Counting curves and Gromov–Witten invariants

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Projective plane

The complex plane $\mathbb{C}^2 = \{(x,y) \mid x,y \in \mathbb{C}\}$ can be compactified by adding a new point, which is called a **point at infinity**, for each direction. In this way two parallel lines meet at the point at infinity corresponding to their common direction. This new space is called the **projective plane** and is denoted by $\mathbb{P}^2_{\mathbb{C}}$.

Points in $\mathbb{P}^2_{\mathbb{C}}$ can be described by the triple of their **homogeneous coordinates** $[x_0 : x_1 : x_2]$, where x_0, x_1, x_2 are arbitrary complex numbers such that at least one among them is non zero. The homogeneous coordinates are defined up to multiplication by non zero complex numbers, e.g. $[1:2:3]$ and $[-2:-4:-6]$ are the same point of $\mathbb{P}^2_{\mathbb{C}}$. The embedding $\mathbb{C}^2 \hookrightarrow \mathbb{P}^2_{\mathbb{C}}$, called chart, is given by $(x,y) \mapsto [1:x:y]$ and the points $[0:x_1:x_2]$ constitute the line of the points at infinity.

Distinguishing between points at infinity and points not at infinity is not relevant, since you may always change your coordinates in order to make your favourite line be the line at infinity. $\mathbb{P}^2_{\mathbb{C}}$ is a compact topological space in a natural way and has the structure of complex manifold.

Counting plane curves

Question. How many rational plane curves of degree d pass through $3d-1$ general points?

Fix an integer $d > 0$. Fix points $p_1, \dots, p_{3d-1}$ in general position in the projective plane $\mathbb{P}^2_{\mathbb{C}}$. Look at the following number:

$$N_d := \#\{C \subseteq \mathbb{P}^2_{\mathbb{C}} \text{ rational curve of degree } d \text{ and such that } p_1, \dots, p_{3d-1} \in C\}.$$

First examples. N_1 is the number of lines passing through 2 general points; therefore $N_1 = 1$. N_2 is the number of conics passing through 5 general points. Since a homogenous polynomial of degree 2 in 3 variables has 6 coefficients, $N_2 = 1$.

In general, N_d is a positive number and $N_3 = 12$, $N_4 = 620$, $N_5 = 87304$. These are very non trivial results and they were the only known numbers until 1994 when **Maxim Kontsevich** came out with a surprising recursive formula:

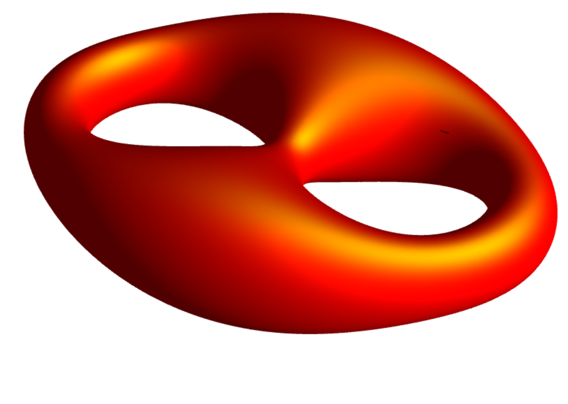
$$N_d = \sum_{\substack{d = d_1 + d_2 \\ d_1, d_2 \in \mathbb{N}^+}} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right).$$

His revolutionary approach introduced the **moduli space of stable maps** $\overline{\mathcal{M}}_{0,3d-1}(\mathbb{P}^2_{\mathbb{C}}, d)$. It is a compact topological space with some extra algebraic structure and its points correspond bijectively to morphisms $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ of degree d with a choice of $3d-1$ points on the domain $\mathbb{P}^1_{\mathbb{C}}$. In other words, the points of $\overline{\mathcal{M}}_{0,3d-1}(\mathbb{P}^2_{\mathbb{C}}, d)$ are parameterisations of plane rational curves of degree d. In this way, the number N_d can be interpreted as the degree-zero part of a homology class on $\overline{\mathcal{M}}_{0,3d-1}(\mathbb{P}^2_{\mathbb{C}}, d)$ and is the first example of a **Gromov–Witten invariant**:

$$N_d = \langle [\mathrm{pt}], \dots, [\mathrm{pt}] \rangle^{\mathbb{P}^2_{\mathbb{C}}}_{0,3d-1,d} = \int_{[\overline{\mathcal{M}}_{0,3d-1}(\mathbb{P}^2_{\mathbb{C}}, d)]^{\mathrm{vir}}} \prod_{i=1}^{3d-1} \mathrm{ev}_i^\star([\mathrm{pt}]).$$

Rational curves

Algebraic curves can have many shapes, as the underlying topological space of an algebraic curve is a compact connected (possibly singular) space of real dimension 2. If they are smooth, their **genus**, i.e. the number of holes of the doughnut, is an important invariant.



The Riemann sphere $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the unique smooth curve of genus 0. In other words, there exists only one complex structure on the sphere $\mathbb{S}^2$.

An algebraic curve C is called **rational** if it is close to be $\mathbb{P}^1_{\mathbb{C}}$ in the following sense: there must exist a non-constant morphism $\mathbb{P}^1_{\mathbb{C}} \to C$. Here, by morphism we mean a continuous function of topological spaces that is defined locally by polynomials.

All lines and conics are rational. Most of the plane curves of degree > 2 are not rational, therefore rational curves are somehow special.

Plane algebraic curves

The equation $a_1 x + a_2 y + a_0 = 0$ gives a line in the plane $\mathbb{C}^2$, if $a_1 \neq 0$ or $a_2 \neq 0$. In order to work in the projective plane we have to take the homogeneous equation $a_0 x_0 + a_1 x_1 + a_2 x_2 = 0$. The set of points of $\mathbb{P}^2_{\mathbb{C}}$ satisfying this equation is called a line in $\mathbb{P}^2_{\mathbb{C}}$ or a **plane curve of degree 1**.

Degree 2 polynomial equations, like $x^2 + y^2 = 1$, $y = x^2$, or $x^2 - y^2 = 1$, define conics. Their projective completions, e.g. the zero loci of $x_1^2 + x_2^2 = x_0^2$, $x_0 x_2 = x_1^2$, or $x_1^2 - x_2^2 = x_0^2$, are called conics in $\mathbb{P}^2_{\mathbb{C}}$ or **plane curves of degree 2**.

In general, fix an integer $d > 0$ and a homogeneous polynomial F in the variables x_0, x_1, x_2, with complex coefficients, and degree d. Then the set of points of $\mathbb{P}^2_{\mathbb{C}}$ satisfying the equation $F = 0$ is called a **plane curve of degree d**.

About the general theory

What we said above can be broadly generalised. Indeed, one may want to know the number of curves of genus g (and fixed homology class $\beta \in \mathrm{H}_2(X, \mathbb{Z})$) lying on a projective variety X and passing trough some submanifolds $Y_1, \dots, Y_n$. This number is denoted by $\langle [Y_1], \dots, [Y_n] \rangle^X_{g,n,\beta}$ and called **Gromov–Witten invariant**. One expects to get these numbers computing integrals on the moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ of genus g stable maps with target X. However, they are extremely hard to compute, usually rational (which is rather unsatisfactory!) and the so called "degenerate contributions" cause various unwanted behaviours.

Counting curves problems are a central topic of research in Algebraic Geometry and Mirror Symmetry and there is still a lot to understand, discover, adjust ...