”Nonlinear Degenerate Equations of Porous Medium type”

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Applied PDEs Course. Lectures 1-2

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Outline

1. **Diffusion**
   - Heat equation
   - Linear Parabolic Equations
   - Nonlinear equations
2. **Mathematical Theory of the Porous Medium Diffusion**
   - Applied motivation
   - Literature
   - Concepts of solution
   - Barenblatt profiles
   - Asymptotic behaviour

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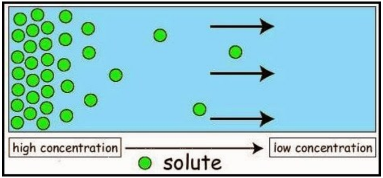
Diffusion

Diffusion equations describe how a continuous medium (say, a population) spreads to occupy the available space.

- Models come from all kinds of applications: fluids, chemicals, bacteria, animal populations, the momentum of a viscous (Newtonian) fluid diffuses, there is diffusion in the stock market,...



Diffusion of particles in a water solution

- So the question is : what is diffusion for a mathematician? how to analyze diffusion mathematically?
  This question has received two quite different answers in recent history.

The two ways to diffusion

The two answers:

- First direction: Is diffusion more or less related to random walk ? This is a correct answer, and this approach leads to Brownian motion and Stochastic Processes, with the famous Ito equation:

$$dx = bdt + \frac{1}{2}\sigma dW.$$

- Second direction: how to explain it with "standard mathematics" based on Analysis? The answer is PDEs of parabolic type, as explained by Kolmogorov in the 1930s. The mother equation is the Heat Equation:

$$\partial_t u = \Delta u.$$

- Understanding this double way has been the source of much effort and the work goes on today.
- Here we will follow the way of Analysis with PDEs, inaugurated by Joseph Fourier (1807, 1822) in an apparently different context, Heat Propagation.

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Expanding the basic model

Some of the problems we face today

- How much of it can be explained with linear models, how much is essentially nonlinear? which are the most relevant mathematical models?
- The stationary states of diffusion belong to an important world, elliptic equations. Elliptic equations, linear and nonlinear, have many relatives: diffusion, fluid mechanics, waves of all types, quantum mechanics, ...
- The Laplacian Δ is really the King of Differential Operators. The fractional Laplacian is close family. How strong is the theory and application of the so-called nonlocal or long-range operators that include the fractional Laplacian family?
- Are we able to treat complex systems and describe their behaviour with the combination of the tools we have?

Main tools : Modelling, Analysis, Stochastics, Asymptotics and Numerics.

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The heat equation origins

- We begin our presentation with the Heat Equation $\boxed{u_t = \Delta u}$ and the analysis proposed by Fourier, 1807, 1822 [1] (Fourier decomposition, spectrum). The mathematical models of heat propagation and diffusion have made great progress both in theory and application.
  They have had a strong influence on 5 areas of Mathematics: PDEs, Functional Analysis, Inf. Dim. Dyn. Systems, Diff. Geometry and Probability. And on and from Physics.

- The heat flow analysis is based on two main techniques: integral representation (convolution with a Gaussian kernel) and mode separation:

$$u(x,t) = \sum T_i(t) X_i(x)$$

where the $X_i(x)$ form the spectral sequence

$$-\Delta X_i = \lambda_i X_i.$$

This is the famous linear eigenvalue problem, Spectral Theory.

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The heat equation semigroup and Gauss

- When heat propagates in free space the natural problem is the initial value problem
$$u_t = \Delta u, \qquad u(x,0) = f(x) \tag{1}$$
which is solved by convolution with the evolution version of the Gaussian function
$$G(x,t) = (4\pi t)^{-n/2} \exp\left(-|x|^2/4t\right). \tag{2}$$
Note that G has very nice analytical properties for $t > 0$, but note that $G(x,0) = \delta(x)$, a Dirac mass. G works as a kernel (Green, Gauss). (G is the Fundamental Solutions. This is a key idea that we would like to copy, they are different in stationary and evolution problems. The concept is problematic in some nonlinear PDEs and very useful in some of them. G is self-similar).
- The maps $S_t : u_0 \mapsto u(t) := u_0 * G(\cdot, t)$ form a linear continuous semigroup of contractions in all L^p spaces $1 \le p \le \infty$.

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Regularity and asymptotics

- **Regularity**. Solutions in the standard class are unique, exist globally in time and they are C^∞ smooth in space and time. For nonnegative data they are strictly positive.

- **Asymptotic behaviour as $t \to \infty$, convergence to the Gaussian**. Under very mild conditions on u_0 it is proved that

$$\lim_{t\to\infty} t^{n/2}(u(x,t) - M\,G(x,t)) = 0 \qquad (3)$$

uniformly, if $M = \int u_0(x)\,dx$. For convergence in L^p less is needed. Thus,

$$\lim_{t\to\infty} \|(u(x,t) - M\,G(x,t)\|_1 = 0 \qquad (4)$$

This is the famous Central Limit Theorem in its continuous form (Probability).

(we will try to repeat those questions over and over; the answers vary with the models)

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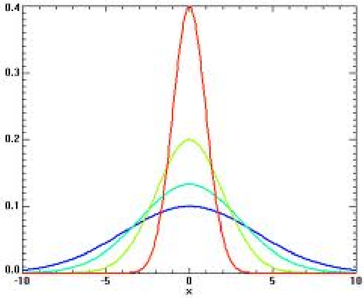
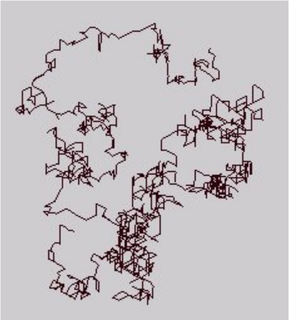
Heat equation graphs. Conflicting views

- The comparison of ordered dissipation vs underlying chaos





Left, the evolution to a nice Gaussian

Right, a sample of random walk, origin of Brownian motion

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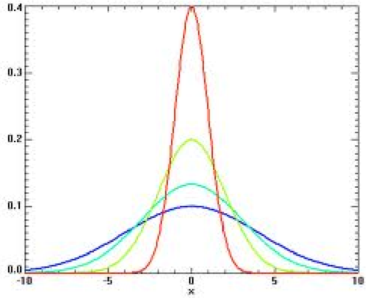

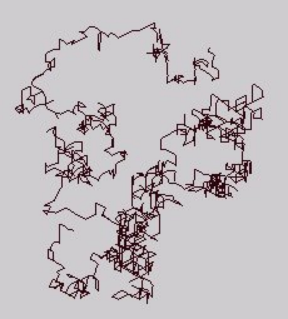

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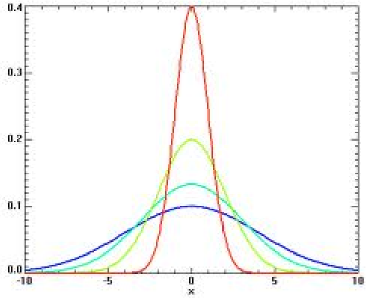

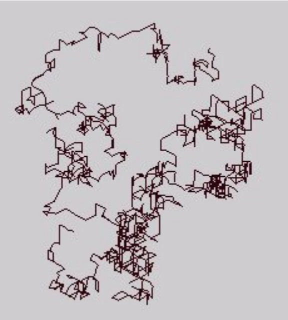

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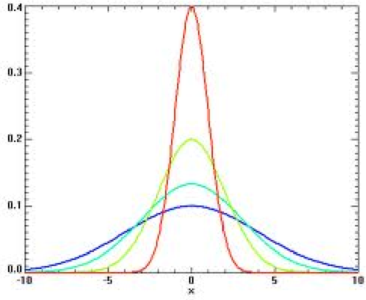

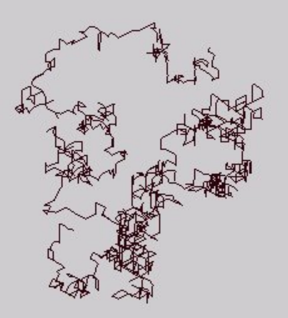

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Linear heat flows

Until well into the XXth century diffusion was almost exclusively heat equation, a part of the classical theory of PDEs. From 1822 until 1950 the heat equation has motivated
(i) Fourier analysis decomposition of functions (and set theory),
(ii) development of other linear equations
$\Longrightarrow$ Theory of Parabolic Equations

$$u_t = \sum a_{ij}\partial_i\partial_j u + \sum b_i \partial_i u + cu + f$$

Main inventions in Parabolic Theory:
(1) a_{ij}, b_i, c, f regular $\Rightarrow$ Maximum Principles, Schauder estimates, Harnack inequalities; C^α spaces (Hölder); potential theory; generation of semigroups.
(2) coefficients only continuous or bounded $\Rightarrow$ $W^{2,p}$ estimates, Calderón-Zygmund theory, weak solutions; Sobolev spaces.

The probabilistic approach: Diffusion as an stochastic process: Bachelier, Einstein, Smoluchowski, Wiener, Levy, Ito,...

$$dX = bdt + \sigma dW$$

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Nonlinear heat flows

- In the last 50 years emphasis has shifted towards the Nonlinear World. Maths more difficult, more complex and more realistic.
  My group works in the areas of Nonlinear Diffusion and Reaction Diffusion.
  I will talk about the theory mathematically called Nonlinear Parabolic PDEs. General formula

$$u_t = \sum \partial_i A_i(u, \nabla u) + \sum B(x, u, \nabla u)$$

- Typical nonlinear diffusion: $u_t = \Delta u^m$
- Typical reaction diffusion: $u_t = \Delta u + u^p$

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The Nonlinear Diffusion Models

- The Stefan Problem (Lamé and Clapeyron, 1833; Stefan 1880)

$$SE : \begin{cases} u_t = k_1 \Delta u & \text{for } u > 0, \\ u_t = k_2 \Delta u & \text{for } u < 0. \end{cases} \quad TC : \begin{cases} u = 0, \\ \mathbf{v} = L(k_1 \nabla u_1 - k_2 \nabla u_2). \end{cases}$$

  Main feature: the free boundary or moving boundary where $u = 0$. TC= Transmission conditions at $u = 0$.

- The Hele-Shaw cell (Hele-Shaw, 1898; Saffman-Taylor, 1958)

$$u > 0, \ \Delta u = 0 \quad \text{in} \quad \Omega(t); \quad u = 0, \ \mathbf{v} = L\partial_n u \quad \text{on} \quad \partial\Omega(t).$$

- The Porous Medium Equation →*(hidden free boundary)*

$$u_t = \Delta u^m, \quad m > 1.$$

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$$u > 0,\ \Delta u = 0 \quad \text{in} \quad \Omega(t); \quad u = 0,\ \mathbf{v} = L\partial_n u \quad \text{on} \quad \partial\Omega(t).$$

- The Porous Medium Equation →*(hidden free boundary)*

$$u_t = \Delta u^m, \quad m > 1.$$

- The p-Laplacian Equation, $u_t = \mathrm{div}\,(|\nabla u|^{p-2}\nabla u)$.

The Nonlinear Diffusion Models

- The Stefan Problem (Lamé and Clapeyron, 1833; Stefan 1880)

$$SE : \begin{cases} u_t = k_1 \Delta u & \text{for } u > 0, \\ u_t = k_2 \Delta u & \text{for } u < 0. \end{cases} \quad TC : \begin{cases} u = 0, \\ \mathbf{v} = L(k_1 \nabla u_1 - k_2 \nabla u_2). \end{cases}$$

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- The Standard Blow-Up model (Kaplan, 1963; Fujita, 1966)

$$u_t = \Delta u + u^p$$

Main feature: If $p > 1$ the norm $\|u(\cdot, t)\|_\infty$ of the solutions goes to infinity in finite time. Hint: Integrate $u_t = u^p$.
Problem: *what is the influence of diffusion / migration?*

- General scalar model

$$u_t = \mathcal{A}(u) + f(u)$$

- The system model: $\overrightarrow{\mathbf{u}} = (u_1, \cdots, u_m) \to$ chemotaxis.
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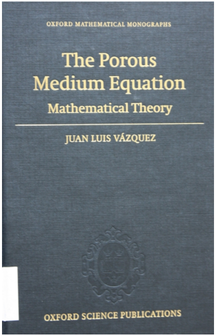

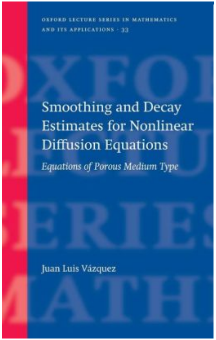

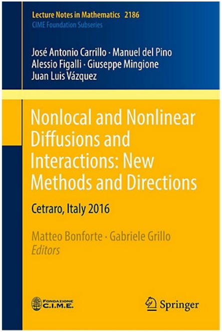

2006-2007 and 2017

Outline

The Porous Medium Equation

If you go to Wikipedia and look for the Diffusion Equation you will find

$$\frac{\partial \phi(\vec{r},t)}{\partial t} = \nabla \cdot \left(D(\phi,\vec{r})\nabla\phi(\vec{r},t)\right)$$

It is not difficult from here to conclude that the simplest model of nonlinear diffusion equation is

$$u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1}[= m|u|^{m-1}]$$

It degenerates at $u = 0$ if $m > 1$, $\Longrightarrow$ slow diffusion

On the contrary, if $m < 1$ it is singular at $u = 0$ if $\Longrightarrow$ fast diffusion

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Applied motivation for the PME

- Flow of gas in a porous medium (Leibenzon, 1930; Muskat 1933) $m = 1 + \gamma \geq 2$

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{v}) = 0, \\ \mathbf{v} = -\frac{k}{\mu}\nabla p, \quad p = p(\rho). \end{cases}$$

Second line left is the Darcy law for flows in porous media (Darcy, 1856).
Porous media flows are potential flows due to averaging of Navier-Stokes on the pore scales.
To the right, put $p = p_o\,\rho^\gamma$, with $\gamma = 1$ (isothermal), $\gamma > 1$ (adiabatic flow).

$$\rho_t = \operatorname{div}\left(\frac{k}{\mu}\rho\nabla p\right) = \operatorname{div}\left(\frac{k}{\mu}\rho\nabla(p_o\rho^\gamma)\right) = c\Delta\rho^{\gamma+1}.$$

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$$\frac{d}{dt}\int_{\Omega} c\rho T\,dx = \int_{\partial\Omega} k(T)\nabla T \cdot \mathbf{n} dS.$$

  Put $k(T) = k_o T^n$, apply Gauss law and you get

$$c\rho \frac{\partial T}{\partial t} = \mathrm{div}(k(T)\nabla T) = c_1 \Delta T^{n+1}.$$

  $\rightarrow$ When k is not a power we get $\boxed{T_t = \Delta\Phi(T)}$ with $\Phi'(T) = k(T)$.
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- Regularity of solutions: *is there a limit? C^k for some k?*
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References

About PME

- J. L. Vázquez, ”The Porous Medium Equation. Mathematical Theory”, Oxford Univ. Press, 2007, 624 pages

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Concept of solution

There are many concepts of generalized solution of the PME:

- Classical solution: only in nondegenerate situations, $u > 0$.
- Limit solution: physical, but depends on the approximation (?).
- Weak solution Test against smooth functions and eliminate derivatives on the unknown function; it is the mainstream; (Oleinik, 1958)

$$\int\int (u\,\eta_t - \nabla u^m \cdot \nabla \eta)\,dxdt + \int u_0(x)\,\eta(x,0)\,dx = 0.$$

Very weak

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More on concepts of solution

Solutions are not always weak:

- Strong solution. More regular than weak but not classical: weak derivatives are L^p functions. *Big benefit: usual calculus is possible.*
- Semigroup solution / mild solution. The typical product of functional discretization schemes: $u = \{u_n\}_n$, $u_n = u(\cdot, t_n)$,

$$u_t = \Delta\Phi(u), \qquad \frac{u_n - u_{n-1}}{h} - \Delta\Phi(u_n) = 0$$

Now put $f := u_{n-1}$, $u := u_n$, and $v = \Phi(u)$, $u = \beta(v)$:

$$-h\Delta\Phi(u) + u = f, \qquad \boxed{-h\Delta v + \beta(v) = f.}$$

"Nonlinear elliptic equations"; Crandall-Liggett Theorems   Ambrosio, Savarè, Nochetto

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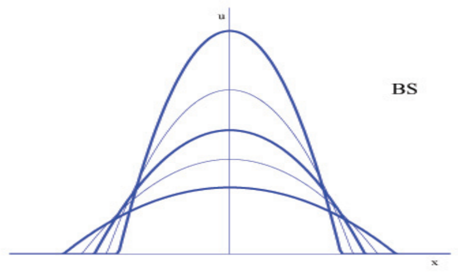
Barenblatt profiles (ZKB)

- These profiles are the alternative to the Gaussian profiles. They are source solutions. *Source means that $u(x,t) \to M\,\delta(x)$ as $t \to 0$.*
- Explicit formulas (1950):

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \left(C - k\xi^2\right)_+^{1/(m-1)}$$



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Free boundary at distance $|x| = ct^{\beta}$

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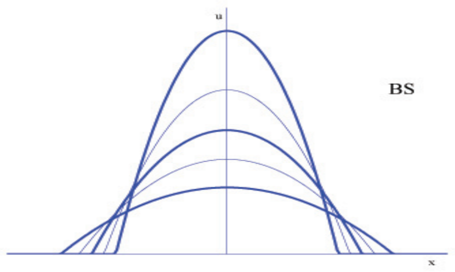

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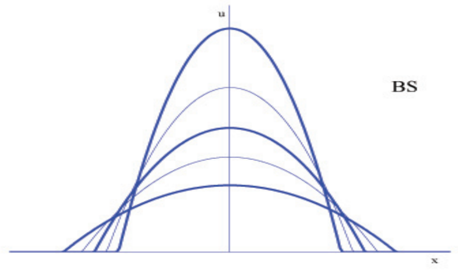

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Regularity results

- The universal estimate holds (Aronson-Bénilan, 79):

$$\boxed{\Delta v \geq -C/t.}$$

$v \sim u^{m-1}$ is the pressure.

- (Caffarelli-Friedman, 1982) C^α regularity: there is an $\alpha \in (0,1)$ such that a bounded solution defined in a cube is C^α continuous.
- If there is an interface Γ, it is also C^α continuous in space time.
- How far can you go?
  Free boundaries are stationary (metastable) if initial profile is quadratic near $\partial\Omega$: $u_0(x) = O(d^2)$. This is called waiting time. Characterized by JLV in 1983. *Visually interesting in thin films spreading on a table.*

  Existence of corner points possible when metastable, $\Rightarrow$ *no* C^1
  Aronson-Caffarelli-V. Regularity stops here in $n = 1$

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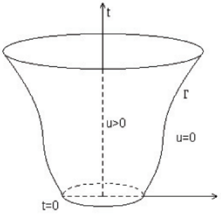
Free Boundaries in several dimensions



A regular free boundary in n-D

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- A free boundary with a hole in 2D, 3D is the way of showing that focusing accelerates the viscous fluid so that the speed becomes infinite. This is blow-up for $\mathbf{v} \sim \nabla u^{m-1}$. The setup is a viscous fluid on a table occupying an annulus of radii r_1 and r_2. As time passes $r_2(t)$ grows and $r_1(t)$ goes to the origin. As $t \to T$, the time the hole disappears.
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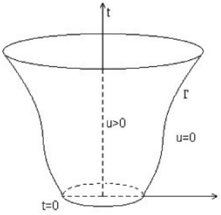

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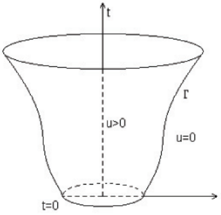

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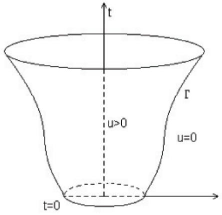

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FDE profiles

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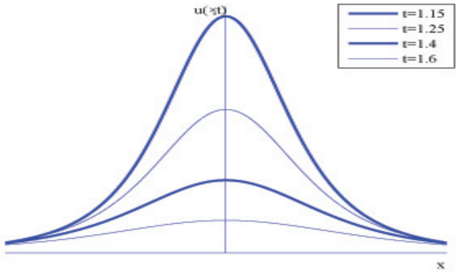

$$\alpha = \frac{n}{2-n(1-m)}$$
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Solutions for $m < 1$ with fat tails (polynomial decay; anomalous distributions) .They are very important in Probability, associated to Levy flights.

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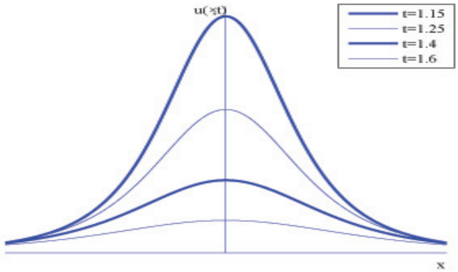

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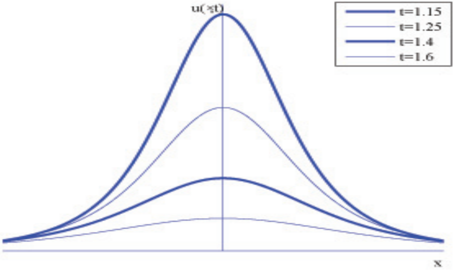

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Asymptotic behaviour I
Nonlinear Central Limit Theorem

Choice of domain: $\mathbb{R}^n$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^n)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x)\,dx + \iint f\,dxdt$.

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] *Let $B(x,t;M)$ be the Barenblatt with the asymptotic mass M and $f = 0$; u converges to B after renormalization*

$$t^\alpha |u(x,t) - B(x,t)| \to 0$$

For every $p \geq 1$ we have

$$\|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p-1).$$

The case $p = 1$ works for all $f \in L^1_{x,t}$.

Note: α *and* $\beta = \alpha/n = 1/(2 + n(m-1))$ *are the zooming exponents as in* $B(x,t)$.

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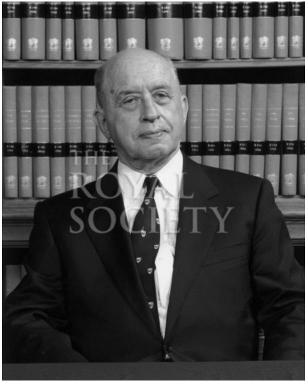

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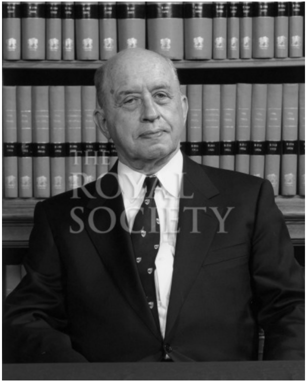

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Asymptotic behaviour. III

- The rates. Carrillo-Toscani 2000. Using entropy functional with entropy dissipation control you can prove decay rates when $\int u_0(x)|x|^2\,dx < \infty$ (finite variance):

$$\|u(t) - B(t)\|_1 = O(t^{-\delta}),$$

  We would like to have $\delta = 1$. This problem is still open for $m > 2$. New results by JA Carrillo, Markowich, McCann, Del Pino, Lederman, Dolbeault, Vazquez et al. include $m < 1$.

- Eventual geometry, concavity and convexity Result by Lee and Vazquez (2003): Here we assume compact support.There exists a time after which the pressure is concave, the domain convex, the level sets convex and

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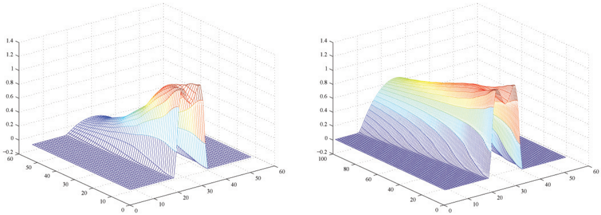
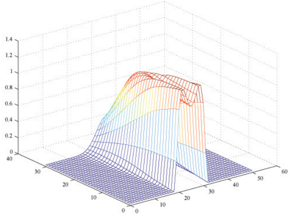
Asymptotic behaviour IV. Concavity

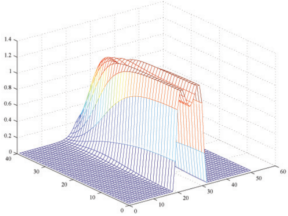
- The eventual concavity results of Lee and Vazquez



Eventual concavity for PME in 3D and in 1D



Eventual concavity for HE



Eventual concavity for FDE

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