Rough calculus on manifolds

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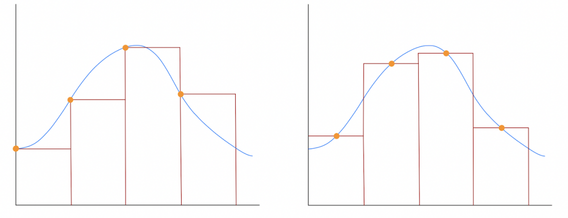
Maths PhD symposium
Imperial College (virtual)
July 14, 2021

Back to Riemann sums

Consider the Riemann integral

$$\int_a^b f(x)\mathrm{d}x := \lim_{|\pi|\to 0} \sum_i f(x_i)(x_{i+1} - x_i)$$
$$:= \lim_{|\pi|\to 0} \sum_i f\left(\frac{x_i + x_{i+1}}{2}\right)(x_{i+1} - x_i)$$



Itô vs. Stratonovich

- $\mathrm{d}Y_t^k = \mu^k(Y_t,t)\mathrm{d}t + \sigma_\gamma^k(Y_t,t)\underbrace{\mathrm{d}X_t^\gamma}_{\sim\mathcal{N}(0,I\mathrm{d}t)} \quad , \quad Y_0 = y_0$

  $\mu, \sigma_1, \ldots, \sigma_m \in \Gamma T\mathbb{R}^n$
- $Y_t^k = y_0 + \int_0^t \mu^k(Y_t,t)\mathrm{d}t + \int_0^t \sigma_\gamma^k(Y_t,t)\mathrm{d}X_t^\gamma$

  $\mu, \sigma_1, \ldots, \sigma_m \in \Gamma T\mathbb{R}^n$
- X Brownian motion, a.s. non-differentiable: must integrate;
- But X a.s. not bounded variation: Riemann-Stieljes sums a.s. do not converge;
- Luckily, convergence does hold in probability...
- ... but the result depends on the evaluation point!

$$\mathbb{P}\lim_{|\pi|\to 0} \sum_{[s,t]\in\pi} H_s(X_t - X_s) =: \int H\mathrm{d}X \qquad \text{(Itô)}$$

$$\neq \mathbb{P}\lim_{|\pi|\to 0} \sum_{[s,t]\in\pi} H_{\frac{s+t}{2}}(X_t - X_s) =: \int H \circ \mathrm{d}X \quad \text{(Stratonovich)}$$

Rough paths

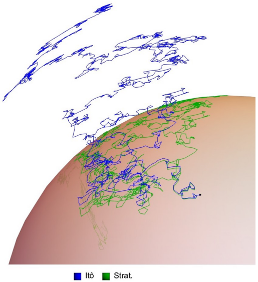
- In stochastic analysis there are many different inequivalent integration theories;
- Rough paths: a rigorous, general description of what an integration theory might look like;
- Idea: postulate the values of the (otherwise undefined)

$$\boldsymbol{X}_{st}^{\gamma_1,\ldots,\gamma_n} := \int_{s<u_1<\ldots<u_n<t} \mathrm{d}X_{u_1}^{\gamma_1} \cdots \mathrm{d}X_{u_n}^{\gamma_n}$$

  and use these to define more general integrals - and thus differential equations - via Taylor-type expansions;
- $\boldsymbol{X}$ must satisfy: (i) **regularity**, (ii) **additivity**, and optionally (iii) **integration by parts**;
- Rough path theory gives meaning to $\int \cdot \, \mathrm{d}X$ for many processes X beyond Brownian motion (Gaussian, Markov,...),
- and in doing so separates probability and analysis (solution map $\boldsymbol{X} \mapsto Y$ is continuous).

Rough paths on smooth manifolds

- My research: transfer the theory to the curved setting;
- If we drop (iii) weird things happen: $\mathrm{d}f(X) = f'(X)\mathrm{d}X$ no longer holds!
- This causes problems: $\mathrm{d}Y^k = V^k_\gamma(Y)\mathrm{d}\boldsymbol{X}^\gamma$ no longer coordinate-invariant;
- Extrinsic viewpoint: if V_γ // $M \subset \mathbb{R}^d$, Y no longer stays on M



Rough paths on smooth manifolds II

- Change of variable formula for $1/3 < \alpha$-Hölder **non-geometric** rough paths:

$$\mathrm{d}f(X) = \partial_\gamma f(X)\mathrm{d}\boldsymbol{X}^\gamma + \frac{1}{2}\partial_{\alpha\beta} f(X)\mathrm{d}\widehat{X}^{(\alpha\beta)}$$

$$\widehat{X}^{(\alpha\beta)}_{st} := X^\alpha_{st} X^\beta_{st} - \boldsymbol{X}^{\alpha\beta}_{st} - \boldsymbol{X}^{\alpha\beta}_{st}$$

(Itô calculus: $\widehat{X} = [X]$ quadratic variation);

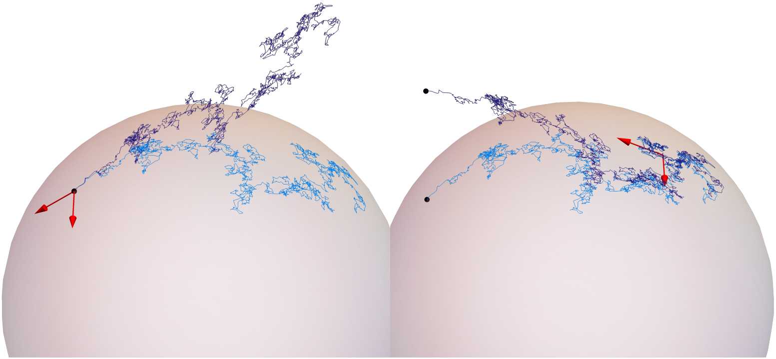
- Transfer principle: given a connection ∇

$$\mathrm{d}_\nabla \boldsymbol{X}^\gamma = \mathrm{d}\boldsymbol{X}^\gamma + \frac{1}{2}\Gamma^\gamma_{\alpha\beta}(X)\mathrm{d}\widehat{X}^{(\alpha\beta)}$$

transforms like a vector;

- Use this to develop rough version of classical differential geometry: path integrals, differential equations, parallel transport and Cartan development (which require a connection on TTM).

Rough paths on smooth manifolds III



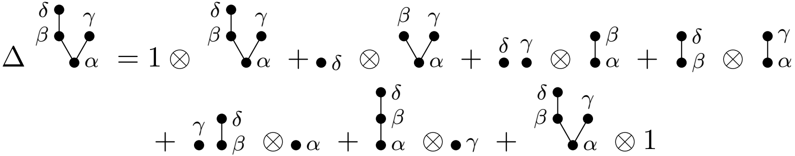
Cartan development of a 2D Brownian path on a sphere

Branched rough paths on Hopf algebras

- $1/3 \geq \alpha$-Hölder non-geometric rough paths are defined on the **Connes-Kreimer Hopf algebra** of non-planar rooted trees;
- Without integration by parts, Taylor expansions must involve more terms, e.g.

$$\boldsymbol{X}_{st} \text{ “}=\text{” } \int_s^t \bigg(\int_{u<v<w<t} \mathrm{d}X_w^\delta \mathrm{d}X_v^\beta \bigg) \cdot \bigg(\int_u^t \mathrm{d}X_z^\gamma \bigg) \mathrm{d}X_u^\alpha$$

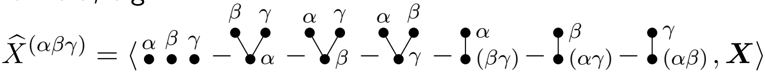
- The coproduct $\Delta \colon \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is defined in terms of cuts, e.g.



and is used to express additivity: $\boldsymbol{X}_{st}^\tau = \langle \Delta\tau, \boldsymbol{X}_{su} \otimes \boldsymbol{X}_{ut} \rangle$.

A transfer principle for branched rough paths

- What does a transfer principle for branched rough paths $\mathrm{d}_\nabla \boldsymbol{X}$ look like?
- Bracket extension $\widehat{\boldsymbol{X}}^{(\gamma_1,\dots,\gamma_n)}$ needed for change of variable formula, e.g.



- Given ∇ torsion-free, define local tensors $\nabla^m_{\beta_1,\dots,\beta_n} \in \Gamma TM^{\otimes m}$ by $\nabla_\gamma \coloneqq \partial_\gamma$ and

$$\begin{cases} \nabla^1_{\beta_1,\dots,\beta_n} \coloneqq \nabla_{\beta_1} \cdots \nabla_{\beta_n} & \\ \nabla^m_{\beta_1,\dots,\beta_n} \coloneqq 0 & m > n \\ \nabla^m_{\beta_1,\dots,\beta_n} \coloneqq \nabla_{\beta_1} \nabla^m_{\beta_2,\dots,\beta_n} + \nabla_{\beta_1} \otimes \nabla^{m-1}_{\beta_2,\dots,\beta_n} & 2 \le m \le n \end{cases}$$

and Christoffel symbols $\nabla^m_{\beta_1,\dots,\beta_n} \eqqcolon \Gamma^{\alpha_1,\dots,\alpha_m}_{\beta_1,\dots,\beta_n} \partial_{\alpha_1} \otimes \cdots \otimes \partial_{\alpha_m}$

A transfer principle for branched rough paths II

Theorem (Transfer principle for branched rough integrals)

The differential

$$\mathrm{d}_\nabla \widehat{\boldsymbol{X}}^{\alpha_1,\ldots,\alpha_m} := \Gamma^{\alpha_1,\ldots,\alpha_m}_{\beta_1,\ldots,\beta_n}(X)\mathrm{d}\widehat{\boldsymbol{X}}^{\beta_1,\ldots,\beta_n}$$

transforms as a tensor, and $\mathrm{d}_\nabla \boldsymbol{X}^\gamma$ *can be used to define the path integral* $\int f(X)\mathrm{d}_\nabla \boldsymbol{X}$ *for a one-form* f*. Moreover, the following change of variable formula holds:*

$$f(X_s) - f(X_t) = \frac{1}{n!}\int_s^t \nabla_{\gamma_1,\ldots,\gamma_n} f(X)\mathrm{d}_\nabla \widehat{\boldsymbol{X}}^{\gamma_1,\ldots,\gamma_n}$$

- The above transfer principle gives meaning to

$$\mathrm{d}_{\nabla^N}\boldsymbol{Y} = V(Y,X)\mathrm{d}_{\nabla^M}\boldsymbol{X}$$

when $\boldsymbol{X}$ is quasi-geometric.

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