Mathematical Problems in Climate Dynamics,
Nelder Fellow Lectures

3 March 2014

Lecture V
Advanced Spectral Methods for Time Series Analysis

Michael Ghil

Ecole Normale Supérieure, Paris, and University of California, Los Angeles

Please visit these sites for more info.

http://www.atmos.ucla.edu/tcd/

http://www.environnement.ens.fr/

Overall Outline

- **Lecture I: Observations and planetary flow theory (GFD(⌘))**
- **Lecture II: Atmospheric LFV(*) & LRF(**)**
- **Lecture III: EBMs(+), paleoclimate & "tipping points"**
- **Lecture IV: The wind-driven ocean circulation**

➔ **Lecture V: Advanced spectral methods–SSA(±) *et al.***
- **Lecture VI: Nonlinear & stochastic models—RDS(❖)**

(⌘) GFD = Geophysical fluid dynamics
(*) LFV = Low-frequency variability
(**) LRF = Long-range forecasting
(+) EBM = Energy balance model
(±) SSA = Singular-spectrum analysis
(❖) RDS = Random dynamical system

Advanced Spectral Methods, Nonlinear Dynamics and the Nile River

Motivation

1. Climatic time series have typically broad spectral peaks, on top of a continuous, "warm-colored" background ➔ ***Method***
2. Connections to nonlinear dynamics ➔ ***Theory***
3. Need for stringent statistical tests ➔ ***Toolkit***
4. Applications for analysis and prediction ➔ ***Examples***

Joint work with many people: M.R. Allen, M.D.Dettinger, Y. Feliks, A. Groth, K. Ide, N. Jiang, C.L. Keppenne, D. Kondrashov, M. Kimoto, M.E. Mann, K.-C. Mo, M.C. Penland, G. Plaut, A.W. Robertson, A. Saunders, D. Sornette, S. Speich, C. Taricco, Y. Tian, Y.S. Unal, G. Vivaldo, P. Yiou *i.a.*

Motivation & Outline

1. Data sets in the geosciences are often short and contain errors: this is both an obstacle and an incentive.

2. Phenomena in the geosciences often have both regular ("cycles") and irregular ("noise") aspects.

3. Different spatial and temporal scales:
   one person's noise is another person's signal.

4. Need both deterministic and stochastic modeling.

5. Regularities include (quasi-)periodicity ➔ spectral analysis via "classical" methods (see SSA-MTM Toolkit).

6. Irregularities include scaling and (multi-)fractality ➔ "spectral analysis" via Hurst exponents, dimensions, etc. (see Multi-Trend Analysis, MTA)

7. Does some combination of the two, + deterministic and stochastic modeling, provide a pathway to prediction?

For details and publications, please visit these two Web sites:

TCD http://www.atmos.ucla.edu/tcd/ ➔ key person – Dmitri Kondrashov!

E2-C2 http://www.ipsl.jussieu.fr/~ypsce/py_E2C2.html

Outline

- Time series analysis
  - The “smooth” and “rough” part of a time series
  - Oscillations and nonlinear dynamics
- Singular spectral analysis (SSA)
  - Principal components in time and space
  - The SSA-MTM Toolkit
- The Nile River floods
  - Longest climate-related, instrumental time series
  - Gap filling in time series
  - NAO and SOI impacts on the Nile River
- Concluding remarks
  - Cautionary remarks (“garde-fous”)
  - References

Climatic Trends & Variability

- Standard view — Binary thinking, dichotomy:

  Trend — Predictable (completely), deterministic, reassuring, good;

  Variability — Unpredictable (totally), stochastic, disconcerting, bad.

- In fact, these two are but extremes of a spectrum of, more or less predictable, types of climatic behavior, between the totally boring & the utterly surprising.

- (Linear) Trend = Stationary >

  Periodic > Quasi-periodic >

  Deterministically aperiodic >

  Random Noise

- Here ">" means "better, more predictable", &

  Variability = Periodic + Quasi-periodic +

  Aperiodic + Random

Time Series in Nonlinear Dynamics

The 1980s — decade of greed & fast results

(LBOs, junk bonds, fractal dimension).

Packard *et al*. (1980), Roux *et al*. (1980);

Mañe (1981), Ruelle (1981), Takens (1981);

- Method of delays: $\ddot{x}_i = f_i(x_1, ..., x_n) \Leftrightarrow x^{(n)} = F(x^{(n-1)}, ..., x)$

$$\ddot{x} = F(x, \dot{x}) \Rightarrow \begin{cases} \dot{x} = y, \\ \dot{y} = F(x, y) \end{cases}$$

Differentiation ill-posed ⇒ use differences instead!

1st Problem — smoothness:

Whitney embedding lemma doesn't apply to most attractors (e.g.,Lorenz)

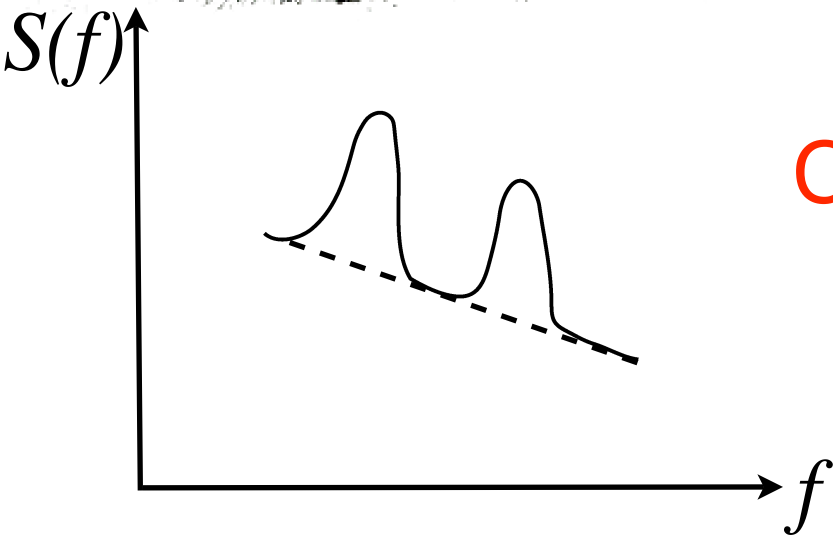
2nd Problem — noise;

3rd Problem — sampling: long recurrence times.

- Some rigorous results on convergence:

Smith (1988, *Phys. Lett. A*), Hunt (1990, *SIAM J. Appl. Math.*)

Spectral Density (Math)/Power Spectrum (Science & Engng.)



Continuous background
+ peaks

- **Wiener-Khinchin (Bochner) Theorem** **Blackman-Tukey Method**

$$R(s) = \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} x(t)x(t+s)dt$$

$$S(f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(s)e^{-ifs}ds \equiv \hat{R}(s)$$

i.e., the **lag-autocorrelation function** & the **spectral density**

are **Fourier transforms** of each other.

Power Law for Spectrum

S(f) ~ f^{-p} + poles

i.e. linear in log-log coordinates

For a 1st-order Markov process or "red noise" $p = 2$

"Pink" noise, $p = 1$ ($1/f$, flicker noise)

"White" noise, $p = 0$

Low-order dynamical (deterministic) systems

have exponential decay of *S(f)* (linear in log-linear coordinates)

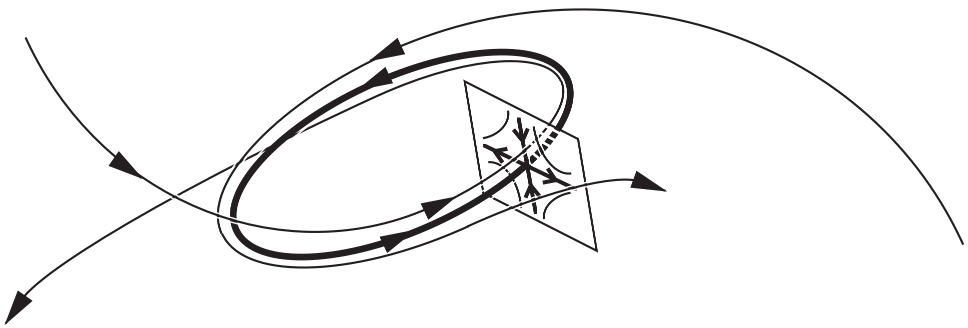
e.g. for Smale horseshoe $\forall k \quad \exists 2^k$ unstable orbits of period *k*

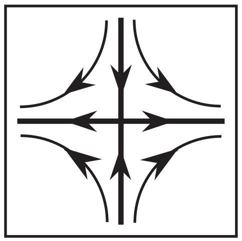
N.B. Bhattacharaya, Ghil & Vulis (1982, *J. Atmos. Sci*.) showed a spectrum $S \sim f^{-2}$ for a nonlinear PDE with delay (doubly infinite-dimensional)

Power Law for Spectrum (cont'd)

- Hypothesis: "***Poles***" correspond to the least unstable periodic orbits

"***unstable limit cycles***" "***Poincaré section***"





- Major clue to the physics

  that underlies the dynamics

- N.B. Limit cycle not necessarily elliptic, i.e. not

$$(x, y) = (a_f sin(ft), b_f cos(ft))$$

Outline

- Time series analysis
  - The “smooth” and “rough” part of a time series
  - Oscillations and nonlinear dynamics
- Singular spectral analysis (SSA)
  - Principal components in time and space
  - The SSA-MTM Toolkit
- The Nile River floods
  - Longest climate-related, instrumental time series
  - Gap filling in time series
  - NAO and SOI impacts on the Nile River
- Concluding remarks
  - Cautionary remarks (“garde-fous”)
  - References

Singular Spectrum Analysis (SSA)

Spatial EOFs

x -- *space*

$$\phi(x,t) = \sum a_k(t) e_k(x)$$

$$C_\phi(x,y) = E\phi(x,\omega)\phi(y,\omega) = \frac{1}{T}\int_o^T \phi(x,t)\phi(y,t)\,dt$$

$$C_\phi e_k(x) = \lambda_k e_k(x)$$

Colebrook (1978); *Weare & Nasstrom* (1982);
Broomhead & King (1986: BK); *Fraedrich* (1986)

BK+VG: Analogy between Mañe-Takens embedding and the Wiener-Khinchin theorem

SSA

s -- *lag*

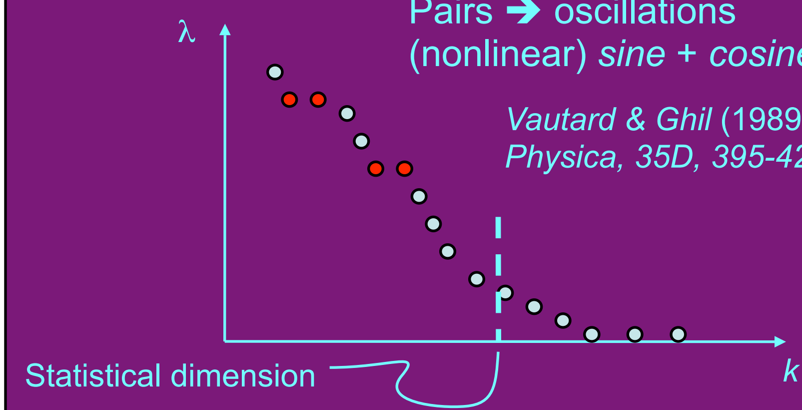
$$X(x+s) = \sum a_k(t) e_k(s)$$

$$C_X(s) = EX(t+s,\omega)\phi(s,\omega) = \frac{1}{T}\int_o^T X(t)X(t+s)\,dt$$

$$C_X e_k(s) = \lambda_k e_k(s)$$

Pairs ➔ oscillations
(nonlinear) *sine* + *cosine* pair

Vautard & Ghil (1989: VG)
Physica, 35D, 395-424



Power Spectra & Reconstruction

- **A. Transform pair**:

$$X(t+s) = \sum_{k=1}^{M} a_k(t) e_k(s), \; e_k(s) - EOF$$

The e_k's are **adaptive filters**,

$$a_k(t) = \sum_{s=1}^{M} X(t+s) e_k(s), \; a_k(t) - PC$$

the a_k's are **filtered time series**.

B. Power spectra

$$S_X(f) = \sum_{k=1}^{M} S_k(f); \quad S_k(f) = R_k(s); \quad R_k(s) \approx \frac{1}{T} \int_0^T a_k(t) a_k(t+s) dt$$

C. Partial reconstruction

$$X^K(t) = \frac{1}{M} \sum_{k \in K} \sum_{s=1}^{M} a_k(t-s) e_k(s);$$

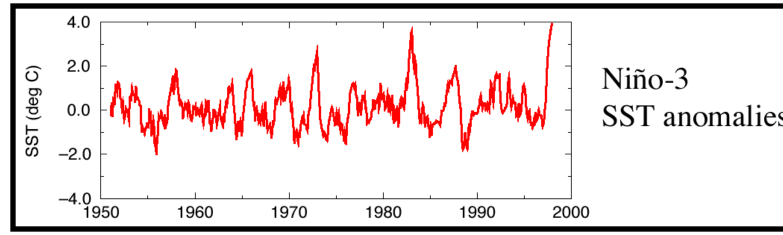
in particular: $K = \{1, 2, \ldots.., S\}$ or $K = \{k\}$ or $K = \{l, l+1; \lambda_l \approx \lambda_{l+1}\}$

Singular Spectrum Analysis (SSA)

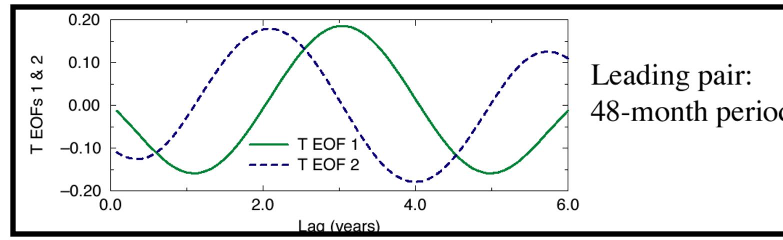
SSA decomposes (geophysical & other)
time series into
Temporal EOFs (T-EOFs) and
Temporal Principal Components (T-PCs),
based on the series' lag-covariance matrix

Selected parts of the series can be
reconstructed, via
Reconstructed Components (RCs)

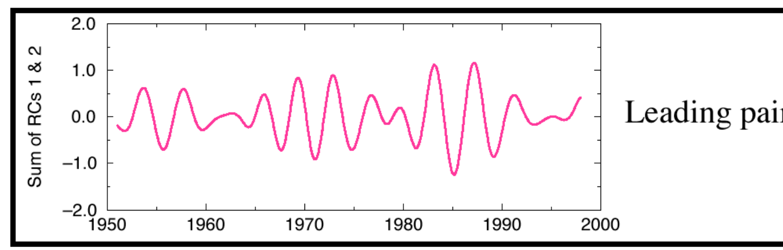
Time series



T-EOFs



RCs



- SSA is good at isolating oscillatory behavior via paired eigenelements.
- SSA tends to lump signals that are longer-term than the window into
  - one or two trend components.

Selected References:
Vautard & Ghil (1989, *Physica* D);
Ghil *et al*. (2002, *Rev. Geophys.)*

SSA for Southern Oscillation Index (SOI)

SOI = mean monthly values of Δp_s (Tahiti – Darwin)

Results ("undigested") from 1933–1988 time interval (*)

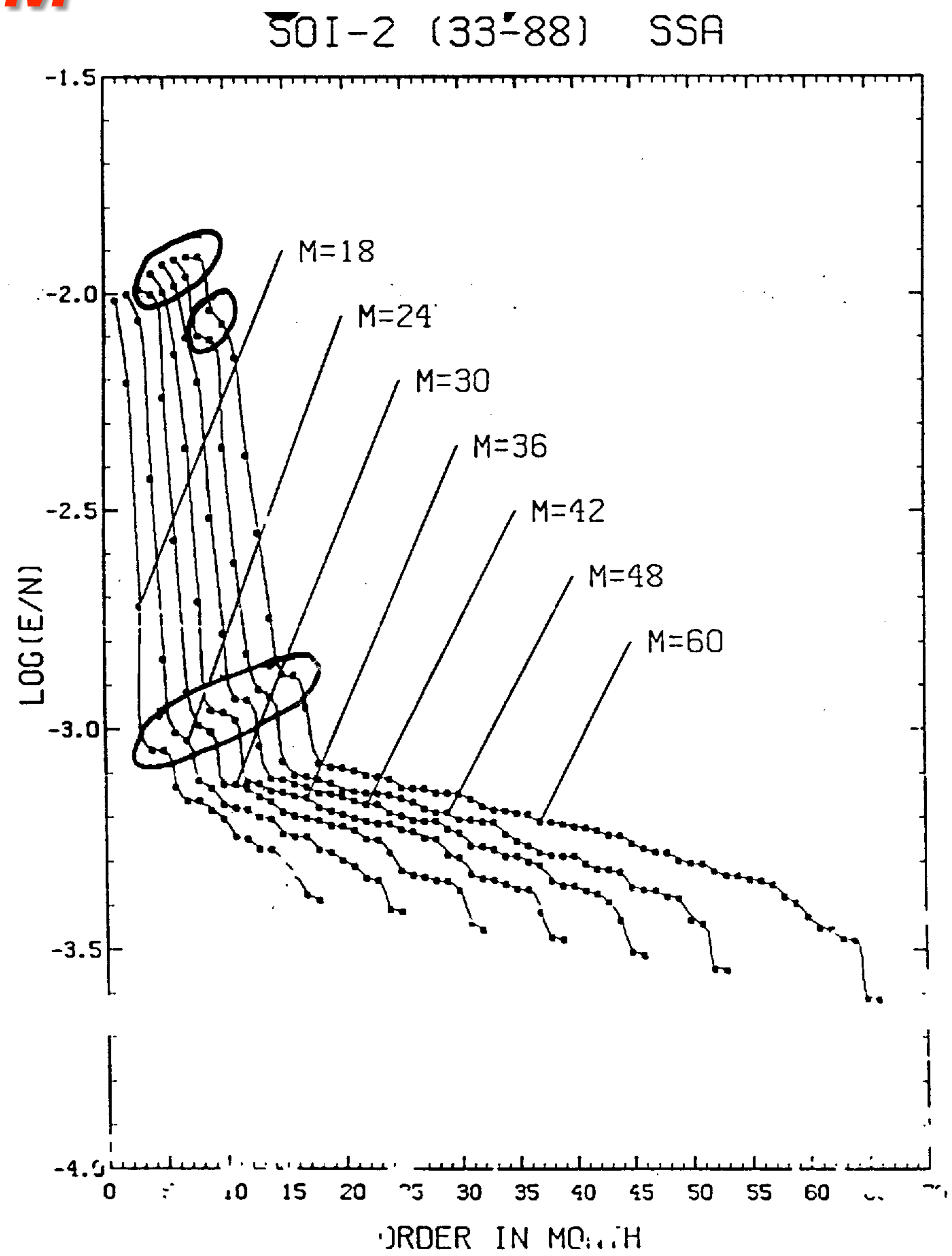
1. For $18 < M < 60$ months, singular spectra show a clear break at $5 < S < 17$ (= "deterministic" part; $M - S$ = "noise");
2. 3 pairs of EOFs stand out:
   EOFs 1 + 2 (27%), 3 + 4 (19.7%), and 9 + 10 (3%);
3. the associated periods are ~
   60 mos. ("ENSO"), 30 mos. (QBO"), and 5.5 mos. (?!)

(*) E. M. ("Gene") Rasmusson, X. Wang, and C.F. Ropelewski, 1990: The biennial component of ENSO variability. *J. Marine Syst.*, **1**, 71–96.

Variable window size *M*

Sampling interval – $\tau_s = 1$ month

Window width $M\tau_s$:
$18\tau_s < \tau_w < 60\tau_s$ or
1.5 yr $< \tau_w < 5$ yr.



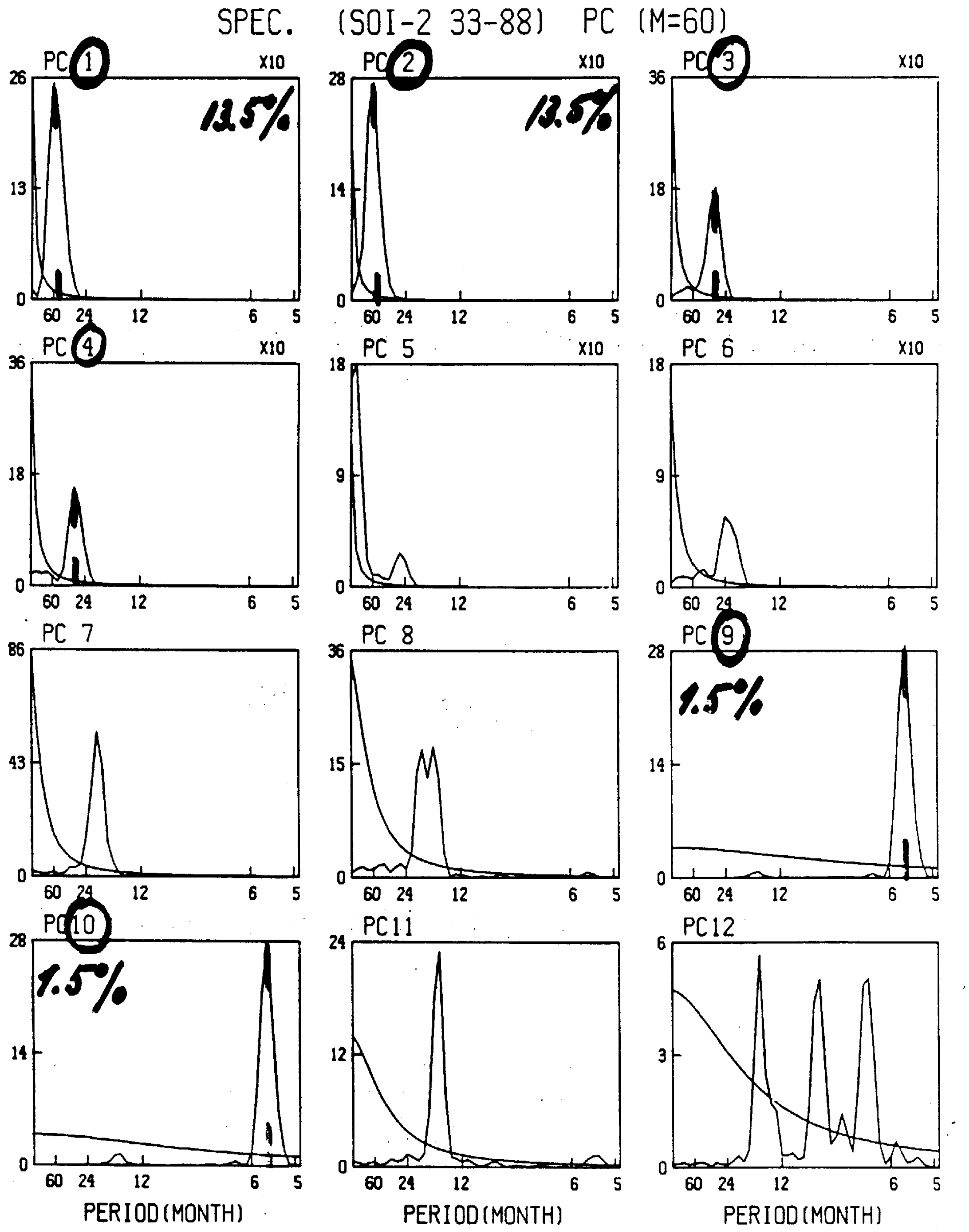
Spectral peaks
(*M* = 60)

Each principal component (PC) is Fourier analyzed separately; individual variance indicated as well.

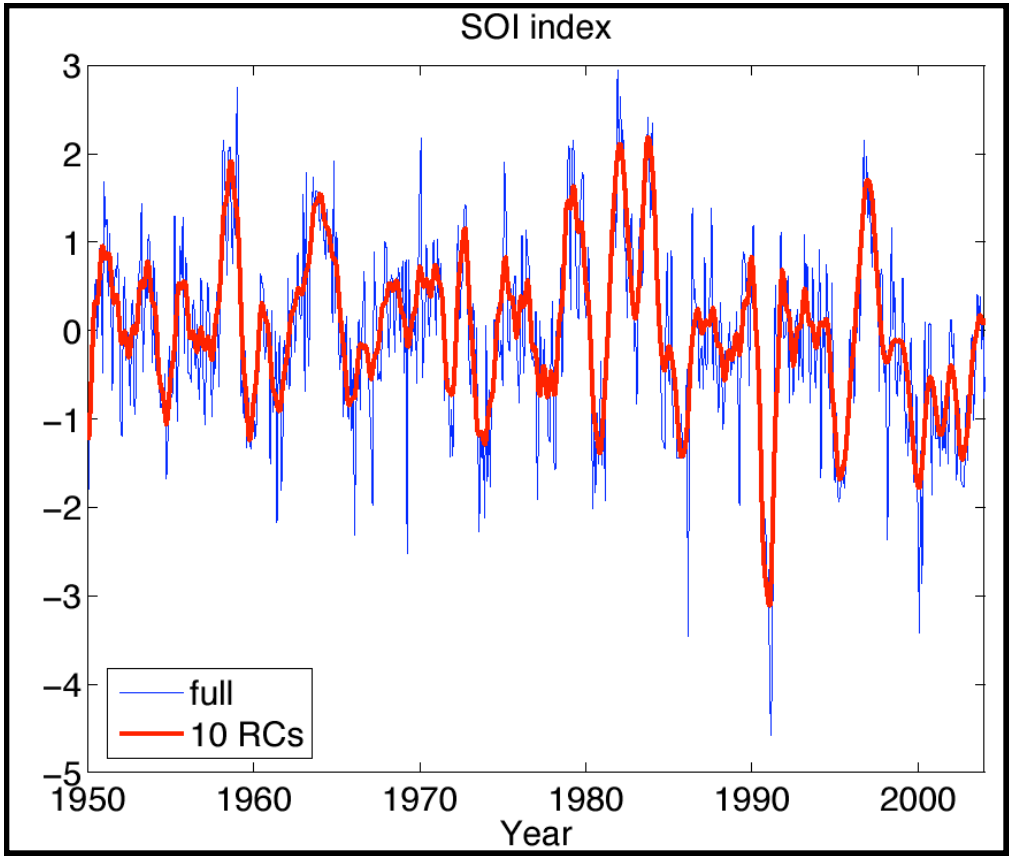
PCs (1+2) – period = 60 months, low-frequency or “ENSO” or quasi-quadrennial (QQ) component;

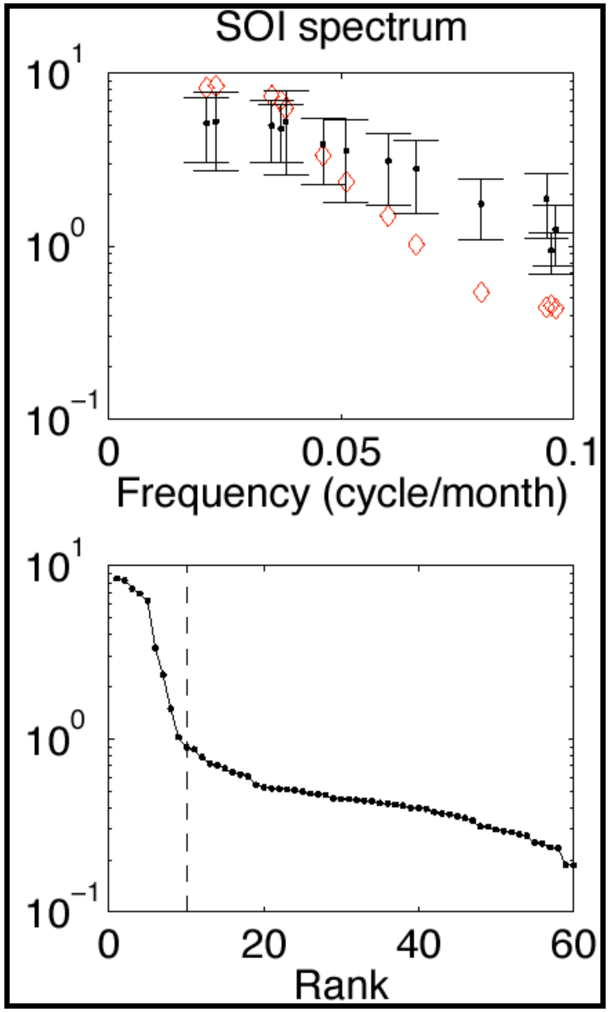
PCs (3+4) – period = 30 months quasi-biennial (QB) component;

PCs (9+10) – period = 5.5 months



Singular Spectrum Analysis (SSA) and M-SSA (cont'd)

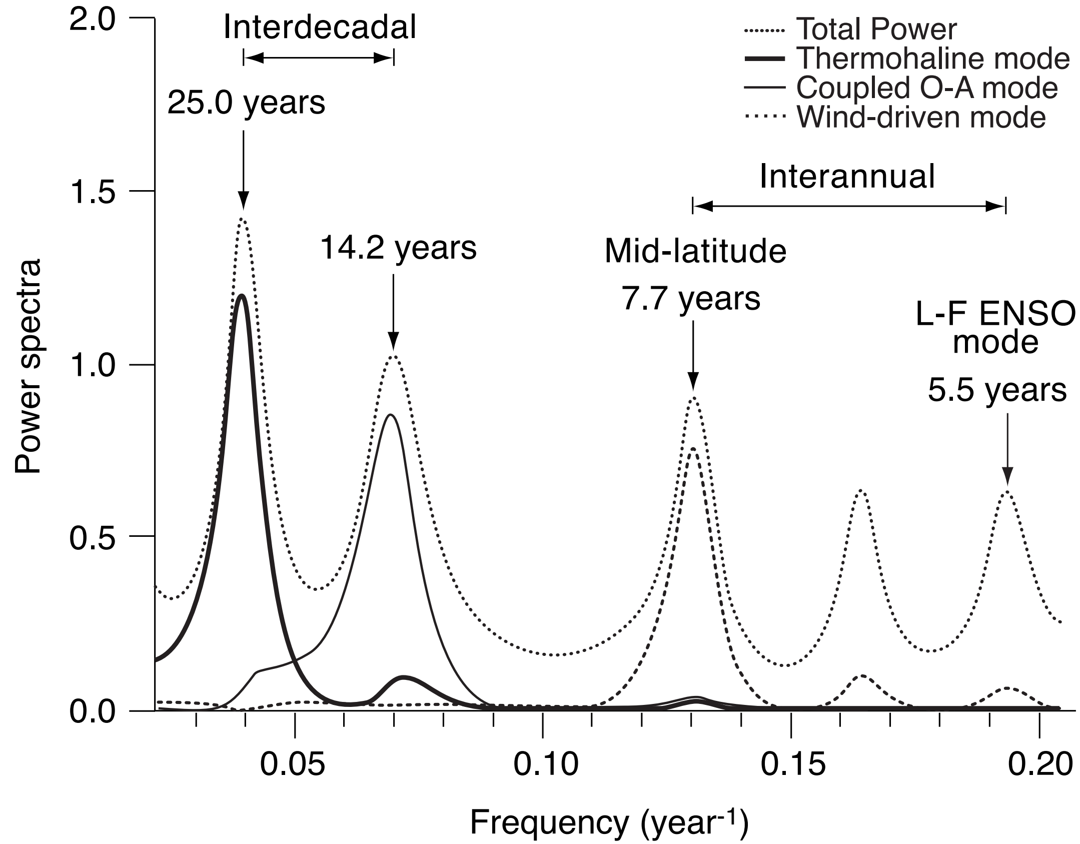




- Break in slope of SSA spectrum distinguishes "***significant***" from "***noise***" EOFs
- Formal Monte-Carlo test (Allen and Smith, 1994) identifies 4-yr and 2-yr ENSO oscillatory modes. A window size of M = 60 is enough to "resolve" these modes in a monthly SOI time series

SSA (prefilter) + (low-order) MEM

- "Stack" spectrum



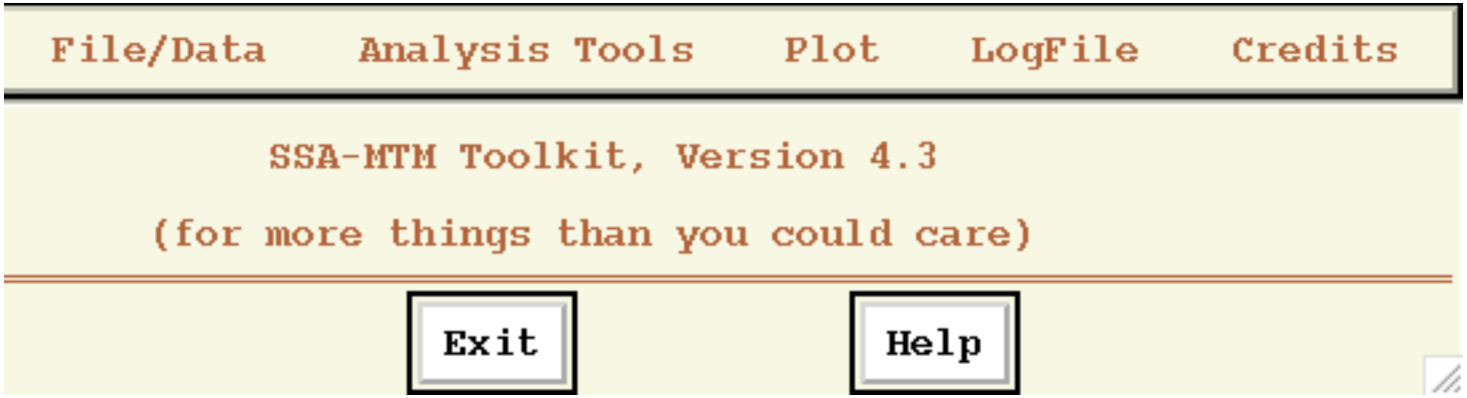
In good agreement with MTM peaks of **Ghil & Vautard (1991, *Nature*)** for the Jones *et al.* (1986) temperatures & stack spectra of Vautard *et al.* (1992, *Physica D*) for the IPCC "consensus" record (both global), to wit 26.3, 14.5, 9.6, 7.5 and 5.2 years.

Peaks at 27 & 14 years also in Koch sea-ice index off Iceland (Stocker & Mysak, 1992), etc.
Plaut, Ghil & Vautard (1995, *Science*)

A Free Toolkit for Spectral Analysis

The SSA-MTM Toolkit:

- Developed at UCLA, with collaborations on 3 continents, since 1994.
- GUI based, for linux, unix and MacOSX platforms.
- Latest developments by D. Kondrashov (UCLA).
- Hundreds of downloads at every new version.
- Available at: www.atmos.ucla.edu/tcd/ssa

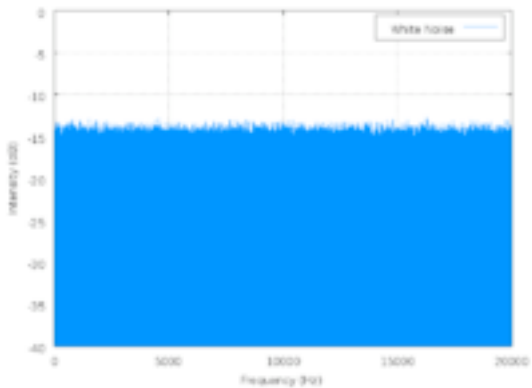


- Ported to Sun, Dec, SGI, PC Linux, and Mac OS X
- Graphics support for IDL and Grace
- Precompiled binaries are available at www.atmos.ucla.edu/ tcd/ssa
- Includes ***Blackman-Tukey FFT***, ***Maximum Entropy Method***, ***Multi-Taper Method (MTM)***, ***SSA and M-SSA***.
- Spectral estimation, decomposition, reconstruction & prediction.
- Significance tests of "**oscillatory modes**" vs. "**noise.**"

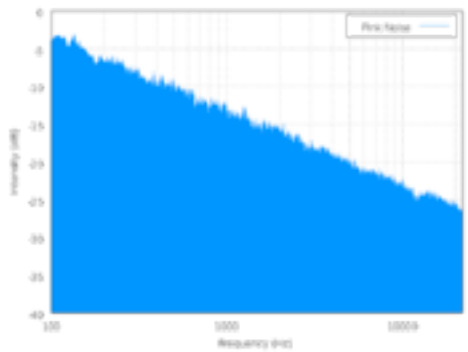
General Goals

- Reduce the variance of the spectral estimate of a time series, based on the periodogram (MTM), correlogram (BT) or other (SSA).
- Estimate peak frequencies to “fingerprint” limit cycles of the underlying dynamical system.
- Provide confidence intervals when such behavior is blurred by noise.

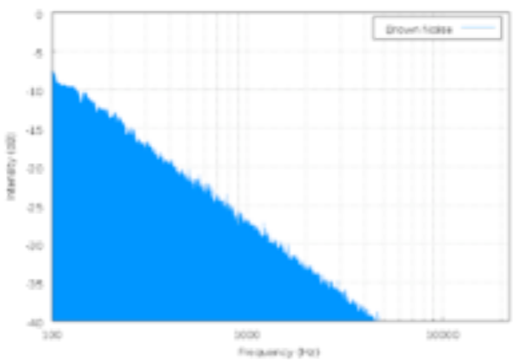
Noise "colors"



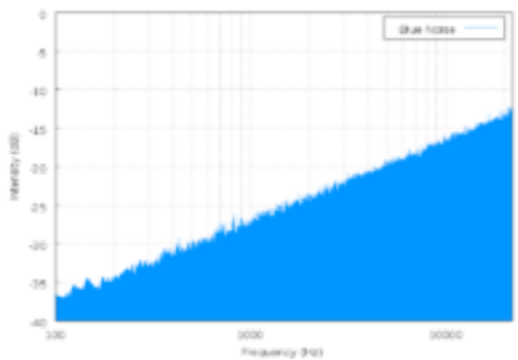
White noise, $S \sim f^0$



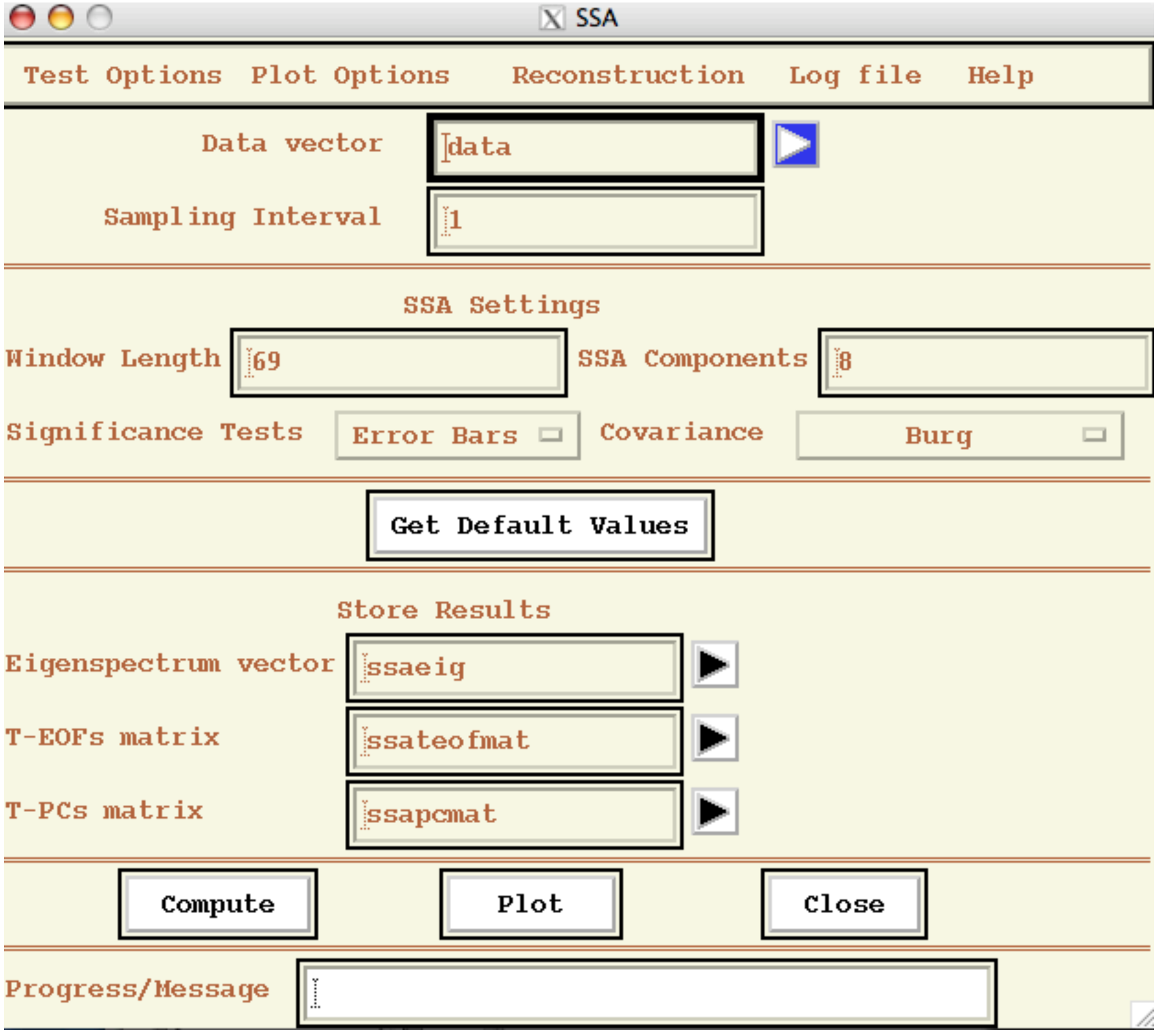
Pink (or 1/*f*) noise, $S \sim f^{-1}$



Red (or Brown) noise, $S \sim f^{-2}$



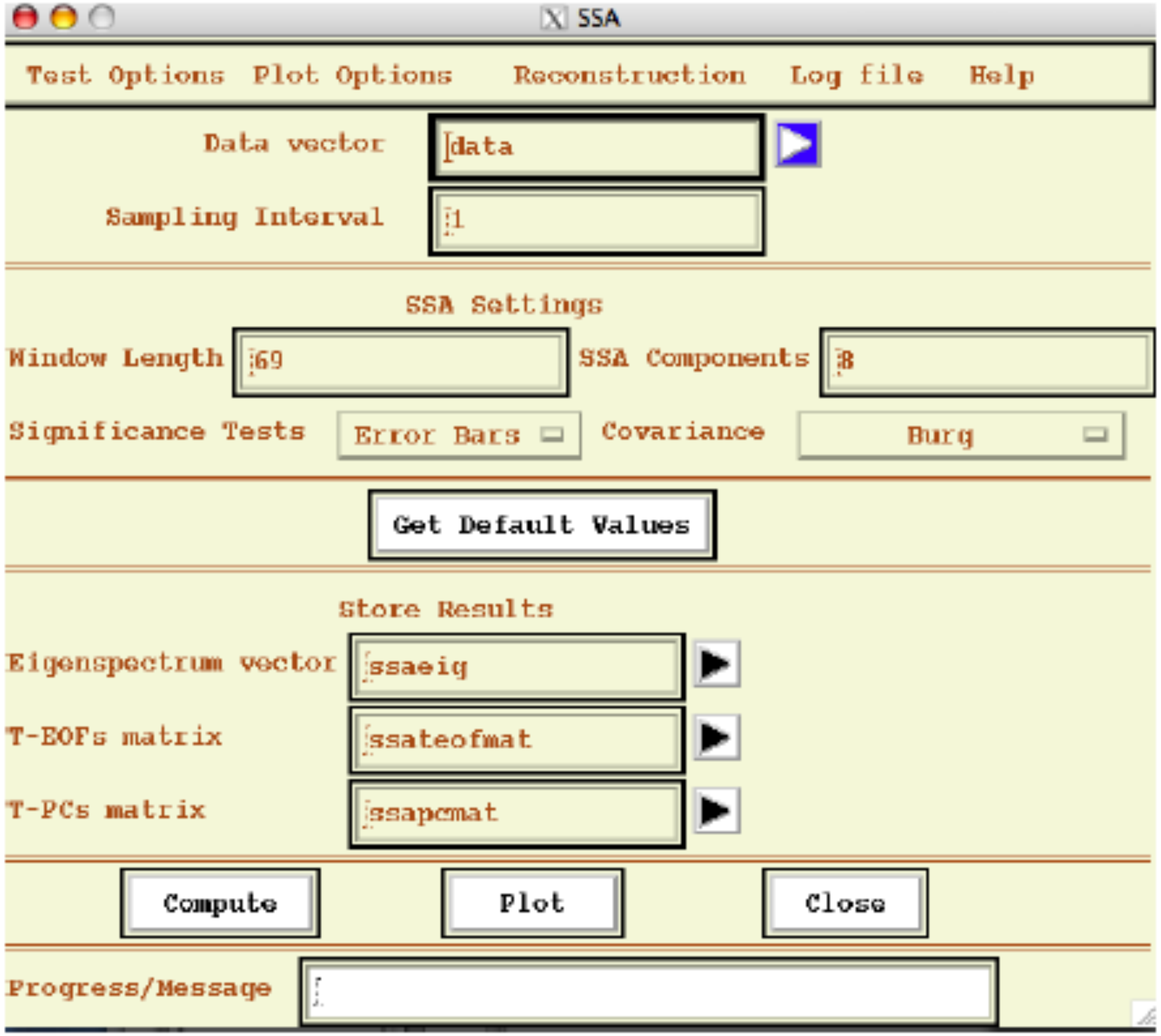
Blue noise, $S \sim f^{+1}$



- **Free!!!**
- Data management with ***named vectors & matrices.***
- ***Default values*** button.

Targeted audience

- Non-specialists in time series analysis
  - Reasonable default options
  - Reads ASCII files
- Non-specialists in computer management
  - Precompiled binaries
  - User-friendly interface



Type of noise used in the toolkit

- Red noise:
  - AR(1) random process: $X(t+1) = aX(t) + b(t)$
  - Decreasing spectrum (due to inertia)

$$\boxed{C_X(\tau) = \frac{\sigma^2 a^{|\tau|}}{1-a^2}}$$

$$P_X(f) = C_X(0)\frac{1-a^2}{1-2a\cos\ 2\pi f + a^2}$$

Outline

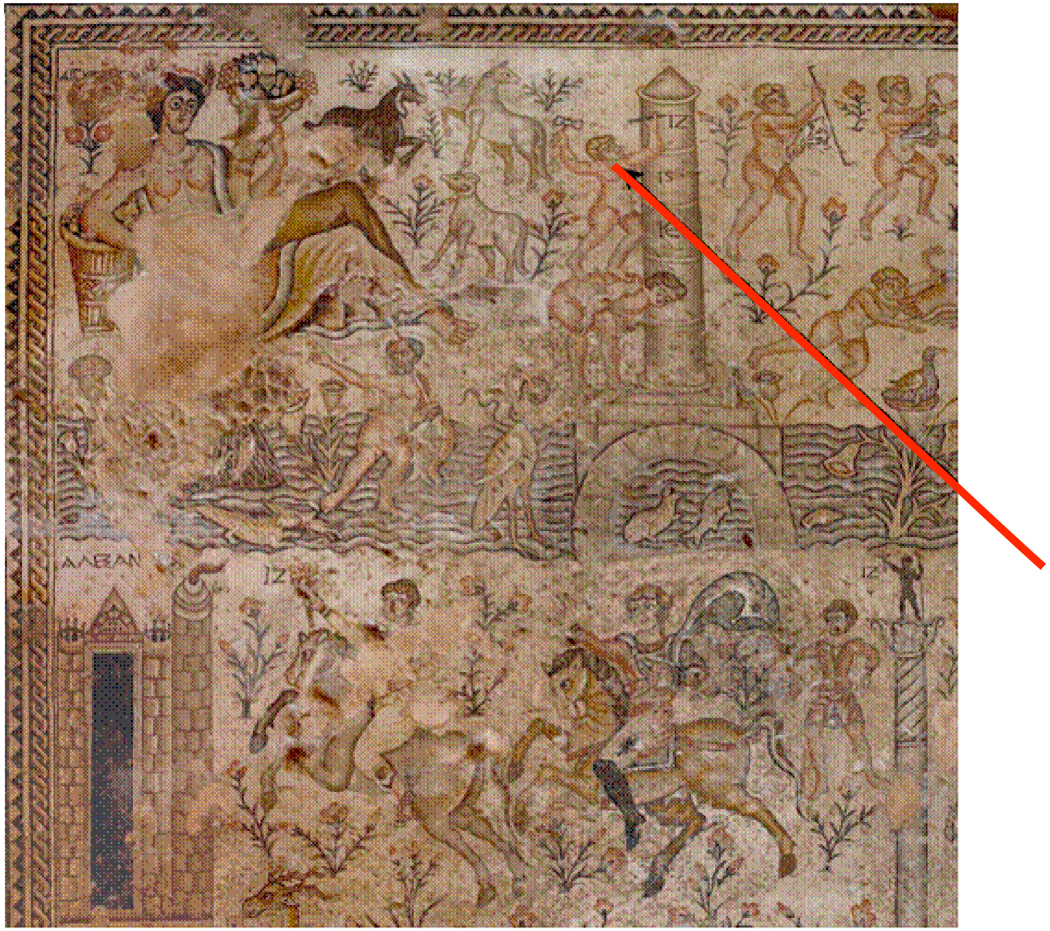
- Time series analysis
  - The “smooth” and “rough” part of a time series
  - Oscillations and nonlinear dynamics
- Singular spectral analysis (SSA)
  - Principal components in time and space
  - The SSA-MTM Toolkit
- The Nile River floods
  - Longest climate-related, instrumental time series
  - Gap filling in time series
  - NAO and SOI impacts on the Nile River
- Concluding remarks
  - Cautionary remarks (“garde-fous”)
  - References

The Nile River Records Revisited:

How good were Joseph's predictions?

Michael Ghil, ENS & UCLA
Yizhak Feliks, IIBR & UCLA,
Dmitri Kondrashov, UCLA

Why are there data missing?



Hard Work

- Byzantine-period mosaic from Zippori, the capital of Galilee (1st century B.C. to 4th century A.D.); photo by Yigal Feliks, with permission from the Israel Nature and Parks Protection Authority)

What to do about gaps?

- Most of the advanced ***filling-in*** methods are different flavors of ***Optimal Interpolation*** (**OI**: Reynolds & Smith, 1994; Kaplan 1998).

Drawbacks: they either (i) require error statistics to be specified *a priori*; or (ii) derive it **only** from the interval of dense data coverage.

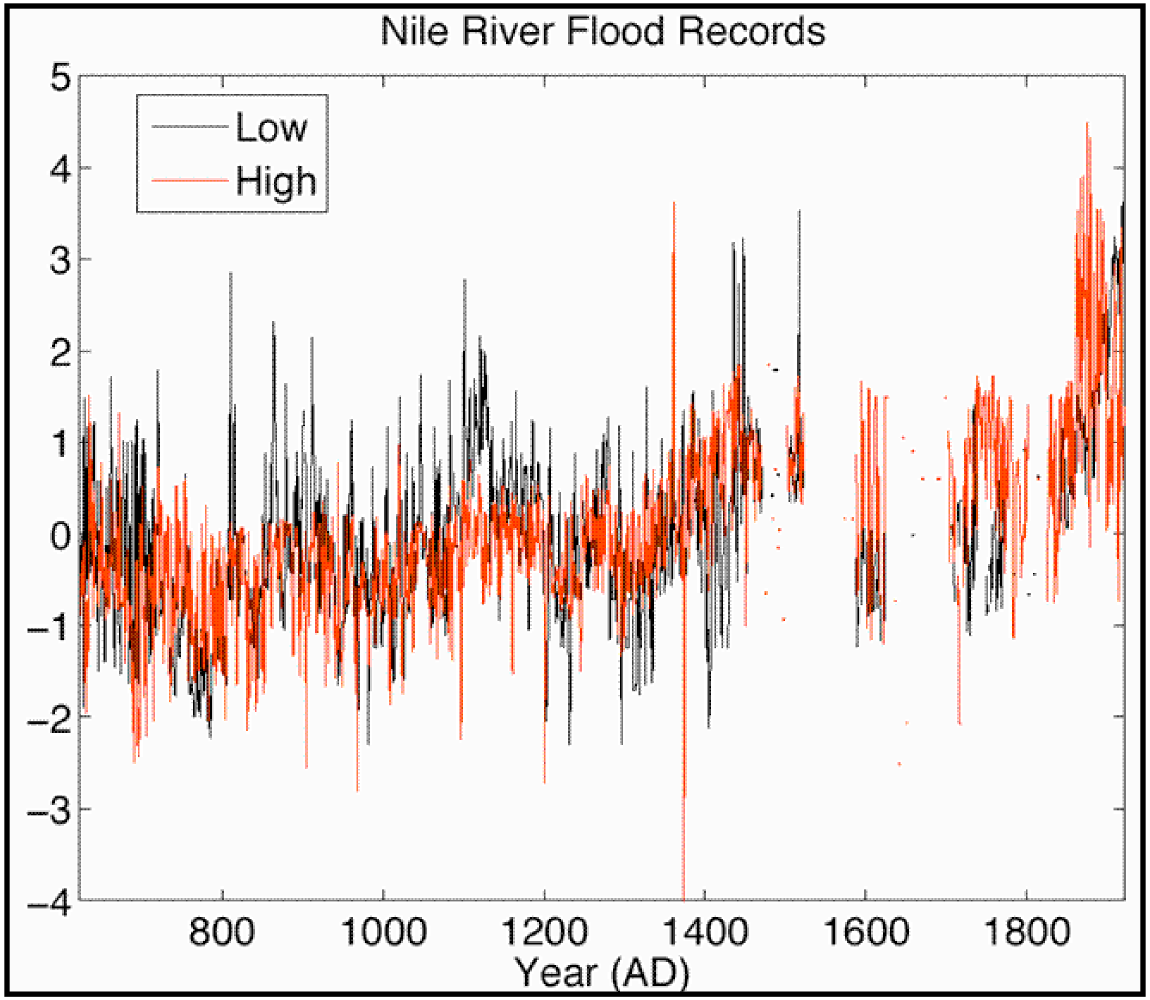
EOF Reconstruction (Beckers & Rixen, 2003): (i) iteratively compute **spatial-covariance** matrix using **all the data**; (ii) determine via cross-validation "**signal**" EOFs and use them to fill in the missing data; accuracy is similar to or better than **OI** (Alvera-Azcarate *et al*. 2004).

Drawbacks: uses **only** spatial correlations **=>** cannot be applied to very **gappy** data.

We propose ***filling in*** **gaps by applying iterative SSA (or M-SSA):**

Utilize both spatial and temporal correlations of data => can be used for highly **gappy** data sets; simple and easy to implement!

Historical records are full of “gaps”....



Annual maxima and minima of the water level at the nilometer on Rodah Island, Cairo.

SSA (M-SSA) Gap Filling

Main idea: utilize both spatial and temporal correlations to iteratively compute self-consistent lag-covariance matrix; M-SSA with $M = 1$ is the same as the EOF reconstruction method of Beckers & Rixen (2003)

Goal: keep "signal" and truncate "noise" — usually a few leading EOFs correspond to the dominant oscillatory modes, while the rest is noise.

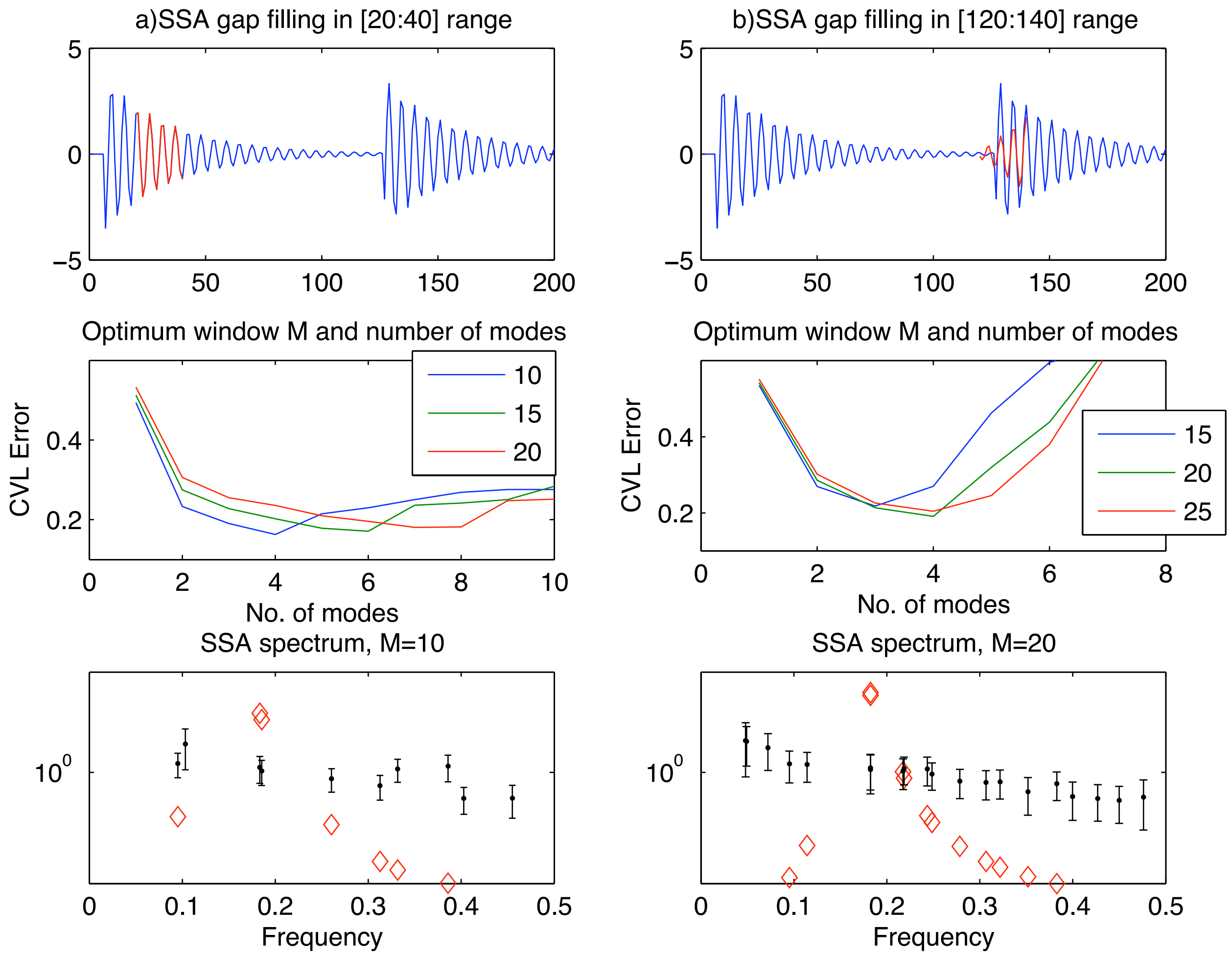
(1) for a given window width M: center the original data by computing the unbiased value of the mean and set the missing-data values to zero.

(2) start iteration with the first EOF, and replace the missing points with the reconstructed component (RC) of that EOF; repeat the SSA algorithm on the new time series, until convergence is achieved.

(3) repeat steps (1) and (2) with two leading EOFs, and so on.

(4) apply cross-validation to optimize the value of M and the number of dominant SSA (M-SSA) modes K to fill the gaps: a portion of available data (selected at random) is flagged as missing and the RMS error in the reconstruction is computed.

Synthetic I: Gaps in Oscillatory Signal



- Very good gap filling for smooth modulation; OK for sudden modulation.

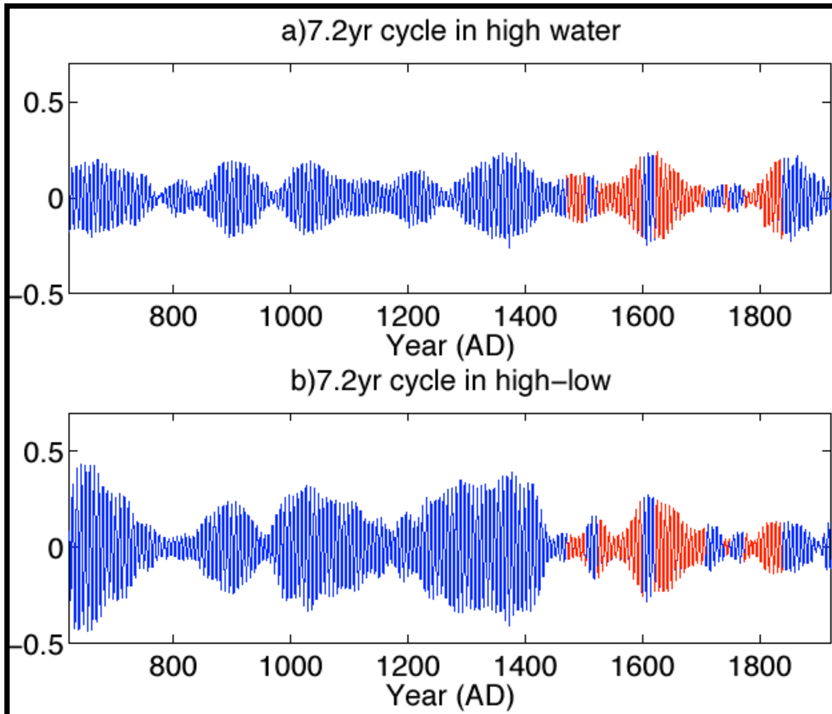
Table 1a: Significant oscillatory modes in short records (A.D. 622–1470)

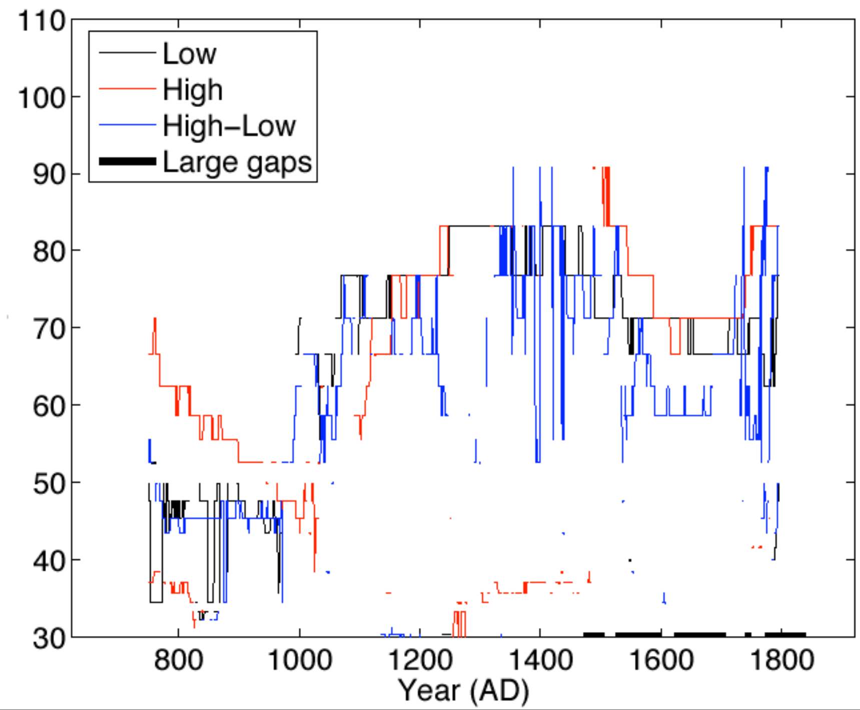
Periods	Low	High	High-Low
40–100yr	**64** (9.3%)	**64** (6.9%)	**64** (6.6%)
20–40yr		[32]	
10–20yr	**12.2** (5.1%), **18.0** (6.7%)		**12.2** (4.7%), **18.3** (5.0%)
5–10yr	**6.2** (4.3%)	**7.2 (4.4%)**	**7.3 (4.4%)**
0–5yr	**3.0** (2.9%), **2.2** (2.3%)	**3.6** (3.6%), **2.9** (3.4%), **2.3** (3.1%)	**2.9** (4.2%),

Table 1b: Significant oscillatory modes in extended records (A.D. 622–1922)

Periods	Low	High	High-Low
40–100yr	**64** (13%)	**85** (8.6%)	**64** (8.2%)
20–40yr		**23.2** (4.3%)	
10–20yr	[12], **19.7** (5.9%)		**12.2** (4.3%), **18.3** (4.2%)
5–10yr	[6.2]	**7.3** (4.0%)	**7.3** (4.1%)
0–5yr	**3.0** (4%), **2.2** (3.3%)	**4.2** (3.3%), **2.9** (3.3%), **2.2** (2.9%)	[4.2], **2.9** (3.6%), **2.2** (2.6%)

Significant Oscillatory Modes

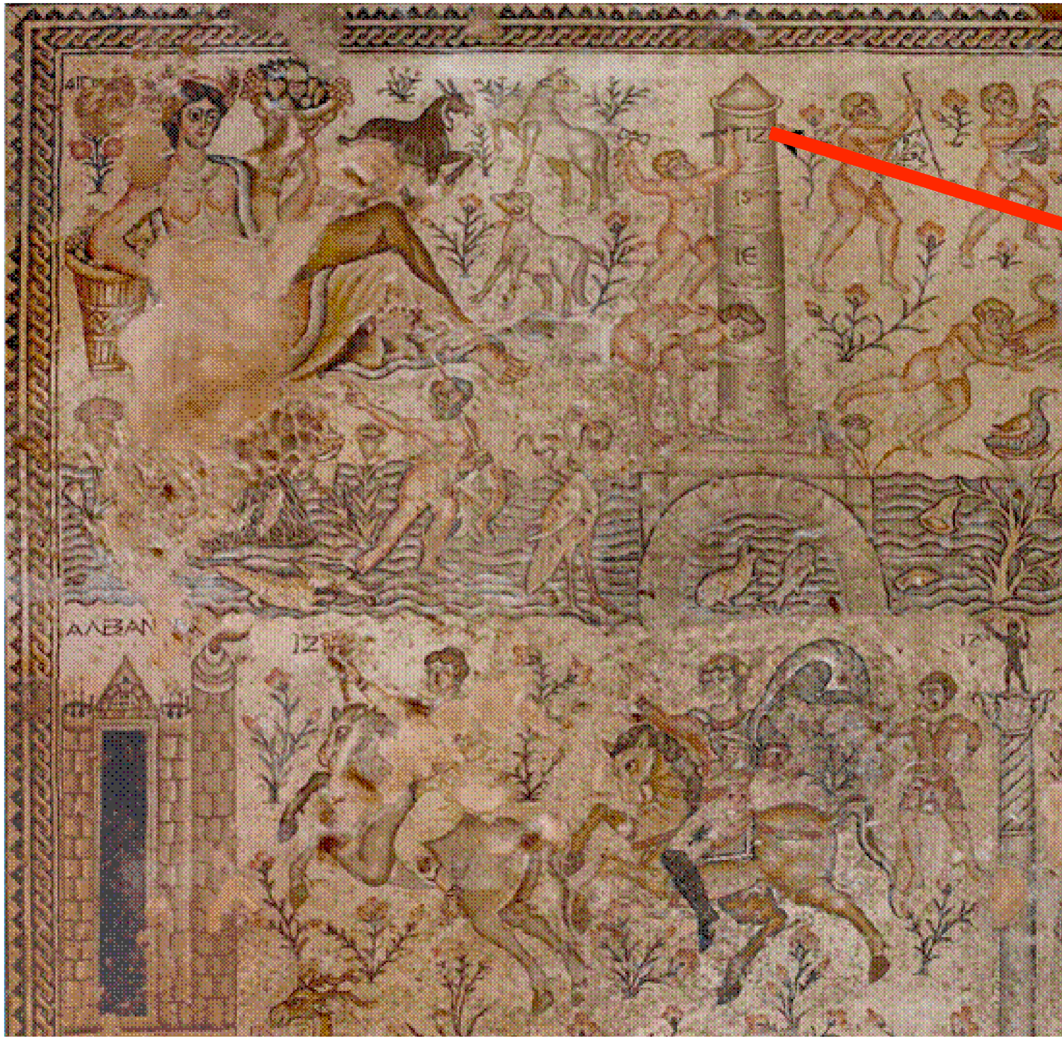




SSA reconstruction of the 7.2-yr mode in the extended Nile River records: (a) high-water, and (b) difference. Normalized amplitude; reconstruction in the large gaps in red.

Instantaneous frequencies of the oscillatory pairs in the low-frequency range (40–100 yr). The plots are based on multi-scale SSA [Yiou *et al*., 2000]; local SSA performed in each window of width $W = 3M$, with $M = 85$ yr.

How good were Joseph's predictions?



Pretty good!

Outline

- Time series analysis
  - The “smooth” and “rough” part of a time series
  - Oscillations and nonlinear dynamics
- Singular spectral analysis (SSA)
  - Principal components in time and space
  - The SSA-MTM Toolkit
- The Nile River floods
  - Longest climate-related, instrumental time series
  - Gap filling in time series
  - NAO and SOI impacts on the Nile River
- Concluding remarks
  - Cautionary remarks (“garde-fous”)
  - References

Significance tests (“garde-fous”) in SSA

To check a spectral feature, e.g., an oscillatory pair:

1. Find pair for given data set $\{X_n: n = 1,2, \ldots N\}$ and window width M.
2. Apply statistical significance tests (MC-SSA, etc.).
3. Check robustness of pair by changing M, sampling interval τ_s, etc.
3. Apply additional methods (MTM, wavelets, etc.) and their tests to $\{X_n\}$.
4. Obtain additional time series pertinent to the same phenomenon $\{Y_m\}$, etc.
5. Apply steps (1)–(3) to these data sets.
6. Use multi-channel SSA (M-SSA) and other multivariate methods to check mutual dependence between $\{X_n\}$, $\{Y_m\}$, etc.
7. Based on steps (1)–(6), try to provide a physical explanation of the mode.
8. Use (7) to predict an as-yet-unobserved feature of the data sets.
9. If this new feature is found in new data, go on to next problem.
10. If not, go back to an earlier step of this list.

(*) **Ghil, M.**, M. R. Allen, M. D. Dettinger, K. Ide, D. Kondrashov, M. E. Mann, A. W. Robertson, A. Saunders, Y. Tian, F. Varadi, and P. Yiou, 2002: Advanced spectral methods for climatic time series, *Rev. Geophys.*, **40**(1), pp. **3**.1–**3**.41, doi: 10.1029/2000RG000092.

Spectral analysis of time series

Problem 7

a. Apply SSA and one or two other advanced spectral methods to
your favorite time series.

b. Follow the "ten commandments" of time series analysis.

Some references

Broomhead, D. S., King, G. P., 1986a. Extracting qualitative dynamics from experimental data. *Physica D*, **20**, 217–236.

Ghil, M., and R. Vautard, 1991: Interdecadal oscillations and the warming trend in global temperature time series, *Nature*, **350**, 324–327.

Ghil, M., M. R. Allen, M. D. Dettinger, K. Ide, D. Kondrashov, M. E. Mann, A. W. Robertson, A. Saunders, Y. Tian, F. Varadi, and P. Yiou, 2002: Advanced spectral methods for climatic time series*, Rev. Geophys.*, **40**(1), pp. **3**.1–**3**.41, doi: 10.1029/2000RG000092.

Karhunen, K., 1946. Zur Spektraltheorie stochastischer Prozesse. *Ann. Acad. Sci. Fenn. Ser. A1, Math. Phys.*, **34**.

Loève, M., 1978. *Probability Theory*, Vol. II, 4th ed., Graduate Texts in Mathematics, vol. 46, Springer-Verlag.

Plaut, G., **M. Ghil** and R. Vautard, 1995: Interannual and interdecadal variability in 335 years of Central England temperatures, *Science*, **268**, 710–713.

Vautard, R., and **M. Ghil**, 1989: Singular spectrum analysis in nonlinear dynamics, with applications to paleoclimatic time series, *Physica D*, **35**, 395–424.

Reserve slides

Diapos de réserve

Singular

Spectrum

Analysis

Cours « Séries temporelles en écologie et épidémiologie »

STEM (AgroParisTech, ENS, MNHN, P-6, P-11)

Singular Spectrum Analysis (SSA) and the SSA-MTM Toolkit

Michael Ghil

CERES-ERTI, etc.

Based on joint work with many students, post-docs, and colleagues over the years; please see http://www.environnement.ens.fr/ and http://www.atmos.ucla.edu/tcd/ for further details.

Monte Carlo SSA

(Allen et Smith, *J. Clim.*, 1995)

Goal: Assess whether the SSA spectrum can reject the null hypothesis that the time series is red noise.

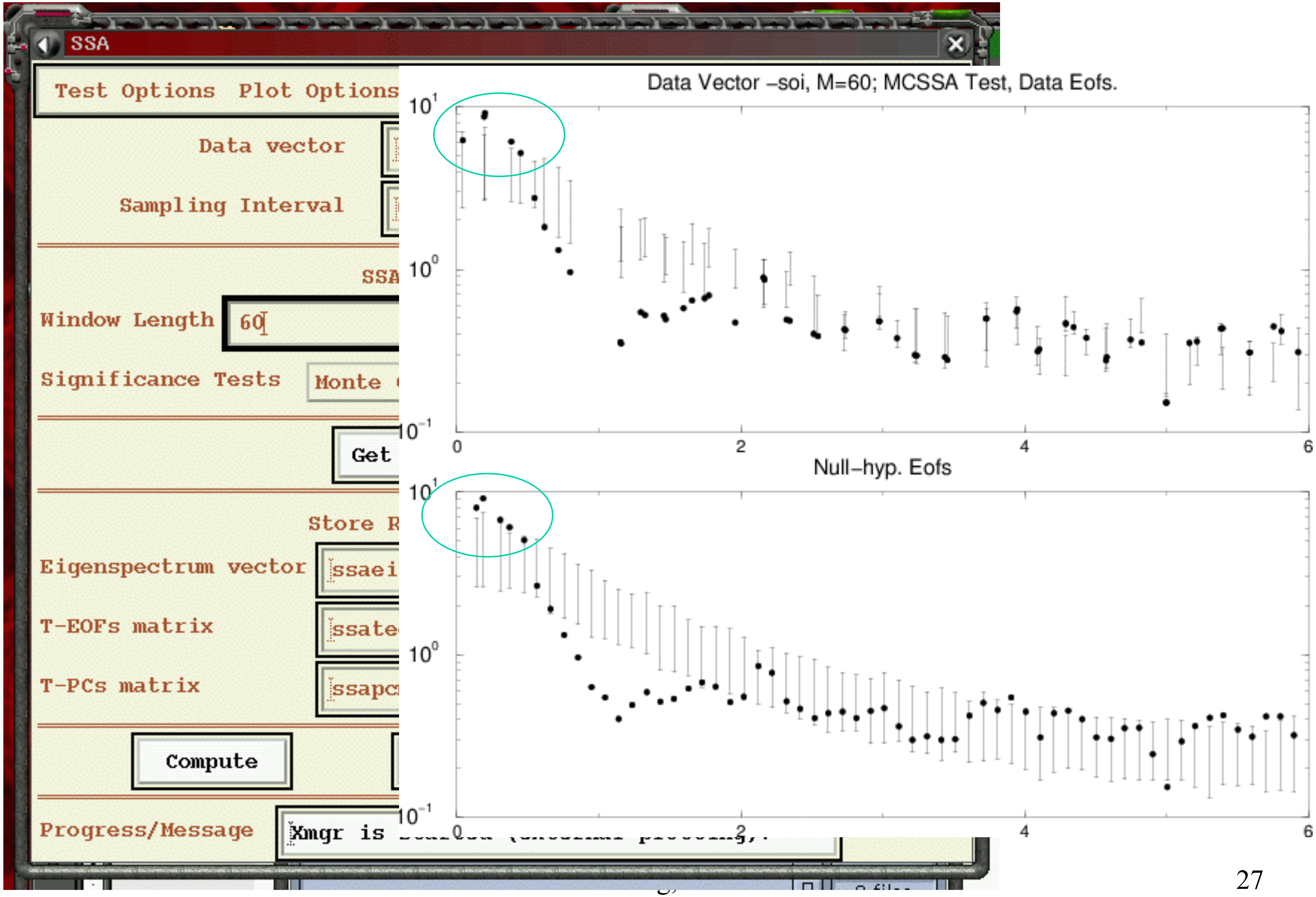
Procedure:

- Estimate red noise parameters with same variance and auto-covariance as the observed time series $X(t)$
- Compare the pdf of the projection of the noise covariance onto the data EOFs:

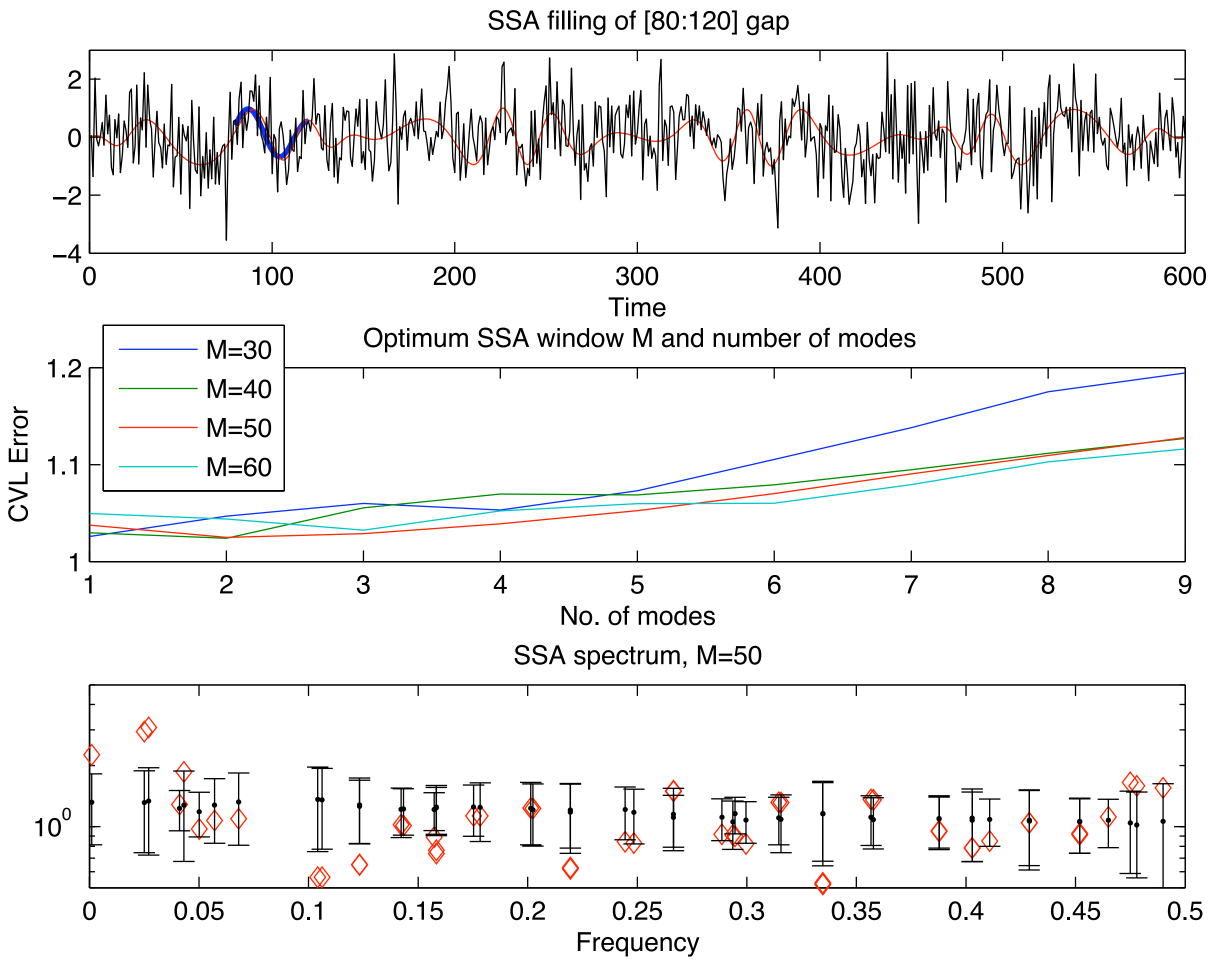
$$\Lambda_B = R_X^t \underbrace{C_R}_{\text{Covar. bruit rouge}} \overbrace{R_X}^{\text{EOFs données}}$$

The null hypothesis is rejected using the pdf of Λ_B.

Monte Carlo SSA: red noise test

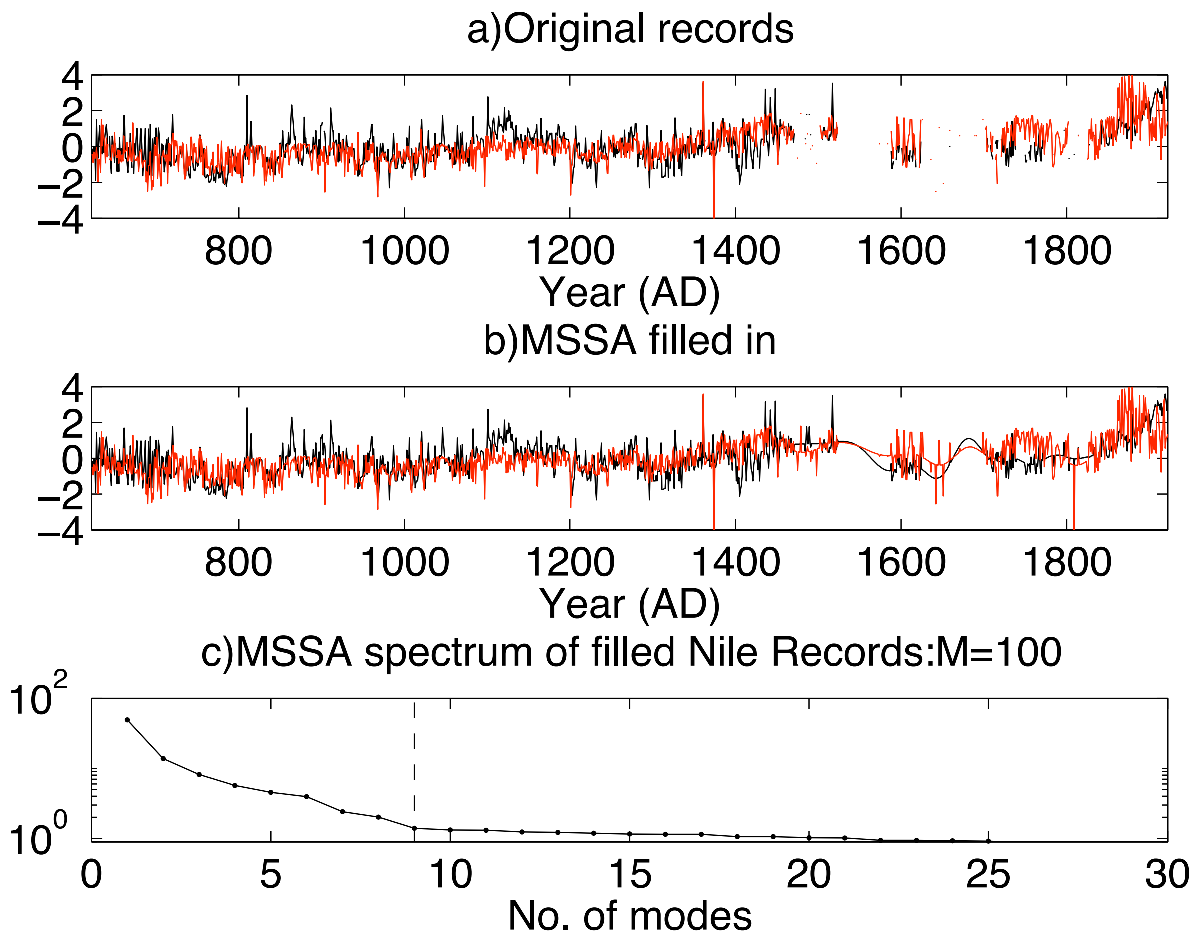


Synthetic II: Gaps in Oscillatory Signal + Noise



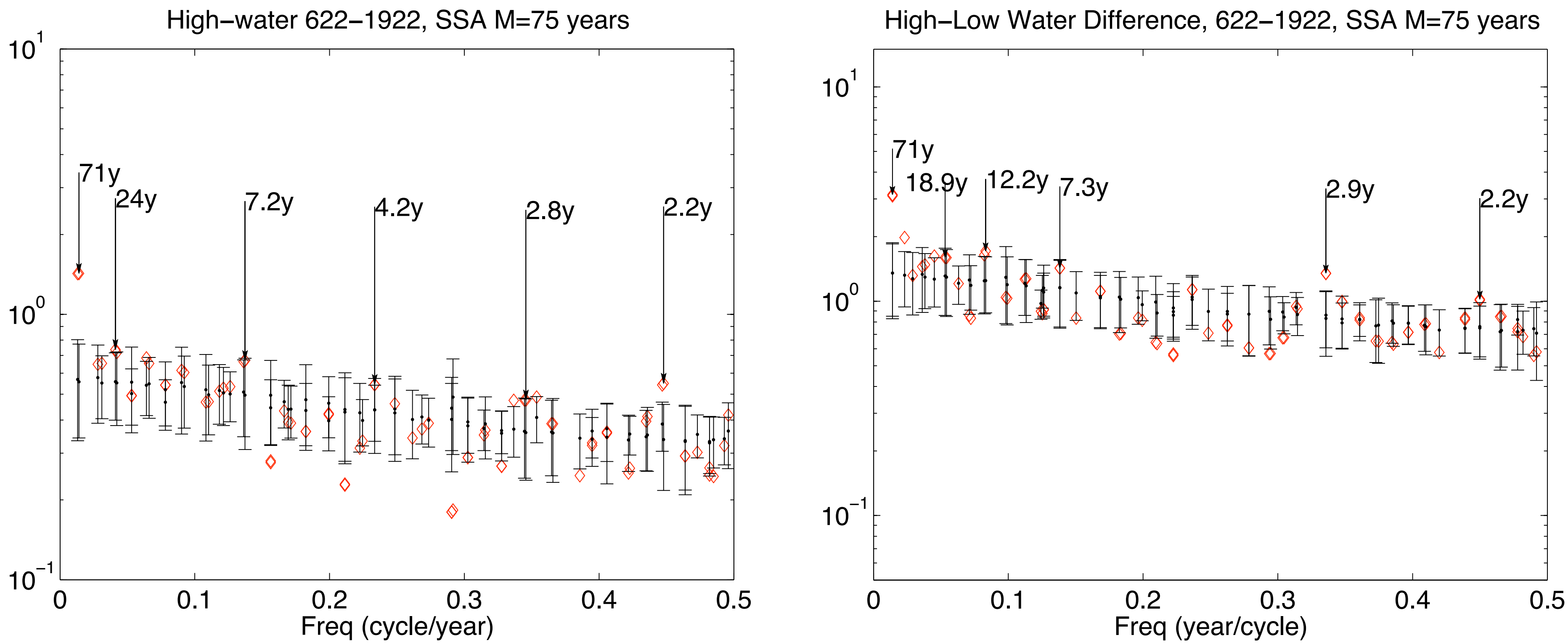
$$x(t) = sin(\frac{2\pi}{300}t) * cos(\frac{2\pi}{40}t + \frac{\pi}{2}sin\frac{2\pi}{120}t)$$

Nile River Records



- High level
- Low level

MC-SSA of Filled-in Records

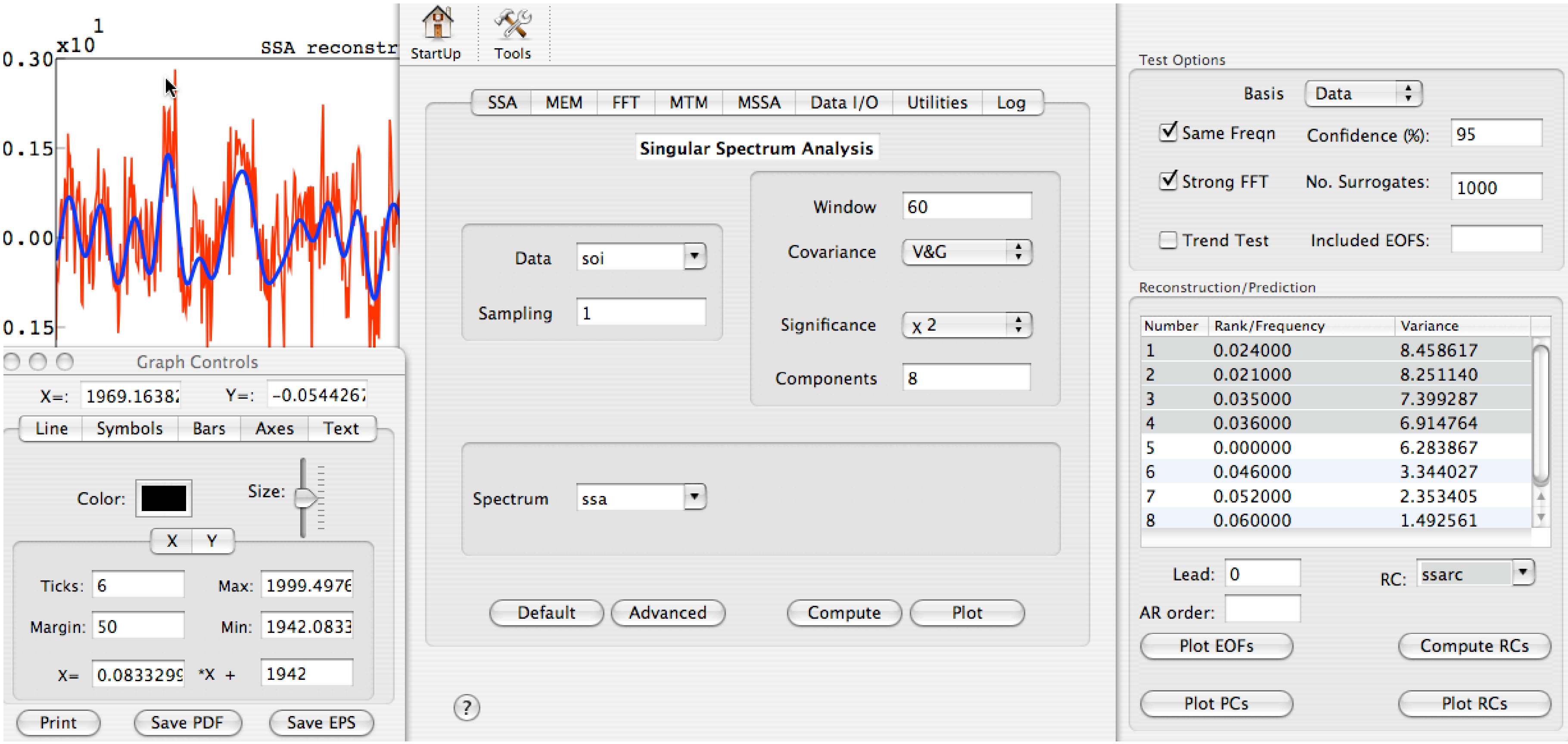


SSA results for the extended Nile River records;

arrows mark highly significant peaks (at 95%), in both SSA and MTM.

The Nile River Basin initiative will greatly modify the flow along the longest & best-documented river system in the world …

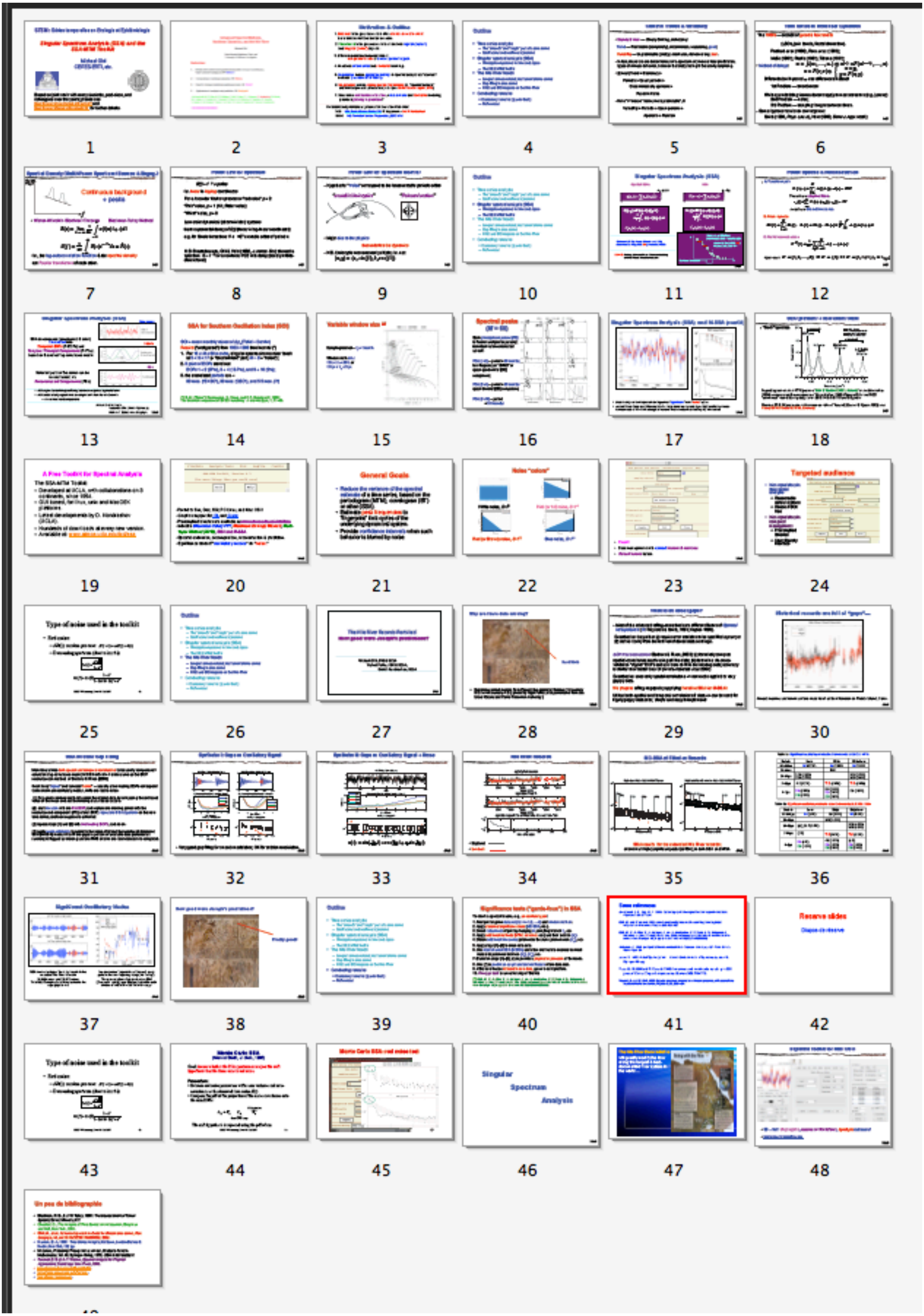
kSpectra Toolkit for Mac OS X



- **$$ ... but: *Project files*, *Automator WorkFlows*, *Spotlight* and more!**
- **www.spectraworks.com**

Un peu de bibliographie

- Blackman, R. B., & J. W. Tukey, 1958: *The Measurement of Power Spectra*, Dover, Mineola, N.Y.
- Chatfield, C., *The Analysis of Time Series: An Introduction*, Chapman and Hall, New York, 1984.
- Ghil, M., *et al*., Advanced spectral methods for climatic time series, *Rev. Geophys.,* 40, doi:10.1029/2001RG000092, 2002.
- Hannan, E. J., 1960: *Time Series Analysis*, Methuen, London/Barnes & Noble, New York, 152 pp.
- M. Loève, *Probability Theory. Vol. II, 4th ed.,* Graduate Texts in Mathematics, Vol. 46, Springer-Verlag, 1978*, ISBN 0-387-90262-7.*
- Percival, D. B.,& A. T. Walden, *Spectral Analysis for Physical Applications*, Cambridge Univ. Press, 1993.
- http://www.ipsl.jussieu.fr/CLIMSTAT/ .
- http://www.atmos.ucla.edu/tcd/ssa .
- http://www.r-project.org .


A Free Toolkit for Spectral Analysis
General Goals
Targeted audience
Type of noise used in the toolkit
Reserve slides
Singular
Spectrum
Analysis
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42
43
44
45
46
47
48