CONDITIONED STOCHASTIC STABILITY OF EQUILIBRIUM STATES ON REPELLERS

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Backstory. Dynamical systems typically admit (infinitely) many invariant measures. To identify the relevant invariant measure (i.e. the one we would expect to see performing a naive experiment) the theory of *stochastic stability* considers random perturbations of the original system and studies whether the unique stationary measure of the induced Markov process converges to an invariant measure as the strength of the perturbation vanishes. The limiting object is a stochastically stable invariant measure.

Fact I. Stochastic stability has only been studied in the setting of attractors and for a single type of potential, giving rise to SRB and physical measures.

Fact II. Long transients also exhibit shared statistics but classical stochastic stability tools are not applicable. In some sense, "there is nothing to study in the limit when everything eventually escapes".

Abstract. We propose a notion of conditioned stochastic stability of invariant measures on repellers: we consider whether quasi-ergodic measures of absorbing Markov processes, generated by random perturbations of the deterministic dynamics and conditioned upon survival in a neighbourhood of a repeller, converge to an invariant measure in the zero-noise limit. Under suitable choices of the random perturbation, we find that equilibrium states on uniformly expanding repellers are conditioned stochastically stable.

CONDITIONED RANDOM DYNAMICS (1)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a Markov process evolving on M. We may stop or **kill** this process in two different ways:

i) Hard killing: The process enters a **cemetery** state $\partial \subset M$.

ii) Soft killing: For a given **weight function** $\phi : M \setminus \partial \to \mathbb{R}^-$, at every state X_n the process is killed with probability $e^{\phi(X_n)}$.

The process: We denote this (ϕ-weighted) process by X_n^ϕ, where

$$X_{n+1}^\phi = \begin{cases} X_{n+1}, & \text{with prob } e^{\phi(X_n)}, \\ \partial, & \text{with prob } 1 - e^{\phi(X_n)}. \end{cases}$$

Eventually, $X_n \in \partial$ so the stationary measure of the process is δ_∂, so...

Question: What is the statistical behaviour of X_n^ϕ before being absorbed by the cemetery state ∂?

Set $\tau^\phi := \min\{n : X_n^\phi \in \partial\}$, the (random) time the process is killed.

We can ask two different questions about the statistics of X_n^ϕ:

i) For a measurable subsets $A \subset M$, how does

$$\mathbb{P}_x\left[X_n^\phi \in A \,|\, \tau^\phi > n\right] := \frac{\mathbb{P}_x^\phi[X_n^\phi \in A]}{\mathbb{P}_x^\phi[X_n^\phi \notin \partial]}$$

behave as $n \to \infty$? This is the so-called **Yaglom limit**.

ii) For a measurable observable $f : M \setminus \partial \to \mathbb{R}$, how does

$$\mathbb{E}_x^\phi\left[\frac{1}{n}\sum_{i=0}^{n-1} f \circ X_i^\phi \,\middle|\, \tau^\phi > n\right]$$

behave as $n \to \infty$? These are **conditioned Birkhoff averages**.

QUASI-STATIONARY AND QUASI-ERGODIC MEASURES (2)

Consider X_n evolving in $S \subset M$ and let $\partial = M \setminus S$ be the cemetery state. We pay attention to these two measures:

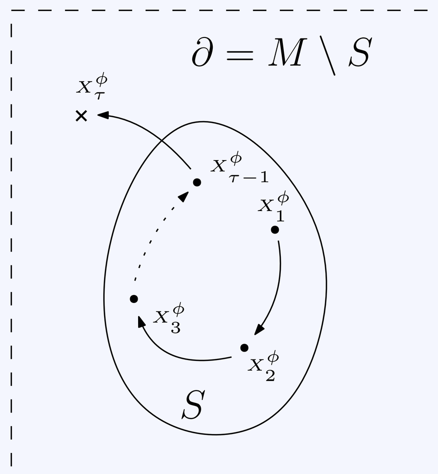
i) **(QSM)** A Borel probability measure μ on S is a **quasi-stationary measure** of the weighted Markov process X_n^ϕ if

$$\int_S e^{\phi(y)} P(y, \mathrm{d}x)\mu(\mathrm{d}x) = \lambda^\phi \mu(\mathrm{d}x)$$

and $\lambda^\phi = \int_S e^{\phi(x)} P(x, S)\mu(\mathrm{d}x) > 0$ is the **growth rate** of μ for X_n^ϕ on S. Under suitable conditions, quasi-stationary measures provide the Yaglom limit [1].

ii) **(QEM)** A probability measure ν on S is a **quasi-ergodic measure** of the ϕ-weighted Markov process X_n^ϕ if for any bounded measurable function $f : S \to \mathbb{R}$ it holds that

$$\mathbb{E}_x^\phi\left[\frac{1}{n}\sum_{i=0}^{n-1} f \circ X_i^\phi \,\middle|\, \tau^\phi > n\right] := \frac{1}{\mathbb{E}_x[e^{S_n\phi}\mathbb{1}_{\{\tau>n\}}]}\mathbb{E}_x\left[e^{S_n\phi}\mathbb{1}_{\{\tau>n\}}\frac{1}{n}\sum_{i=0}^{n-1} f \circ X_i\right] \xrightarrow{n\to\infty} \int_S f(y)\nu(\mathrm{d}y), \quad \nu\text{-a.s.}$$



conditioned upon $\tau^\phi > n$

`probs` $\neq$ **`freqs`**
(QSM) (QEM)

How are they different? If a system allows for escape (killing) it is no longer true that the `probability` of being in a subset equals the `frequency` of entering that set. Conditioned upon not dying, QSMs provide the probability and are "agnostic of the past" while QEDs provide the frequency, which depends on the surviving paths.

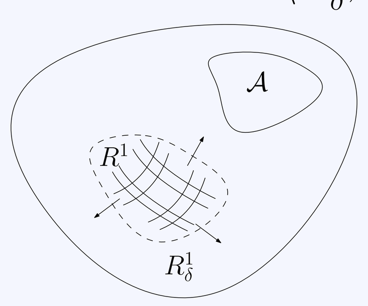
TRANSIENT DYNAMICS (3)

The theory of transient dynamics aims to understand a system's behaviour before stabilising, say by entering an attractor $\mathcal{A}$.

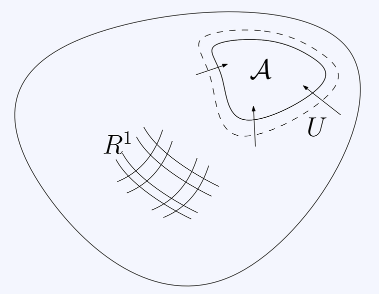
If a transient is long enough ("sticky") then there is a chance that transient orbits exhibit **shared statistics**, e.g. expected time to jump into the attractor, distribution of the path before escape, etc.

Transient behaviour is usually governed by **repelling invariant sets**. These are subsets $R^1, \ldots, R^k$ of $\Lambda = \bigcap_{n\geq 0} T^{-n}(M \setminus \partial)$, where

i) **Local transient.** $\partial = M \setminus R_\delta^i, \quad R_\delta^i \supset R^i$



ii) **Global transient.** $\partial = U \supset \mathcal{A}$



Hypothesis: We assume that the set Λ is **uniformly expanding**, i.e. there exist $C, r > 0$ such that for all $x \in \Lambda$, $\|\,\mathrm{d}T^n(x)^{-1}\| < C(1+r)^{-n}$ for every $n \geq 1$ and that there exists a neighbourhood V of Λ in M such that $T^{-1}(\Lambda) \cap V = \Lambda$.

THERMODYNAMIC FORMALISM (4)

Consider a map $T : M \to M$ and a **potential** $\psi \in C^\alpha, \psi : M \to \mathbb{R}$.

Let $\Lambda \subset M$ be T-invariant, i.e. $T(\Lambda) = \Lambda$.

We define the **topological pressure** of the triple (T, Λ, ψ) as

$$P_{\text{top}}(T, \Lambda, \psi) = \sup_{\mu\in\mathcal{I}(T,\Lambda)} P_\mu(T, \Lambda, \psi) = \sup_{\mu\in\mathcal{I}(T,\Lambda)} h_\mu(T) + \int \psi \mathrm{d}\mu,$$

with h_μ the Kolmogorov-Sinai (metric) entropy and $\mathcal{I}(T, \Lambda)$ the set of invariant measures of T on Λ.

We call ν an **equilibrium state** if $P_{\text{top}}(T, \Lambda, \psi) = P_\nu(T, \Lambda, \psi)$. These are the main objects of study in the theory of **thermodynamic formalism** [2].

Note: Equilibrium states are physically relevant in the sense that they minimise the *free energy* of the system [3].

Stochastic stability meets Thermodynamic formalism:
Often it holds that the stochastically stable measure μ_0 is the equilibrium state associated with the potential $\psi = -\log|\det \mathrm{d}T|$.

This also aligns with physical and SRB measures in some settings but stochastic stability has never considered potentials.

STOCHASTIC STABILITY (5)

A map T may admit **many** invariant measures. Can we identify a relevant invariant measure? Or in particular...

Are there measures that are "stable" if we add some **noise** to the system?

Consider the Markov process generated by **random perturbations** of T:

$$T : M \to M \qquad T_\omega : M \to M$$
$$x_{n+1} = T(x_n) \qquad X_{n+1} = T_\omega(X_n)$$

with $\text{dist}_{C^2}(T, T_\omega) < \varepsilon$.

Noise "washes out" isolated invariant sets and "spreads" the dynamics and, typically, the random process X_n admits a **unique stationary measure** μ_ε.

If $\mu_\varepsilon \xrightarrow{\varepsilon\to 0} \mu_0$ (in weak*), then μ_0 is a **stochastically stable** invariant measure of T [4]:

$$\text{stationary measure} \qquad \mu_\varepsilon \xrightarrow[w^*]{\varepsilon\to 0} \mu_0 \qquad \text{invariant measure}$$

Important: Stochastically stable measures are always supported on topological attractors.

EQUILIBRIUM STATES ON REPELLERS (6)

Theorem. (Ruelle [5]) Let $R^1, \ldots, R^k \subset \Lambda$ be as in (3). On each R^i, for every α-Hölder potential $\psi : R^i \to \mathbb{R}$, there exist **unique T-invariant equilibrium state** ν^ψ for the potential ψ on R^i.

Note: These are the measures we end up approximating although this is not our motivation.

CONDITIONED STOCHASTIC STABILITY (7)

Instead of stationary measures, we consider convergence of **quasi-ergodic measures** of (ϕ-weighted) absorbing Markov processes:

$$\text{quasi-ergodic measure} \qquad \nu_\varepsilon^\phi \xrightarrow[w^*]{\varepsilon\to 0} \nu_0^\phi \qquad \text{invariant measure}$$

If so, we say that ν_0^ϕ is **conditioned stochastically stable**.

RESULTS (8)

Standing Hypothesis: We assume that on each R^i, for each potential ψ, there exists a unique equilibrium state ν^ψ as in (6).

Theorem A1. (Local hard-killing) There exists a unique T-invariant measure ν_0 on R which is conditioned stochastically stable on every sufficiently small neighbourhood of R.

Theorem A2. (Local hard-killing + soft-killing) Given a Hölder weight function ϕ, there exists a unique T-invariant measure ν^ψ on R which is conditioned ϕ-weighted stochastically stable on every sufficiently small neighbourhood of R. Moreover, ν^ψ is the unique equilibrium state associated with the potential $\psi = \phi - \log|\det \mathrm{d}T|$ on R, i.e. $\nu^\psi = \nu_0^\phi$.

Theorem B. (global hard-killing + soft-killing) Given a C^2 map T, a Hölder weight function ϕ, and a suitable open set $\partial \subset M$, assume that

i) $T|_\Lambda : \Lambda \to \Lambda$ is uniformly expanding, with Λ as in (3),

ii) $\Lambda \subset \text{Int}(M \setminus \partial)$, and

iii) $T : \Lambda \to \Lambda$ admits a unique equilibrium state ν^ψ associated with the potential $\psi = \phi - \log|\det \mathrm{d}T|$, which is mixing.

Then ν^ψ is conditioned ϕ-weighted stochastically stable on $M \setminus \partial$, i.e. $\nu^\psi = \nu_0^\phi$.

NEXT STEPS (9)

i) We can extend these results to systems with hyperbolic repellers and not only uniformly expanding. The machinery needed is significantly more convoluted (we use anisotropic spaces from [6]).

ii) How do these techniques generalise non-uniformly hyperbolic systems or uniformly hyperbolic maps with singularities?

iii) Can we find an example where the thermodynamic formalism does not provide equilibrium states but conditioned stochastic stability does?

References

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Questions or Comments?
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