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Spectral Theory for Bilinear Control Systems

Fritz Colonius
Universität Augsburg

Introduction

A bilinear control systems has the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} u_i(t) A_i x(t) = A(u)x, \;\; u(t) = (u_i(t))_{i=1,\ldots,m} \in \Omega,$$

with $d \times d$-matrices $A_0, A_1, \ldots, A_m \in \mathbb{R}^{d\times d}$ and compact convex control range $\Omega \subset \mathbb{R}^m$.
We will consider the associated control flow and controllability properties as well as exponential stability properties.
Crucial insight will be gained by analyzing the projection to (real) projective space $\mathbb{P}^{d-1}$.

A rather different analysis of bilinear control systems can be found in
D.L. Elliott, Bilinear Control Systems, 2009
or in the work of Luiz San Martin and others based on semigroups in Lie groups.

The linear control flow

As in the general case, a bilinear control system defines a control flow on $\mathcal{U} \times \mathbb{R}^d$, given by

$$\Phi(t, u, x) = (\theta_t u, \varphi(t, x, u)), t \in \mathbb{R}.$$

The special property of this control flow is its linearity with respect to x,

$$\Phi(t, u, \alpha x + \beta y) = \alpha \Phi(t, u, x) + \beta \Phi(t, u, y), \alpha, \beta \in \mathbb{R}.$$

The state space $\mathcal{U} \times \mathbb{R}^d$ has the structure of a (topologically trivial) vector bundle with compact metric base space $\mathcal{U}$.
Furthermore, we know that the periodic points are dense for the shift θ, hence the base space is chain transitive.

Projective space

Linearity of $\Phi(t,u,x)$ in x immediately implies that one gets an induced flow on $\mathcal{U}\times\mathbb{P}^{d-1}$.
$\mathbb{P}^{d-1}$ may be obtained by identifying opposite points on the unit sphere.
For a solution $x(t)=\varphi(t,x_0,u)$ of $\dot{x}=A(u)x$ one obtains with $s(t)=\frac{x(t)}{\|x(t)\|}$, where $\|x(t)\|=\sqrt{\langle x(t),x(t)\rangle}$,

$$\dot{s}(t)=\left[A(u)-s(t)^T A(u)s(t)\cdot I\right]s(t).$$

In fact,

$$\begin{aligned}\dot{s} &= \frac{\dot{x}\,\|x\|-x\,\langle\dot{x},x\rangle/\|x\|}{\|x\|^2}=\frac{A(u)x\,\|x\|-x\,\langle A(u)x,x\rangle/\|x\|}{\|x\|^2}\\ &= \left[A(u)-s(t)^T A(u)s(t)\cdot I\right]s(t).\end{aligned}$$

Abbreviating $h(s,u)=\left[A(u)-s^T A(u)s\cdot I\right]s$ we can write this as

$$\dot{s}(t)=h(s(t),u(t)) \text{ on } \mathbb{S}^{d-1}.$$

The subtracted term $\left[s^T A(u)s\right]s$ is the radial component of $A(u)s$.

Exponential growth rates

The exponential growth rate or Lyapunov exponent of a solution for (u, x_0) is

$$\lambda(u, x_0) = \limsup_{t\to\infty} \frac{1}{t} \log \|\varphi(t, x_0, u)\| .$$

Somewhat surprisingly, also the Lyapunov exponents are determined by the induced system on projective space,

$$\lambda(u, x_0) = \limsup_{t\to\infty} \frac{1}{t} \int_0^t q(u(\tau), s(\tau)) d\tau \text{ with } q(u, s) := s^\top A(u) s.$$

Selgrade's Theorem

Theorem. Let Φ be a continuous linear flow on on a vector bundle with compact chain transitive base space $\mathcal{U} \times \mathbb{R}^d$. Then for the induced flow $\mathbb{P}\Phi$ on $\mathcal{U} \times \mathbb{P}^{d-1}$ the induced flow has finitely many chain recurrent components $\mathcal{M}_1, \ldots, \mathcal{M}_\ell, 1 \leq \ell \leq d$.

Every $\mathcal{M}_i$ defines an invariant subbundle via

$$\mathcal{V}_i := \mathbb{P}^{-1}(\mathcal{M}_i) = \{(u, x) \in \mathcal{U} \times \mathbb{R}^d \,|(u, \mathbb{P}x) \in \mathcal{M}_i\}$$

and the following decomposition into a Whitney sum holds

$$\mathcal{U} \times \mathbb{R}^d = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_\ell.$$

Example

Consider

$$\dot{x} = Ax.$$

For an eigenvector x corresponding to a real eigenvalue μ of A the point $\mathbb{P}x$ is an equilibrium in $\mathbb{P}^{d-1}$.

More generally, let $\lambda_1, \ldots, \lambda_\ell$ be the pairwise different real parts of the eigenvalues of A and denote by $V(\lambda_i)$ be the direct sum of all generalized eigenspaces for the eigenvalues with real part equal to λ_i. Then the $\mathcal{M}_i := \mathbb{P}V_i$ are the chain recurrent components and

$$\mathbb{R}^d = \bigoplus_{i=1}^{\ell} V(\lambda_i) = \bigoplus_{i=1}^{\ell} \mathbb{P}^{-1}\mathcal{M}_i.$$

The chain control sets

Corollary. For a bilinear control system $\dot{x} = A(u)x, u(t) \in \Omega$, there are $1 \leq \ell \leq d$ chain control sets E_i for the induced system in projective space and there is a Whitney decomposition

$$\mathcal{U} \times \mathbb{R}^d = \bigoplus_{i=1}^{\ell} \mathbb{P}^{-1}\mathcal{E}_i,$$

where the $\mathcal{E}_i$ are the lifts of the chain control sets E_i in $\mathbb{P}^{d-1}$,

$$\mathcal{E}_i = \left\{(u, p) \in \mathcal{U} \times \mathbb{P}^{d-1} \,|s(t) \in E_i, t \in \mathbb{R}, \text{ for } \dot{s} = h(u, s), s(0) = p\right\}.$$

Questions:

- Proof of Selgrade's theorem?

- How are the Lyapunov exponents related to the chain control sets?

- Do the chain control sets coincide with the control sets in projective space?

- What about the control sets in $\mathbb{R}^d$?

- Consequences for stability and stabilizability?

On the proof of Selgrade's theorem

This is based on the relation between chain recurrence, Morse decompositions and attractor-repeller pairs.

Recall:

A **Morse decomposition** of a flow is given by $\{\mathcal{M}_i \,|\, i = 1, \ldots, \ell\}$ with nonvoid, pairwise disjoint and compact isolated invariant sets s.t.

(i) $\forall x \in X : \omega(x), \alpha(x) \subset \bigcup_{i=1}^{\ell} \mathcal{M}_i$;

(ii) there are no cycles.

If the number of chain recurrent components is finite, this corresponds to the finest Morse decomposition. In particular, if the number of chain control sets in a compact invariant set is finite, this corresponds to the finest Morse decomposition of the control flow.

Definition. For a flow on a compact metric space X an attractor A is a compact invariant set with a nbhd N such that

$$A = \omega(N) := \{ y \in X \,|\, \exists (x_n) \in N, \exists t_n \to \infty : y = \lim x_n \cdot t_n \}$$

A compact invariant set R is a repeller if there is a nbhd N^* such that

$$R = \alpha(N^*) := \{ y \in X \,|\, \exists (x_n) \in N^*, \exists t_n \to -\infty : y = \lim x_n \cdot t_n) \}.$$

Proposition. For every attractor,

$$R := \{ x \in X \,|\, \omega(x) \cap A = \varnothing \}$$

is a repeller, called the complementary repeller.

Morse decompositions and attractor-repeller pairs

Theorem. Let $\mathcal{M}_i, i = 1, \dots, \ell$, be subsets of X. Equivalent are:
(i) $\{\mathcal{M}_i \,|\, i = 1, \dots, \ell\}$ form a Morse decomposition;
(ii) there is an increasing sequence of attractors

$$\varnothing = A_0 \subset A_1 \subset \cdots \subset A_n = X$$

such that $\mathcal{M}_{n-i} = A_{i+1} \cap A_i^*$ for $0 \leq i \leq n-1$.

Proof of Selgrade's theorem

Steps of the proof: Show that

- an attractor for the projectivized flow $\mathbb{P}\Phi$ on $\mathcal{U} \times \mathbb{P}^{d-1}$ defines a subbundle of $\mathcal{U} \times \mathbb{P}^{d}$-
- an attractor-repeller pair defines an invariant subbundle decomposition for the linear flow Φ on $\mathcal{U} \times \mathbb{R}^{d}$.
- then one can use the dimension of the subbundles to show that there is a finest Morse decomposition into Morse sets $\mathcal{M}_i$, hence
- this are the chain recurrent components in $\mathcal{U} \times \mathbb{P}^{d}$.
- defining a decomposition of $\mathcal{U} \times \mathbb{R}^{d}$ into invariant subbundles $\mathcal{V}_i := \mathbb{P}^{-1}(\mathcal{M}_i)$.

The Morse spectrum of the bilinear system I

Recall: For $\varepsilon, T > 0$ an (ε, T)-chain ζ in $\mathcal{U} \times \mathbb{P}^{d-1}$ is given by

$$n \in \mathbb{N}, T_0, T_1, \ldots, T_{n-1} > T, (u_0, p_0), \ldots, (u_n, p_n) \in \mathcal{U} \times \mathbb{P}^{d-1}$$

such that

$$d(\Phi(T_i, (u_i, p_i)), (u_{i+1}, p_{i+1})) < \varepsilon \text{ for all } i.$$

Chain exponent of ζ

$$\lambda(\zeta) = \left(\sum_{i=1}^{n-1} T_i\right)^{-1} \sum_{i=1}^{n-1} \left(\log \|\varphi(T_i, x_i, u_i)\| - \log \|x_i\|\right),$$

The Morse spectrum is

$$\sum\nolimits_{Mo} = \left\{\lambda \in \mathbb{R}, \exists \varepsilon_n \to 0, T_n \to \infty, (\varepsilon_n, T_n)\text{-chains } \zeta_n : \lim \lambda(\zeta_n) = \lambda\right\}.$$

The Morse spectrum of the bilinear system II

Results:

(i) $\sum_{Mo} = \bigcup_{i=1}^{\ell} \sum_{Mo}(\mathcal{M}_i)$
(ii) Each $\sum_{Mo}(\mathcal{M}_i)$ consists of a closed interval $[\kappa_i^*, \kappa_i]$.
(iii) For $i < j$ we have $\kappa_i^* < \kappa_j^*$ and $\kappa_i < \kappa_j$.
(iv) $\sum_{Ly} \subset \sum_{Mo}$ and the κ_i^*, κ_i are actually Lyapunov exponents.
(v) The Lyapunov exponents are dense in $\sum_{Mo}$.

(Un)stable subbundle

The upper spectral interval $\Sigma_{Mo}(\mathcal{M}_\ell) = [\kappa_\ell^*, \kappa_\ell]$ determines the robust stability of $\dot{x} = A(u(t))x$ (and stabilizability of the system if the set $\mathcal{U}$ is interpreted as a set of admissible control functions).

The **stable, center, and unstable subbundles** of $\mathcal{U} \times \mathbb{R}^d$ are defined as

$$L^- = \bigoplus_{j:\ \kappa_j < 0} \mathbb{P}^{-1}\mathcal{M}_j,\ L^0 = \bigoplus_{j:\ 0 \in [\kappa_j^*, \kappa_j]} \mathbb{P}^{-1}\mathcal{M}_j, L^+ = \bigoplus_{j:\ \kappa_j^* > 0} \mathbb{P}^{-1}\mathcal{M}_j.$$

Corollary. The zero solution of $\dot{x} = A(u(t))x$, $u \in \mathcal{U}$, is exponentially stable for all $u \in \mathcal{U}$ iff $\kappa_\ell < 0$ iff $L^- = \mathcal{U} \times \mathbb{R}^d$.

Stability radius

The maximal spectral value $\kappa_\ell(\rho)$ is continuous in ρ. Hence we can define the (asymptotic-) stability radius of this family as

$$r = \inf\{\rho \geq 0 \,|\, \exists u_0 \in \mathcal{U}^\rho : \dot{x}^\rho = A(u_0(t))x^\rho \text{ is not exp.stable}\}.$$

The linear oscillator

The linear oscillator with uncertain restoring force:

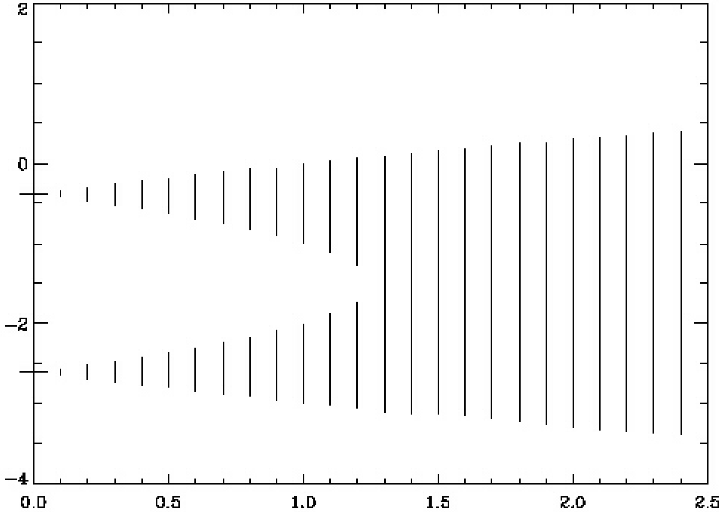
$$\ddot{x} + b\dot{x} + [1 + u(t)]x = 0, \text{ with } u(t) \in [-\rho, \rho], b = 1.5 > 0.$$

or, in state space form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u(t) \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with $u(t) \in [-\rho, \rho]$ and $b > 0$. (For $b \leq 0$ the system is unstable even for constant perturbations.)

Spectral intervals for the linear oscillator



Control sets and chain control sets

The projected system on $\mathbb{P}^{d-1}$ has $0 < k \leq \ell \leq d$ control sets $\mathbb{P}D_j$ with nonvoid interior. Generically, these are the chain control sets. Every chain control set contains (at least one) control set.

The control sets of the bilinear system in $\mathbb{R}^d$ are exactly those cones over the control sets $\mathbb{P}D_j$ for which $0 \in \left(\kappa_j^*, \kappa_j\right)$.

The bilinear system is completely controllable in $\mathbb{R}^d \setminus \{0\}$ iff the projected system is completely controllable and $0 \in \left(\kappa^*, \kappa\right)$.

Stability and stabilizability

- The bilinear control system is exponentially stable for all $u \in \mathcal{U}$ iff $\kappa_\ell < 0$ (robust stability)

- The system is exponentially stabilizable by feedback iff $\kappa_\ell^* < 0$.

Concluding remarks

Bilinear control systems may be viewed as linear flows on vector bundles.

Their topological analysis via chain transitivity, Morse decompositions and attractors leads to a spectral theory which allows us to find results on controllability, stability and stabilizability.