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Harmonic Analysis on Groups

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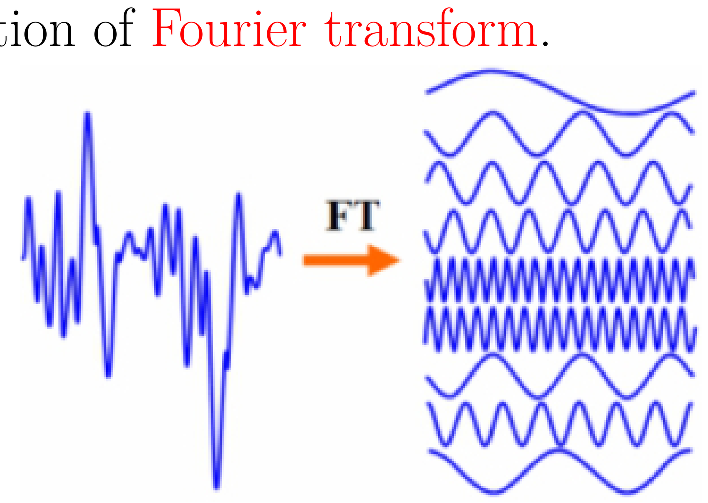
Objectives

- The abstract objective is to present the main ideas to **extend classical Harmonic Analysis** to the more abstract setting of locally-compact groups.
- To illustrate the ideas we apply them to a remarkable example of a locally yet not globally compact group, that is, the **Heisenberg Group** $\mathbb{H}_n$.

Classical Harmonic Analysis

Harmonic Analysis is a branch of mathematics concerned with the representation of a function as a superposition of simpler functions, based on the study of the notion of Fourier transform.



This is a powerful method to tackle many problems from applied sciences, such as engineering, neurovision and image/sound processing. The main tool in this theory is the concept of Fourier transform: let $f \in L^2(\mathbb{R}^d)$, then the Fourier transform of f is the function $\hat{f} : \mathbb{R}^d \to \mathbb{C}$ such that $\forall \xi \in \mathbb{R}^d$ we have:

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) dx. \qquad (1)$$

Furthermore, under suitable hypotheses on f, it is possible to represent it in terms of its Fourier transform:

$$f(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} \hat{f}(\xi) d\xi,$$

and the following Plancherel formula holds:

$$\|f\|_{L^2(\mathbb{R})} = \|\hat{f}\|_{L^2(\mathbb{R})}.$$

The trigonometric polynomials $\{e_\xi := e^{ix\cdot\xi}\}_{\xi\in\mathbb{R}^d}$ appearing in (1) are the 'simpler functions' that allow us to obtain the desired representation.

Generalisation to a group G

Requirements

- We want to replace $\mathbb{R}^d$ with a suitable group G.
- We want to integrate over the chosen group G, therefore we need a measure on it. It has to satisfy an analogue of the translation invariance of the Lebesgue measure dx, that is
$$|A + x| = |A|, \qquad \forall A \subset \mathbb{R}^d$$
- We want to find an analogue of the trigonometric functions
$$e_\xi : \mathbb{R}^d \to \mathbb{C} \sim \mathcal{U}(\mathbb{C}),$$
such that
$$e_\xi(x+y) = e_\xi(x)e_\xi(y), \qquad \forall x, y \in \mathbb{R}^d.$$
Note that we are associating to $\mathbb{R}^d$ a new space $\widehat{\mathbb{R}^d} = \{e_\xi \,|\, \xi \in \mathbb{R}^d\} \sim \mathbb{R}^d$.

Ingredients

- We choose $(G, \cdot)$ to be a locally compact group.
- It is possible to show that on every locally compact group, there is a left-invariant measure $d\mu$, called Haar measure [2].
- Let $\mathcal{H}_\pi$ be a Hilbert space of any dimension d_π. Then consider the unitary representation π of G on $\mathcal{H}_\pi$, that is, the homomorphism
$$\pi : G \to \mathcal{U}(\mathcal{H}_\pi).$$
A subset $V \subset \mathcal{H}_\pi$ is said to be invariant for a representation π if $\pi(x)V \subset V$ for all $x \in G$. We look for irreducible representations, i.e., those for which the only invariant spaces are $\{0\}$ and $\mathcal{H}_\pi$. Then we can define the unitary dual of G to be
$$\widehat{G} = \{[\pi] \,|\, \pi \text{ is a unitary, irreducible rep. of } G\}.$$

Group Fourier Transform

Given $f \in L^2(G)$ for every $[\pi] \in \widehat{G}$ we can define the operator $\hat{f}(\pi) : \mathcal{H}_\pi \to \mathcal{H}_\pi$ where $\mathcal{H}_\pi$ is the Hilbert space associated to π:

$$\hat{f}(\pi) := \int_G \pi^*(x) f(x) d\mu.$$

More precisely, for every $v_1, v_2 \in \mathcal{H}_\pi$ we can write:

$$\left(\hat{f}(\pi)v_1, v_2\right)_{\mathcal{H}_\pi} = \int_G (\pi^*(x)v_1, v_2)_{\mathcal{H}_\pi} f(x) dx.$$

The Heisenberg group

Let us now consider an example of a non-compact, locally compact group, that is, the Heisenberg group:

$$G = \mathbb{H}_n \sim \mathbb{R}^{2n+1},$$

endowed with the non-commutative group law:

$$(x, y, t)(x', y', t') := (x + x', y + y', t + t' + \frac{1}{2}(xy' - x'y)).$$

The Haar measure of $\mathbb{H}_n$ coincides with the Lebesgue measure of $\mathbb{R}^{2n+1}$. All the unitary, irreducible representations of $\mathbb{H}_n$ are given by

$$\pi_\lambda : \mathbb{H}_n \to \mathcal{U}(L^2(\mathbb{R}^n)),$$

where

$$\pi_\lambda(x, y, t)\phi(u) = [\pi_1(x, \lambda y, \lambda t)](u) = e^{i\lambda(t + \frac{1}{2}xy + yu)}\phi(u + x),$$

for $\lambda \in \mathbb{R}/\{0\}$. These π_λ are called Schrödinger representations, and $\widehat{\mathbb{H}_n} = \{[\pi_\lambda] \,|\, \lambda \in \mathbb{R} \setminus \{0\}\}$.

Group Fourier Transform on $\mathbb{H}_n$

We define the group Fourier transform on $\mathbb{H}_n$ as an operator-valued function: for a function $f \in L^1(\mathbb{H}_n)$ the associated Fourier transform is a family of operators acting on the Hilbert space $L^2(\mathbb{R}^n)$. More precisely, for every $\lambda \in \mathbb{R} \setminus \{0\}$, the group Fourier transform at π_λ is given by

$$\hat{f}(\pi_\lambda)\varphi(u) = \int_{\mathbb{H}_n} f(x, y, t) e^{i\lambda(t + \frac{1}{2}xy + uy)} \varphi(u + x) dx dy dt.$$

for every $\varphi \in L^2(\mathbb{R}^n)$ and $u \in \mathbb{R}^n$. We also have the analogue to the Euclidean case:

$$\|f\|^2_{L^2(\mathbb{H}_n)} = \int_{\mathbb{R}} \|\pi_\lambda(f)\|^2_{HS} \frac{|\lambda|^n d\lambda}{(2\pi)^{n+1}} = \|\hat{f}\|^2_{L^2(\widehat{\mathbb{H}}_n)}.$$

Applications & Open Questions

- The Heisenberg group is one of the most important structures studied in sub-Riemannian geometry. The latter is a valuable tool to study neurovision problems [1].
- Moreover, considering reductive groups, one finds that abstract harmonic analysis has applications in number theory, e.g., to prove analytic continuation of the Selberg zeta-function [2].
- We have used this theory to study analogues of Gevrey spaces on groups with hypo-elliptic operators. On the Heisenberg group we have a characterisation equivalent to the Euclidean case. The case on a general group is still open.

References

[1] G. Citti, A. Sarti, *Neuromathematics of Vision*, Springer, 2014.

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[3] E. M. Stein, *Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, 1993.

[4] `https://1millionmonkeystyping.files.wordpress.com/2014/07/fourier5.jpg`