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Control sets, the control flow and relations to random systems

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Introduction

We will associate to control-affine systems a continuous dynamical system which allows us to use methods from the theory of dynamical systems on metric spaces in order to obtain results on controllability properties.

This is, in particular, based on ideas due to C. Conley involving chain transitivity, Morse decompositions and attractor-repeller pairs.

Discrete-time systems

Consider

$$x_{k+1} = f(x_k, u_k),\ \ u_k \in \Omega,\ \text{ for } k \in \mathbb{N} = \{0, 1, ...\},$$

where $f : M \times \Omega \to M$ is continuous on metric spaces M and Ω.
A control function u is an element of $\Omega^{\mathbb{N}}$ (or $\Omega^{\mathbb{Z}}$), the solutions are $\varphi(k, x, u)$.
The shift $\theta : \Omega^{\mathbb{N}} \to \Omega^{\mathbb{N}}$ is $\quad \theta((u_k)_{k\geq 0}) = (u_{k+1})_{k\geq 0}$.
Define the skew product map

$$S : \Omega^{\mathbb{N}} \times M \to M, S(u, x) = (\theta u, f(x, u_0)).$$

Then

$$S^k(u, x) = (\theta^k u, \varphi(k, x, u))$$

and φ is a cocycle, i.e.,

$$\varphi(k + \ell, x, u) = \varphi(k, \varphi(\ell, x, u), \theta^\ell u),\ \text{ for } k, \ell \in \mathbb{N}$$

Discrete-time systems

Proposition. The shift θ and the map S define continuous dynamical systems. If Ω is compact, also $\Omega^{\mathbb{N}}$ is compact.

Proof. Compactness of $\Omega^{\mathbb{N}}$ follows by Tychonov. Continuity of θ follows since the sets

$$W = W_0 \times W_1 \times \cdots \times W_N \times \Omega \times \cdots \subset \Omega^{\mathbb{N}}$$

with $W_i \subset U$ open form a subbasis of the product topology and the preimages

$$\theta^{-1} W = U \times W_0 \times W_1 \times \cdots \times W_N \times U \times \cdots$$

are open. S is continuous by continuity of f.

Continuous-time systems

Consider control-affine systems

$$\begin{aligned}\dot{x}(t) &= f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)),\\ u &\in \mathcal{U} = \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m) \mid u(t) \in \Omega \subset \mathbb{R}^m\}\end{aligned}$$

with trajectories $\varphi(t, x, u), t \in \mathbb{R}$. A special case are bilinear systems

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} u_i(t) A_i x(t), \text{ with } A_i \in \mathbb{R}^{d\times d}.$$

Define the shift on $\mathcal{U}$ by $(\theta_t u)(s) = u(t+s),\ s \in \mathbb{R}$. Then

$$\Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{R}^n \to \mathcal{U} \times \mathbb{R}^n,\ (t, u, x) \to \Phi_t(u, x) = (\theta_t u, \varphi(t, x, u))$$

is a skew product flow,

$$\varphi(t+s, x, u) = \varphi(t, \varphi(s, x, u), \theta_s u) \text{ for } t, s \in \mathbb{R},$$

hence

$$\Phi(t+s, x, u) = (\theta_{t+s} u, \varphi(t+s, x, u)) = \Phi_t \circ \Phi_s(u, x).$$

The shift

Proposition. Assume that $\Omega \subset \mathbb{R}^m$ is convex and compact. Then
(i) Then $\mathcal{U} = \{u \in L^\infty(\mathbb{R}, \mathbb{R}^m) \mid u(t) \in \Omega \subset \mathbb{R}^m\}$ is weak* compact and metrizable in $L^\infty = \left(L^1\right)^*$.
(ii) The shift θ is continuous, the periodic points are dense.

Proof. (i) $\mathcal{U}$ is a convex, bounded closed subset of L^∞, hence, by Alaoglu's Theorem compact and metrizable. The periodic functions are dense: Let $u \in \mathcal{U}$ and $\varepsilon > 0$.

$$\forall x \in L^1 \; \exists T > 0 : \int_{\mathbb{R} \setminus [-T,T]} \|x(t)\| \, dt < \varepsilon / \mathrm{diam}\Omega.$$

Define $u_p(t) = u(t)$ on $[-T, T]$ and extend periodically. Then

$$\left| \int_{\mathbb{R}} [u(t) - u_p(t)]^\top x(t) dt \right| \leq \mathrm{diam}\Omega \int_{\mathbb{R} \setminus [-T,T]} \|x(t)\| \, dt.$$

(ii) Continuity of the shift in the L^1-topology on $\mathcal{U}$ follows, since the shift in L^1 is continuous.

Shadowing

da Silva and Kawan DCDS (2016) have shown that the shift on $\mathcal{U}$ satisfies the following shadowing property:

For every $\varepsilon > 0$ there is $\delta > 0$ such that for every sequence $(u^k)_{k\in\mathbb{Z}}$ in $\mathcal{U}$ with $d(\theta_1 u^k, u^{k+1}) \leq \delta$ there is $u \in \mathcal{U}$ with

$$d(\theta_k u, u^{k+1}) \leq \varepsilon.$$

If the chain $(u^k)_{k\in\mathbb{Z}}$ is periodic, u can be chosen as a periodic function.

The Control Flow

Theorem. For a control affine system with compact and convex control range Ω, the control flow

$$\Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{R}^n \to \mathcal{U} \times \mathbb{R}^n,\ (t, u, x) \to \Phi_t(u, x) = (\theta_t u, \varphi(t, x, u))$$

is continuous.

Proof for bilinear control systems $\dot{x} = A_0 x + \sum_{i=1}^m u_i(t) A_i x$.
Let $t^n \to t^0, u^n \to u^0$ and $x^n \to x^0$ and abbreviate $\varphi^n(t) = \varphi(t^n, x^n, u^n)$.
Using Arzela-Ascoli, let $\varphi^n(\cdot) \to \psi(\cdot)$. Then on $[0, t^0 + 1]$

$$\varphi^n(t) = x^n + \int_0^t A_0 \varphi^n(s) + \sum_{i=1}^m u_i^n(s) A_i \left[\varphi^n(s) - \psi(s)\right] + \sum_{i=1}^m u_i^n(s) A_i \psi(s) ds$$

and by weak* convergence $\int_0^t \sum_i u_i^n(s) A_i \psi(s) ds \to \int_0^t \sum_i u_i^0(s) A_i \psi(s) ds$.

Hence $\psi = \varphi^0$.

Relations to controllability

A flow Φ on a compact metric space X is topologically transitive if there is $x \in X$ with

$$X = \{y = \lim_{k\to\infty} \Phi(t_k, x) \,|\, t_k \to \infty\}.$$

It is topologically mixing if for all open $V, W \subset X$ there is $T > 0$ with

$$\Phi(T, V) \cap W \neq \varnothing.$$

Note: Topological mixing $\Rightarrow$ topological transitive.

Recall: A control set D is a maximal set such that for all $x \in D$ there is $u \in \mathcal{U}$ with $\varphi(t, x, u) \in D$, $t \geq 0$, and

$$D \subset \overline{\mathcal{R}(x)} \text{ for all } x \in D.$$

A point x is locally accessible if for all $T > 0$

$$\mathrm{int}\mathcal{R}_{\leq T}(x) \neq \varnothing \text{ and } \mathrm{int}\mathcal{C}_{\leq T}(x) \neq \varnothing.$$

Relations to controllability

The lift of a control set D with nonvoid interior is

$$\mathrm{cl}\{(u,x) \in \mathcal{U} \times M \,|\, \varphi(t,x,u) \in \mathrm{int}D \text{ for all } t \in \mathbb{R}\}.$$

Theorem. Assume local accessibility.
(i) The lift of a control set D with nonvoid interior is a maximal topologically mixing set.
(ii) Conversely, every maximal topologically transitive set whose projection to M has nonvoid interior is the lift of a control set.

Proof. (i) Needs the subbasis of the topology on $\mathcal{U}$.
(ii).Use local accessibility!

Let Φ be a continuous flow on a compact metric space X.
For $\varepsilon, T > 0$ an (ε, T)-chain ζ from $x \in X$ to $y \in X$ is given by

$$n \in \mathbb{N}, x_0 = x, x_1, \ldots, x_n = y, T_0, T_1, \ldots, T_{n-1} > T$$

such that

$$d(\Phi(T_i, x_i), x_{i+1}) < \varepsilon \text{ for all } i.$$

A set $K \subset M$ is chain transitive if for all $x, y \in K$ and all $\varepsilon, T > 0$ there is an (ε, T)-chain from x to y.
A maximal chain transitive set is called chain recurrent component.

Remark. Conley's Fundamental Theorem implies that the control flow Φ is gradient-like outside the maximal chain transitive sets.$\mathcal{E}_i$.

Examples.

We return to control systems.

Definition. A **chain control set** $E \subset M$ is a maximal set with
(i) for all $x \in E$ there is $u \in \mathcal{U}$ with $\varphi(t, x, u) \in E$ for all $t \in \mathbb{R}$;
(ii) for all $x, y \in E$ and all $\varepsilon, T > 0$ there is a controlled (ε, T)-chain from x to y given by

$$n \in \mathbb{N}, x_0 = x,\ x_1, \ldots, x_n = y,\ u_0, \ldots, u_{n-1} \in \mathcal{U}, T_0, \ldots, T_{n-1} > T \text{ with}$$

$$\mathrm{d}(\varphi(T_i, x_i, u_i), x_{i+1}) < \varepsilon \text{ for all } i.$$

Chain control sets

We return to control-affine systems

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)),\ u \in \mathcal{U} = \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m) \,|\, u(t) \in \Omega\}$$

with control flow

$$\Phi : \mathbb{R} \times \mathcal{U} \times M \to \mathcal{U} \times M,\ (t, u, x) \to (\theta_t u, \varphi(t, x, u)).$$

Theorem. For every compact chain control set E the lift

$$\mathcal{E} := \{(u, x) \in \mathcal{U} \times M \,|\, \varphi(t, x, u) \in E,\ t \in \mathbb{R}\}$$

is a chain recurrent component for the control flow Φ and conversely.

For the **proof** observe that the projection of a chain transitive set for Φ to M yields controlled (ε, T)-chains. For the converse one has to construct (ε, T)-chains in $\mathcal{U} \times M$ from controlled (ε, T)-chains.

Alternative characterization

A **Morse decomposition** of a flow is given by $\{\mathcal{M}_i \,|\, i = 1, \ldots, \ell\}$ with nonvoid, pairwise disjoint and compact isolated invariant sets s.t.
(i) $\forall x \in X : \omega(x), \alpha(x) \subset \bigcup_{i=1}^{\ell} \mathcal{M}_i$;
(ii) there are no cycles.

Example

$$\dot{x} = x(x-1)(x-2)^2(x-3).$$

Morse decompositions are e.g.

$$\begin{aligned}
\mathcal{M}_1 &= \{0\} \preceq \mathcal{M}_2 = [1,3] \\
\mathcal{M}_1 &= \{0\} \preceq \mathcal{M}_3 = \{1\} \succeq \mathcal{M}_2 = [2,3] \\
\mathcal{M}_1 &= \{0\} \cup [2,3] \preceq \mathcal{M}_2 = \{1\}.
\end{aligned}$$

with finest Morse decomposition

$$\{0\} \preceq \{1\} \succeq \{2\} \succeq \{3\}.$$

Theorem. If for a flow on a compact metric space the number of chain recurrent components is finite, this corresponds to the finest Morse decomposition.

In particular, if the number of chain control sets in a compact invariant set is finite, this corresponds to the finest Morse decomposition of the control flow.

Parameter dependence

Under appropriate compactness assumptions, chain control sets depend upper semicontinuously on parameters, and control sets depend lower semicontinuously on parameters (in the Hausdorff metric).

Theorem. Fix α_0 and suppose that D^{α_0}.is a control set such that $\mathrm{cl} D^{\alpha_0} = E^{\alpha_0}$ is a chain control set. Then there are control sets D^{α} and chain control sets E^{α} with

$$\lim_{\alpha \to \alpha_0} \mathrm{cl} D^{\alpha} = \mathrm{cl} D^{\alpha_0} = E^{\alpha_0} = \lim_{\alpha \to \alpha_0} E^{\alpha}.$$

Thus we see that abrupt changes in the behavior can be expected only if control sets and chain control sets are different.

Chain control sets versus control sets I

Next we turn to conditions which ensure that a chain control set is the closure of a control set.

Theorem. Consider different control ranges $U^{\rho} = \rho \cdot U$ with $\rho \geq 0$, and assume the following ρ-**inner-pair condition**:

For all x, all $\rho' > \rho \geq 0$ and all $u \in \mathcal{U}^{\rho}$ there is $T > 0$ with

$$\varphi(T, x, u) \in \mathrm{int}\mathcal{R}^{\rho'}(x).$$

Then for all but at most countably many ρ-values and all control sets

$$\mathrm{cl}D^{\rho} = E^{\rho}.$$

Gayer (2003) could prove the following sufficient condition for the ρ-inner pair condition.

The ρ-inner pair condition holds.for all systems of the form

$$\ddot{x} + g(t, x, \dot{x}) = h(t, x, \dot{x})u(t)$$

with g and h T-periodic in t and $h(t, x, \dot{x}) > 0$.

For the proof one plans a trajectory and solves for the control u (some

Chain control sets versus control sets II

An alternative are hyperbolicity conditions for the control flow which imply the shadowing property.

FC/Du (2003), da Silva and Kawan (2016)

Stochastic Perturbations: Degenerate Markov Diffusions

Consider

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} \xi_i(t,\omega) f_i(x) \text{ on } M$$

with $\xi = h(\eta)$ and **background noise**

$$\mathrm{d}\eta = g_0(\eta)dt + \sum_{j=1}^{l} g_j(\eta) \circ dW_j \text{ on } N \text{ compact,}$$

η stationary, ergodic, and $h : N \to U$ surjective, $U \subset \mathbb{R}^m$ compact, convex.

Associated deterministic system

Consider the control system

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i(t) f_i(x) \quad \text{on } M, \; u \in \mathcal{U},$$

where

$$\mathcal{U} := \{u : \mathbb{R} \to \mathbb{R}^m \,|u(t) \in U \;\text{ for all } t \text{, locally integrable}\}$$

for a given $U \subset \mathbb{R}^m$ with trajectories $\varphi(t, x_0, u)$, $t \in \mathbb{R}$. Let

$$\mathcal{R}^{+}_{\leq T}(x) = \{\varphi(t, x, u) \,|t \in [0, T] \text{ and } u \in \mathcal{U}\}.$$

Associated deterministic system

Consider the control system

$$\dot{x} = f_0(x) + \sum_{i=1}^m u_i(t) f_i(x) \quad \text{on } M,\ u \in \mathcal{U},$$

where

$$\mathcal{U} := \{u : \mathbb{R} \to \mathbb{R}^m \,|u(t) \in U \text{ for all } t, \text{ locally integrable}\}$$

for a given $U \subset \mathbb{R}^m$ with trajectories $\varphi(t, x_0, u)$, $t \in \mathbb{R}$. Let

$$\mathcal{R}_{\leq T}(x) = \{\varphi(t, x, u) \,|t \in [0, T] \text{ and } u \in \mathcal{U}\}.$$

Local accessibility means

$$\mathrm{int}\mathcal{R}_{\leq T}(x) \neq \emptyset \text{ and } \mathrm{int}\mathcal{C}_{\leq T}(x) \neq \emptyset \text{ for all } x \text{ and } T > 0.$$

A set $D \subset M$ is called a **control set** (a set of approximate controllability) if it is maximal with

$$D \subset \overline{\mathcal{R}(x)} \text{ for all } x \in D.$$

A set $C \subset M$ is an **invariant control set** if $\overline{C} = \overline{\mathcal{R}(x)}$ for all $x \in C$.

Stochastic System vs. Control System

Analyze the pair process

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} h_i(\eta) f_i(x)$$
$$\mathrm{d}\eta = g_0(\eta)\mathrm{d}t + \sum_{j=1}^{l} g_j(\eta) \circ \mathrm{d}W_j$$

Assume

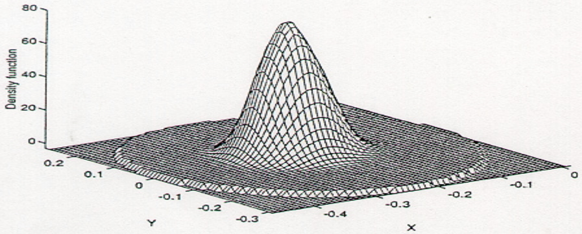
$$\dim \mathcal{LA}\{g_1, \ldots, g_l\}(q) = \dim N \text{ for all } q \in N,$$

$$\dim \mathcal{LA}\left\{ \begin{bmatrix} f_0 + \sum h_i(w) f_i \\ g_0 + \sum w_j g_j \end{bmatrix}, w \in \mathbb{R}^l \right\} \begin{bmatrix} x \\ q \end{bmatrix} = \dim M + \dim N.$$

Theorem The supports of the ergodic measures μ_i are $C_i \times N$ with C_i the invariant control sets. The μ_i are unique and have C^∞ densities.
All other points are transient.

Stroock/Varadhan, Kunita, Kliemann 1987, Arnold/Kliemann 1987, FC/Kliemann 2008

Typical stationary density on an invariant control set



Invariant density, $\rho = 0.05$

support indicated by drop in grid.

Piecewise Deterministic Markov Processes

Let $E = \{0, 1, ..., m\}$ and for any $i \in E$ let $F^i : \mathbb{R}^d \to \mathbb{R}^d$ be a smooth (C^∞) vector field with corresponding flow $\Phi_t^i(x)$, $t \geq 0$.

A **Piecewise Deterministic Markov Process (PDMP)** has the form $Z_t = (X_t, Y_t)$ living on $\mathbb{R}^d \times E$ where the continuous component X_t evolves according to a flow Φ_t^i; the component on E determines which of the flows Φ_t^i is active with random switching times.

Davis (1993)

Piecewise Deterministic Markov Processes

Choice of the flow $\Phi^i, i \in E = \{0, 1, ..., m\}$ on $M \subset \mathbb{R}^d$: Let

$$x \mapsto Q(x) = (Q(x))_{i,j} : \mathbb{R}^d \to \mathbb{R}^{(m+1)\times(m+1)}$$

be continuous with $Q(x)$ irreducible and aperiodic for all x.

Piecewise Deterministic Markov Processes

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be continuous with $Q(x)$ irreducible and aperiodic for all x.
Random switching times T_n: Determined by a homogeneous Poisson process $(N_t)_{t\geq 0}$ with intensity λ, and $U_n = T_n - T_{n-1}$.

Piecewise Deterministic Markov Processes

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be continuous with $Q(x)$ irreducible and aperiodic for all x.
Random switching times T_n: Determined by a homogeneous Poisson process $(N_t)_{t\geq 0}$ with intensity λ, and $U_n = T_n - T_{n-1}$.
The discrete-time process: Let $\tilde{Z}_n = \left(\tilde{X}_n, \tilde{Y}_n\right)$ on $M \times E$ be recursively defined by

$$\tilde{X}_{n+1} = \Phi^{\tilde{Y}_n}(U_{n+1}, \tilde{X}_n)$$

$$\mathbb{P}\left[\tilde{Y}_{n+1} = j \,\middle|\, \tilde{X}_{n+1}, \tilde{Y}_n = i\right] = Q(\tilde{X}_{n+1})_{i,j}.$$

The continuous-time process (by interpolation):

$$Z_t = \left(\Phi^{\tilde{Y}_n}(t - T_n, \tilde{X}_n), \tilde{Y}_n\right) \text{ for } t \in [T_n, T_{n+1}].$$

The associated deterministic control system

Recall that the flows Φ^i are given by the vector fields F^i.

$$\dot{x} = \sum_{i=0}^{m} v_i(t) F^i(x), \qquad t \geq 0.$$

with

$$v(t) = (v_i(t)) \in \left\{ v \in \mathbb{R}^{m+1} \,\middle|\, \sum_{i=0}^{m} v_i = 1, v_i \in \{0, 1\} \right\}.$$

Up to closure, the trajectories of this system coincide with those of the control-affine system

$$\dot{x} = F^0(x) + \sum_{i=1}^{m} u_i(t)[F^i(x) - F^0(x)]$$

with controls taking values in

$$U = \left\{ u \in \mathbb{R}^m \,\middle|\, \sum_{i=1}^{m} u_i \leq 1, u_i \in [0, 1] \right\}.$$

A Decisive Lemma for PDMP

Lemma

For all $T > 0, x \in M, i \in E, \delta > 0$ and every trajectory $\varphi(\cdot, x, u)$ of the control system one finds for start in x and $i \in E$ that there is $\varepsilon > 0$ such that

$$\mathbb{P}_{x,i}\left[\sup_{t\in[0,T]} \|X_t - \varphi(t,x,u)\| \leq \delta\right] \geq \varepsilon.$$

Benaïm, Le Borgne, Malrieu and Zitt (2015)

In the terminology of Arnold and Kliemann (1983) this is a **tube lemma** connecting the stochastic system and the control system.

A consequence of the tube lemma

Corollary
Let C be an invariant control set with nonvoid interior and let $x \in M$ with $\overline{O^+(x)} \cap C \neq \emptyset$.

Then there are $T > 0$ and $\varepsilon > 0$ with

$$\mathbb{P}_{x,i}\left[X_T \in \operatorname{int} C\right] \geq \varepsilon \text{ for all } i \in E.$$

This follows since then x can be steered into the interior of C in finite time.

Characterization of the supports of invariant measures for Piecewise Deterministic Markov Processes (PDMP)

Theorem

Assume that the control system is locally accessible on a compact positively invariant set M.

(i) Then for every ergodic measure μ of the process (Z_t) there is a compact invariant control set C with $\mathrm{supp}\mu = C \times E$.

(ii) Conversely, let C be a compact invariant control set. Then there exists an ergodic measure μ with support equal to $C \times E$ and every invariant measure with support contained in $C \times E$ has support equal to $C \times E$.

This is also true for the discrete-time process $(\tilde{Z}_n)$.

Theorem
Assume that for some x in a compact invariant control set C the Lie algebra $\mathcal{L}\mathcal{A}(F^0, \ldots, F^m)$ has full rank at x.

Then there is a unique invariant measure μ with $\mathrm{supp}\mu = C \times E$ (hence μ is ergodic) and there are $c > 0$ and $0 < \rho < 1$ such that for all $(x, i) \in C \times E$ and $A \subset C$

$$|\mathbb{P}_{x,i}[\tilde{Z}_n \in A] - \mu(A)| \leq c\rho^n, n \in \mathbb{N}.$$

An Example: Lotka-Volterra model with hunting and resting

The model:

$$\dot{x} = \alpha x \left(1 - \frac{1}{K}\right) x - \beta x y$$
$$\dot{y} = -\beta x y + \gamma (L - y)$$

$\frac{1}{\beta}$ corresponds to the hunting time of the predator y,
$\frac{1}{\gamma}$ corresponds to the resting time of the predator y,
normalized via $\frac{1}{\beta} + \frac{1}{\gamma} = 1$.
Coexistence and extinction as hunting (and resting) time undergoes random fluctuations.

Horsthemke, Lefever (84), FC, de la Rubia, Kliemann (96)

The Lotka-Volterra model as a PDMP

The model:

$$\boxed{\begin{array}{c}\dot{x} = \alpha x\left(1 - \frac{1}{K}\right)x - \beta xy \\ \dot{y} = -\beta xy + \gamma(\beta)(L - y)\end{array}}$$

with the normalization$\frac{1}{\beta} + \frac{1}{\gamma(\beta)} = 1$. For

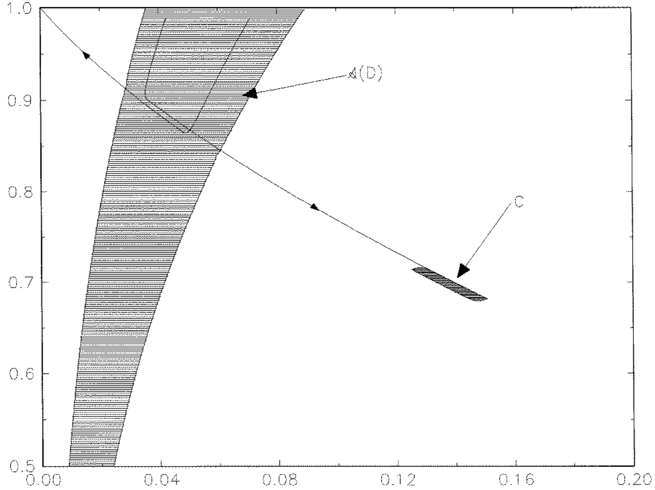
$$K = 0.5, L = 1.0, \alpha = 4.0, \beta > 4.0$$

the rectangle $[0, K] \times [0, L]$ is invariant and the fixed points are $(0, L)$(stable), an unstable and a stable fixed point.

Let β switch randomly between $\beta = 4.1$ and $\beta = 4.2$. Thus $E = \{0, 1\}$,

$$\begin{aligned} F^0 \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \alpha x\left(1 - \frac{1}{K}\right)x - 4.1xy \\ -4.1xy + \gamma(4.1)(L - y) \end{bmatrix}, \\ F^1 \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \alpha x\left(1 - \frac{1}{K}\right)x - 4.2xy \\ -4.2xy + \gamma(4.2)(L - y) \end{bmatrix} \end{aligned}$$

Two invariant measures with supports given by $\{(0, L)\}$ and C.



1.0
0.9
0.8
0.7
0.6
0.5
0.00
0.04
0.08
0.12
0.16
0.20
A(D)
C

Other (more realistic) Lotka-Volterra systems with random switching have been analyzed in detail by

Benaïm and Lobry, Lotka-Volterra with randomly fluctuating environments or "How switching between beneficial environments can make survival harder",

Annals of Applied Probability (2016).

PDMP for a particle in a double well potential

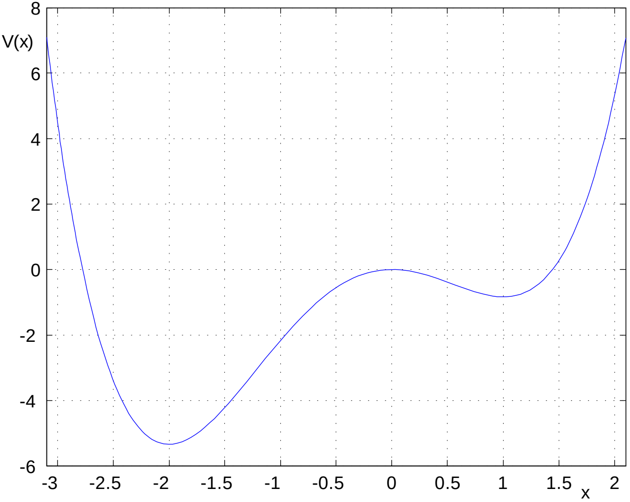
Consider

$$\ddot{x} + \gamma x + \frac{dV}{dx}(x) = 0$$

with

$$V(x) = \frac{1}{2}x^4 + \frac{2}{3}x^3 - 2x^2 \pm \rho x$$

PDMP for a particle in a double well potential



$V(x)$ with $\rho = 0$

PDMP with a double well potential

$$\dot{x} = y$$
$$\dot{y} = -\gamma\ y - x(2x^2 + 2\,x - 4) \pm \rho$$

with $\gamma = 0.1$ and random switching between $\pm\rho$. Here $E = \{0, 1\}$ and

$$F^0 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -\gamma\, y - x\,(2x^2 + 2\,x - 4) + \rho \end{bmatrix},$$
$$F^1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -\gamma\, y - x\,(2x^2 + 2\,x - 4) - \rho \end{bmatrix}.$$

The associated control system is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ -\gamma\, y - x\,(2x^2 + 2\,x - 4) \end{bmatrix} + \begin{bmatrix} 0 \\ u(t) \end{bmatrix},\ u(t) \in [-\rho, \rho].$$

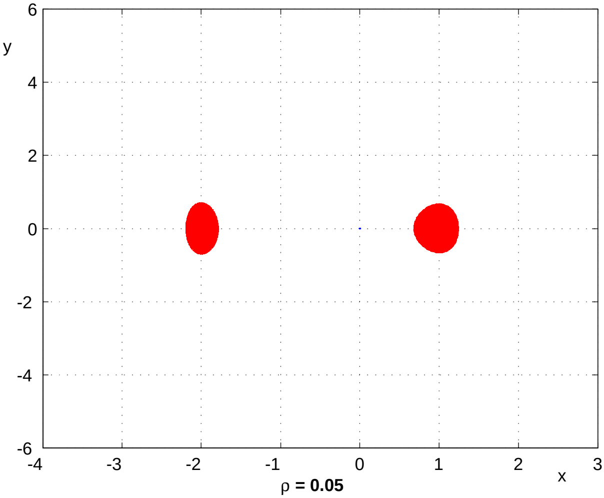
PDMP with a double well potential

For $\rho = 0.05$ there are two invariant control sets $C_1^{0.05}$ and $C_2^{0.05}$ that contain the stable fixed points $(1,0)$ and $(-2,0)$, respectively, of the uncontrolled equation and one non-invariant control set $D^{0.05}$ containing the hyperbolic fixed point $(0,0)$ of the uncontrolled equation.

Increasing the control range, one finds that the control sets $C_1^{\rho_0}$ and D^{ρ_0} merge for some ρ_0 close to 0.085 and form one variant control set.

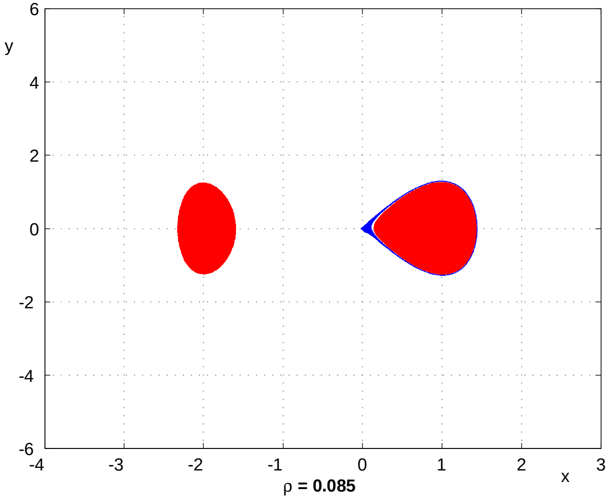
This determines the number of invariant measures for the PDMP and their supports.
Computations: Tobias Gayer with GAIO

Bifurcations: PDMP with a double well potential



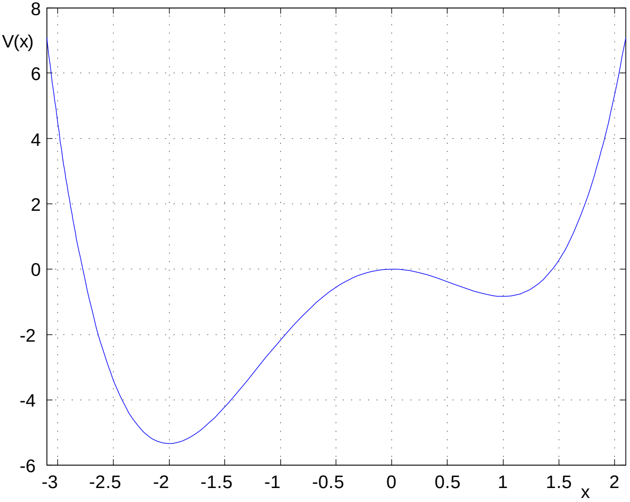
Supports of two invariant measures

PDMP with a double well potential

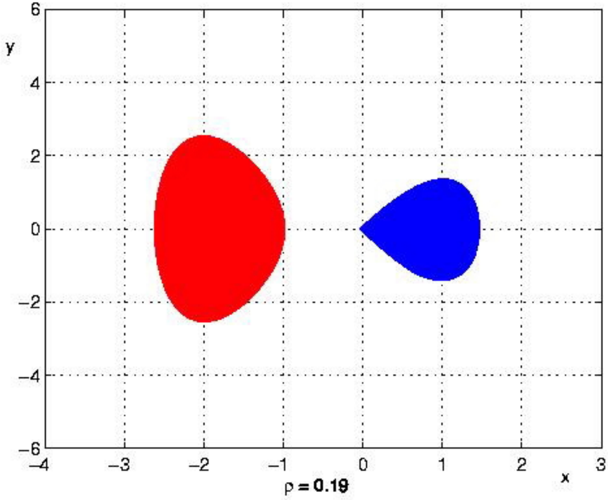


Supports of two invariant measures

PDMP with a double well potential

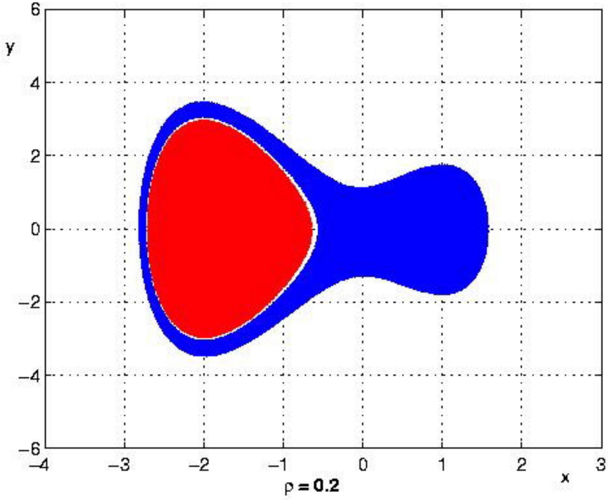


Bifurcations: PDMP with a double well potential



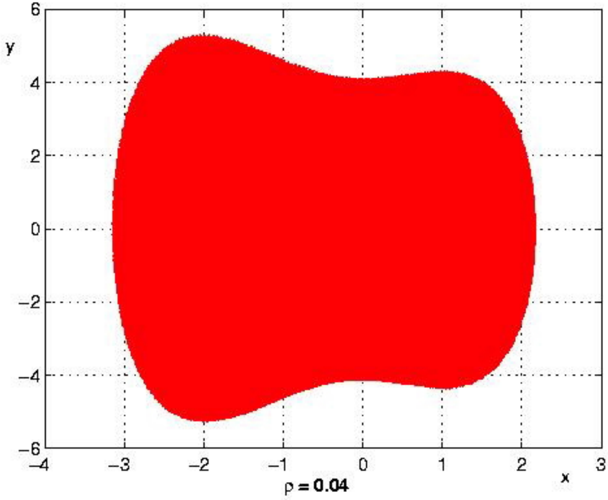
Support of a single invariant measure (in red)

Bifurcations: PDMP with a double well potential



Support of a single invariant measure (in red)

Bifurcations: PDMP with a double well potential



Support of a single invariant measure (in red)

Final Remarks

The concept of control flow allows us to consider the theory of (open loop) control systems as a chapter in the theory of dynamical systems. The control term can also be interpreted as a deterministic perturbation. As a random perturbation, one obtains that for degenerate Markov diffusions and for Piecewise Deterministic Markov Processes (with continuous trajectories) the supports of the invariant measures can be characterized by controllability properties.

In general, PDMP may also allow random jumps. Although control systems allowing discontinuous trajectories have been analyzed in the literature, their controllability properties are apparently unknown.