Mean-Field Games

Lectures at the Imperial College London

3rd Lecture: Solving Mean-Field Games

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Part I. Revisiting McKean-Vlasov FBSDEs

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a. Within the framework of MFG

Program without common noise

• Make use of the results from the first chapter in order to characterize the optimal paths in the fixed point

∘ in the FBSDE formulation of the optimization problem ⇝ replace the environment by the law of the solution

∘ derive an FBSDE of the McKean-Vlasov type of the general form

$$
\begin{aligned}
X_t &= \xi + \int_0^t b\left(X_s, \mathcal{L}(X_s), Y_s, Z_s\right) ds \\
&\quad + \int_0^t \sigma(X_s, \mathcal{L}(X_s), Y_s) dW_s \\
Y_t &= g(X_T, \mathcal{L}(X_T)) + \int_t^T f\left(X_s, \mathcal{L}(X_s), Y_s, Z_s\right) ds \\
&\quad - \int_t^T Z_s dW_s
\end{aligned}
$$

• Choose the coefficients accordingly and solve!

Program with common noise

• Make use of the results from the first chapter in order to characterize the optimal paths in the fixed point

∘ in the FBSDE formulation of the optimization problem $\rightsquigarrow$ replace the environment by the conditional law of the solution

∘ derive an FBSDE of the McKean-Vlasov type of the general form

$$X_t = \xi + \int_0^t b\left(X_s, \mathcal{L}(X_s|W^0), Y_s, Z_s\right) ds$$
$$+ \int_0^t \sigma(X_s, \mathcal{L}(X_s|W^0), Y_s) dW_s + \sigma^0(X_s, \mathcal{L}(X_s|W^0), Y_s) dW_s^0$$
$$Y_t = g(X_T, \mathcal{L}(X_T|W^0)) + \int_t^T f\left(X_s, \mathcal{L}(X_s|W^0), Y_s, Z_s\right) ds$$
$$- \int_t^T Z_s dW_s - \int_t^T Z_s^0 dW_s^0$$

• Choose the coefficients accordingly and solve!

MKV FBSDE for the value function

- Consider, on $(\Omega, \mathbb{F}, \mathbb{P})$, the MKV FBSDE

$$
\begin{aligned}
X_t &= \xi \\
&\quad + \int_0^t b\Big(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Z_s\sigma^{-1}(X_s, \mathcal{L}(X_s)))\Big)\, ds \\
&\quad + \int_0^t \sigma(X_s, \mathcal{L}(X_s))dW_s \\
Y_t &= g(X_T, \mathcal{L}(X_T)) \\
&\quad + \int_t^T f\,(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Z_s\sigma^{-1}(X_s, \mathcal{L}(X_s))))\, ds \\
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  - $\alpha^\star(x, \mu, z)$ is the unique minimizer of $\alpha \mapsto H(x, \mu, \alpha, z)$

- Under assumptions of Chapter 1 $\rightsquigarrow$ solution to MKV FBSDE is MFG equilibrium

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Y_t &= \partial_x g(X_T, \mathcal{L}(X_T)) \\
&\quad + \int_t^T \partial_x H\Big(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Y_s), Y_s\Big)\, ds \\
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- Under assumptions of Chapter 1 $\rightsquigarrow$ solution to MKV FBSDE is MFG equilibrium

Existence and uniqueness in small time

- New two-point-boundary-problem $\rightsquigarrow$
  - Cauchy-Lipschitz theory in small time only
- Example when $\sigma^0 \equiv 0$

$$X_t = \xi + \int_0^t b\left(X_s, \mathcal{L}(X_s), Y_s, Z_s\right) + \int_0^t \sigma(X_s, \mathcal{L}(X_s), Y_s) dW_s$$

$$Y_t = g(X_T, \mathcal{L}(X_T)) + \int_t^T f\left(X_s, \mathcal{L}(X_s), Y_s, Z_s\right) ds - \int_t^T Z_s dW_s$$

- Lipschitz setting
  - b, σ, f and g L-Lipschitz continuous in (x, μ, y, z)
  - Lipschitz in $\mu \leftrightsquigarrow W_2$ Wasserstein distance
  - $(b, f, \sigma, \sigma^0, g)(t, 0, \delta_0, 0, 0)$ bounded
  - $\Rightarrow$ existence and uniqueness provided that $T \leq c(L)$

Part III. McKean-Vlasov FBSDEs

b. Lions derivative overs $\mathcal{P}_2(\mathbb{R}^d)$

Differentiation on $\mathcal{P}_2(\mathbb{R})$

- Consider $\mathcal{U} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$
- Lifted-version of $\mathcal{U}$

$$\hat{\mathcal{U}} : L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d) \ni X \mapsto \mathcal{U}(\mathrm{Law}(X))$$

  - $\mathcal{U}$ differentiable if $\hat{\mathcal{U}}$ Fréchet differentiable (Lions)
  - independent of the choice of $(\Omega, \mathbb{P})$ (rich enough)
- Differential of $\mathcal{U}$
  - Fréchet derivative of $\hat{\mathcal{U}}$ with $\mu = \mathrm{Law}(X)$

$$D\hat{\mathcal{U}}(X) = \partial_\mu \mathcal{U}(\mu)(X), \quad \partial_\mu \mathcal{U}(\mu) : \mathbb{R}^d \ni x \mapsto \partial_\mu \mathcal{U}(\mu)(x) \in \mathbb{R}^d$$

  - Derivative of $\mathcal{U}$ at $\mu \rightsquigarrow \partial_\mu \mathcal{U}(\mu) \in L^2(\mathbb{R}, \mu; \mathbb{R}^d)$

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  - Derivative of $\mathcal{U}$ at $\mu \rightsquigarrow \partial_\mu \mathcal{U}(\mu) \in L^2(\mathbb{R}, \mu; \mathbb{R}^d)$
- Finite dimensional projection

$$\partial_{x_i}\Big[\mathcal{U}\Big(\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big)\Big] = \frac{1}{N}\partial_\mu \mathcal{U}\Big(\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big)(x_i).$$

Examples

- 1st example: $\mathcal{U}(\mu) = \int_{\mathbb{R}^d} h(x) d\mu(x)$
  - two r.v.'s X and Y with values in $\mathbb{R}^d$

$$\begin{aligned}\mathcal{U}(\mathcal{L}(X + \varepsilon Y)) &= \mathbb{E}[h(X + \varepsilon Y)] \\ &= \mathbb{E}[h(X)] + \varepsilon \mathbb{E}\big[\partial h(X) Y\big] + o(\varepsilon)\end{aligned}$$

  - $\partial_\mu \mathcal{U}(\mu)(v) = \partial h(v)$
- 2nd example: $\mathcal{U}(\mu) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(x - y) d\mu(x) d\mu(y)$
  - two r.v.'s X and Y with independent copies X' and Y'

$$\begin{aligned}&\mathcal{U}(\mathcal{L}(X + \varepsilon Y)) \\ &= \mathbb{E}[h(X - X' + \varepsilon(Y - Y'))] \\ &= \mathbb{E}[h(X - X')] + \varepsilon \mathbb{E}\big[\partial h(X - X')(Y - Y')\big] + o(\varepsilon) \\ &= \mathbb{E}[h(X - X')] + \varepsilon \mathbb{E}\big[\partial h(X - X') Y\big] - \varepsilon \mathbb{E}\big[\partial h(X' - X) Y\big] + o(\varepsilon)\end{aligned}$$

  - $\partial_\mu \mathcal{U}(\mu)(v) = \int_{\mathbb{R}^d} \partial h(v - y) d\mu(y) - \int_{\mathbb{R}^d} \partial h(y - v) d\mu(y)$

Connection with W_2 distance

- Let $\mathcal{U}$ be Lions-differentiable with

$$\underbrace{\mathbb{E}[|\partial_\mu U(\mu)(X)|^2]}_{\int_{\mathbb{R}^d} |\partial_\mu U(\mu)(v)|^2 d\mu(v)} \le C^2, \qquad \mathcal{L}(X) = \mu$$

- For $X, X' \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$

$$\begin{aligned}
&\mathcal{U}(\mathcal{L}(X')) - \mathcal{U}(\mathcal{L}(X)) \\
&= \int_0^1 \frac{d}{dt}\mathcal{U}(\mathcal{L}(tX' + (1-t)X))dt \\
&= \int_0^1 \frac{d}{dt}\hat{\mathcal{U}}(tX' + (1-t)X)dt \\
&= \int_0^1 \mathbb{E}[\partial_\mu \mathcal{U}(\mathcal{L}(tX' + (1-t)X))(tX' + (1-t)X)(X' - X)]dt \\
&\le C\mathbb{E}[|X' - X|^2]^{1/2}
\end{aligned}$$

  - take inf over (X, X') with given laws $\rightsquigarrow$ Lipschitz w.r.t. W_2

Part III. McKean-Vlasov FBSDEs

c. Control of McKean-Vlasov and potential games

Rough version of the Pontryagin principle

- Controlled MKV processes (no common noise)

$$dX_t = b(X_t, \mathcal{L}(X_t), \alpha_t)dt + \sigma(X_t, \mathcal{L}(X_t))dW_t$$

  - optimize the cost $J(\alpha) = \mathbb{E}[g(X_T, \mathcal{L}(X_T)) + \int_0^T f(X_t, \mathcal{L}(X_t), \alpha_t)dt]$

- Optimize w.r.t. the measure as well
  - Use the same H and the same $\hat{\alpha}(t, x, \mu, y)$
  - Adjoint equations:

$$\begin{aligned} dX_t &= b(X_t, \mu_t, \hat{\alpha}(t, X_t, \mathcal{L}X_t, Y_t))dt + \sigma dW_t \\ dY_t &= -\partial_x H(X_t, \mathcal{L}(X_t), \hat{\alpha}(X_t, \mathcal{L}(X_t), Y_t), Y_t)dt \\ &\qquad - \text{"}\partial_\mu H(X_t, \mathcal{L}(X_t), \hat{\alpha}(X_t, \mathcal{L}(X_t), Y_t), Y_t)\text{"}dt + Z_t dW_t \\ Y_T &= \partial_x g(X_T, \mathcal{L}(X_T)) + \text{"}\partial_\mu g(X_T, \mathcal{L}(X_T))\text{"} \end{aligned}$$

  - What do "$\partial_\mu H$" and "$\partial_\mu g$" mean?

Right version of the Pontryagin principle

- Adjoint equations take the form

$$
\begin{aligned}
dX_t &= b(X_t, \mathcal{L}(X_t), \hat{\alpha}(t, X_t, \mathcal{L}(X_t), Y_t))dt + \sigma dW_t \\
dY_t &= -\partial_x H(X_t, \mathcal{L}(X_t), \hat{\alpha}(t, X_t, \mathcal{L}(X_t), Y_t), Y_t)dt \\
&\qquad - \mathbb{E}'[\partial_\mu H(X'_t, \mathcal{L}(X_t), \hat{\alpha}(X'_t, \mathcal{L}(X_t), Y'_t))(X_t)]dt + Z_t dW_t \\
Y_T &= \partial_x g(X_T, \mathcal{L}(X_T)) + \mathbb{E}'[\partial_\mu g(X'_T, \mathcal{L}(X_T))(X_T)]
\end{aligned}
$$

  - ∘ (X'_t, Y'_t) independent copy of (X_t, Y_t) on $(\Omega', \mathbb{F}', \mathbb{P}')$

- example $\rightsquigarrow$ social optimization with
  - ∘ $f(\mu, \alpha) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)d\mu(x)d\mu(y) + \frac{1}{2}|\alpha|^2$, f symmetric
  - ∘ $g(\mu) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(x-y)d\mu(x)d\mu(y)$, g symmetric
  - ∘ $b(\alpha) = \alpha$

$$
\partial_\mu H(\cdot) = \partial_\mu f(\mathcal{L}(X_t))(X_t) = \mathbb{E}'[\partial f(X_t - X'_t)] = \partial_{|x=X_t}\mathbb{E}'[f(x - X'_t)]
$$

∘ same equilibrium as MFG with $\int_{\mathbb{R}^d} f(x-y)d\mu(y) + \frac{1}{2}|\alpha|^2 \rightsquigarrow$ potential game!

Part II. Solving MFG without common noise

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a. Schauder fixed point theorem

Objective

- Assume $\sigma^0 \equiv 0$ and provide solution to

$$X_t = \xi + \int_0^t b\left(X_s, \mathcal{L}(X_s), Y_s, Z_s\right) + \int_0^t \sigma(X_s, \mathcal{L}(X_s), Y_s) dW_s$$

$$Y_t = g(X_T, \mathcal{L}(X_T)) + \int_t^T f\left(X_s, \mathcal{L}(X_s), Y_s, Z_s\right) ds - \int_t^T Z_s dW_s$$

- Assumption that

when $(\mathcal{L}(X_t))_{0 \le t \le T}$ replaced by some fixed input $(\mu_t)_{0 \le t \le T}$

$\Rightarrow$

existence and uniqueness of a solution to the FBSDE
in environment $(\mu_t)_{0 \le t \le T}$

- Example: implement the results from Chapter 1!
    - apply the two characterizations for stochastic optimal control

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- Example: implement the results from Chapter 1!
    - apply the two characterizations for stochastic optimal control

How to attack existence?

- Preliminary remark: no hope for solving MFG by Picard fixed theorem
  - at least under classical Lipschitz assumptions only
  - expect small time
  - need refined assumptions
- First step $\rightsquigarrow$ existence only
  - forget about uniqueness!
  - use a fixed point theorem without uniqueness!
- Use Schauder's fixed point theorem
  - see statement in the next slide
  - need a structure with a compactness
  - in the framework of MFG $\rightsquigarrow$ fixed point is on probability measures $\rightsquigarrow$ nice compactness criterion!

Statement of the Schauder fixed point theorem

- Generalisation of Brouwer's theorem from finite to infinite dimension
- Let $(V, \|\cdot\|)$ be a normed linear space
  - $\emptyset \neq E \subset V$ with E closed and convex
  - $\phi : E \to E$ continuous such that $\phi(E)$ is relatively compact
  - $\Rightarrow$ existence of a fixed point to ϕ
- In MFG $\rightsquigarrow$ what is V, what is E, what is ϕ?
  - recall that MFG equilibrium is a flow of measures $(\mu_t)_{0 \leq t \leq T}$
$$E \subset \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d))$$
  - need to embed into a linear structure
$$\mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d)) \subset \mathcal{C}([0,T], \mathcal{M}_1(\mathbb{R}^d))$$
  - $\mathcal{M}_1(\mathbb{R}^d)$ set of signed measures ν with $\int_{\mathbb{R}^d} |x| d|\nu|(x) < \infty$

Compactness on the space of probability measures

- Equip $\mathcal{M}_1(\mathbb{R}^d)$ with a norm $\|\cdot\|$ and restrict to $\mathcal{P}_1(\mathbb{R}^d)$ such that
  - convergence of $(\nu_n)_{n\geq 1}$ in $\mathcal{P}_1(\mathbb{R}^d)$ implies weak convergence

$$\forall f \in C_b(\mathbb{R}^d, \mathbb{R}), \quad \lim_{n\to\infty} \int_{\mathbb{R}^d} f d\nu_n = \int_{\mathbb{R}^d} f d\nu$$

  - if $(\nu_n)_{n\geq 1}$ has uniformly bounded moments of order $p > 2$

$$\text{Unif. square integrability} \Rightarrow W_2(\nu_n, \nu) \to 0$$

  - says that the input in the coefficients varies continuously!

$$b(x, \nu_n, y, z),\ \sigma(x, \nu_n),\ \sigma^0(x, \nu_n),\ f(x, \nu_n, y, z),\ g(x, \nu_n)$$

- Conversely, if $(\nu_n)_{n\geq 1}$ has bounded moments of order $p > 2$
  - $(\nu_n)_{n\geq 1}$ admits a weakly convergent subsequence
  - then convergence for W_2 by unit. integrability and for $\|\cdot\|$ also

Application to MKV FBSDE

- Choose E as continuous $(\mu_t)_{0\le t\le T}$ from $[0,T]$ to $\mathcal{P}_2(\mathbb{R}^d)$

$$\sup_{0\le t\le T}\int_{\mathbb{R}^d}|x|^4 d\mu_t(x)\le K \qquad \text{for some } K$$

- Construct $\phi \rightsquigarrow$ fix $(\mu_t)_{0\le t\le T}$ in E and solve

$$X_t = \xi + \int_0^t b\left(X_s,\mu_s,Y_s,Z_s\right) + \int_0^t \sigma(X_s,\mu_s,Y_s)dW_s$$

$$Y_t = g(X_T,\mu_T) + \int_t^T f\,(X_s,\mu_s,Y_s,Z_s)\,ds - \int_t^T Z_s dW_s$$

  - let $\phi\Big(\boldsymbol{\mu}=(\mu_t)_{0\le t\le T}\Big) = (\mathcal{L}(X_t^{\boldsymbol{\mu}}))_{0\le t\le T}$

- Assume bounded coefficients and $\mathbb{E}[|\xi|^4]<\infty$
  - choose K such that $\mathbb{E}[|X_t^{\boldsymbol{\mu}}|^4]\le K$

$$\Rightarrow E \text{ stable by } \phi$$

  - $W_2(\mathcal{L}(X_t^{\boldsymbol{\mu}}),\mathcal{L}(X_s^{\boldsymbol{\mu}}))\le C\mathbb{E}[|X_t^{\boldsymbol{\mu}}-X_s^{\boldsymbol{\mu}}|^2]^{1/2}\le C|t-s|^{1/2}$

Conclusion

- Consider continuous $\boldsymbol{\mu} = (\mu_t)_{0 \le t \le T}$ from $[0,T]$ to $\mathcal{P}_2(\mathbb{R}^d)$

  ∘ for any $t \rightsquigarrow (\phi(\boldsymbol{\mu}))_t$ in a compact subset of $\mathcal{P}_2(\mathbb{R}^d)$

  ∘ $[0,T] \ni t \mapsto (\phi(\boldsymbol{\mu}))_t$ is uniformly continuous in $\boldsymbol{\mu}$

  ∘ by Arzelà-Ascoli $\Rightarrow$ output lives in a compact subset of $E \subset C([0,T], \mathcal{P}_2(\mathbb{R}^d))$ (and thus of $C([0,T], \mathcal{M}_1(\mathbb{R}^d))$)

- Continuity of ϕ on E

  ∘ stability of the solution of FBSDEs with respect to a continuous perturbation of the environment

  ∘ under assumption of Chapter 1 and continuity w.r.t. environment $\rightsquigarrow$ answer is yes

- ϕ is continuous and compact range $\Rightarrow$ existence of a fixed point

Part II. Solving MFG without common noise

b. Statements and refinements

Non-degenerate setting

- Growth conditions

$$|b(x,\mu,\alpha)| \leq C(1+|\alpha|),\ |(\sigma,\sigma^{-1},g)(x,\mu)| \leq C$$
$$|f(x,\mu,\alpha)| \leq C(1+|\alpha|^2)$$

- Lipschitz condition

$$|(b,\sigma,\sigma^{-1},g)(x',\mu,\alpha') - (b,\sigma,\sigma^{-1},g)(x,\mu,\alpha)| \leq C(|x'-x| + |\alpha'-\alpha|)$$
$$|f(x',\mu,\alpha) - f(x,\mu,\alpha)| \leq C|x'-x|$$
$$|f(x,\mu,\alpha') - f(x,\mu,\alpha)| \leq C(1+|\alpha|+|\alpha'|)|\alpha'-\alpha|$$

- b linear in α and f strictly convex in α $\Rightarrow$ unique minimizer $\alpha^\star(x,\mu,z)$ of the Hamiltonian; and regularity of the minimizer
  - interpretation of the value function
  - for any input $\boldsymbol{\mu} = (\mu_t)_{0\leq t\leq T}$ $\Rightarrow$ unique optimal path with bounded control (comes from the fact that the gradient of HJB is bounded)
- $\Rightarrow$ existence of an MFG equilibrium!

Restricted convex setting

- Use the stochastic Pontryagin principle

  ◦ need to state the conditions for the derivative of the Hamiltonian

  ◦ require b to be linear in $x \Rightarrow$ no a priori bound for $b \rightsquigarrow$ need to adapt the result of Section I

  ◦ simplify $\rightsquigarrow$ take the example when b independent of x (say $b(x, \mu, \alpha) = \alpha$)

- Backward equation ($\partial_x f = \partial_x H$)

$$
\begin{aligned}
Y_t &= \partial_x g(X_T, \mathcal{L}(X_T)) \\
&\quad + \int_t^T \partial_x f \left(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Y_s) \right) ds - \int_t^T Z_s dW_s
\end{aligned}
$$

  ◦ require g convex in x and f convex in (x, α) but $\partial_x g$ and $\partial_x f$ bounded $\rightsquigarrow$ very restrictive!

  ◦ if continuity with respect to $\mu \Rightarrow$ existence of an MFG solution

Mollification procedure

- Convex Lipschitz is not satisfactory
  - use a mollification procedure
- Approximate coefficients (f, g) by coefficients (f_n, g_n) such that
  - f_n and g_n are convex and Lipschitz
  - general procedure for approximating convex functions

$$\Phi^n(x) = \sup_{|y| \le n} \inf_{z \in \mathbb{R}^d} \left[\langle y, x - z \rangle + \Phi(z) \rangle \right]$$

  - solve MFG for $(g_n, f_n) \rightsquigarrow$ equilibrium $(\mu_t^n)_{0 \le t \le T}$
- Converging subsequence of $(\mu_t^n)_{0 \le t \le T}$?
  - new compactness problem in $C([0, T], \mathcal{P}_2(\mathbb{R}^d))$
  - analysis $\rightsquigarrow$ boils down to control

$$\sup_{n \ge 1} \sup_{t \in [0,T]} \int_{\mathbb{R}^d} |x| d\mu_t^n(x) < \infty$$

  - must prevent any blow-up of the means of the equilibria!

Solvability in the convex setting

- Convex setting
  - b linear in (x, α), g convex in x
  - f convex in (x, α) and strictly convex in α
- Local Lipschitz continuity of the cost functionals

$$\begin{aligned}
&\left|f(t, x', \mu', \alpha') - f(t, x, \mu, \alpha)\right| + \left|g(x', \mu') - g(x, \mu)\right| \\
&\leq L\left[1 + |x'| + |x| + |\alpha'| + |\alpha| + \left(\int_{\mathbb{R}^d} |y|^2 d(\mu + \mu')(y)\right)^{1/2}\right] \\
&\qquad \times \left[|(x', \alpha') - (x, \alpha)| + W_2(\mu', \mu)\right],
\end{aligned}$$

- f and g C^1 w.r.t. (x, α) with Lipschitz derivatives
- weak-mean reverting condition

$$\langle x, \partial_x f(t, 0, \delta_x, 0)\rangle \geq -c(1 + |x|) \quad \text{and} \quad \langle x, \partial_x g(0, \delta_x)\rangle \geq -c(1 + |x|)$$

- $\Rightarrow$ existence of an MFG equilibrium!

Linear-quadratic in $d = 1$

- Apply previous results with
  - $b(t,x,\mu,\alpha) = a_t x + a'_t \mathbb{E}(\mu) + b_t \alpha_t$
  - $g(x,\mu) = \frac{1}{2}[qx + q'\mathbb{E}(\mu)]^2 \leftrightsquigarrow$ (mean-reverting) $qq' \geq 0$
  - $f(t,x,\mu,\alpha) = \frac{1}{2}[\alpha^2 + (m_t x + m'_t \mathbb{E}(\mu))^2] \leftrightsquigarrow$ (mean-rev.) $m_t m'_t \geq 0$
- Compare with direct method $\rightsquigarrow$ adjoint equations

$$
\begin{aligned}
dX_t &= [a_t X_t + a'_t \mathbb{E}(X_t) - b_t^2 Y_t]dt + \sigma dW_t \\
dY_t &= -[a_t Y_t + m_t(m_t X_t + m'_t \mathbb{E}(X_t))]dt + Z_t dW_t \\
Y_T &= q[qX_T + q'\mathbb{E}(X_T)]
\end{aligned}
$$

  - take the mean

$$
\begin{aligned}
d\mathbb{E}(X_t) &= [(a_t + a'_t)\mathbb{E}(X_t) - b_t^2 \mathbb{E}(Y_t)]dt \\
d\mathbb{E}(Y_t) &= -[a_t \mathbb{E}(Y_t) + m_t(m_t + m'_t)\mathbb{E}(X_t)]dt \\
\mathbb{E}(Y_T) &= q(q + q')\mathbb{E}(X_T)
\end{aligned}
$$

- existence and uniqueness if $q(q+q') \geq 0$, $m_t(m_t + m'_t) \geq 0$

Part II. Solving MFG without common noise

c. Uniqueness criterion

A counter-example to uniqueness

• Consider the MKV FBSDE

$$\begin{aligned} dX_t &= b(\mathbb{E}(Y_t))dt + dW_t, \quad X_0 = x_0 \\ dY_t &= -f(\mathbb{E}(X_t))dt + Z_t dW_t, \quad Y_T = g(\mathbb{E}(X_T)) \end{aligned}$$

∘ take bounded and Lipschitz coefficients $\leadsto$ existence of a solution

∘ uniqueness may not hold!

∘ completely different of the system with $b(Y_t), f(X_t)$ and $g(X_T)$ for which uniqueness holds true!

• Proof $\leadsto$ take the mean

$$\begin{aligned} d\mathbb{E}(X_t) &= b(\mathbb{E}(Y_t))dt, \quad \mathbb{E}(X_0) = x_0 \\ d\mathbb{E}(Y_t) &= -f(\mathbb{E}(X_t))dt, \quad \mathbb{E}(Y_T) = g(\mathbb{E}(X_T)) \end{aligned}$$

∘ led back to counter-example for FBSDE $\leadsto$ choose b, f and g equal to the identity on a compact subset

Lasry Lions monotonicity condition

- Recall for an FBSDE without noise ($\sigma = \sigma^0 = 0$)

  ◦ existence and uniqueness may hold for the Pontryagin system if convex cost functional

  ◦ convexity $\leftrightsquigarrow$ monotonicity of $\partial_x g$ and $\partial_x H$

  ◦ what is monotonicity condition in the direction of the measure?

- Lasry Lions monotonicity condition

  ◦ b, σ do not depend on μ

  ◦ $f(x,\mu,\alpha) = f_0(x,\mu) + f_1(x,\alpha)$ (μ and α are separated)

  ◦ monotonicity property for f_0 and g w.r.t. μ

$$\int_{\mathbb{R}^d} (f_0(x,\mu) - f_0(x,\mu'))d(\mu - \mu')(x) \geq 0$$
$$\int_{\mathbb{R}^d} (g(x,\mu) - g(x,\mu'))d(\mu - \mu')(x) \geq 0$$

Monotonicity restores uniqueness

- Assume that for any input $\boldsymbol{\mu} = (\mu_t)_{0\le t\le T}$ unique optimal control $\alpha^{\star,\boldsymbol{\mu}}$
  - + existence of an MFG for a given initial condition
- Lasry Lions $\Rightarrow$ uniqueness of MFG equilibrium!
  - if two different $\leadsto \alpha^{\star,\boldsymbol{\mu}} \neq \alpha^{\star,\boldsymbol{\mu}'}$

$$\underbrace{J^{\boldsymbol{\mu}}(\alpha^{\star,\boldsymbol{\mu}})}_{\text{cost under } \boldsymbol{\mu}} < J^{\boldsymbol{\mu}}(\alpha^{\star,\boldsymbol{\mu}'}) \quad \text{and} \quad \underbrace{J^{\boldsymbol{\mu}'}(\alpha^{\star,\boldsymbol{\mu}'})}_{\text{cost under } \boldsymbol{\mu}'} < J^{\boldsymbol{\mu}'}(\alpha^{\star,\boldsymbol{\mu}})$$

so that

$$J^{\boldsymbol{\mu}'}(\alpha^{\star,\boldsymbol{\mu}}) - J^{\boldsymbol{\mu}'}(\alpha^{\star,\boldsymbol{\mu}'}) + J^{\boldsymbol{\mu}}(\alpha^{\star,\boldsymbol{\mu}'}) - J^{\boldsymbol{\mu}}(\alpha^{\star,\boldsymbol{\mu}}) > 0$$
$$J^{\boldsymbol{\mu}'}(\alpha^{\star,\boldsymbol{\mu}}) - J^{\boldsymbol{\mu}}(\alpha^{\star,\boldsymbol{\mu}}) - [J^{\boldsymbol{\mu}'}(\alpha^{\star,\boldsymbol{\mu}'}) - J^{\boldsymbol{\mu}}(\alpha^{\star,\boldsymbol{\mu}'})] > 0$$

$$\mathbb{E}\Big[\underbrace{g(X_T^{\star,\boldsymbol{\mu}},\mu'_T) - g(X_T^{\star,\boldsymbol{\mu}},\mu_T)}_{\int_{\mathbb{R}^d}(g(x,\mu'_T) - g(x,\mu_T))d\mu_T(x)} - \underbrace{\big(g(X_T^{\star,\boldsymbol{\mu}'},\mu'_T) - g(X_T^{\star,\boldsymbol{\mu}'},\mu_T)\big)}_{\int_{\mathbb{R}^d}(g(x,\mu'_T) - g(x,\mu_T))d\mu'_T(x)} + \dots\Big] > 0$$

  - same for $f_0 \Rightarrow$ LHS must be ≤ 0

Example for Lasry Lions

- Examples for $h(x,\mu)$ satisfying

$$\int_{\mathbb{R}^d} (h(x,\mu) - h(x,\mu'))d(\mu - \mu')(x) \geq 0$$

  - if h is independent of x
  - if h given by

$$h(x,\mu) = \langle x, \bar{\mu} \rangle, \quad \bar{\mu} = \int_{\mathbb{R}^d} y d\mu(y)$$

  - if h is given by

$$h(x,\mu) = \int_{\mathbb{R}^d} f(x-y)d\mu(y) \quad \text{and} \quad f \ \text{odd}$$

  - if $d = 1$ and h is independent of x

$$h(x,\mu) = \mu(-\infty, x] \quad \text{and} \quad \mu, \mu' \ \text{have no atoms}$$

Part III. Solving MFG with common noise

Part III. Solving MFG with common noise

a. Strategy of proof

General prospect

- Solve MFG with a common noise
  - need to solve conditional MKV FBSDE

$$
\begin{aligned}
X_t &= \xi + \int_0^t b\Big(X_s, \mathcal{L}(X_s|\boldsymbol{W}^0), Y_s, Z_s\Big)\,ds \\
&\quad + \int_0^t \sigma\Big(X_s, \mathcal{L}(X_s|\boldsymbol{W}^0), Y_s\Big)dW_s + \int_0^t \sigma^0\Big(X_s, \mathcal{L}(X_s|\boldsymbol{W}^0), Y_s\Big)dW_s^0 \\
Y_t &= g\Big(X_T, \mathcal{L}(X_T|\boldsymbol{W}^0)\Big) + \int_t^T f\Big(X_s, \mathcal{L}(X_s|\boldsymbol{W}^0), Y_s, Z_s\Big)\,ds \\
&\quad - \int_t^T Z_s dW_s - \int_t^T Z_s^0 dW_s^0
\end{aligned}
$$

- Again $\rightsquigarrow$ Cauchy Lipschitz theory in small time
  - may adapt the result for MFG without common noise
- How to implement Schauder's fixed point over intervals of arbitrary length?

Need for revisiting the strategy of proof

- Try to follow the same strategy as in the case σ^0
- Fix $\boldsymbol{\mu} = (\mu_t)_{0 \le t \le T}$ random process with values in $\mathcal{P}_2(\mathbb{R}^d)$ and adapted w.r.t. $\boldsymbol{W}^0$ on $(\Omega^0, \mathbb{F}^0, \mathbb{P}^0)$
  - call $\boldsymbol{X}^{\boldsymbol{\mu}} = (X_t^{\boldsymbol{\mu}})_{0 \le t \le T}$ the forward component of the solution to

$$
\begin{aligned}
X_t &= \xi + \int_0^t b\left(X_s, \mu_s, Y_s, Z_s\right) ds \\
&\quad + \int_0^t \left(\sigma(X_s, \mu_s, Y_s) dW_s + \sigma^0(X_s, \mu_s, Y_s) dW_s^0\right) \\
Y_t &= g(X_T, \mu_T) + \int_t^T f\left(X_s, \mu_s, Y_s, Z_s\right) ds - \int_t^T \left(Z_s dW_s + Z_s^0 dW_s^0\right)
\end{aligned}
$$

  - Solve $\mu_t(\omega^0) = \mathcal{L}(X_t^{\boldsymbol{\mu}} | \boldsymbol{W}^0)(\omega^0)$ for any $t \in [0, T]$ and for almost every $\omega^0 \in \Omega^0 \rightsquigarrow$ fixed point in

$$
\left(\mathcal{C}([0, T], \mathcal{P}_2(\mathbb{R}^d))\right)^{\Omega_0}
$$

  - much too big for nice compactness criterion!

Discretization method

- General idea $\rightsquigarrow$ discretize the conditioning in the MKV FBSDE!
  - $\mathcal{L}(X_t|W^0) \rightsquigarrow \mathcal{L}(X_t|\text{finitely supported process})$
  - Π projection mapping onto space grid $\{x_1, \ldots, x_M\} \subset \mathbb{R}^d$
  - $t_1, \ldots, t_N$ a finite time grid $\subset [0, T]$
  - $\hat{W}^0_{t_i} = \Pi(W^0_{t_i})$
- Solve the forward-backward system with

$$\mathcal{L}(X_t|\hat{W}^0_{t_1}, \ldots, \hat{W}^0_{t_i}), \quad t_i \leq t < t_{i+1}$$

- Fixed point strategy

  ◦ input $\boldsymbol{\mu} = (\mu_t)_{0 \leq t \leq T}$ adapted with respect to discrete filtration generated by $(\hat{W}^0_{t_1}, \ldots, \hat{W}^0_{t_N})$

  ◦ solve the fixed point $\mu_t(\hat{W}^0_{t_1}, \ldots, \hat{W}^0_{t_N}) = \mathcal{L}\big(X^{\boldsymbol{\mu}}_t|\hat{W}^0_{t_1}, \ldots, \hat{W}^0_{t_N}\big)$
- Since $(\hat{W}^0_{t_1}, \ldots, \hat{W}^0_{t_N})$ has finite support of size MN $\rightsquigarrow$ fixed point in $(\mathcal{C}([0, T], \mathcal{P}(\mathbb{R})))^{MN}$

Passing to the limit

• For any M and $N \rightsquigarrow \boldsymbol{\mu}^{\star,M,N} = (\mu_t^{\star,M,N})_{0\le t\le T}$ fixed point under the discretized conditioning

∘ call $(\boldsymbol{X}^{\star,M,N}, \boldsymbol{Y}^{\star,M,N}, \boldsymbol{Z}^{\star,M,N}, \boldsymbol{Z}^{0,\star,M,N})$ solution of the corresponding FBSDE

∘ aim at extracting converging subsequence

• Assume tightness $(\boldsymbol{\mu}^{\star,M,N})_{M,N\ge1}$ seen as processes with paths in $\mathcal{C}([0,T],\mathcal{P}_2(^d))$

$$(\mu_t^{\star,M,N})_{0\le t\le T} \underset{\mathcal{L}}{\longrightarrow} \mu^\star \quad \text{up to subsequence}$$

∘ $\Big(X_t^{\star,M,N}, Y_t^{\star,M,N}, \int_0^t Z_s^{\star,M,N} ds, \int_0^t Z_s^{0,\star,M,N} ds\Big)_{0\le t\le T}$ weakly converges to solution of FBSDE in environment $\boldsymbol{\mu}^\star$?

∘ is $\boldsymbol{\mu}^\star$ the flow of conditional measures of the solution?

• Main issue: loose adaptability of $\boldsymbol{\mu}^\star$ with respect to systemic noise in the limit!

Part III. Solving MFG with common noise

b. Weak and strong solutions

Need for a weak solution

• In previous slides $\rightsquigarrow$ loose adaptability of $\boldsymbol{\mu}^\star$ with respect to common noise

∘ set-up is made of $(\Omega^0, \mathbb{F}^0, \mathbb{P}^0)$ and $(\Omega^1, \mathbb{F}^1, \mathbb{P}^1)$

∘ Ω^1 carries idiosyncratic noise and Ω^0 carries both common noise and limit $\boldsymbol{\mu}^\star$ $\rightsquigarrow$ $\mathbb{F}^0$ larger than Brownian filtration!

• Loose martingale representation theorem $\rightsquigarrow$ FBSDE in the limit takes the form

$$X_t = \xi + \int_0^t b\left(X_s, \mu_s^\star, Y_s, Z_s\right) ds + \int_0^t \left(\sigma(X_s, \mu_s^\star) dW_s + \sigma^0(X_s, \mu_s^\star) dW_s^0\right)$$

$$Y_t = g(X_T, \mu_T) + \int_t^T f\left(X_s, \mu_s^\star, Y_s, Z_s\right) ds - \int_t^T Z_s dW_s - \Big(\underbrace{M_T - M_t}_{\text{mart. } \perp \boldsymbol{W}}\Big)$$

∘ conditioning takes the form

$$\mu_t = \mathcal{L}\left(X_t \mid \mathcal{F}_t^0\right) \quad \left(\text{may differ from } \mathcal{L}\left(X_t \mid (W_s^0)_{0 \le s \le t}\right)\right)$$

Strong vs. weak equilibra

- Strong sense
  - probability spaces $(\Omega^0, \mathbb{F}^0, \mathbb{P}^0)$ and $(\Omega^1, \mathbb{F}^1, \mathbb{P}^1)$ are given
  - example $\rightsquigarrow$ canonical spaces

$$\Omega^0 = C([0,T],\mathbb{R}^d) \quad \Omega^1 = \underbrace{\mathbb{R}^d}_{\text{initial condition}} \times C([0,T],\mathbb{R}^d)$$

  - require $(\mu_t^\star)_{0\le t\le T} = \mathcal{L}(X_t|\boldsymbol{W}^0)$
- Weak sense: probability space is not given
  - $\exists$ 2 filtered probability spaces $(\Omega^0, \mathbb{F}^0, \mathbb{P}^0)$ and $(\Omega^1, \mathbb{F}^1, \mathbb{P}^1)$
  - $(W_t^0, \mu_t^\star)_{0\le t\le T}$ is carried on Ω^0, $(X_0, W_t)_{0\le t\le T}$ on Ω^1
  - $\mu_t^\star = \mathcal{L}(X_t|\mathcal{F}_t^0)$ (conditioning is enlarged but independent of W)
- Same type of assumptions as in Section II $\Rightarrow$ existence of weak MFG
- Yamada-Watanabe: strong ! + weak $\exists \Rightarrow$ strong $\exists$
  - reconstruct solutions on the same space

Part III. Solving MFG with common noise

c. Common noise may restore uniqueness

Smoothing effect of common noise

- Lasry Lions conditions $\Rightarrow$ strong uniqueness
- ODEs without uniqueness $\rightsquigarrow$ SDEs with uniqueness!
  - restoration of uniqueness with common noise
- Simple example
  - $b(x, \mu, \alpha) = -x + b(m) + \alpha$, $m = \int x' d\mu(x')$
  - $f(x, \mu, \alpha) = \frac{1}{2}[(x + f(m))^2 + \alpha^2]$
  - $g(x, \mu) = \frac{1}{2}(x + g(m))^2$
- Stochastic Pontryagin $\rightsquigarrow$ strong solution if $Y_t = X_t + \chi_t$

$$
\begin{aligned}
dm_t &= (b(m_t) - 2m_t - \chi_t)dt + dW_t^0, \\
d\chi_t &= -(f + b)(m_t)dt + \zeta_t dW_t^0, \quad \chi_T = g(m_T)
\end{aligned}
$$

  - $m_t = \mathbb{E}[X_t | \boldsymbol{W}^0]$
  - b, f, g smooth bounded + noise $\Rightarrow$ $\exists$ and !
  - without noise $\Rightarrow$! may fail