Bayesian model comparison

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1. A coin is tossed $N = 250$ times and it returns $H = 140$ heads. Evaluate the evidence that the coin is biased using Bayesian model comparison and contrast your findings with the usual (frequentist) hypothesis testing procedure (*i.e.*, testing the null hypothesis that $p_H = 0.5$). Discuss the dependency on the choice of priors.

2. In 1919 two expeditions sailed from Britain to measure the light deflection from stars behind the Sun's rim during the solar eclipse of May 29th. Einstein's General Relativity predicts a deflection angle

$$\alpha = \frac{4GM}{c^2 R},$$

where G is Newton's constant, c is the speed of light, M is the mass of the gravitational lens and R is the impact parameter. It is well known that this result it exaclty twice the value obtained using Newtonian gravity. For $M = M_\odot$ and $R = R_\odot$ one gets from Einstein's theory that $\alpha = 1.74$ arc seconds.

The team led by Eddington reported 1.61 ± 0.40 arc seconds (based on the position of 5 stars), while the team headed by Crommelin reported 1.98 ± 0.16 arc seconds (based on 7 stars).

What is the Bayes factor between Einstein and Newton gravity from those data? Comment on the strength of evidence.

3. Assume that the combined constraints from CMB, BAO and SNIa on the density parameter for the cosmological constant can be expressed as a Gaussian posterior distribution on Ω_Λ with mean 0.7 and standard deviation 0.05. Use the Savage-Dickey density ratio to estimate the Bayes factor between a model with $\Omega_\Lambda = 0$ (i.e., no cosmological constant) and the ΛCDM model, with a flat prior on Ω_Λ in the range $0 \le \Omega_\Lambda \le 2$. Comment on the strength of evidence in favour of ΛCDM.

4. If the cosmological constant is a manifestation of quantum fluctuations of the vacuum, QFT arguments lead to the result that the vacuum energy density ρ_Λ scales as

$$\rho_\Lambda \sim \frac{c\hbar}{16\pi} k_{\mathrm{max}}^4 \tag{1}$$

where k_{max} is a cutoff scale for the maximum wavenumber contributing to the energy density[1]. Adopting the Planck mass as a plausible cutoff scale (i.e., $k_{\mathrm{max}} = c/\hbar M_{\mathrm{Pl}}$) leads to "the cosmological constant problem", i.e., the fact that the predicted energy density

$$\rho_\Lambda \sim 10^{76}\ \mathrm{GeV}^4 \tag{2}$$

is about 120 orders of magnitude larger than the observed value, $\rho_{\mathrm{obs}} \sim 10^{-48}\ \mathrm{GeV}^4$.

Repeat the above estimation of the evidence in favour of a non-zero cosmological constant, adopting this time a flat prior in the range $0 \le \Omega_\Lambda/\Omega_\Lambda^{\mathrm{obs}} < 10^{120}$. What is the meaning of this result? What is

[1] See e.g. Carroll & Press, *Ann. Rev. Astron. Astrophys.* 30:499-542, 1992.

the required observational accuracy (as measured by the posterior standard deviation) required to override the Occam's razor penalty in this case?

It seems that it would be very difficult to create structure in a universe with $\Omega_\Lambda \gg 100$, and so life (at least life like our own) would be unlikely to evolve. How can you translate this "anthropic" argument into a quantitative statement, and how would it effect our estimate of Ω_Λ and the model selection problem?

5. **Optional – Evidence for a flat Universe from supernovae type Ia**[2]

   This problem follows up the cosmological parameter estimation problem from supernovae type Ia done in Day 2.

   (a) If you have not already done so, generalise your MCMC code to a non-flat Universe, including the cosmological constant energy density, Ω_v, as a free parameter.
   Adopt uniform priors: $\Omega_\mathrm{m} \sim U(0,2)$ and $\Omega_\mathrm{v} \sim U(0,2)$.

   (b) Produce a 2D marginalised posterior pdf in the $(\Omega_\mathrm{m}, \Omega_\mathrm{v})$ plane.

   (c) Produce a 1D marginalised posterior pdf for the curvature parameter, $\Omega_\kappa = 1 - \Omega_\mathrm{v} - \Omega_\mathrm{m}$, paying attention to normalising it to unity probability content.
   What is the shape of the prior on Ω_κ implied by your choice of a uniform prior on $\Omega_\mathrm{m}, \Omega_\mathrm{v}$?

   (d) Use the Savage-Dickey density ratio formula to estimate from the above 1D posterior the evidence in favour of a flat Universe, $\Omega_\kappa = 0$, compared with a non-flat Universe, $\Omega_\kappa \neq 0$, with prior $P(\Omega_\kappa) = U(-1,1)$.
   Discuss the dependency of your result on the choice of the above prior range.

[2] For a more thorough treatment, see M. Vardanyan, R. Trotta and J. Silk (2009) Mon. Not. R. Astron. Soc. 397 , 431-444 (2009) and M. Vardanyan, R. Trotta and J. Silk (2011) MNRASLett 413, 1, 2011, L91ÐL95.