Market impact models and optimal execution algorithms

Fabrizio Lillo

https://fabriziolillo.wordpress.com

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Outline

- June 7: Microstructure of double auction markets
  1. Market impact(s): origin and phenomenology
  2. Impact of single trades
  3. Order flow: phenomenology and models
- June 9: Market impact models.
  1. Transient impact models
  2. History dependent impact models
  3. Order book models
- June 16: Market impact of large trades and optimal execution.
  1. Phenomenology of large trade executions
  2. Models of optimal execution
- References

Elements of price formation

- The mechanism of price formation stems from the complicated interplay between incoming orders (and cancellations) and price change due to these orders
- Market reaction to trades, termed **market impact**, describes how much price change immediately and in the near future in response to order $\rightarrow$ mechanical vs induced response
- The **order flow** is composed by market orders, limit orders and cancellations and depends on the state of the book as well on the past price history

What is market impact?

- Market impact refers to the "correlation" between an incoming order (to buy or to sell) and the subsequent price change.
- Market impact induces extra costs. Indeed, large volumes must typically be fragmented and executed incrementally. The total cost of this large trade is quickly dominated, as sizes become large, by the average price impact
- Monitoring and controlling impact has therefore become one of the most active domains of research in quantitative finance since the mid-nineties.
- Volume dependence of impact (By how much do larger trades impact prices more than smaller trades?), and temporal behavior of impact (is the impact of a trade immediate and permanent, or does the impact decay after one stops trading?).
- Impact is a dynamical quantity since it depends on the available liquidity, but also on the recent history of my trades.

Why is there market impact?

- **Agents successfully forecast short term price movements and trade accordingly.** This does result in measurable correlation between trades and price changes, even if the trades by themselves have absolutely no effect on prices at all. If an agent correctly forecasts price movements and if the price is about to rise, the agent is likely to buy in anticipation of it.'Noise induced' trades, with no information content, have no price impact.
- **The impact of trades reveals some private information.** The arrival of new private information causes trades, which cause other agents to update their valuations, leading to a price change. But if trades are anonymous and there is no easy way to distinguish informed traders from non informed traders, then all trades must impact the price since other agents believe that at least of fraction of these trades contains some private information, but cannot decide which ones.
- **Impact is a purely statistical effect.** Imagine for example a completely random order flow process, that leads to a certain order book dynamics (see, e.g. "zero-intelligence" models). Conditional to an extra buy order, the price will on average move up if everything else is kept constant. Fluctuations in supply and demand may be completely random, unrelated to information, but a well defined notion of price impact still emerges. In this case impact is a completely mechanical – or better, statistical – phenomenon.

Information and market impact: the Kyle (1985) model

- Strategic asymmetric information model.
- Three agents: One market maker (MM), One informed trader, Many noise traders
- The terminal (liquidation) value v of an asset is normally distributed with mean p_0 and variance Σ_0.
- The informed trader knows v and enters a demand x (volume).
- Noise traders submit a net order flow u, which is Gaussian distributed with mean zero and variance σ_u^2
- The MM observes the total demand $y = x + u$ and then sets a price p. All the trades are cleared at p, any imbalance is exchanged by the MM.

Kyle model: one period (continued)

- The informed trader wants to trade as much as possible to exploit her informational advantage
- However the MM knows that there is an informed trader and if the total demand is large (in absolute value) she is likely to incur in a loss. Thus the MM protects herself by setting a price that is increasing in the net order flow.
- The solution to the model is an expression of this trade-off

Informed trader

- The informed trader conjectures that the MM uses a linear price adjustment rule $p = \lambda y + \mu$, where λ is inversely related to liquidity.

- The informed trader's profit is

$$\pi = (v - p)x = x[v - \lambda(u + x) - \mu] \tag{1}$$

  and the expected profit is

$$E[\pi] = x(v - \lambda x - \mu) \tag{2}$$

- The informed traders maximizes the expected profit, i.e.

$$x = \frac{v - \mu}{2\lambda} \tag{3}$$

- In Kyle's model the informed trader can loose money, but on average she makes a profit

Market maker

- The MM conjectures that the informed trader's demand is linear in v, i.e. $x = \alpha + \beta v$
- Knowing the optimization process of the informed trader, the MM solves

$$\frac{v - \mu}{2\lambda} = \alpha + \beta v \tag{4}$$

which gives

$$\alpha = -\frac{\mu}{2\lambda} \qquad \beta = \frac{1}{2\lambda} \tag{5}$$

- As liquidity drops the informed agent trades less
- The MM observes y and sets

$$p = E[v|y] \tag{6}$$

- Use conditional expectation for normal variables to find

$$E[v|y] = E[v|u + \alpha + \beta v] \tag{7}$$

Equilibrium solution

- The impact is linear and the liquidity increases with the amount of noise traders

$$p = p_0 + \frac{1}{2}\sqrt{\frac{\Sigma_0}{\sigma_u^2}}y \tag{8}$$

- The informed agent trades more when she can hide her demand in the noise traders demand

$$x = (v - p_0)\sqrt{\frac{\sigma_u^2}{\Sigma_0}} \tag{9}$$

- The expected profit of the informed agent depends on the amount of noise traders

$$E[\pi] = \frac{(v - p_0)^2}{2}\sqrt{\frac{\sigma_u^2}{\Sigma_0}} \tag{10}$$

- The noise traders loose money and the MM breaks even (on average)

Kyle model in multiple periods in a nutshell

- There are N auctions, each taking place at times $0 = t_0 < t_1 < ... < t_N = 1$
- The liquidation value of the asset is v, normally distributed with mean p_0 and variance Σ_0
- The quantity traded by noise traders in auction n is $\Delta u_n = u_n - u_{n-1}$, where u_n is a Brownian motion with zero mean and instantaneous variance σ_u^2
- x_n is the aggregate position of the informed after after the nth auction and $\Delta x_n = x_n - x_{n-1}$ is the quantity traded in this auction
- Each auction is divided in two steps:
  1. The informed and the noise traders place the aggregate demand $\Delta x_n + \Delta u_n$
  2. The market maker sets the liquidation price p_n

Kyle model in multiple periods in a nutshell

- The informed trader's trading strategy is a vector of functions $X =< X_1, ...X_N >$, where $x_n = X_n(p_1, ..., p_{n-1}, v)$
- The market maker pricing rule is a vector of function $P =< P_1, ..., P_N >$, where $p_n = P_n(x_1 + u_1, ..., x_n + u + n)$
- The profit of the informed on position acquired at auctions $n, ..., N$ is $\pi_n = \sum_{k=n}^{N}(v - p_k)x_k$
- A *sequential auction equilibrium* is a pair X, P such that
  - Profit maximization. $\forall n = 1, .., N$ and $\forall X'$ s.t. $X'_1 = X_1, ...X'_{n-1} = X_{n-1}$ it is
  $$E[\pi_n(X, P)|p_1, ..., p_{n-1}, v] \geq E[\pi_n(X', P)|p_1, ..., p_{n-1}, v] \tag{11}$$
  - Market efficiency. $\forall n = 1, .., N$ it is
  $$p_n = E[v|x_1 + u_1, ..., x_n + u_n] \tag{12}$$
- A linear equilibrium is a sequential auction equilibrium in which the functions X and P are linear
- A recursive linear equilibrium is a linear equilibrium s.t. $\exists \lambda_1, ...\lambda_N$ s.t. $\forall n = 1, .., N$

$$p_n = p_{n-1} + \lambda_n(\Delta x_n + \Delta u_n) \tag{13}$$

Kyle model in multiple periods in a nutshell

Theorem

There exists a unique linear equilibrium and this equilibrium is a recursive linear equilibrium. In this equilibrium there are constants $\beta_n, \lambda_n, \alpha_n, \delta_n, \Sigma_n$ such that

$$\Delta x_n = \beta_n (v - p_{n-1}) \Delta t_n \tag{14}$$
$$\Delta p_n = \lambda_n (\Delta x_n + \Delta u_n) \tag{15}$$
$$\Sigma_n \equiv var[v | \Delta x_1 + \Delta u_1, \ldots \Delta x_n + \Delta u_n] = (1 - \beta_n \lambda_n \Delta t_n) \Sigma_{n-1} \tag{16}$$
$$E[\pi_n | p_1, \ldots, p_{n-1}, v] = \alpha_{n-1} (v - p_{n-1})^2 + \delta_{n-1} \tag{17}$$

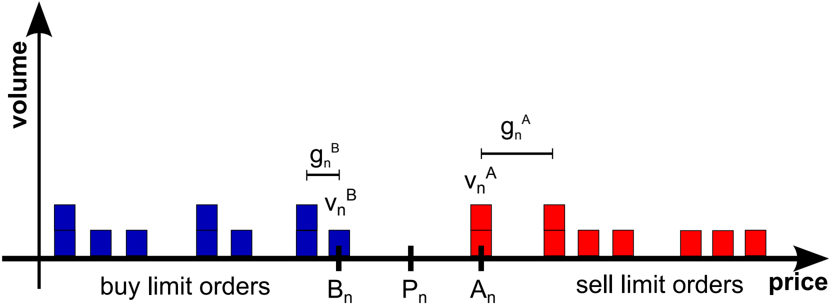
Predictions:

- The informed agent "slices and dices" (splits) her order flow in order to hide in the noise trader order flow
- Linear price impact
- Uncorrelated total order flow
- Permanent and fixed impact
- Variance of fundamental value v declines (but does not go to zero unless $N \to \infty$)

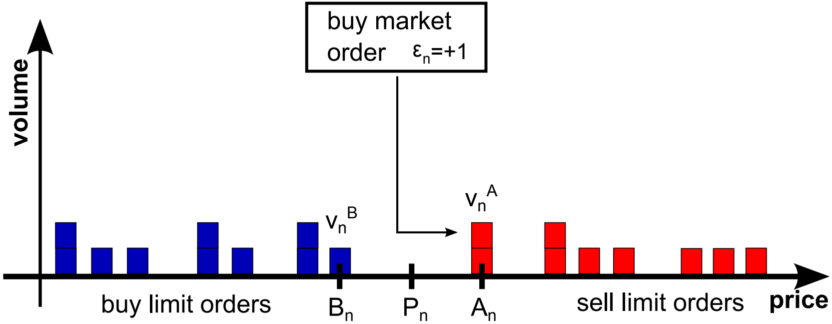
Market impact as a statistical effect: the zero intelligence model

- Many financial markets work through a continuous double auction mechanism
- No designated market makers, everyone can provide or take liquidity
- Identity of traders is not visible
- Sometimes multiple order books competing for liquidity (market fragmentation)
- Variety of rules (fees, tick size, transparency, latency, etc), but a common set of order types.

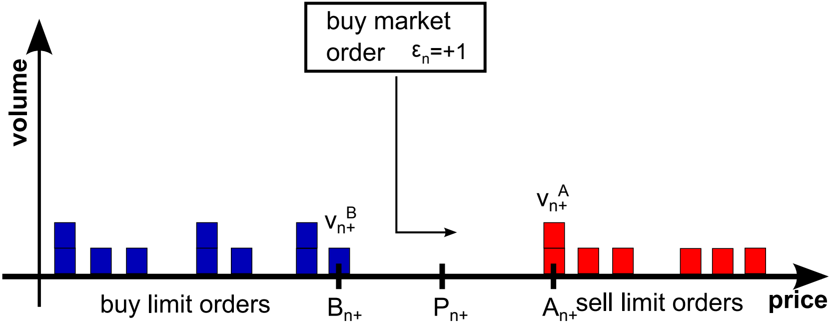
Dynamics of Limit Order Book



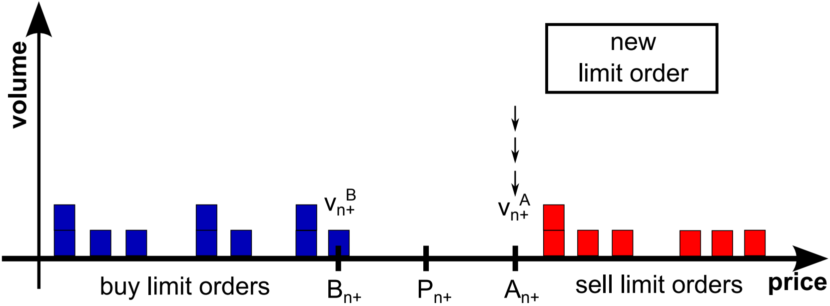
Dynamics of Limit Order Book



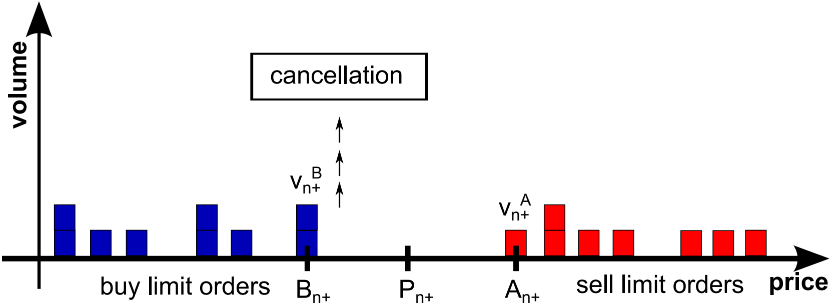
Dynamics of Limit Order Book



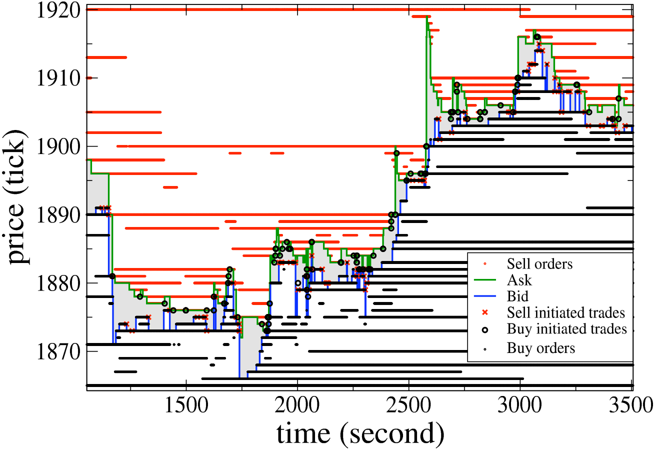
Dynamics of Limit Order Book



Dynamics of Limit Order Book



Dynamics of the Limit Order Book

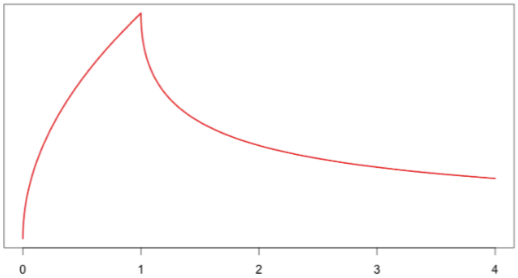


From Ponzi et al. 2009

Zero intelligence model

- Daniels et al. (2003), Smith et al. (2003), Cont and De Larrard (2013)
- Order book is modeled as a discrete price grid of constant minimum price increment w (the tick size).
- Limit order placement follows a Poisson process of rate λ per unit price and unit event time.
- Market orders arrive at a rate μ per unit event time.
- Each existing limit order has the same probability ν per unit event time to be cancelled.
- Donier et al. (2015) add a term of random reassessment of limit orders (see next lecture)

Impact trajectory of a VWAP execution in a zero intelligence model



Under some assumptions (vanishingly small cancellation rate and small participation rate of the VWAP execution) the price dynamics is

$$p(t) = K \int_0^t \frac{1}{\sqrt{4D(t-s)}} ds \qquad (18)$$

(Donier et al 2015)

Which market impact?

There are different types of price impact (often confused even in the specialized literature)

- Impact of an *individual market order* of size v (or more generally of a limit order or even of a cancellation)
- The correlation of the average price change in a given time interval T with the *total* market order imbalance in the same interval (i.e. the sum of the signed volume $\pm v$ of *all individual trades*.)
- Cross impact, i.e. how do trades on asset A impact the price of asset B.
- The impact of a given order of size Q, executed with many trades in a given direction, originating from the same agent.

Terminology

Large orders executed incrementally have different names in the literature

- Large trades
- Large orders
- Hidden orders
- Packages
- Algorithmic executions
- Metaorders (Bouchaud et al.)
- ......

A benchmark model: linear and permanent impact (Kyle)

The average price variation due to a signed volume ϵv is given by:

$$\Delta p = \lambda \epsilon v, \tag{19}$$

where λ is the inverse of liquidity and ϵ is $+1$ (-1) for buyer (seller) initiated trade. It is direct to show that

- The impact of individual trades is linear in volume and permanent i.e.

$$R_{so}(T) = E\left[(p_T - p_0) \cdot \epsilon_0\right] = \lambda\, E[v], \tag{20}$$

- The impact of aggregated order flow is linear in the volume imbalance

$$p_T = p_0 + \lambda \sum_{n=0}^{N-1} \epsilon_n v_n + \sum_{n=0}^{N-1} \eta_n, \tag{21}$$

- The price impact of a metaorder of total volume Q is linear

$$R_{mo}(T|Q) = E[(p_T - p_0) \cdot \epsilon_{mo} | \sum_{n \in mo} v_n = Q] = \lambda Q, \tag{22}$$

- The time correlation properties of returns are "inherited" by those of order flow $\rightarrow$ market efficiency

Empirical facts: individual impact

Empirical data consistently shows a *sublinear* (concave) volume dependence of impact of individual orders

$$E[\Delta p|v] \equiv R_{so}(T=1|v) \propto v^{\psi}; \qquad \psi \in [0.1, 0.3], \tag{23}$$

or even a logarithmic dependence $R_{so}(T=1|v) \propto \ln v$.

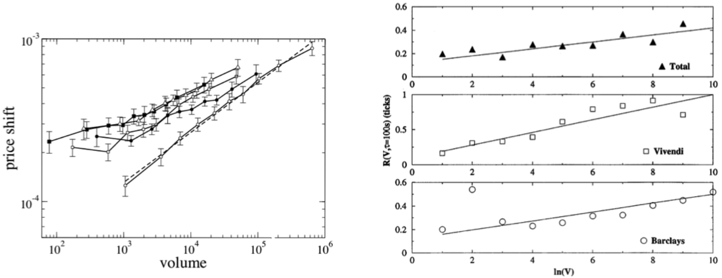


Figure: Impact of individual market orders for London Stock Exchange (left, from Lillo and Farmer 2004) and Paris Bourse (right, from Bouchaud and Potters 2002)

Empirical facts: individual impact (II)

By considering stocks with different market capitalization C on the immediate impact, one can show that impact of individual transactions can be approximately rescaled

$$R_{so}(T = 1|v) \approx C^{-0.3} F\left(C^{0.3} \frac{v}{\overline{v}}\right), \tag{24}$$

where $\overline{v}$ is the average volume per trade for a given stock, and $F(u)$ a master function that behaves as a u^{ψ} for small arguments.

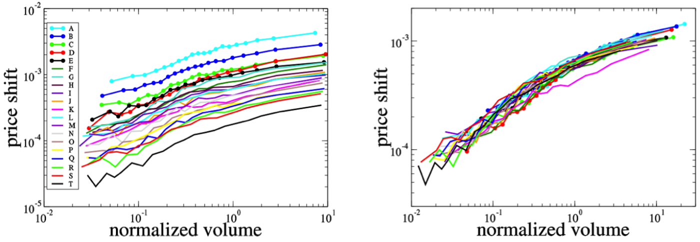
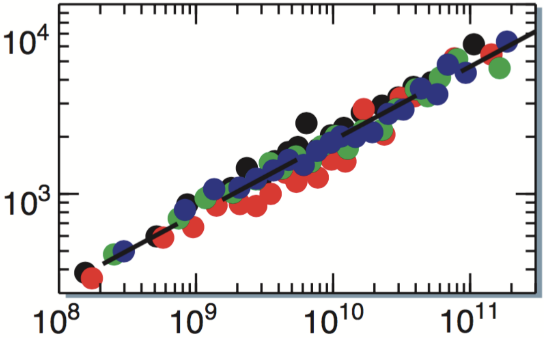


Figure: Impact of individual transactions of groups of stocks with different capitalization (left) and the same curves after rescaling (right) (Lillo et al. 2003).

Empirical facts: individual impact (III)



Different colors are different years (Lillo et al. 2003).

Kyle lambda (illiquidity) scales as $\lambda \sim C^{-0.4}$ (note that the dependence of the average volume on market cap has been already considered).

Price impact of single trades from order book shape?

- Let $V(r) = \int_0^r u(x)dx$ indicate the cumulative number of shares (depth) up to price return r.
- A market order of size v will produce a return

$$r = V^{-1}(v) \tag{25}$$

- For example if $V(r) \propto r^{\eta}$, it is $r \propto v^{1/\eta}$
- A linear order book, $u(x) = kx$ would lead to a square root impact, $r \propto \sqrt{v}$
- We will see that this explanation does not work for individual trades.
- It has been suggested as a mechanism for explaining the impact of metaorder executions (Bouchaud et al, see last lecture), but the book is the latent one rather than the real one.

The average shape of the limit order book

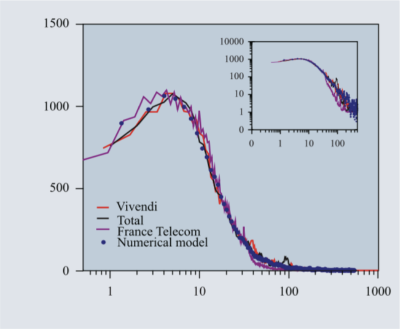


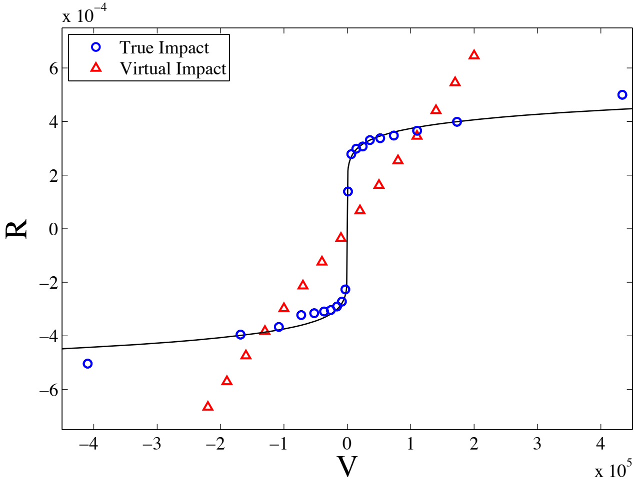
Figure 2. Average volume of the queue in the order book for the three stocks, as a function of the distance Δ from the current bid (or ask) in a log–linear scale. Both axes have been rescaled in order to collapse the curves corresponding to the three stocks. The thick dots correspond to the numerical model explained below, with $\Gamma = 10^{-3}$ and $p_m = 0.25$. Inset: same data in log–log coordinates.

From Bouchaud et al 2002. Note that this is the sample average shape of the order book which might be different from the typical shape.

Real and virtual impact

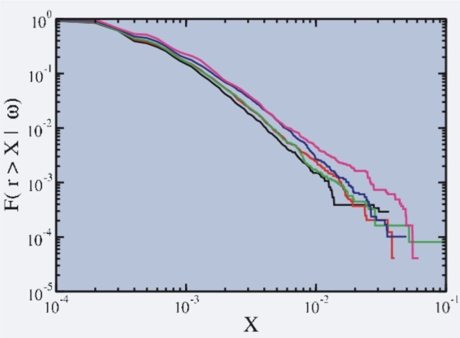
- Is this explanation in terms of the relation between price impact and the limit order book shape correct ?
- The basic assumptions are:
  - The traders placing market orders trade their desired volume irrespectively from what is present on the limit order book
  - The limit order book is filled in a continuous way, i.e. all the price levels are filled with limit orders
- We test the first assumption by measuring the virtual price impact, i.e. the price shift that would have been observed in a given instant of time if a market order of size q arrived in the market
- We test the second assumption by considering the fluctuations of market impact

Real and virtual impact for individual trades



Fluctuations of impact of individual trades

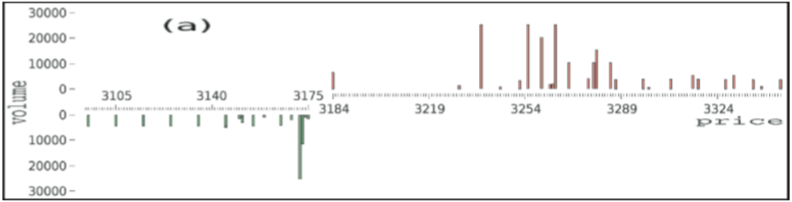
- Let us decompose the conditional probability of a return r conditioned to an order of volume q as
$$p(r|v) = (1 - g(v))\delta(r) + g(v)f(r|v) \tag{26}$$
- Compute the cumulative probability of a price return r conditioned to the volume q and to the fact that price moves
$$F(r > X|v) = \int_X^\infty f(r|v)dr \tag{27}$$
- Following analysis mostly on small tick stocks (e.g. AAPL)



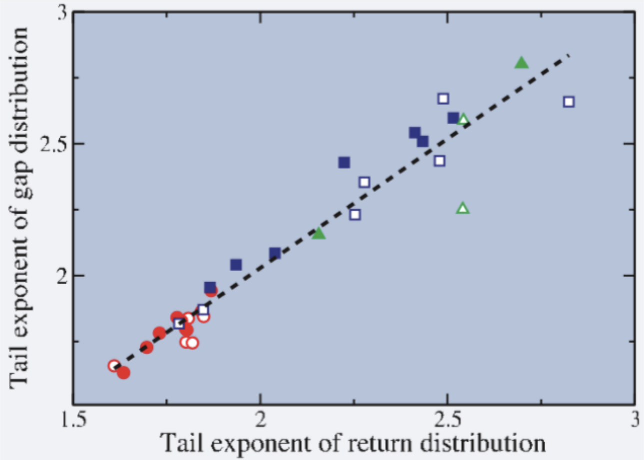
The orders are sorted by size into five groups with roughly the same number of orders in each group. Ranging from small orders to large orders, the curves are black, red, green, blue, and magenta (from Farmer et al 2004)

- The role of the transaction volume is negligible conditional to the fact that the price moves.
- The volume is important in determining whether the price moves or not.
- The fluctuations in market impact are important.

The role of liquidity fluctuations



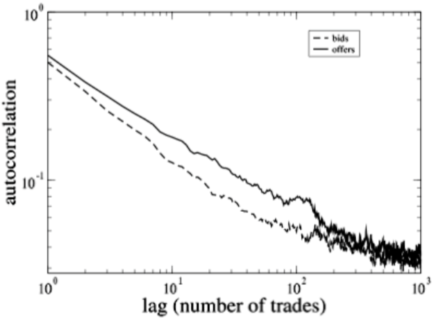
Gaps in the order book and return distribution



From Farmer et al 2004. Red circles for low liquidity stocks, blue squares for medium liquidity stocks, and green triangles for high liquidity stocks. Empty symbols refer to sell market orders and filled symbols to buy market orders.

Liquidity fluctuations are key determinants of fat tails of return distribution

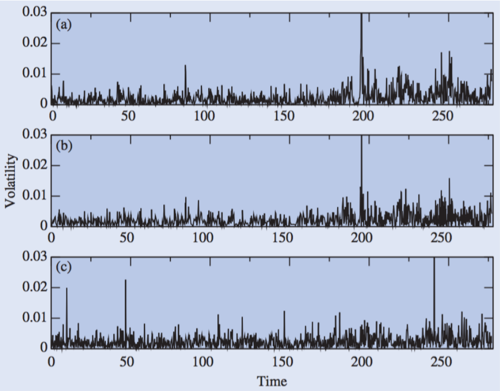
Persistence of illiquidity (gaps)



From Lillo and Farmer 2005. Autocorrelation function of the first gap size for bids and offers of AstraZeneca in a log-log plot.

- Size of gaps, measuring illiquidity, are very autocorrelated, consistent with long memory processes.
- Similar result for spread.
- The state of the order book is extremely persistent

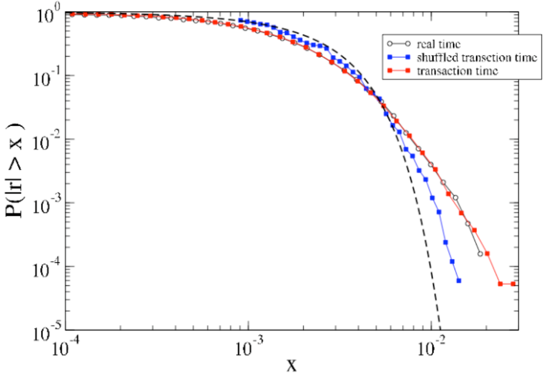
Fat tails: volume or liquidity fluctuations?



From Gillemot et al 2006. (a) real time (15min). (b) transaction time (24 transactions $\sim$ 15 min) (c) shuffled real time (fluctuating number of trades but randomized liquidity fluctuations).

- Returns in different time clocks: transaction, volume (Clark, Ane and Geman)
- In recent years liquidity fluctuations are more important than number of trades (or volume) in determining price fluctuations.
- Effect much stronger for small tick size assets.

A shuffling experiment on a longer time scale



From Gillemot et al 2006. Cumulative distribution of absolute logreturns for Procter & Gamble stock under different time clocks. The circles refer to 15-minute returns, the squares refer to returns aggregated with a fixed number of transactions, and the triangles show the cumulative distribution obtained by randomly shuffling individual transaction returns and then aggregating them in a way that matches the number of transactions in each real-time interval. The dashed line corresponds to a normal distribution.

Aggregated market impact

Aggregated market impact

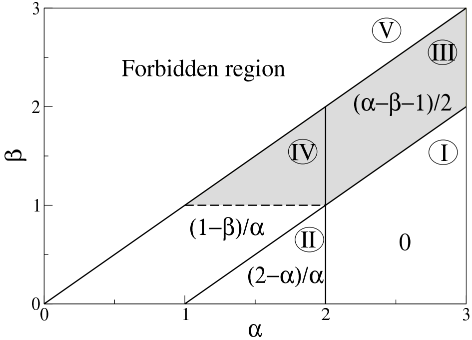
- Impact over a time interval, e.g. 5 min or 100 trades.
- In transaction time we consider $R(Q,N) = E[p_{t+n} - p_t | Q_N = Q]$ where $Q_N = \sum_{n=0}^{N-1} \epsilon_n v_n \equiv \sum_{n=0}^{N-1} w_n$.
- Assuming an uncorrelated order flow and a single trade impact function $r = f(w) + \eta$, we have

$$R(Q,N) = \int dw_1 \dots dw_N p(w_1) \dots p(w_N) \sum_{i=1}^{N} f(w_i) \delta(Q - \sum_{i=1}^{N} w_i)$$
$$= \frac{N}{2\pi} \frac{1}{P_N(Q)} \int d\lambda e^{(N-1)h(\lambda)} g(\lambda) e^{-i\lambda Q}$$

  where $h(\lambda)$ is the logarithm of the Fourier transform of the volume distribution and $g(\lambda)$ is the Fourier transform of the product of the volume distribution and the impact function.
- Moreover $P_N(Q)$ is the probability density that the total signed volume in the N trades is Q.
- Example: $f(w) \propto sign(w)|w|^{\beta}$ and $p(w) \sim w^{-\alpha-1}$, then for large N the aggregate impact behaves for small Q as

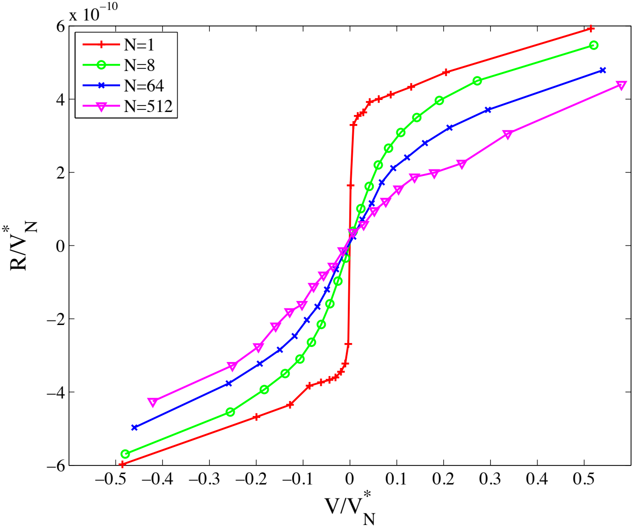
$$R(Q,N) \sim \frac{Q}{N^{\kappa}} \tag{28}$$

Aggregated market impact



Value of the exponent κ as a function of the impact exponent β and of the tail exponent of the volume distribution α

Aggregated market impact



(Re-scaled) Aggregate impact for the stock AstraZeneca at LSE

Aggregated market impact

- For a realistic range of values, the model predicts that around $Q = 0$ the aggregate impact is linear.
- The linearity region increases with the aggregation scale N (length of the interval)
- The slope decreases with N
- This behavior is reproduced also by a model with correlated order flow and transient impact (TIM model, see below)
- The linear relation between order imbalance and returns at an aggregated level is often interpreted as a manifestation of the permanent component of the impact. In this context is a purely mechanical/statistical effect.

Order flow

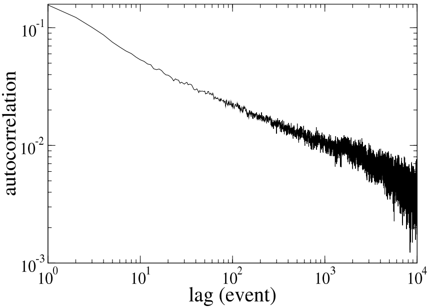
Order flow

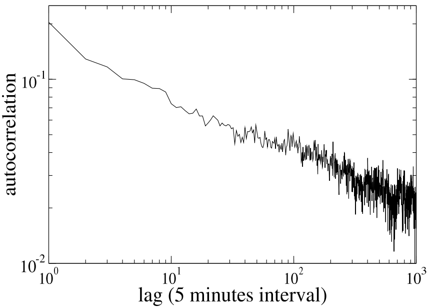
- We focus here on orders that trigger transactions, i.e. *market orders*
- A buy market order moves the price up and a sell market order moves the price down (on average)
- The flow of market orders reflects the supply and demand of shares
- A market order is characterized by a volume v and a sign $\epsilon = +1$ for buy orders and $\epsilon = -1$ for sell orders.
- We consider the time series in market order time, i.e. time advances of one unit when a new market order arrives.
- The unconditional sample autocorrelation function of signs is

$$C(\tau) = \frac{1}{N}\sum_t \epsilon_t \epsilon_{t+\tau} - \left(\frac{1}{N}\sum_t \epsilon_t\right)^2,$$

where N is the length of the time series.

Autocorrelation of market order flow

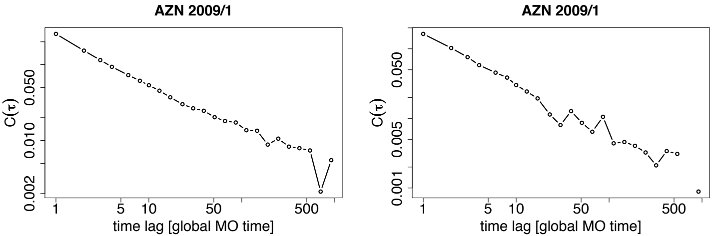




From Lillo and Farmer (2004)
Similar plots observed in many different markets, different periods, different asset classes.

Market order flow is very persistent in time

It has been shown (Bouchaud *et al.*, 2004, Lillo and Farmer, 2004) that the time series of market order signs is a long memory process.



$C(\tau)$ of market order signs ϵ (left) and signed volumes ϵv (right).
The autocorrelation function decays asymptotically as

$$C(\tau) \sim \tau^{-\gamma} = \tau^{2H-2}$$

where H is the Hurst exponent. For the investigated stocks $H \simeq 0.75$ (i.e. $\gamma \simeq 0.5$).

Long memory processes

- Let $\gamma(k)$ be the autocovariance function of a time series X_t. A process is long memory if in the limit $k \to \infty$ it is

$$\gamma(k) \sim k^{-\gamma} L(k) \qquad \gamma \in (0,1) \tag{29}$$

where $L(k)$ is a slowly varying function.

- The Hurst exponent is $H = 1 - \gamma/2$
- Equivalently the spectral density diverges for low frequencies $\omega \to 0$ as

$$g(\omega) \simeq \omega^{1-2H} L(\omega) \tag{30}$$

- The integrated process is superdiffusive $Var(\sum_{s=0}^{t} X_s) \sim t^{2H}$
- Examples: fractional ARIMA (fARIMA)

$$(1-L)^d X_t = \epsilon_t \qquad d = H - 1/2 \tag{31}$$

fractional Brownian motion (in continuous time)

- Frequently observed in finance: volatility, volume, spread,....

What is the origin of long-memory in order flow?

Two explanations have been proposed

- Herding among market participants (LeBaron and Yamamoto 2007). Agents herd either because they follow the same signal(s) or because they copy each other trading strategies. Direct vs indirect interaction
- Order splitting (Lillo, Mike, and Farmer 2005). To avoid revealing true intentions, large investors break their trades up into small pieces and trade incrementally (Kyle, 1985). Convert heavy tail of large orders volume distributions in correlated order flow.

Is it possible to quantify **empirically** the contribution of herding and order splitting to the autocorrelation of order flow?
Note that this is part of the question on the origin of *diagonal effect* raised in Biais, Hillion and Spatt (1995).

Decomposing the autocorrelation function

Assume we know the identity of the investor placing any market order.

- For each investor i we define a time series of market order signs ϵ_t^i which is equal to zero if the market order at time t was not placed by investor i and equal to the market order sign otherwise
- The autocorrelation function can be rewritten as

$$C(\tau) = \frac{1}{N}\sum_t \sum_{i,j} \epsilon_t^i \epsilon_{t+\tau}^j - \left(\frac{1}{N}\sum_t \sum_i \epsilon_t^i\right)^2$$

Decomposing the autocorrelation function

We rewrite the acf as $C(\tau) = C_{split}(\tau) + C_{herd}(\tau)$ where

$$C_{split}(\tau) = \sum_i \left(P^{ii}(\tau) \left[\frac{1}{N^{ii}(\tau)} \sum_t \epsilon_t^i \epsilon_{t+\tau}^i \right] - \left[P^i \frac{1}{N^i} \sum_t \epsilon_t^i \right]^2 \right)$$

$$C_{herd}(\tau) = \sum_{i \neq j} \left(P^{ij}(\tau) \left[\frac{1}{N^{ij}(\tau)} \sum_t \epsilon_t^i \epsilon_{t+\tau}^j \right] - P^i P^j \left[\frac{1}{N^i} \sum_t \epsilon_t^i \right] \left[\frac{1}{N^j} \sum_t \epsilon_t^j \right] \right)$$

N^i is the number of market orders placed by agent i, $P^i = N^i/N$, $N^{ij}(\tau)$ is the number of times that an order from investor i at time t is followed by an order from investor j at time $t+\tau$, and $P^{ij}(\tau) = N^{ij}(\tau)/N$

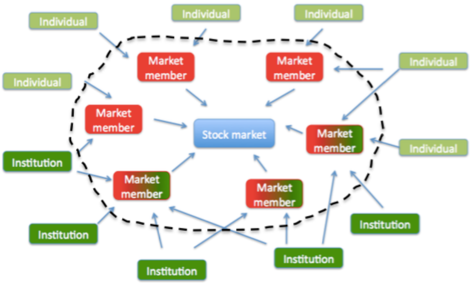
Market member (brokerage) data

- The investigated markets are:
- Spanish Stock Exchange (BME) 2001-2004
- London Stock Exchange (LSE) 2002-2004

❑ Firms are credit entities and investment firms which are members of the stock exchange and are entitled to trade in the market.

❑ Roughly 200 Firms in the BME and LSE (350/250 in the NYSE)

VALOR	VOLUMEN	PRECIO	SOCCOM	SOCVEN	HORA	FECHA
TEF	236	2187	9405	9858	90108	01/06/2000
TEF	1764	2187	9405	9487	90108	01/06/2000
ANA	110	3800	9839	9855	90109	01/06/2000
CAN	37	2194	9839	9578	90109	01/06/2000
CAN	151	2200	9839	9412	90109	01/06/2000
VIS	214	700	9821	9561	90109	01/06/2000
SOL	286	1299	9839	9838	90110	01/06/2000
ALB	104	2710	9839	9843	90110	01/06/2000
ALB	29	2719	9839	9419	90110	01/06/2000
ACX	97	3689	9839	9843	90111	01/06/2000
AGS	120	1445	9839	9487	90111	01/06/2000
AGS	110	1448	9839	9485	90111	01/06/2000
ACS	107	2930	9839	9863	90111	01/06/2000
SCH	11226	1045	9858	9880	90112	01/06/2000
CTE	96	1935	9839	9832	90112	01/06/2000
CTE	50	1955	9839	9872	90112	01/06/2000
CTE	14	1958	9839	9426	90112	01/06/2000
FER	237	1296	9839	9560	90112	01/06/2000
SGC	50	3980	9820	9560	90113	01/06/2000
ACR	161	1139	9839	9487	90113	01/06/2000
ACR	47	1140	9839	9845	90113	01/06/2000
DRC	20	803	9839	9573	90114	01/06/2000
DRC	267	805	9839	9484	90114	01/06/2000
AUM	111	1649	9839	9474	90114	01/06/2000



- Investigation at the level of market members and not of the agents (individuals and institutions
- The dataset covers the whole market
- The resolution is at the level of individual trade (no temporal aggregation)

Market members' persistence in activity

The activity of market members (independently from their trading direction) is characterized by the persistence

$$\tilde{P}^{ii}(\tau) = P^{ii}(\tau) - (P^i)^2$$

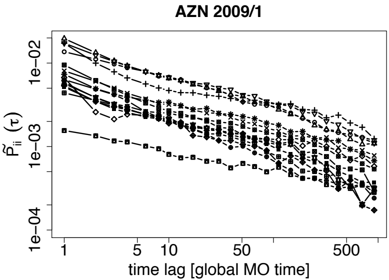


Figure: The diagonal terms of persistence in activity, i.e., $P^{ii}(\tau) - [P^i]^2$ of MO placement for the 15 most active participant codes, the first half of 2009 for AZN.

Market member activity is highly clustered in (transaction) time. I.e. there is some degree of predictability that a member active now will be active in the near future.

Herding or splitting?

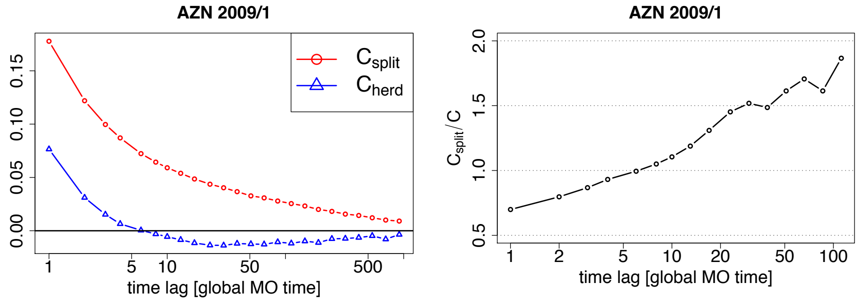


Figure: Left panel. The splitting and the herding term of the correlation of MO signs (the two terms sum up to $C(\tau)$) for the first half year of 2009 for AZN. Right panel. The splitting ratio of MO signs (defined as the ratio of the splitting term in the correlations and the entire correlation) for the first half year of 2009 for AZN.

From Toth et al. 2015.

Splitting dominates herding at the broker level (especially for large lags)

Antiherding

$C_{herd}(\tau) < 0$ is statistically significant when $10 \lesssim \tau \lesssim 80$. Why?

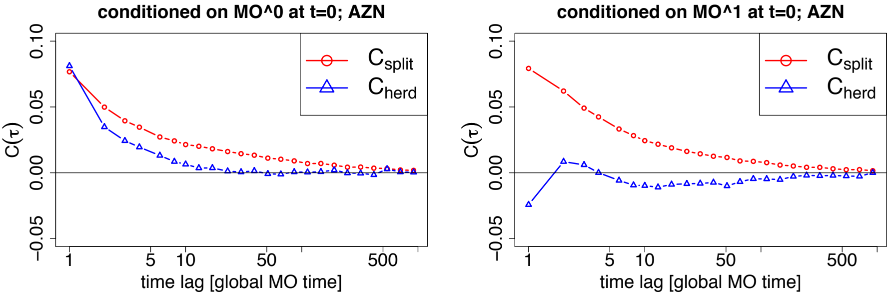


Figure: Splitting and herding component of the MO sign acf conditional to the event at time t, a market order that does not changes the price (left panel) and a market order that does change the price (right panel).

The antiherding phenomenon is due to investors refraining from placing market orders in the same direction of a recent market order of another investor that changed the price.

Brokers vs agents: two stylized models

- Herding and the splitting behavior should be observed at the investor level not at broker level, but our data do not allow to do this
- The relation between investors and brokers is in general complex and not fully explored empirically.
- In Toth et al (2015) we develop two stylized agent based models that take into account possible variations of the broker-investor relation and of the mechanism responsible for the long memory.
    - Long memory is generated exogenously by an autocorrelated signal and brokers have a different probability of trading following the signal
    - Long memory is generated endogenously and the apparent splitting at the broker level comes both from the heterogeneity of brokers activity and from the correlated choice of brokers by agents which are close in the network of influence.
- No splitting in both models
- Main conclusion: **Agent based models of the broker-investor relation are able to reproduce the splitting herding decomposition only at a cost of having an unrealistic level of heterogeneity of activity among brokers**

A preliminary investigation of real agents

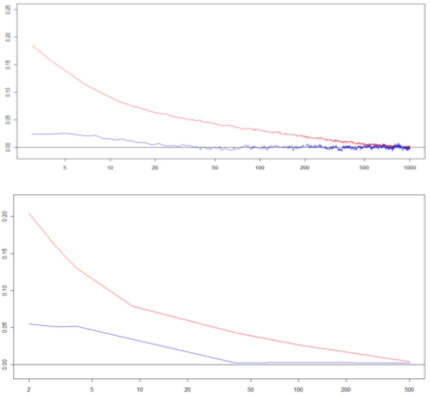


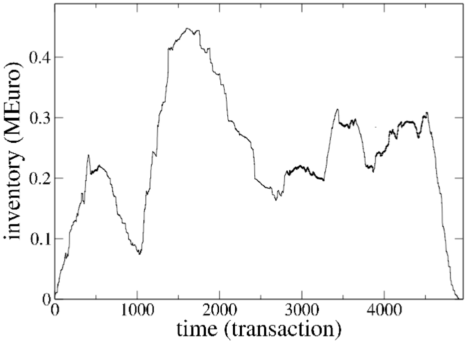
Figure: Splitting and herding component for brokers (top) and agents/accounts (bottom) of a European stock.

Direct evidence for order splitting

- We have seen that correlated order flow is mostly due to order splitting.
- We want to find direct evidence of splitting, characterize the large trades and the splitting characteristics, and to measure the market impact of these large orders.
- The difficulty is, of course, data.
- Some studies use proprietary data of a large financial institution
- We follow a different approach: **statistical identification of large trades from market member data.**

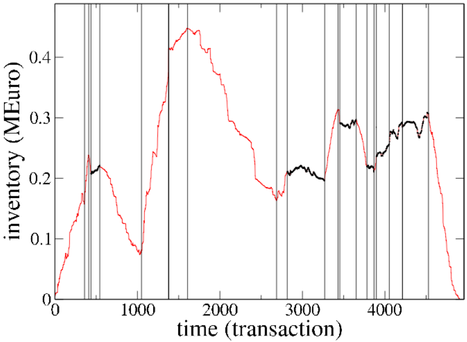
An example: inventory time series

Credit Agricole trading Santander



- Clear trends are visible
- The identification of large trades (*metaorders*) must be statistical: a typical regime switching problem

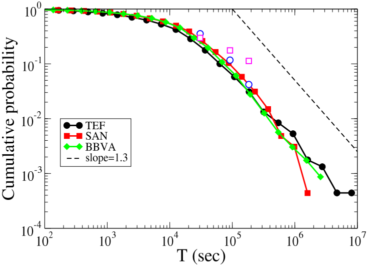
Credit Agricole trading Santander

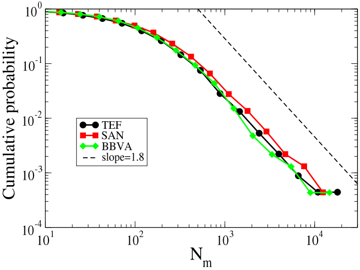


Different algorithms:

- Modified t-test (G. Vaglica, F. Lillo, E. Moro, and R. N. Mantegna, Physical Review E **77**, 036110 (2008).)
- Hidden Markov Model (G. Vaglica, F. Lillo, and R. N. Mantegna, New Journal of Physics, **12** 075031 (2010)).

Scaling of metaorder size





- Metaorder size is asymptotically power law distributed
- Different "size" measures (number of trades, time, total volume) roughly agree on the tail exponent.
- Which model can generate this power law distribution?

The Lillo-Mike-Farmer (2005) order flow model

- M funds that want to trade one metaorder each of a size L_i $(i = 1, .., M)$ taken from a distribution p_L, $(L \in \mathbb{N}^+)$.
- The sign of the each metaorder is taken randomly and at each time step one fund is picked randomly with uniform probability.
- The selected fund initiates a trade of the sign of its metaorder, and the size of the metaorder is reduced by one unit.
- When the metaorder is completely traded, a new one is drawn from p_L and assigned a random sign.
- The distribution p_L of metaorder size with the autocorrelation function of trade signs. In particular if the distribution is Pareto

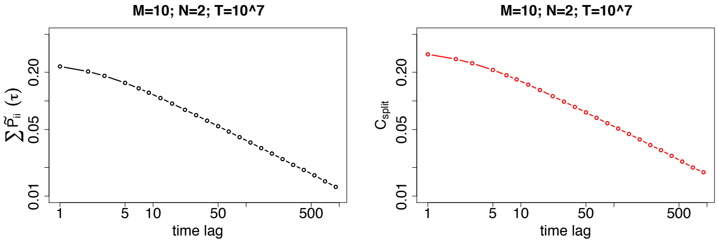
$$p_L = \frac{1}{\zeta(\alpha)} \frac{1}{L^{1+\alpha}}$$

where $\zeta(\alpha)$ is the Riemann zeta function, then the autocorrelation function of trade signs decays asymptotically as

$$\rho_s(\ell) = E[\epsilon_n \epsilon_{n+\ell}] \sim \frac{M^{\alpha-2}}{\ell^{\alpha-1}}$$

The Lillo-Mike-Farmer (2005) order flow model (II)

- The model connects the exponent of the autocorrelation function of order signs with the tail exponent of metaorder distribution, since $\gamma = 1 - \alpha$.
- There is a growing empirical evidence that the distribution of metaorder size is asymptotically Pareto distributed with a tail exponent close to $\alpha = 1.5$ (Lillo et al. 2005, Gabaix et al. 2006, Vaglica et al. 2008, Bershova et al. 2013).
- Hence the model predicts that $\gamma = \alpha - 1$, i.e. $\gamma \simeq 0.5$, as observed empirically.
- Numerical simulations



Indirect validation of the splitting model

In many markets there are two alternative methods of trading

- The on-book (or downstairs) market is public and execution is completely automated (Limit Order Book)
- The off-book (or upstairs) market is based on personal bilateral exchange of information and trading.

We assume that revealed orders are placed in the on-book market, whereas off-book orders are proxies of metaorders

Indirect validation of the splitting model

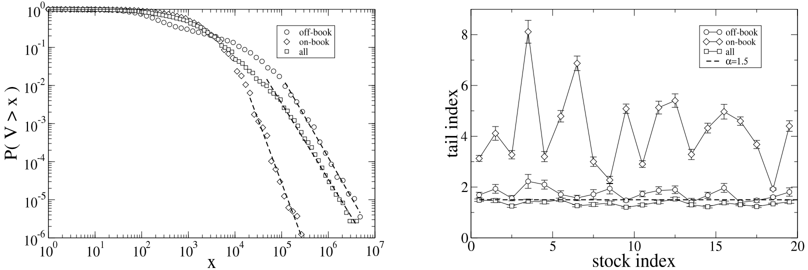
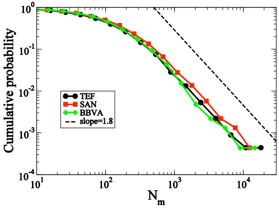


Figure: From Lillo et al 2005. Left. Volume distributions of off-book trades (circles), on-book trades (diamonds), and the aggregate of both (squares). The dashed black lines have the slope found by the Hill estimator and are shown for the largest one percent of the data. Right. Hill estimator of the tail exponent.

The fitted exponent $\alpha \simeq 1.5$ for the metaorder size and the market order sign autocorrelation exponent γ are consistent with the order splitting model ($\gamma = \alpha - 1$).

Direct validation of the splitting model



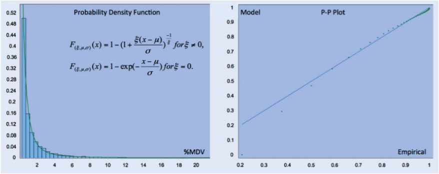


Figure 2. (a) Log–log plot of number of orders vs. order size expressed as %MDV. The table below shows statistics for linear fit model implying that the probability of order size S decays as a power function of S. (b) Distribution of order sizes (as %MDV): Fit to Generalized Pareto with shape parameter $\xi = 0.64$, suggesting $\beta \sim 1/\xi = 1.56$.

- Metaorder size is asymptotically power law distributed (left from Vaglica et al 2008)
- The tail exponent is consistent with the splitting model
- Recently Bershova and Rakhlin (2013) found a tail exponent of 1.56 by investigating metaorders of clients of AllianceBernstein (right)

Heterogeneity of time scales of agents in financial markets

There is a growing evidence of a power law tailed distribution of time scales of agents

- Distribution of duration of metaorders has a tail exponent of ~ 1.5 (Vaglica et al 2008, Bershova and Rakhlin 2013)
- A multiscale GARCH introduced by Borland and Bouchaud (2005) where agents use stop loss on a given time horizon is consistent with data if the distribution of time scale is power law with tail exponent ~ 1.2
- A simple optimization argument for limit order execution shows that fat tail in limit order prices is consistent with a power law tailed time horizon distribution with exponent ~ 1.5 (Lillo 2007)
- A censored data analysis of the time to fill of limit orders can be used to obtain the distribution of intended lifetime of limit orders. Empirical data are consistent with a power law distribution with tail exponent ~ 1.6 (Eisler et al 2009).

Building a predictor: The DAR(p) model

DAR(p) model: a generalization of autoregressive models for discrete valued variates

$$X_n = V_n X_{n-A_n} + (1 - V_n) Z_n,$$

$$Z_n \sim \Xi, \qquad V_n \sim \mathcal{B}(1, \chi), \qquad P(A_n = i) = \phi_i, \quad \sum_{i=1}^{p} \phi_i = 1$$

- Autocorrelation function $\rho_k = Corr(X_n, X_{n+k})$ satisfies:

$$\rho_k = \chi \sum_{i=1}^{p} \phi_i \rho_{k-i}, \qquad k \geqslant 1$$

- Model predictor conditional on $\Omega_{n-1} = \{X_{n-1}, \ldots, X_{n-p}\}$:

$$\hat{X}_{n+s} \equiv \mathbb{E}[X_{n+s} | \Omega_{n-1}] = \chi \sum_{i=1}^{p} \phi_i Y_{n+s-i} + \mathbb{E}[Z](1 - \chi), \quad Y_{n+s-i} = \begin{cases} \hat{X}_{n+s-i} & \text{for } i \leqslant s \\ X_{n+s-i} & \text{for } i > s \end{cases}$$

Predictability of order flow has significantly increased from 2004 to 2009

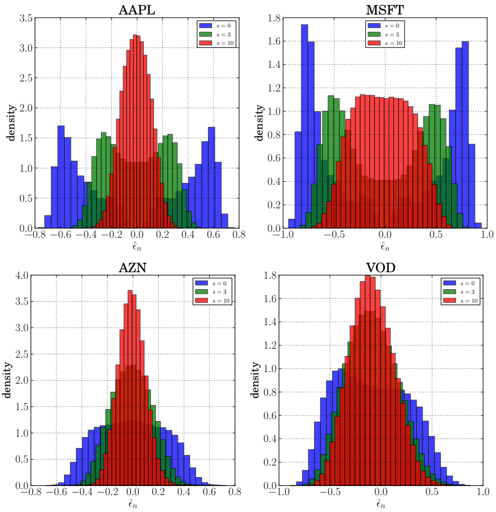


Figure: From Taranto et al. 2014. Distributions of the sign predictor for the stocks AAPL, MSFT, AZN, VOD and $s = 0, 3, 10$ with a DAR(p) model with $p = 500$.

Predictability of order flow and metaorder execution

- The exact predictor discriminates orders due to the active metaorder to those due to the noisy background.
- If n trades of the current metaorder has been already traded, the probability that the metaorder continues is (Farmer et al. 2013)

$$\mathcal{P}_n = \frac{\sum_{i=n+1}^{\infty} p_i}{\sum_{i=n}^{\infty} p_i}$$

- For example, if the metaorder size distribution is Pareto

$$\mathcal{P}_n = \frac{\zeta(1+\alpha, 1+n)}{\zeta(1+\alpha, n)} \simeq \left(\frac{n}{n+1}\right)^{\alpha} \sim 1 - \frac{\alpha}{n}$$

- Let us suppose that the active metaorder is a buy and the participation rate is π. The probability that the next order is a buy is

$$p_n^+ = \frac{1-\pi}{2} + \pi\left(\mathcal{P}_n + \frac{1-\mathcal{P}_n}{2}\right) = \frac{1+\pi\mathcal{P}_n}{2}$$

- If s_n indicates the sign of the active metaorder at time n

$$p_n^+ = \frac{1+s_n\pi\mathcal{P}_n}{2}$$

Since $\hat{\epsilon}_n = 2p_n^+ - 1$, it is

$$\hat{\epsilon}_n = s_n\pi\mathcal{P}_n \sim s_n\pi(1-n^{-1})$$

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