0, h]$, for some $Z \sim \mathrm{N}(0, 1)$:

$$\hat{X}_h := x + \sigma(x)\sqrt{h}Z.$$

Idea of proof

Consider one time step of SDE: $\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t$, with $X_0 = x$.

1. Euler scheme on $[0, h]$, for some $Z \sim \mathrm{N}(0, 1)$:

$$\hat{X}_h := x + \sigma(x)\sqrt{h}Z.$$

2. Weight H_h^{ψ} with $\psi \equiv 1$ (i.e. $H_h^{\psi} := Z/\sqrt{h}$.)

Idea of proof

Consider one time step of SDE: $\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t$, with $X_0 = x$.

1. Euler scheme on $[0, h]$, for some $Z \sim \mathrm{N}(0, 1)$:

$$\hat{X}_h := x + \sigma(x)\sqrt{h}Z.$$

2. Weight H_h^{ψ} with $\psi \equiv 1$ (i.e. $H_h^{\psi} := Z/\sqrt{h}$.)
3. Taylor expand $u(h, \hat{X}_h)$ around $(0, x)$.

Idea of proof

Consider one time step of SDE: $\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t$, with $X_0 = x$.

1. Euler scheme on $[0, h]$, for some $Z \sim \mathrm{N}(0, 1)$:

$$\hat{X}_h := x + \sigma(x)\sqrt{h}Z.$$

2. Weight H_h^{ψ} with $\psi \equiv 1$ (i.e. $H_h^{\psi} := Z/\sqrt{h}$.)
3. Taylor expand $u(h, \hat{X}_h)$ around $(0, x)$.
4. Consider $\mathbb{E}[H_h^{\psi} u(h, \hat{X}_h)]$ - collect powers of Z, recalling

$$\mathbb{E}[Z^k] = \begin{cases} 0 & \text{if } k \text{ is odd;} \\ \prod_{j=1}^{k/2}(2j-1) & \text{if } k \text{ is even.} \end{cases}$$

Theorem 5 (Higher order Δ)

Fix $l \in \mathbb{N}$. Consider a weak Taylor scheme of order $l+1$, on an equidistant mesh π, such that $|\pi| = h$, value function u is sufficiently smooth, and let $\psi \in \mathcal{B}^{l}_{[0,1]}$. Then,

$$\mathbb{E}\left[H_h^{\psi} g(\hat{X}_T)\right] = L^{(1)} u(0,x) + \mathcal{O}(h^{l+1}).$$

Theorem 5 (Higher order Δ)

Fix $l \in \mathbb{N}$. Consider a weak Taylor scheme of order $l+1$, on an equidistant mesh π, such that $|\pi| = h$, value function u is sufficiently smooth, and let $\psi \in \mathcal{B}^{l}_{[0,1]}$. Then,

$$\mathbb{E}\left[H_h^{\psi} g(\hat{X}_T)\right] = L^{(1)} u(0,x) + \mathcal{O}(h^{l+1}).$$

- To prove result, express $\mathbb{E}\left[H_h^{\psi} u(t_n, \hat{X}_{t_n})\right]$ as

$$\mathbb{E}\left[H_h^{\psi} u(h, \hat{X}_h)\right] + \mathbb{E}\left[H_h^{\psi} \sum_{i=1}^{n-1} \left\{ u(t_{i+1}, \hat{X}_{t_{i+1}}) - u(t_i, \hat{X}_{t_i}) \right\}\right].$$

- Deal with first term from previous lemma, and bound telescoping terms from the smoothness of the value function.

Flavour of techniques

- Iterated Itô integrals, and weak Taylor schemes [KP92].
- Expansions introduced by [TT90].
- Choose weights for state-space Greeks.
- Refine H_h^{ψ} for higher order schemes.

Higher order schemes

- Consider N simulations, and fix the step size to $h := 1/N^{\zeta}$.
- Approximate Δ, with $\mathbb{E}\left[H_h^{\psi} g(\hat{X}_T)\right]$.

Higher order schemes

- Consider N simulations, and fix the step size to $h := 1/N^{\zeta}$.
- Approximate Δ, with $\mathbb{E}\left[H_h^{\psi} g(\hat{X}_T)\right]$.

r (Scheme)	Weight	ζ	MSE	Complexity	Slope
1 (Euler)	$\psi \equiv 1 \in \mathcal{B}^0_{[0,1]}$	$1/3$	$\mathcal{O}(N^{-2/3})$	$\mathcal{O}(N^{4/3})$	$-1/2$
2 (WT2)	$\psi \in \mathcal{B}^1_{[0,1]}$	$1/5$	$\mathcal{O}(N^{-4/5})$	$\mathcal{O}(N^{6/5})$	$-2/3$
3 (WT3)	$\psi \in \mathcal{B}^2_{[0,1]}$	$1/7$	$\mathcal{O}(N^{-6/7})$	$\mathcal{O}(N^{8/7})$	$-3/4$

Table 1: Implementation for higher order Δ.

- $\mu(x) \equiv 0$, $\sigma(x) \equiv 1 + \sin^2(x)$, $g(x) \equiv \arctan(x)$.
- $(X_0, T) = (0.3, 1)$, $(\zeta_1, \zeta_2, \zeta_3) = (1/3, 1/5, 1/7)$.

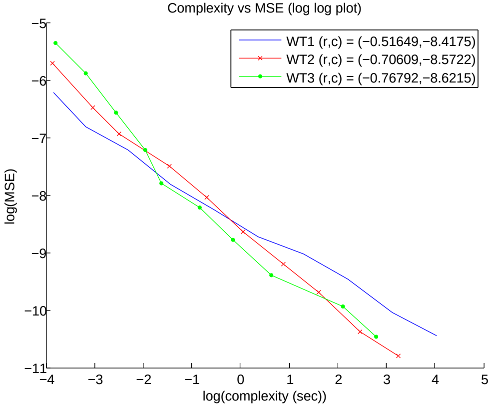

Figure 1: Higher order Δ and ψ.

- $\approx$ 20 seconds for WT3 vs $\approx$ 60 seconds for WT1!

Extrapolating Δ

- Approximation $\hat{X}^h$ is with a grid $|\pi| = h$.

Extrapolating Δ

- Approximation $\hat{X}^h$ is with a grid $|\pi| = h$.
- Show that $\mathbb{E}\left[H_h^{\psi} g(\hat{X}_T^h)\right] = L^{(1)}u(0,x) + c_1 h + \mathcal{O}(h^2)$.

Extrapolating Δ

- Approximation $\hat{X}^h$ is with a grid $|\pi| = h$.
- Show that $\mathbb{E}\left[H_h^{\psi} g(\hat{X}_T^h)\right] = L^{(1)} u(0,x) + c_1 h + \mathcal{O}(h^2)$.
- Approximation $\hat{X}^{2h}$ is with a grid $|\pi| = 2h$.

Extrapolating Δ

- Approximation $\hat{X}^h$ is with a grid $|\pi| = h$.
- Show that $\mathbb{E}\left[H_h^{\psi} g(\hat{X}_T^h)\right] = L^{(1)} u(0,x) + c_1 h + \mathcal{O}(h^2)$.
- Approximation $\hat{X}^{2h}$ is with a grid $|\pi| = 2h$.

Theorem 6 (Romberg extrapolation)

$$2\mathbb{E}\left[H_h^{\psi} g(\hat{X}_T^h)\right] - \mathbb{E}\left[H_{2h}^{\psi} g(\hat{X}_T^{2h})\right] = L^{(1)} u(0,x) + O(h^2).$$

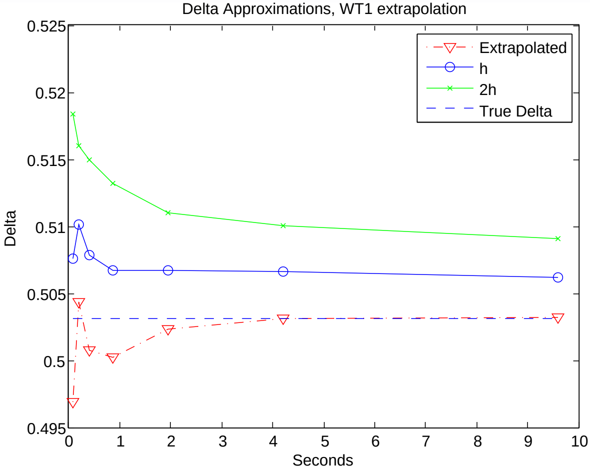

Figure 2: Extrapolated Δ, the value with stepsize h, $2h$ and the true Δ. Euler scheme, $\zeta = 1/5$.

Δ extrapolated

Similar expansion for higher order Romberg extrapolation using better $\psi \in \mathcal{B}^l_{[0,1]}$ and weak Taylor expansions.

r (Scheme)	Weight	ζ	MSE	Complexity	Slope
1 (Euler)	$\psi \equiv 1$	1/5	$\mathcal{O}(N^{-4/5})$	$\mathcal{O}(N^{6/5})$	-2/3
2 (WT2)	$\psi_{s,1}$	1/7	$\mathcal{O}(N^{-6/7})$	$\mathcal{O}(N^{8/7})$	-3/4
3 (WT3)	$\psi_{s,2}$	1/9	$\mathcal{O}(N^{-8/9})$	$\mathcal{O}(N^{10/9})$	-4/5

Table 2: Implementation for the extrapolated Δ.

Extrapolated Δ using WT1 and WT2

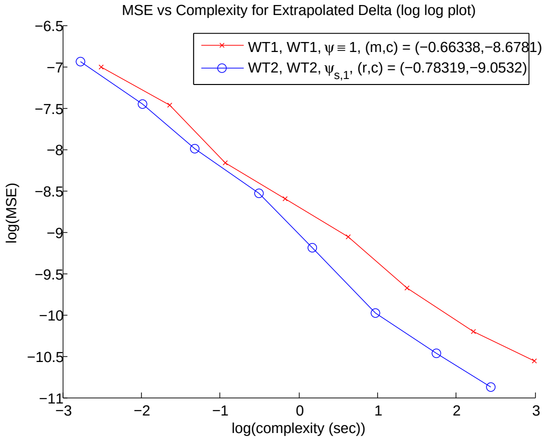

Figure 3: MSE for extrapolated Δ vs Complexity.

Heston Delta

- The Heston model can be represented with i.i.d. Brownian motions $W^{(1)} = (W_t^{(1)})_{t\geq 0}$ and $W^{(2)} = (W_t^{(2)})_{t\geq 0}$ as

$$\mathrm{d}\begin{pmatrix} S_t \\ X_t \end{pmatrix} = \begin{pmatrix} rS_t \\ \kappa(\theta - X_t) \end{pmatrix} \mathrm{d}t + \begin{pmatrix} \sqrt{X_t}S_t & 0 \\ 0 & \xi\sqrt{X_t} \end{pmatrix} \begin{pmatrix} \mathrm{d}W_t^{(1)} \\ \mathrm{d}W_t^{(2)} \end{pmatrix}$$

where $(S_0, X_0) = (x, v)$.

- For an Euler scheme:

$$\Delta = \mathbb{E}\left[g(X_T) \frac{(H_h^{\psi})_{(1)}}{x\sqrt{v}} \right] + \mathcal{O}(h),$$

where $(H_h^{\psi})_1$ is defined with $\psi \in \mathcal{B}^0_{[0,1]}$ and $W_{\cdot}^{(1)}$.

Explicit and drift-implicit schemes

- $(\kappa, \theta, \xi, r, x, v) = (1.15, 0.04, 0.2, 0, 100, 0.04)$.
- Mean reversion $\omega := 2\kappa\theta/\xi^2 = 2.3$.
- Call option with strike $K = 100$, and $T = 1$.

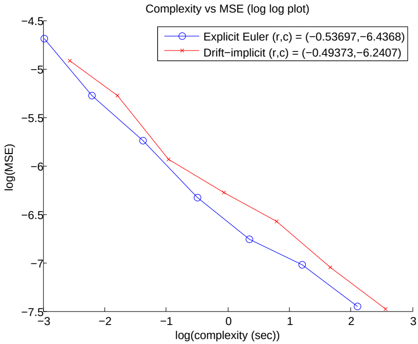

Figure 4: MSE for Heston Δ, $\zeta = 1/3$.

Γ of an option

- Second order sensitivity with respect to initial underlying price, x;

$$\Gamma := \partial_{xx}\mathbb{E}\left[g(X_T)\right].$$

Γ of an option

- Second order sensitivity with respect to initial underlying price, x;

$$\Gamma := \partial_{xx}\mathbb{E}\left[g(X_T)\right].$$

- Find family of functions with desirable properties.

Γ of an option

- Second order sensitivity with respect to initial underlying price, x;

$$\Gamma := \partial_{xx}\mathbb{E}\left[g(X_T)\right].$$

- Find family of functions with desirable properties.
- Previous ideas for higher order approximations and extrapolation.

A class of approximate Γ weights

Definition 7 (ϕ-functions)

Fix $l \in \mathbb{N}^+$. Define $\mathcal{G}^l_{[0,1]}$ as the set of bounded, measurable functions $\phi : [0,1] \to \mathbb{R}$ such that

$$\int_0^1 \phi(s) s ds = 1, \tag{3}$$

and if $l \geq 2$, then for all $k \in \mathbb{N}^+$ such that $2 \leq k \leq l$,

$$\int_0^1 \phi(s) s^k ds = 0. \tag{4}$$

A class of approximate Γ weights

Definition 7 (ϕ-functions)

Fix $l \in \mathbb{N}^+$. Define $\mathcal{G}^l_{[0,1]}$ as the set of bounded, measurable functions $\phi : [0,1] \to \mathbb{R}$ such that

$$\int_0^1 \phi(s) s \mathrm{d}s = 1, \tag{3}$$

and if $l \geq 2$, then for all $k \in \mathbb{N}^+$ such that $2 \leq k \leq l$,

$$\int_0^1 \phi(s) s^k \mathrm{d}s = 0. \tag{4}$$

- Higher order weights are of the form

$$\Gamma_h^\phi := \frac{1}{h^2} \int_{s=0}^h \phi\left(\frac{s}{h}\right) W_s \mathrm{d}W_s,$$

Summary for Γ

- Taylor expanding sufficiently, and using the smoothness of the value function eventually yields:

$$\begin{aligned}\mathbb{E}\left[\Gamma_h^{\phi} u(h,\hat{X}_h)\right] &= \sigma^2 \partial_{xx} u(0,x) + \sigma\sigma' \partial_x u(0,x) + \mathcal{O}(h) \\ &= \sigma^2(x)\Gamma + \sigma(x)\sigma'(x)\Delta + \mathcal{O}(h).\end{aligned} \tag{5}$$

Summary for Γ

- Taylor expanding sufficiently, and using the smoothness of the value function eventually yields:

$$\begin{aligned}\mathbb{E}\left[\Gamma_h^{\phi} u(h, \hat{X}_h)\right] &= \sigma^2 \partial_{xx} u(0,x) + \sigma\sigma' \partial_x u(0,x) + \mathcal{O}(h) \\ &= \sigma^2(x)\Gamma + \sigma(x)\sigma'(x)\Delta + \mathcal{O}(h).\end{aligned} \tag{5}$$

- Deal with telescoping terms.

Summary for Γ

- Taylor expanding sufficiently, and using the smoothness of the value function eventually yields:

$$\mathbb{E}\left[\Gamma_h^{\phi} u(h, \hat{X}_h)\right] \begin{aligned} &= \sigma^2 \partial_{xx} u(0, x) + \sigma\sigma' \partial_x u(0, x) + \mathcal{O}(h) \\ &= \sigma^2(x)\Gamma + \sigma(x)\sigma'(x)\Delta + \mathcal{O}(h). \end{aligned} \tag{5}$$

- Deal with telescoping terms.
- Equation (5), includes the $\Gamma := \partial_{xx} u(0, x)$ of interest.

Γ using a weak Taylor scheme order 2:

Theorem 8 (Γ)

Value function u is sufficiently smooth, $\phi \in \mathcal{G}^1_{[0,1]}$, and WT2 scheme, equidistant time grid $|\pi| = h$. Then,

$$\mathbb{E}\left[\Gamma^{\phi}_{h} g(\hat{X}_T)\right] = \sigma(x)^2\Gamma + \sigma'(x)\sigma(x)\Delta + \mathcal{O}(h).$$

Γ using a weak Taylor scheme order 2:

Theorem 8 (Γ)

Value function u is sufficiently smooth, $\phi \in \mathcal{G}^1_{[0,1]}$, and WT2 scheme, equidistant time grid $|\pi| = h$. Then,

$$\mathbb{E}\left[\Gamma^{\phi}_{h} g(\hat{X}_T)\right] = \sigma(x)^2\Gamma + \sigma'(x)\sigma(x)\Delta + \mathcal{O}(h).$$

- Similarly, higher order Γ approximations can be obtained.

- In Table 3, implementation for higher order schemes for Γ using different schemes, and weights.

Scheme	Weight	ζ	MSE	Complexity	Slope
WT2	$\phi \equiv 2 \in \mathcal{G}^1_{[0,1]}$	$1/4$	$\mathcal{O}(N^{-1/2})$	$\mathcal{O}(N^{5/4})$	$-2/5$
WT3	$\phi_{s,2} \in \mathcal{G}^2_{[0,1]}$	$1/6$	$\mathcal{O}(N^{-2/3})$	$\mathcal{O}(N^{7/6})$	$-4/7$

Table 3: Implementation and MSE for the Gamma.

Weak Taylor 2 scheme, using $\phi \equiv 2$

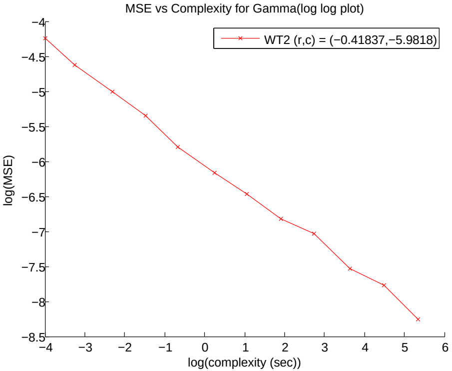

Figure 5: MSE for the Γ. Parameters as in Table 3 (i.e. $\zeta = 1/4$).

Γ extrapolation

Extrapolation using constants A, B:

$$A\mathbb{E}\left[\Gamma_h^{\phi} g(\hat{X}_T^h)\right] - B\mathbb{E}\left[\Gamma_{2h}^{\phi} g(\hat{X}_T^{2h})\right] = \text{Value} + \mathcal{O}(h^{l+1}).$$

ϕ	Value	Scheme	A	B	ζ	MSE	Slope
$\mathcal{G}^1_{[0,1]}$	$\hat{L}_x^{(1,1)} u_0$	Euler	2	1	1/6	$\mathcal{O}(N^{-2/3})$	-4/7
$\mathcal{G}^1_{[0,1]}$	$L^{(1,1)} u_0$	WT2	2	1	1/6	$\mathcal{O}(N^{-2/3})$	-4/7
$\mathcal{G}^2_{[0,1]}$	$L^{(1,1)} u_0$	WT3	4/3	1/3	1/8	$\mathcal{O}(N^{-3/4})$	-2/3

Table 4: Parameters for approximating Γ using extrapolation, using different ζ and schemes.

Γ extrapolation

Extrapolation using constants A, B:

$$A\mathbb{E}\left[\Gamma_h^{\phi} g(\hat{X}_T^h)\right] - B\mathbb{E}\left[\Gamma_{2h}^{\phi} g(\hat{X}_T^{2h})\right] = \text{Value} + \mathcal{O}(h^{l+1}).$$

ϕ	Value	Scheme	A	B	ζ	MSE	Slope
$\mathcal{G}^1_{[0,1]}$	$\hat{L}_x^{(1,1)} u_0$	Euler	2	1	1/6	$\mathcal{O}(N^{-2/3})$	-4/7
$\mathcal{G}^1_{[0,1]}$	$L^{(1,1)} u_0$	WT2	2	1	1/6	$\mathcal{O}(N^{-2/3})$	-4/7
$\mathcal{G}^2_{[0,1]}$	$L^{(1,1)} u_0$	WT3	4/3	1/3	1/8	$\mathcal{O}(N^{-3/4})$	-2/3

Table 4: Parameters for approximating Γ using extrapolation, using different ζ and schemes.

Remark 2

Extrapolating for the Γ *using an Euler scheme yields* $\hat{L}_x^{(1,1)} = \sigma^2(x)\Gamma$, *which does not include the* Δ *term.*

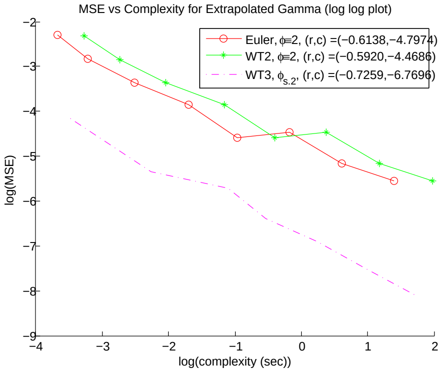

Figure 6: log log plot of the MSE vs Complexity for the Γ using extrapolation. Euler scheme and WT2 with $\phi \equiv 2$, and $(A, B) = (2, 1)$. Third plot is WT3, using $\psi_{s,2}$ and $(A, B) = (4/3, 1/3)$. See Table 4.

Thank you for listening

Bibliography I

J.-F. Chassagneux and D. Crisan.
Runge–Kutta schemes for backward stochastic differential equations.
The Annals of Applied Probability, 24(2):679–720, 2014.

P. Kloeden and E. Platen.
Numerical solution of stochastic differential equations, volume 23.
Springer Verlag, 1992.

D. Talay and L. Tubaro.
Expansion of the global error for numerical schemes solving stochastic differential equations.
Stochastic analysis and applications, 8(4):483–509, 1990.