Superreplication under Volatility Uncertainty for measurable claims

Ariel Neufeld

ETH Zurich

Joint work with Marcel Nutz
Columbia University

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Financial market without model uncertainty

- $(\Omega, \mathcal{F}, \mathbb{F})$ filtered measurable space, $0 < T < \infty$ maturity
- $(S_t)_{0 \leq t \leq T}$ (adapted) price process $\longrightarrow$ i.e. S_t price of asset at time t.
- Probability measure $P \longrightarrow$ law of the price process is **given**.
- Set $\mathcal{H}$ of (admissible) trading strategies
  $\mathcal{H} := \Big\{ (H_t)_{0 \leq t \leq T}$ predictable s.t. $\int H\, dS$ is P-$\mathbb{F}$-supermartingale$\Big\}$
  $\longrightarrow$ to avoid e.g. doubling strategies
- **No arbitrage condition:** Reasonable to assume that $\mathcal{M}^{e,m}_{loc} \neq \emptyset$
  where $\mathcal{M}^{e,m}_{loc} := \big\{ Q \approx P \,\big|\, S$ is a Q-$\mathbb{F}$ local martingale$\big\}$

Superreplication price without uncertainty

- claim $\xi : \Omega \to \mathbb{R}$ being $\mathcal{F}_T$-measurable

Definition: Superreplication price

$$\pi(\xi) := \inf \left\{ x \in \mathbb{R} \,\middle|\, \exists\, H \in \mathcal{H} \text{ s.t. } x + \int_0^T H_t \, dS_t \geq \xi \;\; P\text{-a.s.} \right\}$$

- $\pi(\xi)$ is fair price selling ξ (e.g. as bank) without risk
  i.e. it is smallest initial capital required to produce at time T, without risk, something that is enough to guarantee payoff ξ.

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Theorem: (classical valuation duality in mathematical finance)

Let $\xi : \Omega \to \mathbb{R}$ being $\mathcal{F}_T$-measurable with $\sup_{Q \in \mathcal{M}^{e,m}_{loc}} E^Q[|\xi|] < \infty$. Then:

- $\pi(\xi) = \sup_{Q \in \mathcal{M}^{e,m}_{loc}} E^Q[\xi]$
- $\exists\, H \in \mathcal{H}$ such that $\pi(\xi) + \int_0^T H_t\, dS_t \geq \xi \;\; P$-a.s.,
  i.e. the infimum is attained.

Model Uncertainty

- In reality one does *not* know the exact law of the price process

  $\longrightarrow$ start with $(\Omega, \mathcal{F}, \mathbb{F})$, price process S and maturity $T < \infty$

  But: Instead of a **single** probability measure P,

  $\longrightarrow$ consider a **set** of probability measures $\mathfrak{P}$

- $\mathfrak{P}$ stands for the possible laws of the price process S

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  $$\mathcal{H} := \Big\{ H \text{ predictable s.t. } \int H\, dS \text{ is } P\text{-}\mathbb{F}\text{-supermartingale, } \forall P \in \mathfrak{P} \Big\}$$

- **No arbitrage condition:** Assume that $\mathcal{M}^{e,m}_{loc}(P) \neq \emptyset$ for all $P \in \mathfrak{P}$ where $\mathcal{M}^{e,m}_{loc}(P) := \{ Q \approx P \,|\, S \text{ is a } Q\text{-}\mathbb{F} \text{ local martingale} \}$

Superreplication price under model uncertainty

- claim $\xi : \Omega \to \mathbb{R}$ being $\mathcal{F}_T$-measurable

Definition: Superreplication price under model uncertainty

$$\pi_{\mathfrak{P}}(\xi) := \inf \left\{ x \in \mathbb{R} \,\middle|\, \exists\, H \in \mathcal{H} \text{ s.t. } x + \int_0^T H_t \, dS_t \geq \xi \;\; P\text{-a.s. } \forall P \in \mathfrak{P} \right\}$$

- $\pi_{\mathfrak{P}}(\xi)$ is smallest initial capital required to produce at time T without risk, *under each possible law $P \in \mathfrak{P}$*, something that is enough to guarantee payoff ξ.

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Conjecture

Let $\xi : \Omega \to \mathbb{R}$ $\mathcal{F}_T$-measurable with $\sup_{P \in \mathfrak{P}} \sup_{Q \in \mathcal{M}^{e,m}_{loc}(P)} E^Q[|\xi|] < \infty$. Then:

- $\pi_{\mathfrak{P}}(\xi) = \sup_{P \in \mathfrak{P}} \sup_{Q \in \mathcal{M}^{e,m}_{loc}(P)} E^Q[\xi]$
- $\exists\, H \in \mathcal{H}$ such that $\pi_{\mathfrak{P}}(\xi) + \int_0^T H_t \, dS_t \geq \xi$ P-a.s. for all $P \in \mathfrak{P}$

Our framework

- $(\Omega, \mathcal{F}) := \big(C_0([0,\infty);\mathbb{R}^d),\, \mathcal{B}(C_0([0,\infty);\mathbb{R}^d))\big)$, P_0 Wiener measure
- $(B_t)_{t\geq 0}$ canonical process, $\mathbb{F}$ defined by $\mathcal{F}_t := \sigma\big(B_s; s \leq t\big)$
- $$\mathfrak{P}_S := \Big\{ P_0 \circ \Big(\int_0^\cdot \alpha_t^{1/2}\, dB_t \Big)^{-1} \,\Big|\, \alpha \text{ progressive with values in } \mathbb{S}^{>0} \text{ s.t. } \int_0^T |\alpha_s|\, ds < \infty \ \ P_0\text{-a.s.} \ \ \forall T \in \mathbb{R}_+ \Big\}$$
- Particular $\mathbb{G}$ with $\mathbb{F} \subseteq \mathbb{G} \subseteq \bar{\mathbb{F}}^P$, $\forall P \in \mathfrak{P}_S$

  (where $\bar{\mathbb{F}}^P$ is the P-augmentation of $\mathbb{F}$)
- $0 < T < \infty$ maturity
- Price process $(S_t)_{0\leq t\leq T} := (B_t)_{0\leq t\leq T}$

Why $\mathfrak{P}_S$?

$$\mathfrak{P}_S := \left\{ P_0 \circ \left(\int_0^\cdot \alpha_t^{1/2}\, dB_t \right)^{-1} \,\middle|\, \alpha \text{ progressive with values in } \mathbb{S}^{>0} \text{ s.t. } \int_0^T |\alpha_s|\, ds < \infty \;\; P_0\text{-a.s.} \;\; \forall T \in \mathbb{R}_+ \right\}$$

- Stands for *volatility uncertainty*
- Corresponds to *Peng's G-expectation*

  i.e. for given bounded, closed $\Theta \subseteq \mathbb{S}^{>0}$, define $G(c) := \frac{1}{2} \sup_{\theta \in \Theta} \theta c$

  $\longrightarrow \mathcal{E}^G(\cdot) = \sup_{P \in \mathfrak{P}_S^\Theta} E^P[\,\cdot\,]$,

  where $\mathfrak{P}_S^\Theta := \left\{ P_0 \circ \left(\int_0^\cdot \alpha_t^{1/2}\, dB_t \right)^{-1} \in \mathfrak{P}_S \,\middle|\, \alpha \in \Theta \;\; P_0 \times dt\text{-a.e.} \right\}$

Properties of $\mathfrak{P}_S$

Observe: $\mathfrak{P}_S$ is *non-dominated*!

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Lemma: Properties of each $P \in \mathfrak{P}_S$ (known)

- $(B_t)_{t\geq 0}$ is P-$\mathbb{G}$ local martingale
- The P-augmentation $\bar{\mathbb{F}}^P$ is right-continuous
- P satisfies *predictable representation property*: i.e.

  for any right-continuous $(\bar{\mathbb{F}}^P, P)$-local martingale M there is predictable H

  such that $M = M_0 + \int H\,dB$, P-a.s.

Consequences:

- Market $(\Omega, \mathcal{F}, \mathbb{G}, P, B, T)$ is complete for every $P \in \mathfrak{P}_S$
- $\mathcal{M}^{e,m}_{loc}(P) = P$ for all $P \in \mathfrak{P}_S$

Superreplication under Volatility uncertainty

Conjecture in our framework

$$\pi_{\mathfrak{P}_S}(\xi) := \inf\left\{ x \in \mathbb{R} \,\middle|\, \exists\, H \in \mathcal{H} \text{ s.t. } x + \int_0^T H_t\, dB_t \geq \xi \;\; P\text{-a.s. } \forall P \in \mathfrak{P}_S \right\}$$

Let $\xi : \Omega \to \mathbb{R}$ being $\mathcal{G}_T$-measurable with $\sup_{P \in \mathfrak{P}_S} E^P[|\xi|] < \infty$. Then:

- $\pi_{\mathfrak{P}_S}(\xi) = \sup_{P \in \mathfrak{P}_S} E^P[\xi]$
- $\exists\, H \in \mathcal{H}$ such that $\pi_{\mathfrak{P}_S}(\xi) + \int_0^T H_t\, dS_t \geq \xi$ P-a.s. for all $P \in \mathfrak{P}$,

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Known results:

- Conjecture holds true if ξ satisfies some **continuity** properties (see e.g. Soner, Touzi, Zhang)

Not clear: General case, i.e. when $\xi : \Omega \to \mathbb{R}$ being $\mathcal{G}_T$-measurable *without* any continuity assumption?

Main Theorem

Theorem: (N. and Nutz)

Let $\xi : \Omega \to \mathbb{R}$ being $\mathcal{G}_T$-measurable with $\sup_{P \in \mathfrak{P}_S} E^P[|\xi|] < \infty$. Then:

- $\pi_{\mathfrak{P}_S}(\xi) = \sup_{P \in \mathfrak{P}_S} E^P[\xi]$
- $\exists\, H \in \mathcal{H}$ such that $\pi_{\mathfrak{P}_S}(\xi) + \int_0^T H_t \, dB_t \geq \xi$ P-a.s. for all $P \in \mathfrak{P}_S$, i.e. the infimum is attained.

Remarks:

- Conjecture holds true
- Can replace $\mathfrak{P}_S$ by $\mathfrak{P}_S^\Theta$ whenever $\Theta \subseteq \mathbb{S}^{>0}$ is Borel, where

$$\mathfrak{P}_S^\Theta := \left\{ P_0 \circ \left(\int_0^\cdot \alpha_t^{1/2} \, dB_t \right)^{-1} \in \mathfrak{P}_S \,\middle|\, \alpha \in \Theta \ P_0 \times dt\text{-a.e.} \right\}$$

Step 1) Show that $\sup_{P\in\mathfrak{P}_S} E^P[\xi] \leq \pi_{\mathfrak{P}_S}(\xi)$

- Take any $x \in \mathbb{R}$ with $H \in \mathcal{H}$ s.t. $\xi \leq x + \int_0^T H_t\,dB_t$ P-a.s. $\forall P \in \mathfrak{P}_S$
- $\forall P \in \mathfrak{P}_S$, P-Supermartingale property of $\int H\,dB$ implies $E^P[\xi] \leq x$
- Take $\sup_{P\in\mathfrak{P}_S}$ and inf over all x with the above property

$\longrightarrow$ $\sup_{P\in\mathfrak{P}_S} E^P[\xi] \leq \pi_{\mathfrak{P}_S}(\xi)$

Step 2):

- Find $H \in \mathcal{H}$ s.t. $\xi \leq \sup_{P \in \mathfrak{P}_S} E^P[\xi] + \int_0^T H_t \, dS_t \quad P\text{-a.s. } \forall P \in \mathfrak{P}_S$

$\longrightarrow \sup_{P \in \mathfrak{P}_S} E^P[\xi] \geq \pi(\xi)$ and inf is attained.

Idea: Similar to *Soner, Touzi and Zhang*

- **Assume:** there is process $(Y_t)_{0 \leq t \leq T}$ such that for all $P \in \mathfrak{P}_S$:
  - $Y_0 = \sup_{P \in \mathfrak{P}_S} E^P[\xi]$ P-a.s.
  - $Y_T = \xi$ P-a.s.
  - Y is a P-$\mathbb{G}$ supermartingale

- **Then:** Consider Right limit $\overline{Y} := Y_+$. Then, $\forall P \in \mathfrak{P}_S$
  - $\overline{Y}_0 \leq \sup_{P\in\mathfrak{P}_S} E^P[\xi]$ P-a.s.,
  - $\overline{Y}_T = \xi$ P-a.s.,
  - $\overline{Y}$ is P-$\bar{\mathbb{F}}^P$ supermartingale

- Via Doob-Meyer decomposition and predictable representation property, we obtain for each $P \in \mathfrak{P}_S$:

$$\overline{Y} = \overline{Y}_0 + \int H^P \, dB - K^P, \quad K_0^P = 0, \quad K^P \text{ increasing}$$

Consequence:

$$\xi \leq \sup_{P\in\mathfrak{P}_S} E^P[\xi] + \int H^P \, dB \quad P\text{-a.s. for all } P \in \mathfrak{P}_S$$

$\longrightarrow$ want process H such that $H = H^P$ $\quad P \times dt$-a.s. for all $P \in \mathfrak{P}_S$

- Observe that $d\langle \overline{Y}, B\rangle = H^P d\langle B\rangle \quad P \times dt$-a.s. for all $P \in \mathfrak{P}_S$.
- Observe that quadratic covariation can be constructed *pathwise*
- Define $a := \frac{d\langle B\rangle}{dt} \in \mathbb{S}^{>0} \quad P \times dt$-a.s. for all $P \in \mathfrak{P}_S$
- Define $H := a^{-1} \frac{d\langle \overline{Y}, B\rangle}{dt}$ and show that $H \in \mathcal{H}$

$\longrightarrow \; H \text{ "} = \text{" } \frac{d\langle \overline{Y}, B\rangle}{d\langle B\rangle}$

$\longrightarrow \; H = H^P \quad P \times dt$-a.s. for all $P \in \mathfrak{P}_S$

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$\longrightarrow \sup_{P \in \mathfrak{P}_S} E^P[\xi] \geq \pi(\xi)$ and inf is attained

q.e.d.

How to find process $(Y_t)_{0\leq t\leq T}$?

Problem: How to find process $(Y_t)_{0\leq t\leq T}$ such that $\forall P \in \mathfrak{P}_S$:

- $Y_0 = \sup_{P\in\mathfrak{P}_S} E^P[\xi]$ P-a.s.
- $Y_T = \xi$ P-a.s.
- Y is a P-$\mathbb{G}$ supermartingale

Idea: Want to construct $\mathbb{G}$-adapted process $(Y_t)_{0\le t\le T}$ such that for $s \le t$

- $Y_t = \operatorname{ess\,sup}^P_{P'\in\mathfrak{P}(t;P)} E^{P'}\left[\xi \,\middle|\, \mathcal{G}_t\right] \quad P\text{-a.s.} \quad \text{for all} \quad P \in \mathfrak{P}_S$

- $Y_s = \operatorname{ess\,sup}^P_{P'\in\mathfrak{P}(s;P)} E^{P'}\left[Y_t \,\middle|\, \mathcal{G}_s\right] \quad P\text{-a.s.} \quad \text{for all} \quad P \in \mathfrak{P}_S$

  where $\mathfrak{P}(s;P) := \left\{P' \in \mathfrak{P}_S \,\middle|\, P' = P \text{ on } \mathcal{G}_s\right\}$

" $(Y_t)_{0\le t\le T}$ is the *conditional sublinear expectation* of ξ related to $\mathfrak{P}_S$ satisfying the *time consistency property* "

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Properties of Y:

- $Y_0 := \sup_{P\in\mathfrak{P}_S} E^P[\xi], \qquad Y_T = \xi \;\; P\text{-a.s. for all } P \in \mathfrak{P}_S,$
- $(Y_t)_{0\leq t\leq T}$ is P-$\mathbb{G}$-supermartingale for all $P \in \mathfrak{P}_S$.

Problem: difficult to construct, as $\mathfrak{P}_S$ are *non-dominated*!

Our approach

- Interpret $\sup_{P \in \mathfrak{P}_S} E^P[\xi]$ as control problem (on $C_0[0,\infty)$)

$\longrightarrow$ time consistency $\longleftrightarrow$ Dynamic Programming Principle

- Need conditions on the set of probability measures $\mathfrak{P}$

Definition: Condition (A)

A set of probability measures $\mathfrak{P}$ on $C_0[0,\infty)$ satisfies condition (A) if:

- $\mathfrak{P}$ is analytic, (i.e. image of a Borel space under a Borel map)
- $\mathfrak{P}$ satisfies "invariance property"
- $\mathfrak{P}$ satisfies "pasting property"

$\mathfrak{P}_S$ satisfies condition (A)

Theorem: Nutz and van Handel

Let $\mathfrak{P}$ satisfies condition (A), then:

There exists a $\mathbb{G}$-adapted process $(Y_t)_{0\leq t\leq T}$ such that for any $s \leq t$

- $Y_t = \operatorname{ess\,sup}^P_{P'\in\mathfrak{P}(t;P)} E^{P'}\left[\xi \,\middle|\, \mathcal{G}_t\right]$ P-a.s. for all $P \in \mathfrak{P}$
- $Y_s = \operatorname{ess\,sup}^P_{P'\in\mathfrak{P}(s;P)} E^{P'}\left[Y_t \,\middle|\, \mathcal{G}_s\right]$ P-a.s. for all $P \in \mathfrak{P}$

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Proposition: N. and Nutz

- $\mathfrak{P}_S$ satisfies Condition (A).

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