Markov-Chain Approximation Method for a Class of Lévy Processes

Francesco Mina, Martijn Pistorius

Department of Mathematics, Imperial College London

The more we jump – the more we get – if not more quality, then at least more variety. J. Gleick, *Faster*

Aim and Approach

In recent years Lévy processes have become increasingly popular in the mathematical finance world since the introduction of jumps in the modelling of asset prices has led to many tractable and attractive models that perform significantly better than earlier models. Discretization schemes for stochastic processes lie at the heart of computational finance too. The aim of this work is to introduce a discretization scheme based on a continuous-time Markov chain for a class of Lévy processes, and we investigate the convergence properties of this scheme for one dimensional distributions.

Existing results: the diffusion case.

The problem of introducing a discretization scheme based on a continuous-time Markov chain for the Black-Scholes diffusion process has been widely studied. In the seminal paper [1] the authors established binomial trees as the paradigm for the constructive understanding of pricing theory. The key issue of the rate of convergence of the discrete option price to its continuous limit is studied in [3]. Furthermore, in [6] it is proved that the probability kernel $\mathbf{P}_t^h(x,y)$ of the discretized process converges at the rate $O(h^2)$ to the probability density function $p_t(x,y)$ of the diffusion process. Note that this convergence is uniform in the state variables x and y and that the rate of convergence is optimal. However, lattice models for jump processes are more subtle to implement than the analogous models for diffusions. If the underlying process has jumps, the hopping range is not limited to the nearest neighbours, and it doesn't suffice to match drift and volatility.

Introduction

Suppose that we are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A Lévy process (Y_t) taking values in $\mathbb{R}^d$ is essentially a stochastic process having stationary and independent increments. Any Lévy process has a specific form for its characteristic function; more precisely, for all $t \geq 0$, $u \in \mathbb{R}^d$, we have

$$\mathbb{E}[e^{i(u,Y(t))}] = e^{t\eta(u)}$$

where η is the Lévy symbol of (Y_t), whose precise form is given by the Lévy-Khintchine formula:

$$\eta(u) = i(b,u) - \frac{1}{2}(u, au) + \int_{\mathbb{R}^d\setminus\{0\}} [e^{i(u,y)} - 1 - i(u,y)\chi_{\{|y|<1\}}]\,\nu(dy).$$

The measure on $\mathbb{R}^d\setminus\{0\}$ appearing in the characteristic exponent is called the Lévy measure. On an intuitive level, $\nu((a,b])$ represents the expected number of jumps of size $a < x \leq b$.

Infinitesimal Generator

The infinitesimal generator $\mathcal{L}$ of Y_t is the operator defined by

$$\mathcal{L}f(x) = \lim_{t\to 0}\frac{\mathbb{E}^x[f(Y_t)] - f(x)}{t}$$

where $\mathbb{E}^x[\cdot]$ denotes the expectation of Y_t under the condition $Y_0 = x$. The set of all functions $f : \mathbb{R} \to \mathbb{R}$, such that the limit exists for all $x \in \mathbb{R}$, is the domain of the generator, and is denoted by $\mathcal{D}(\mathcal{L})$. It is a well known fact that the infinitesimal generator $\mathcal{L}$ of the Lévy process can be calculated in terms of the Lévy symbol: if we denote by $S(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing functions, then for $f \in S(\mathbb{R}^d)$ we have

$$(\mathcal{L}f)(x) = (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{i(u,x)}\eta(u)\hat{f}(u)\,du.$$

The resemblance to the definition of a pseudo-differential operator is not coincidental ...

Setup

Now consider the $\mathbb{R}$-valued process defined by

$$Y_t = Y_0 + \mu t + \sigma W_t + X_t$$

where (W_t) is a Wiener process and (X_t) is a compound Poisson process independent of (W_t). Since this Lévy process has a non-zero Gaussian component, it admits at $t > 0$ a density for $Y_t = y$, given $Y_0 = x$, given by

$$p_t(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{\left[i\mu p - \frac{\sigma^2 p^2}{2} + \int_{\mathbb{R}\setminus\{0\}}(e^{ipx}-1)\,\nu(dx)\right]t}\, e^{ip(x-y)}\,dp,$$

where ν is the Lévy measure.

Hypothesis on the Lévy measure

Assume that the Lévy measure is absolutely continuous with respect to the Lebesgue measure, with

$$\nu(dx) = f(x)dx.$$

Furthermore, f is bounded in a neighborhood of the origin, is twice differentiable, and has an integrable second derivative.

The approximating stochastic process (Y_t^h) is a continuous-time Markov chain that will be specified in terms of its infinitesimal generator. Define the operators $\nabla_h, \Delta_h : l^2(h\mathbb{Z}) \to l^2(h\mathbb{Z})$ in the following way:

$$(\nabla_h\phi)(x) := \frac{\phi(x+h) - \phi(x-h)}{2h}$$

$$(\Delta_h\phi)(x) := \frac{\phi(x+h) + \phi(x-h) - 2\phi(x)}{h^2}$$

where $x \in h\mathbb{Z}$. Also, set

$$\Lambda_h(x,y) := f(h(m-n))h$$

where $y = hm$, $x = hn$. The mapping

$$\mathcal{L}_h := \mu\nabla_h + \frac{\sigma^2}{2}\Delta_h + \Lambda_h$$

is a densely defined unbounded operator. The operator $\mathcal{L}_h$ is a genuine Markov generator which defines the continuous-time Markov chain (Y_t^h). In fact, (Y_t^h) turns out to be a Lévy process itself!

Spectral Representation

Take $\mathcal{F}_h$ to be the semi-discrete Fourier transform. We have the following spectral representation

$$\mathcal{F}_h\mathcal{L}_h\mathcal{F}_h^{-1}(\Phi)(p) = F_{\mathcal{L}_h}(p)\Phi(p), \quad \Phi \in L^2([-\frac{\pi}{h}, \frac{\pi}{h}])$$

where $F_{\mathcal{L}_h}$ is a complex function of the real variable p. For $t > 0$, set $\mathbf{P}_t^h := \exp(t\mathcal{L}_h)$, and for any elements $x, y \in h\mathbb{Z}$, set

$$\mathbf{P}_t^h(x,y) := \left(\mathbf{P}_t^h(\delta_y)\right)(x).$$

$\mathbf{P}_t^h(x,y)$, also known as the probability kernel, is equal to the conditional probability $\mathbb{P}(Y_t^h = y | Y_0^h = x)$. Note that some difficulties arise when one is to define $\exp(t\mathcal{L}_h)$, since $\mathcal{L}_h$ is a (densely defined) unbounded operator. Some calculations lead to the formula

$$\mathbf{P}_t^h(x,y) = \frac{h}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{F_{\mathcal{L}_h}(p)t}\, e^{ip(x-y)}\,dp,$$

where dp is the Lebesgue measure on $\mathbb{R}$.

Theorem: Convergence Estimate

Let $t > 0$, then, under the previous assumptions, the following estimate holds

$$\left|p_t(x,y) - \frac{1}{h}\mathbf{P}_t^h(x,y)\right| = O(h^2) \qquad h \to 0$$

and the error term is independent of x and y.

References

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