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ROBUST OPTION PRICING

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A thesis submitted for the degree of

MSc in Mathematics and Finance, 2019-2020

Declaration

The work contained in this thesis is my own work unless otherwise stated.

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08/09/2020

Abstract

Robust pricing is a concept of generating practical value bounds, where the actual price of the derivative lies within, based on the market information instead of relying heavily on any specific model for the dynamics of the underlying price process. This thesis aims to explore three methods of robust pricing which lower the model risk: Uncertain volatility model(UVM), Martingale optimal transport(MOT) and Model-independent methods for path-dependent options, and to check their validity via numerical implementations. The standard UVM and Lagrangian UVM will be studied for the value bounds with optimal hedging strategy. The MOT problem can be reduced into a linear program(LP): the primal or the dual problem, then the solution to LP tells the best possible value bounds and hedging portfolio. Approaches to finding optimal value bounds for the Lookback, Digital and Barrier options will also be discussed.

The numerical results for these three methods prove their effectiveness, that is the market prices indeed lie within the value bounds generated. Furthermore, the value bounds are reasonably narrow so that arbitrage opportunities could be detected when the derivative is overpriced or underpriced. A comparison between UVM and MOT will be conducted. In general, the performance of UVM is slightly better than that of MOT. The reason behind it and other factors which may influence the final results will be analysed.

Acknowledgements

I would first like to thank my supervisor, Dr. Pietro Siorpaes, for his support and insightful advice for my thesis, especially in this difficult situation this year.

I would also like to thank my grandfather, who offered me the irreplaceable support during my growth. And I would like to thank my whole family, who always care and help me unconditionally.

And I would like to thank my girlfriend, Yue, who has been with me for four years, for her kindness and sincerity. And I would like to thank every friend whom I have ever encountered in my life.

Contents

List of Figures

List of Tables

Introduction

The standard method of pricing options is to derive prices of options such as Call and Put under the assumption that the behaviour of underlying prices follows certain models, with risk-neutral arguments. By choosing reasonable models and parameters, the derived prices are expected to approximate the prices at which derivatives are actually traded. However, the model risk may arise due to inadequate models. Sometimes models may oversimplify the reality such as volatility surfaces which cannot be generated by the simple Black-Scholes model. Instead, the overfitting problem may appear due to complex models containing a large number of parameters.

In order to reduce the model risk, that is to lessen the exposure to potentially deficient models, alternative approaches to option pricing will be introduced in this paper. These methods, called 'Robust Pricing', depends rarely, or even not on the assumption that the real world is correctly described by any proposed models; they mainly rely on the information from the market. The aim is to produce optimal value bounds where the price of derivatives should lie within. At the same time, some hedging strategies will also be discussed to improve the bounds of the price.

This paper contains three parts: i) Uncertain volatility models, ii) Martingale optimal transport, iii) Robust pricing for path-dependent options with alternative methods. The first part introduces the uncertain volatility model, which assumes the real volatilities lie within a value band without depending on any deterministic models. Then the optimal value bounds for derivatives can be obtained by utilising extremes of volatilities and the Black-Scholes-Barenblatt equation. The Lagrangian UVM which optimises the hedging strategy for a best possible value bound will also be studied. The second part describes the optimal transport problem and subsequently the martingale optimal transport problem. The methods of reducing MOT into two linear programs(LP): the primal and the dual, and their solutions for finding the optimal value bounds with hedging strategy, will be presented. The last part will mainly focus on model-independent approaches to pricing path-dependent options: Lookback, Digital and Barrier options. The objective of these three methods is to generate robust upper and lower value bounds containing the actual derivative prices almost surely. The analysis of these methods in a theoretical and numerical sense will be conducted. The source code for numerical implementations in this thesis can be found HERE.

Literature review
The framework of uncertain volatility models mainly relies on the work of Avellaneda, Levy and Paras in [21]and [22]. The approach is first to study the standard UVM, thereby introducing the Lagrangian UVM, which utilises portfolios containing hedging instruments to obtain optimal value bounds. The hedging strategies in λ-UVM are also enhanced by Karoui and Quenez[26] and Paras[3].

The martingale optimal transport part mostly depends on the book Model-free Hedging: A Martingale Optimal Transport Viewpoint, of Henry-Labordère[32]. Baker[8] and Guo and Obłój[12] introduces the methods of solving the MOT primal. The study on MOT dual problem can also be found in [32].

The alternative method of robust pricing for Lookback options is discussed based on the view of Hobson[10]. Brown, Hobson and Rogers[13] present pricing methods for Digital and Barrier options.

Many other research papers are also studied and contribute to the completion of this thesis. Some of them will be cited specifically throughout the thesis.

Chapter 1

Pricing and hedging derivatives using uncertain volatility models

1.1 Introduction

Volatility is the key parameter in pricing financial derivatives. Many efforts have been made to estimate it. For example, the volatility of returns of underlying assets is assumed to be constant in the Black-Scholes model. However, this is not observed in real markets, and we can find volatility smile or skew from the graph of the implied volatility instead. Local volatility models, which regard the volatility as a function of both the current asset level and of time, might be a solution to this problem. Stochastic volatility models is also an approach to model derivatives accurately; they treat the underlying security's volatility as a random process.

Apart from the above methods, there is an alternative way to deal with the volatility: the uncertain volatility models. When using this method, the value of volatility is assumed to lie between two extreme values σ_{min} and σ_{max}, instead of following any deterministic or stochastic model. These two bounds could be, for instance deduced from extreme values of the implied volatility. Then we are able to derive a value band where the real portfolio price will lie within. Besides, the model of underlying assets is presumed to be in the form $dS_t = rdt + \sigma_t dW_t$, so that the discounted stock process is still a martingale under the risk-neutral probability measures.

In this chapter, the framework of the uncertain volatility models will be demonstrated firstly. Methods of pricing and hedging financial derivatives under this framework will be subsequently discussed. Finally, some applications of uncertain volatility models will be shown via numerical implementations.

1.2 The uncertain volatility model

1.2.1 Basics

In this section, the basic definition of Uncertain Volatility Model(UVM) will be discussed. And firstly, we will follow the common assumption that the price of an underlying is in a form of Itô process without any dividend:

$$\mathrm{d}S_t/S_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t,$$

where σ_t and μ_t are non-anticipative drift and volatility respectively, and W_t is a Brownian motion. We will assume the drift μ_t is constant and only study the volatility σ_t. After some transformations such as change of numeraire, we can obtain an equivalent martingale measure(EMM) under which the discounted stock price is a true martingale:

$$\mathrm{d}S_t/S_t = r\mathrm{d}t + \sigma_t \mathrm{d}W_t. \tag{1.2.1}$$

And we suppose the value of volatility lies between two extreme values:

$$\sigma_{min} \le \sigma_t \le \sigma_{max}. \tag{1.2.2}$$

1.2.2 Calibration

Intuitively, we may encounter unrealistic option prices if the volatility takes values in an extensive range. However, spurious arbitrage opportunities may appear if the volatility band $[\sigma_{min},\ \sigma_{max}]$ is set to be too narrow when we conduct the calibration[22].

According to classical valuation theory, we know that the spot volatilities is mathematically related to implied volatilities calculated by Black-Scholes formula. Furthermore, if we assume that spot volatilities lie within a band, then implied volatilities of options traded would lie within predetermined bounds over certain periods. This means the assumption of volatilities bands indicates an upper bound of the purchase price and a lower bound of the sale price of traded options.[22]

If we assume σ_{min} and σ_{max} are constant, then

$$\sigma_{min} \leq \sigma_{impl}(t,\ T) \leq \sigma_{max}.$$

where $\sigma_{impl}(t,\ T)$ is the implied volatility of an option with maturity T at time t.

If we assume that the bands are time-dependent deterministic functions

$$\sigma_{min} = \sigma_{min}(t), \quad \sigma_{max} = \sigma_{max}(t),$$

then we will have

$$\frac{1}{T-t}\int_t^T \sigma_{min}^2(u)\mathrm{d}u \leq \sigma_{impl}^2(t,\ T) \leq \frac{1}{T-t}\int_t^T \sigma_{max}^2(u)\mathrm{d}u$$

In summary, we could determine the volatility band to be either constant or time-dependent. The critical point in calibration is that the historical option prices should be utilised so that we can obtain a range of implied volatilities. By doing this, we need to ensure implied volatilities lie within the bands that we propose. At the same time, bands are supposed to be reasonably narrow.

1.2.3 Derivatives pricing under the UVM framework

We will now introduce the method of pricing derivatives under the framework of the uncertain volatility model. Our aim in this section is to drive bounds for the value of a derivative.

Suppose the derivative security on a stock with price process $(S_t)_{t\geq 0}$ is path-dependent and specified by a stream of cash flows

$$F_1(S_{t_1}), F_2(S_{t_2}), F_3(S_{t_3}), ..., F_N(S_{t_N}),$$

where $F_j(\cdot)$ are payoffs due at each settlements date $t_1 < t_2 < t_3 < ... < t_N$. And we denote the class of all equivalent martingale measures to be $\mathcal{P}$ such that equation (1.2.1) holds for volatilities lying within the band (1.2.2), and non arbitrage opportunity exists.

If we calibrate the volatility band accurately, then the value of this derivative $V(t, S_t)$ is supposed to lie between the following bounds:

$$V^+\ (S_t,\ t)\ = \sup_{P\in\mathcal{P}} \mathrm{E}^P[\sum_{j=1}^{N} \mathrm{e}^{-r(t_j-t)} F_j(S_{t_j})], \tag{1.2.3}$$

and

$$V^-(S_t,\ t) = \inf_{P\in\mathcal{P}} \mathrm{E}^P[\sum_{j=1}^{N} \mathrm{e}^{-r(t_j-t)} F_j(S_{t_j})], \tag{1.2.4}$$

where measure P belongs to the set of all EMMs $\mathcal{P}$ and E_t^P is the conditional expectation under P given information up to time t[21].

The above two functions could be calculated by solving dynamical programming differential equations if we regard it as a stochastic control problems with control variable σ_t[27]. Therefore, when we consider the simple case that $N = 1, t_1 = T, F_1 = F$, then these extreme functions can be transformed into a final-value problem[21]:

$$\frac{\partial V(S,t)}{\partial t} + r\left(S\frac{\partial V(S,t)}{S} - V(S,\ t)\right) + \frac{1}{2}\sigma^2\left[\frac{\partial^2 V(S,t)}{\partial S^2}\right]S^2\frac{\partial^2 V(S,t)}{\partial S^2} = 0, \tag{1.2.5}$$

with terminal condition

$$V(S,T) = F(S).$$

And V^+ could be obtained by setting

$$\sigma[X] = \begin{cases} \sigma_{max} & \text{if } X \geq 0. \\ \sigma_{min} & \text{if } X < 0. \end{cases}, \tag{1.2.6}$$

in equation (1.2.5).Similarly, we obtain V^- by setting

$$\sigma[X] = \begin{cases} \sigma_{max} & \text{if } X \leq 0. \\ \sigma_{min} & \text{if } X > 0. \end{cases}. \tag{1.2.7}$$

If we consider general case that there are multiple payoffs, we firstly solve equation(1.2.5) for $t_{N-1} < t \leq t_N$ with terminal condition $V(S, t_N) = F_N(S)$. Then at time t_{N-1}, we set the value function to be

$$V(S, t_{N-1}) = \lim_{t\to 0} V(S, t_{N-1} + t) + F_{N-1}(S).$$

The first term on the right hand side represents the value at date t_{N-1} right after the cash flow $F_{N-1}(S_{t_{N-1}})$ is paid out. Then we use equation (1.2.5) to calculate the value at time t_{N-2} etc.

It is worth noting that this non-linear PDE (1.2.5) is, in fact, the Black-Scholes-Barenblatt(BSB) equation, which is commonly used when the volatility is uncertain and the generalisation of standard Black-Scholes PDE where the volatility is constant($\sigma_{min} = \sigma_{max}$). We will demonstrate how to price derivatives using this equation via the trinomial tree method in Section 1.3.

1.2.4 Delta-hedging

In the previous section, we have defined the bounds for the value of a derivative under several assumptions. It is easy to find that the value of the derivative in equation (1.2.3) is the worst-case scenario of volatility path since the stream of cash flows reaches its largest possible value under an equivalent martingale measure among $\mathcal{P}$. This means

$$\sigma_t = \sigma\left[\frac{\partial^2 V^+(S_t,t)}{\partial S^2}\right],$$

so $\sigma[\cdot]$ satisfies equation (1.2.6).

If we are interested in conducting a standard Black-Schole style delta-hedging strategy to replicate this derivative, we can construct a self-financing portfolio containing long positions on $\triangle_t$ shares of the stock

$$\triangle_t = \frac{\partial V^+(S_t,t)}{\partial S},$$

and B_t units of bonds

$$B_t = V^+(S_t,t) - S_t \times \frac{\partial V^+(S_t,t)}{\partial S}.$$

Since the volatility is assumed to lie within the band, this portfolio with initial value $V^+(S_t,t)$, which is the largest possible cash flows caused by the volatility path, will have a non-negative final value after paying out all cash flows. This means a short position in the derivative will be risklessly hedged.

Moreover, according to Avellaneda, Levy and Paras[21], this strategy is optimal since its initial cost is the least possible value among all other dominating strategies using only stocks and bonds. And the initial cost cannot be lower due to the worst-case scenario.

1.2.5 Risk Diversification

One of the important characteristics of the UVM is that it allows us to quantify the diversification of volatility risk in portfolios containing derivatives. Let us consider a portfolio that consists of two derivatives with discounted payoffs

$$\Phi\ (S_t,\ t)\ = \sum_{j=1}^{N} \mathrm{e}^{-r(t_j - t)} F_j(S_{t_j}),$$

and

$$\Psi\ (S_t,\ t)\ = \sum_{k=1}^{N'} \mathrm{e}^{-r(t'_k - t)} G_k(S_{t'_k}),$$

With the knowledge of previous sections, we could regard V^+ as the offer price and V^- as the bid price of a derivative, and $V^+ - V^-$ is the bid-ask spread. Also, it is easy to show that

$$\sup_{P\in\mathcal{P}} \mathrm{E}^P[\Phi + \Psi] \leq \sup_{P\in\mathcal{P}} \mathrm{E}^P[\Phi] + \sup_{P\in\mathcal{P}} \mathrm{E}^P[\Psi],$$

and

$$\inf_{P\in\mathcal{P}} \mathrm{E}^P[\Phi + \Psi] \geq \inf_{P\in\mathcal{P}} \mathrm{E}^P[\Phi] + \inf_{P\in\mathcal{P}} \mathrm{E}^P[\Psi].$$

This means the best risk-avoiding offer price of this portfolio is lower than the sum of the separate optimal offer price of two derivatives. Likewise, the bid price of the portfolio is higher than the sum of individual bid prices. So the bid-ask spread becomes narrow.

This can be explained from the viewpoint of option portfolios in the UVM. Because options prices in the BSB equation are obtained with either σ_{min} or σ_{max} and they are convex in S, the price bounds for standard options are the Black-Scholes prices directly calculated by extreme volatilities. Whereas the portfolio containing several options has more complex convexity. It will be priced differently since the BSB equation uses volatility path that results in optimal non-arbitrage offer/bid prices[21], which means pricing the whole portfolio is more efficient than pricing and adding up each component. One can also imagine intuitively that volatilities of different options offset each other, thereby narrowing the volatility bands, so the extreme values of portfolios in BSB equation are reduced.

1.3 The Lagrangian uncertain volatility model(λ-UVM)

In this section, we will illustrate the methods of hedging with derivatives such as options. The Lagrangian Uncertain Volatility Model(λ-UVM) and its connection with standard UVM will be introduced. Also, the framework of the trinomial tree for the numerical implementation of λ-UVM will also be discussed.

1.3.1 Optimal static hedging

Suppose there is a derivative on a stock with price process $(S_t)_{t\geq 0}$ is path-dependent and characterised by a stream of cash flows

$$F_1(S_{t_1}), F_2(S_{t_2}), F_3(S_{t_3}), ..., F_N(S_{t_N}),$$

where $F_j(\cdot)$ are payoffs due at each settlements date $t_1 < t_2 < t_3 < ... < t_N$, so the discounted payoff of this derivative at time t is $\Phi = \sum_{i=1}^{N} \mathrm{e}^{-r(t_i - t)} F_i(S_{t_i})$. We also assume that there are M European options available for us to hedge, with payoff $G_i(S_{\tau_i})$ and expiration τ_i for $\tau_1 \leq \tau_2 \leq ... \leq \tau_M$. And the prices of these options are $C_1, C_2, ..., C_M$.

Suppose we hold a short position on this derivative and are interested in the optimal number of options to long/short in order to hedge it efficiently. We construct a portfolio of options for hedging, the number of each option is $\lambda_1, \lambda_2, ..., \lambda_M$ respectively. So at time t this portfolio is worth $\Psi = \sum_{j=1}^{M} \lambda_j \mathrm{e}^{-r(\tau_j - t)} G_j(S_{\tau_j})$.

Therefore, the residual liability after we short the derivative and construct the hedging portfolio is

$$\sum_{i=1}^{N} \mathrm{e}^{-r(t_i-t)} F_i(S_{t_i}) - \sum_{j=1}^{M} \lambda_j \mathrm{e}^{-r(\tau_j-t)} G_j(S_{\tau_j}).$$

And the total cost of hedging with the cost of options, in the worst-case scenario, is

$$V^+(S_t, t; \lambda_1, ..., \lambda_M) = \sup_{P\in\mathcal{P}} \mathrm{E}^P \left\{ \sum_{i=1}^{N} \mathrm{e}^{-r(t_i-t)} F_i(S_{t_i}) - \sum_{j=1}^{M} \lambda_j \mathrm{e}^{-r(\tau_j-t)} G_j(S_{\tau_j}) \right\} + \sum_{j=1}^{M} \lambda_j C_j. \quad (1.3.1)$$

The portfolio that we construct will be optimal when the total cost of this hedging strategy is minimised, so we have an optimisation problem.

$$\inf_{\lambda_1,...,\lambda_M} V^+(t,\ S_t;\ \lambda_1,\ \ldots,\ \lambda_M), \quad (1.3.2)$$

and we call it the Lagrangian Uncertain Volatility Model(λ-UVM).

Similarly, if we try to hedge a long position on this derivative with options, the worst-case cost of hedging V^- is

$$V^-(S_t, t; \lambda_1, ..., \lambda_M) = \inf_{P\in\mathcal{P}} \mathrm{E}^P \left\{ \sum_{i=1}^{N} \mathrm{e}^{-r(t_i-t)} F_i(S_{t_i}) - \sum_{j=1}^{M} \lambda_j \mathrm{e}^{-r(\tau_j-t)} G_j(S_{\tau_j}) \right\} + \sum_{j=1}^{M} \lambda_j C_j, \quad (1.3.3)$$

then we have another optimisation problem

$$\sup_{\lambda_1,...,\lambda_M} V^-(t,\ S_t;\ \lambda_1,\ \ldots,\ \lambda_M),$$

Therefore, with $\lambda-$UVM we can deduce the optimal value bounds of a portfolio if we hedge derivatives with options through solving Lagrangian dual-style optimisation problems, and these bounds can be solved by PDEs described in Section 1.2.3. We also assume the vector $(\lambda_1, \lambda_2, ..., \lambda_M)$ is constrained into a reasonable range due to the liquidity considerations. Then λs may take arbitrary positive or negative values within the following constant bounds

$$\Lambda_j^- \leq \lambda_j \leq \Lambda_j^+.$$

According to Avellaneda and Paras[22], the value function V^+ is convex in $(\lambda_1, ..., \lambda_M)$ since the value function is a supremum of linear functions in λ. The value function becomes the standard UVM valuation if the λ vector equals zero, which means we hedge in the cash market. Moreover, the minimum of problem (1.3.2) could be obtained when $\lambda_j \neq 0$. This is because the implied volatility is in the volatility band so that the options can be bought at lower prices and sold at higher prices at σ_{max} or σ_{min}. Therefore, the cost of hedging strategy with efficiency will be less than the expected cost of delta-hedging strategy under the worst-case scenario. Also, the convexity of the value function ensures the uniqueness of the λ vector when the minimum is attained[3].

Unlike the delta hedging, the hedging strategy we propose is static since we do not need to change the option hedge-ratios λ_j. Moreover, this is the only way to hedge options with no gamma risk and to hedge volatility[26]. However, it can be easily enhanced by implementing dynamic strategies. For instance, $\lambda-$UVM can be applied to the liability of the previous trading day with the new options issued and changes in prices, and no additional volatility risk will be added[22].

1.3.2 Trinomial tree framework

The trinomial tree framework that will help us to solve the previous optimisation problems will be introduced in this section. Suppose the stock price S_0 may change after each trading period Δt into three different prices: $S_0 u, S_0 m$ and $S_0 d$. The maturity of derivative is T and we divide it into N steps $t_1, t_2, ..., t_N$ so that each trading period is $\Delta t = \frac{T}{N}$. We construct the model to be recombining such that $ud = m^2$ and $d < m < u$, so the number of possible stock prices at time t is $2t+1$ (rather than 3^t which is the total number of distinct price paths up to time t).

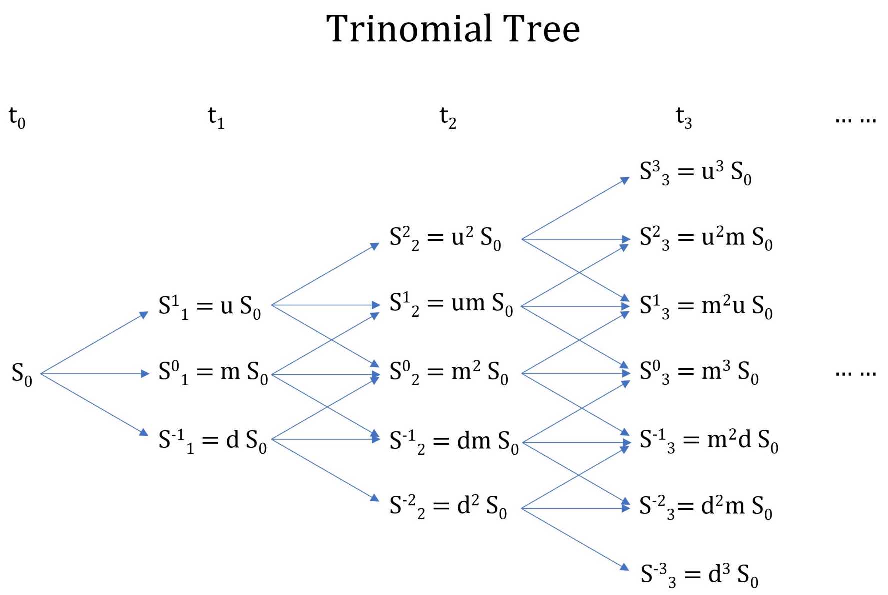

Figure 1.1: Trinomial Tree model.

We notice that the choice of probabilities of the stock price process going to three price levels P_u, P_m, P_d is non-unique, so the trinomial tree model has one degree of freedom at each node. Also, the larger spot volatility will be generated if the probability assigning weight to extreme values is greater than probability assigning weight to the central. Therefore we may choose a reasonable probability set such that the volatility of stock price lies within the band that we expect[22].

According to Avellaneda, Levy and Paras[21], we define the parameters as follows:

$$u = \mathrm{e}^{\sigma_{\max}\sqrt{\triangle t}+r\triangle t}, \quad m = \mathrm{e}^{r\triangle t}, \quad d = \mathrm{e}^{-\sigma_{\max}\sqrt{\triangle t}+r\triangle t},$$

and

$$P_u(p) = p(1 - \frac{\sigma_{max}\sqrt{\triangle t}}{2}), \quad P_m(p) = 1-2p, \quad P_d(p) = p(1+\frac{\sigma_{max}\sqrt{\triangle t}}{2}),$$

where p satisfies

$$\frac{\sigma^2_{min}}{2\sigma^2_{max}} \leq p \leq \frac{1}{2}$$

to exhibit different choices for probabilities, as well as spot volatilities at each node.

1.3.3 UVM algorithm

In order to determine the value bounds of a hedging portfolio, we need to derive the supremum and infimum of the expected residual liability by using the standard UVM:

$$L^+ = \sup_{P\in\mathcal{P}} \mathrm{E}^P \left\{ \sum_{i=1}^{N} \mathrm{e}^{-r(t_i-t)} F_i(S_{t_i}) - \sum_{j=1}^{M} \lambda_j \mathrm{e}^{-r(\tau_j-t)} G_j(S_{\tau_j}) \right\} = \sup_{P\in\mathcal{P}} \mathrm{E}^P \left\{ \sum_{i=1}^{N'} \mathrm{e}^{-r(t_i-t)} \hat{F}_i(S_{t_i}, \lambda_1, ..., \lambda_M) \right\}, \tag{1.3.4}$$

and

$$L^- = \inf_{P\in\mathcal{P}} \mathrm{E}^P \left\{ \sum_{i=1}^{N} \mathrm{e}^{-r(t_i-t)} F_i(S_{t_i}) - \sum_{j=1}^{M} \lambda_j \mathrm{e}^{-r(\tau_j-t)} G_j(S_{\tau_j}) \right\} = \inf_{P\in\mathcal{P}} \mathrm{E}^P \left\{ \sum_{i=1}^{N'} \mathrm{e}^{-r(t_i-t)} \hat{F}_i(S_{t_i}, \lambda_1, ..., \lambda_M) \right\}, \tag{1.3.5}$$

where $N' = \max(N, M)$ and $\hat{F}_i$ represents the cash flow at time step t_i. After we obtain L^+ and L^-, we add the amount $\sum_{j=1}^{M} \lambda_j C_j$ gained by hedging the derivative, to derive the final value bounds V^+ and V^-.

We will construct a N-period recombining trinomial tree model first. We denote the nodes of trinomial tree by pairs (i, j), where i represents the time and j indicates the price level at the node, $i = 0, 1, ..., N$ and $j = -i, ..., i-1, i$. Each node (i, j) will approach one of three possible nodes in the next time step: $(i+1, j+1), (i+1, j)$ and $(i+1, j-1)$. We build the price process at node (i, j) with the parameters defined:

$$S_i^j = S_0^0 \cdot e^{j \cdot \sigma_{max} \sqrt{\Delta t} + i \cdot r \Delta t}.$$

We use $W_i^{+,j}$ and $W_i^{-,j}$ to represent the values of L^+ and L^- at node (i, j) respectively and go through the price S_i^j at each nodes and compute the corresponding payoffs $\hat{F}_i^j = \hat{F}_i(S_i^j)$. We will compute the terminal residual liability W_N^j first, then loop backward in time for $i = N-1, ..., 1, 0$ to compute W_i^j for each $j = -i, ..., i-1, i$ by applying the following algorithm[21]:

$$W_i^{+,j} = \hat{F}_i^j + \mathrm{E}_{i,j}^P \left[W_{i+1}^+ \right] = \hat{F}_i^j + \mathrm{e}^{-r\Delta t} (W_{i+1}^{+,j} + p_+ l_{i+1}^{+,j}) \tag{1.3.6}$$

where

$$l_{i+1}^{+,j} = \left(1 - \frac{\sigma_{max}\sqrt{\Delta t}}{2}\right) W_{i+1}^{+,j+1} + (1 + \frac{\sigma_{max}\sqrt{\Delta t}}{2}) W_{i+1}^{+,j-1} - 2W_{i+1}^{+,j}. \tag{1.3.7}$$

Similarly,

$$W_i^{-,j} = \hat{F}_i^j + \mathrm{E}_{i,j}^P \left[W_{i+1}^- \right] = \hat{F}_i^j + \mathrm{e}^{-r\Delta t} (W_{i+1}^{-,j} + p_- l_{i+1}^{-,j})$$

where

$$l_{i+1}^{-,j} = (1 - \frac{\sigma_{max}\sqrt{\Delta t}}{2}) W_{i+1}^{-,j+1} + (1 + \frac{\sigma_{max}\sqrt{\Delta t}}{2}) W_{i+1}^{-,j-1} - 2W_{i+1}^{-,j}.$$

And p satisfies:

$$p_+ = \begin{cases} 1/2 & \text{if } l_{i+1}^{+,j} \geq 0, \\ \sigma_{min}^2 / 2\sigma_{max}^2 & \text{if } l_{i+1}^{+,j} < 0, \end{cases} \qquad p_- = \begin{cases} 1/2 & \text{if } l_{i+1}^{-,j} < 0, \\ \sigma_{min}^2 / 2\sigma_{max}^2 & \text{if } l_{i+1}^{-,j} \geq 0. \end{cases}$$

When $p = \frac{1}{2}$, the extreme cases(u and d) carry 100% of the probability and the local variance reaches the extreme value. And equation (1.3.7) can be regarded as the discretisation of the second derivative of V[22].

If we would like to deduce the optimal hedging value bounds, we will need to conduct the optimisation in Section 1.3.1. to get the optimal hedging strategy thereby obtaining an optimal value bounds.

1.4 Numerical implementation

1.4.1 UVM: Call options

In this section, we will conduct the numerical implementations of the classic uncertain volatility model. We will consider a simple case of pricing a Call option under uncertain volatility model.

We choose the underlying of the European Call option to be Amazon stock(AMZN) in the NASDAQ market. The data of Call prices quoted in the market are taken on 12^{th} August 2020, and the corresponding spot price of Amazon S_0 is 3118. And the expiration date of this option is on 21^{th} August 2020. We select the range of strike price K to be $2860, 2870, 2880, ..., 3500$.

After determining the parameters of this option, we firstly will derive the implied volatility in order to choose a reasonable volatility band. As we discussed in the previous section, the implied volatility should lie within the volatility band. Therefore we choose $\sigma_{max} = 0.6$ and $\sigma_{min} = 0.1$ to cover the range of implied volatilities. Moreover, according to the market information, we set the risk-free interest rate to be $r = 0.64\%$.

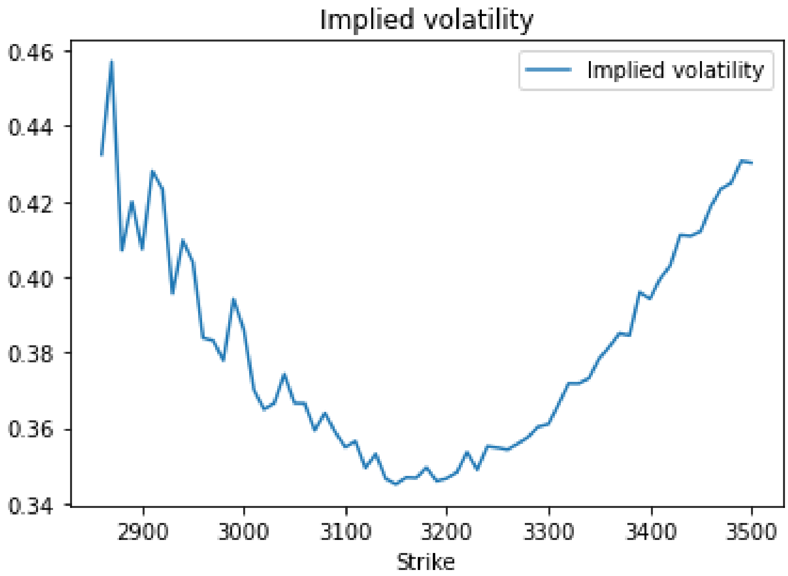

Figure 1.2: Implied volatilities for different strikes.

So far we have determined a plausible volatility band $[\sigma_{min}, \sigma_{max}] = [0.1, 0.6]$, then we will construct a trinomial tree based on the previous algorithm to compute the bounds for the value of this option V^+ and V^-. Notice that we are only price an option by trinomial tree method under UVM framework, so the payoff and cost of hedging instruments $G_j(S_{\tau_j})$ and C_j in equations (1.3.1) and (1.3.3) are zero here.

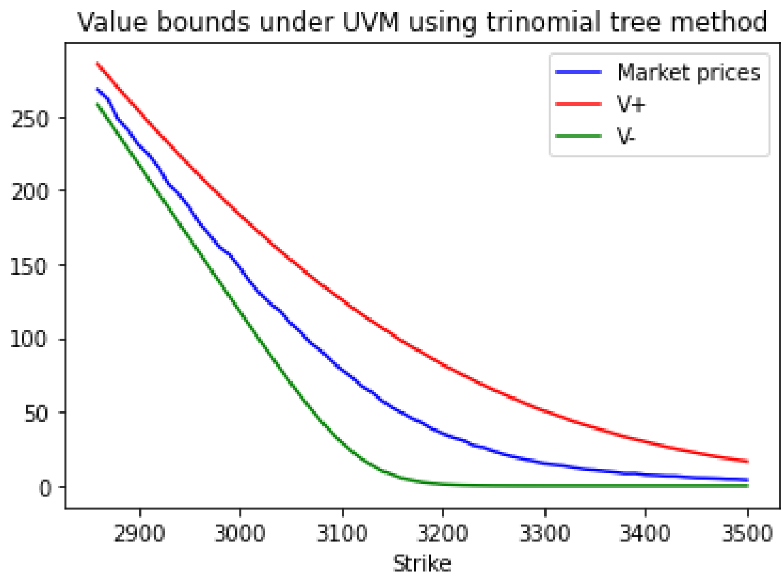

Figure 1.3: Value bounds of option with volatility band [0.1,0.6].

To demonstrate the importance of volatility band calibration, now we will compute the value bounds using the same procedure but different volatility band $[\sigma_{min}, \sigma_{max}] = [0.3, 0.5]$. And we plot the value bounds together to show the effects of different choices of volatility bands

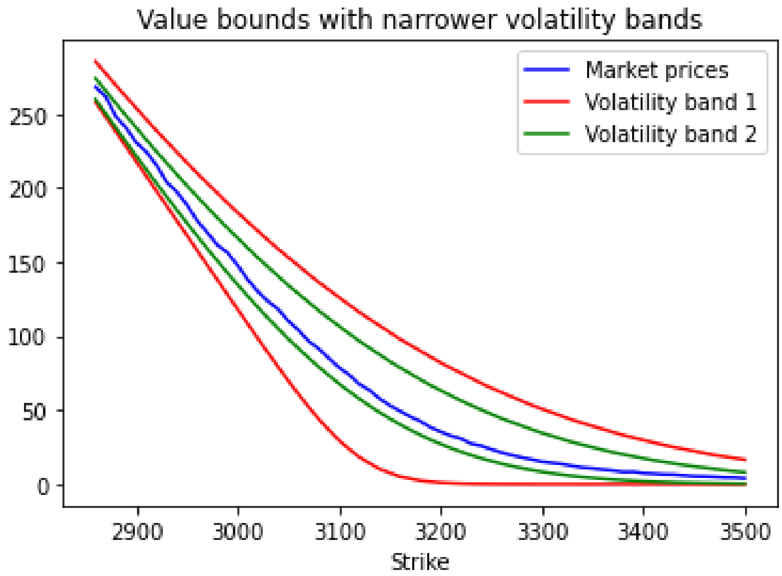

Figure 1.4: Value bounds of option with volatility band [0.1,0.6] and [0.3,0.5].

Obviously, the value bounds get narrower when we use a narrower volatility band; this is in line with our idea before. However, the value spread is still large, compared to the real market bid-ask spread in this case.

In fact, in equation (1.2.5), we find that the implied volatilities of value bounds depend on the sign of $\frac{\partial^2 V^+(S,t)}{\partial S^2}$ and $\frac{\partial^2 V^-(S,t)}{\partial S^2}$. Furthermore, since the derivative we priced is simply a Call option in this example, we know the Gamma of Call options is always positive. Thus, according to equations (1.2.6) and (1.2.7), the volatilities which is used to calculate the derivative value V^+ and V^- are σ_{max} and σ_{min} respectively. This means the upper bound and lower bound of the derivative value computed via UVM algorithm are actually Black-Scholes prices calculated with extreme volatility values. To certify this numerically, we plot the implied volatilities of value bounds computed with the trinomial tree model under UVM framework, together with the implied volatility. The result in the figure below is consistent with our idea.

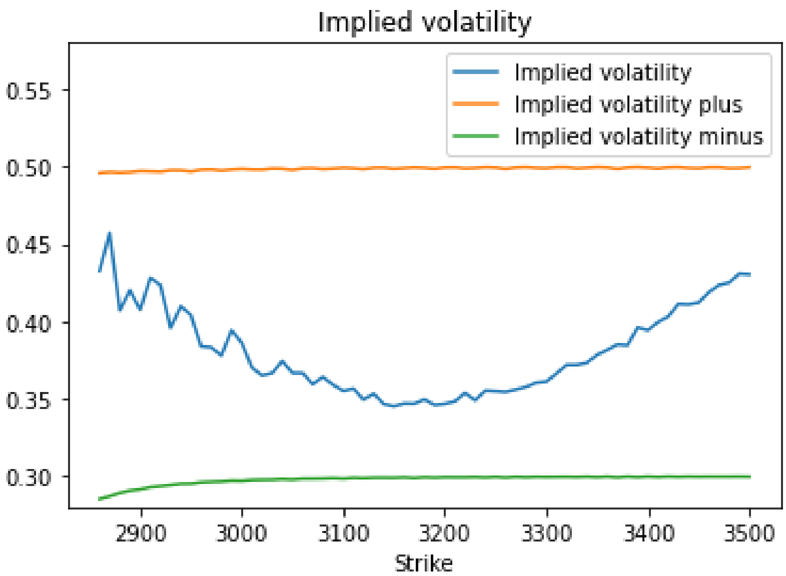

Figure 1.5: Implied volatilities of upper and lower bounds prices.

1.4.2 Lagrangian UVM: Hedging with one Call option

We shall implement the $\lambda-$UVM with the same derivative(Call option) in the previous section. Nevertheless, the difference is that we will use European Call options on the same stock to hedge this derivative here.

Firstly, we will use one unit of a Call option with the same expiration date and strike price $K = 3110$ as the hedging instrument, which is almost the at-the-money call. The volatility band will also be $[\sigma_{min}, \sigma_{max}] = [0.3, 0.5]$. However, this time we should slightly modify the algorithm for constructing the trinomial tree. Since the expiration date of the hedging instrument and derivative is the same, so we only need to consider their payoffs $F_i(S_{t_i})$ and $G_j(S_{\tau_j})$ at the maturity. Moreover, the residual liability computed should be added by the price of hedging instruments C_j in the end. Then we obtain the value bounds with this hedging strategy.

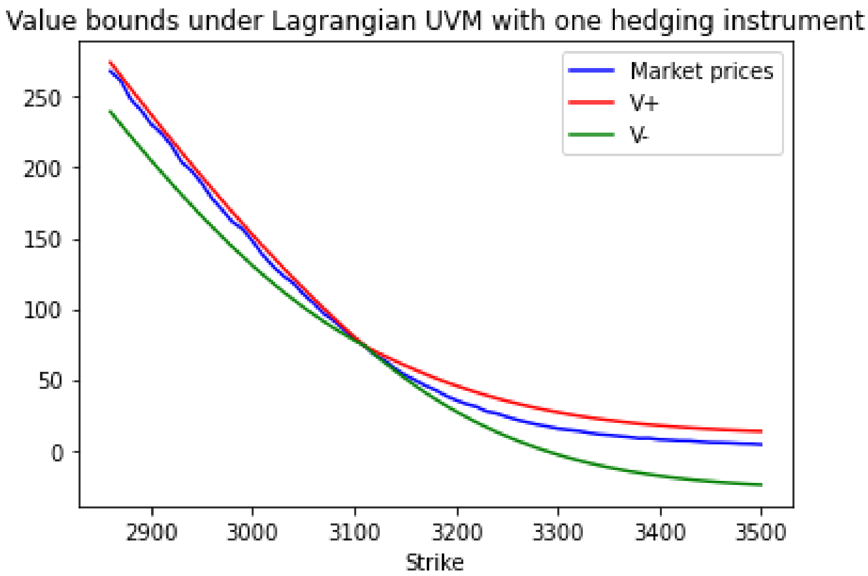

Figure 1.6: Value bounds of the portfolio with one hedging instrument.

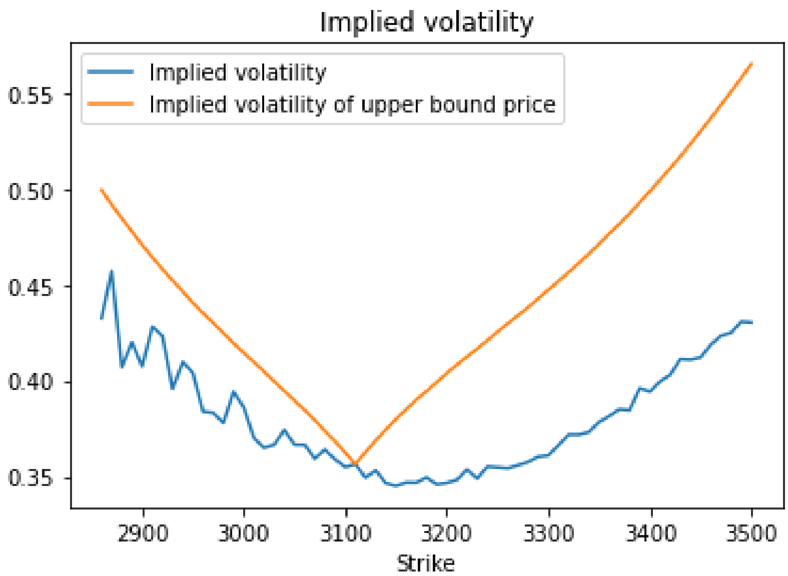

Figure 1.7: Implied volatilities of upper bound price.

As we can observe, the value spread between upper bound and lower bound of this portfolio has been considerably shrunk, compared to that under standard UVM framework in Figure 1.4. This can be interpreted as the improvement made by $\lambda-$UVM on the wide spread of the standard UVM.

And we also plot the implied volatility computed by the upper bound of the value. In contrast to the observation in Figure 1.5, the implied volatility is no longer the extreme value in the volatility band. This is mainly because the BSB equation utilises the volatility path, which seizes the optimal non-arbitrage offer/bid prices.

1.4.3 Lagrangian UVM: Hedging with three Call options

Now we will use one unit of each following three European Call options to hedge the same derivative.

Expiration	Strike	Market price
2020-08-21	2870	261.6
2020-08-21	3110	73.85
2020-08-21	3470	5

Table 1.1: Hedging instruments

Then we perform the same procedure as before to obtain the value bounds of this portfolio and also plot the value spread of this portfolio and the previous portfolio using one hedging instrument. For more straightforward observation, the values of spreads are divided by S_0. It can be observed that the value spread of portfolio with three instruments is wider than that of the one-instrument portfolio when strikes are close to the initial asset price, but narrower as K goes to both sides of the strike range. The reason behind this phenomenon might be that even though we generate the hedging strategy with the optimal combination of the three given instruments, but this three-instrument strategy may not be the best choice to generate the optimal value bound of our derivative. Therefore we may try a portfolio containing a large number of hedging instruments by using $\lambda-$UVM for obtaining narrower value bounds.

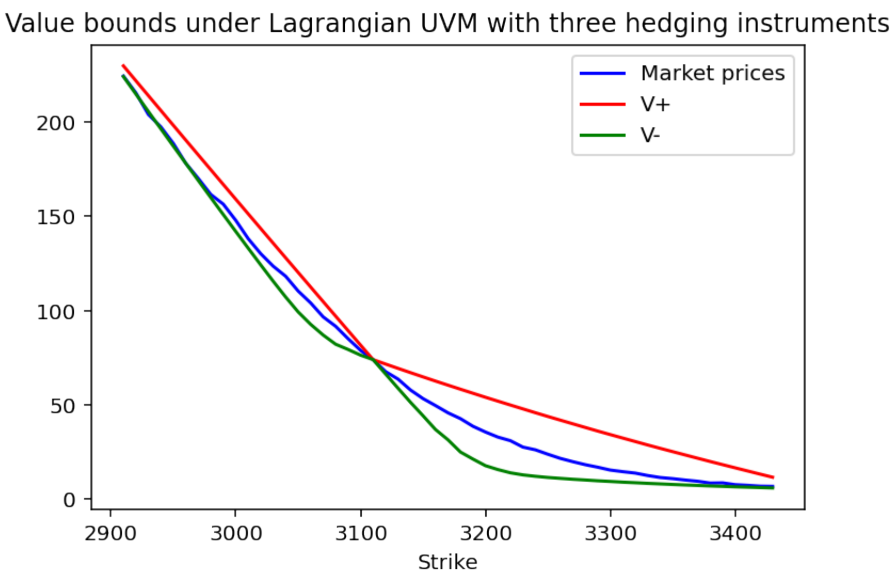

Figure 1.8: Value bounds of the portfolio with three hedging instruments.

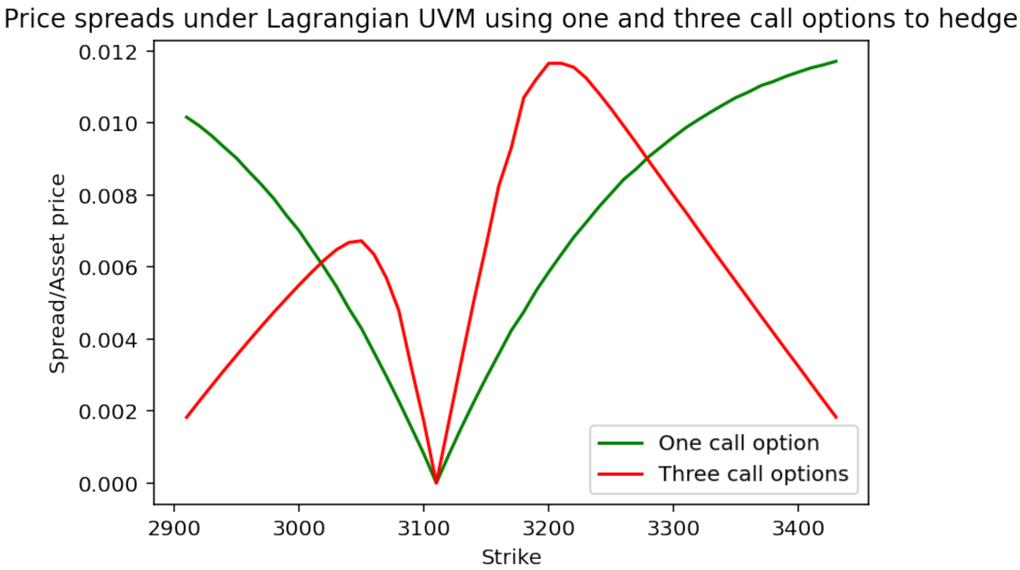

Figure 1.9: Comparison between portfolios with one and three hedging instruments.

1.5 Conclusion

In this chapter, we have discussed the method of managing the volatility under the uncertain volatility model framework. The assumptions we made are that the stock price follows a stochastic Itô process and the volatility lies within a predetermined range. We basically utilised the implied volatility based on the market information to construct a volatility band which covered values of implied volatilities and conducted the calibration. We also discussed how the derivative could be risklessly hedged under UVM and the effects of risk diversification.

Then we introduced the Lagrangian uncertain volatility model which allows us to use options as hedging instruments and the concept of optimal static hedging. The trinomial tree model, a method that helps us to implement the UVM numerically, has also been presented. Consequently, we showed the detailed UVM algorithm, thereby implementing the standard UVM and $\lambda-$UVM with different hedging strategies. By doing this, we showed what value bounds would be with a different choice of the volatility band, as well as the improvement on value bounds made by $\lambda-$UVM. At the same time, we investigated the implied volatility driven from the value bounds we established.

Chapter 2

Martingale optimal transport

2.1 Introduction

In the last chapter, we have introduced the UVM that allows us to find bounds on option prices in the sense of robustness. The assumption in UVM is that the stock price is a Geometric Brownian motion, with unknown volatility lying within given bounds. While in this chapter, we will discuss an alternative way of robust hedging and pricing: Martingale optimal transport(MOT), where the marginals of stock are assumed to be known under any risk-neutral measure in MOT.

We will introduce some basics first. If a market is complete, then the payoff of a derivative could be replicated by self-financing strategies. Moreover, the price of this derivative is equal to the cost of replication under the arbitrage-free condition. Also, there exists a unique equivalent martingale measure(EMM) under which the derivative price equals the expected value of its discounted payoff.

However, if the market is incomplete, there will be multiple EMMs which also leads to a fair price. Like what we have done in Chapter 1, in order to find a reasonable value bound of a portfolio, we will seek for an EMM where the derivative price obtained is in tune with the market information. Along with the standard optimal transport, we will discuss the financial application of MOT in this chapter.

2.1.1 Trading T-Vanilla options

We assume that the T-Vanilla options on an asset S are traded on the market, whose payoff is $\lambda(S_T)$ with a maturity T. In practice, the Vanilla's payoff could be replicated by a long position of some Put and Call T-Vanillas through the Taylor expansion formula [28]:

$$\lambda(S_T) = \lambda(S_0) + \lambda'(S_0)(S_T - S_0) + \int_0^{S_0} \lambda''(K)(K - S_T)^+ dK$$
$$+ \int_{S_0}^{\infty} \lambda''(K)(S_T - K)^+ dK$$

where $(K - S_T)^+$ (resp. $(S_T - K)^+$) is the payoff of a Put (resp. call) option. The second derivative $\lambda''(K)$ can be regarded as a probability function. We also define the pricing operator $\Pi[\cdot]$ and suppose it is linear, so that

$$\Pi\Big[\sum_i \lambda_i (S_T - K_i)^+\Big] = \sum_i \lambda_i \Pi\Big[(S_T - K_i)^+\Big]$$

Also, due to the arbitrage-free condition, we have

$$\Pi[1] = e^{-rT},\ \Pi[S_T] = S_0 \tag{2.1.1}$$

At the same time, $\Pi[(S_T - K)^+]$ should be non-increasing, convex with respect to K and $\Pi[(S_T - K)^+] \geq (S_0 - Ke^{-rT})^+$. According to Riesz's representation theorem, with the condition that the market price of a Call option with strike K approaches to 0 as $K \to \infty$, we then should have a probability measure $\mathbb{P}^{mkt}$ such that[32]

$$C(K) \equiv \Pi[(S_T - K)^+] = \mathrm{E}^{\mathbb{P}^{\mathrm{mkt}}}[e^{-rT}(S_T - K)^+]$$

where $\mathrm{E}^{\mathbb{P}^{\mathrm{mkt}}}[e^{-rT}S_T] = S_0$.

Again by using the linear property, we could obtain the market price of the payoff $\lambda(S_T)$ in terms of market prices of Put and Call T-Vanillas.

$$\begin{aligned}\Pi[\lambda(S_T)] = \mathrm{E}^{\mathbb{P}^{\mathrm{mkt}}}[\lambda(S_T)] = \lambda(S_0) &+ \int_0^{S_0} \lambda''(K)\mathrm{E}^{\mathbb{P}^{\mathrm{mkt}}}[(K - S_T)^+]dK \\ &+ \int_{S_0}^{\infty} \lambda''(K)\mathrm{E}^{\mathbb{P}^{\mathrm{mkt}}}[(S_T - K)^+]dK\end{aligned} \tag{2.1.2}$$

Based on the value $C^i(K)$ of a Call option with strike K and maturity T on asset S_i in the market, we define the T-marginal distributions

$$\mathbb{P}^{mkt}(S_i = K) = \partial_K^2 C^i(K), \quad i = 1, 2. \tag{2.1.3}$$

Note that this second-order derivative exists almost everywhere due to the convexity of $C^i(K)$[11].

For simplicity, we assume the riskless interest rate is zero in the rest of this chapter; this can be easily recovered by introducing a multiplicative factor e^{-rT} to the relevant terms in formulas. We also clarify that $\mathcal{P}(\mathbb{P}^1, \mathbb{P}^2)$ is the set of all probability measures, while $\mathcal{M}(\mathbb{P}^1, \mathbb{P}^2)$ is the set of all martingale measures satisfying the following.

$$\mathcal{M}(\mathbb{P}^1, \mathbb{P}^2) = \{\mathbb{Q} \in \mathcal{P}(\mathbb{P}^1, \mathbb{P}^2) : \mathrm{E}^{\mathbb{Q}}[S_t] = S_0\}.$$

2.2 Optimal Transport Problem

2.2.1 Monge-Kantorovich duality

We will first introduce the optimal transport(OT) in this section due to its close connection to our method in MOT. Also, its interpretation in mathematical finance will be discussed.

Let us begin with two underlying assets S_1 and S_2 with the same maturity T and define a payoff function $c(s_1, s_2)$ in terms of these two assets. And we denote $S_1 \equiv S_T^1$ and $S_2 \equiv S_T^2$.

Assumption 2.2.1. $c : \mathbb{R}_+^2 \to [-\infty,\ \infty)$ is a continuous function such that

$$c^+(s_1, s_2) \le K \cdot (1 + s_1 + s_2)$$

on $(\mathbb{R}_+)^2$ for some constant K.

Then we define the model-independent super-replication price(consistent with Call options on S_1 and S_2) as

Definition 2.2.2.

$$MK_2 \equiv \inf_{\mathcal{P}^*(\mathbb{P}^1, \mathbb{P}^2)} \mathbb{E}^{\mathbb{P}^1}[\lambda_1(S_1)] + \mathbb{E}^{\mathbb{P}^2}[\lambda_2(S_2)], \tag{2.2.1}$$

where $\mathcal{P}^*(\mathbb{P}^1, \mathbb{P}^2)$ is the set of all functions $(\lambda_1, \lambda_2) \in L^1(\mathbb{P}^1) \times L^2(\mathbb{P}^2)$ such that

$$\lambda_1(s_1) + \lambda_2(s_2) \ge c(s_1, s_2) \tag{2.2.2}$$

for $\mathbb{P}^1$-almost all $s_1 \in \mathbb{R}_+$ and $\mathbb{P}^2$-almost all $s_2 \in \mathbb{R}_+$. $\lambda_i(s_i), i = 1, 2$ is the payoff of Vanilla Call options.

The linear program (2.2.1) is called Monge-Kantorovich formulation, and we regard it as the dual problem. The infimum in (2.2.1) could be obtained and its value will not change if the definition of $\mathcal{P}^*(\mathbb{P}^1, \mathbb{P}^2)$ is restricted to some bounded and continuous functions[7]. Thus, the inequality (2.2.2) holds for all $(s_1, s_2) \in \mathbb{R}_+^2$. Note that if $(\lambda_1, \lambda_2) \in \mathcal{P}^*(\mathbb{P}^1, \mathbb{P}^2)$, then $(\lambda_1 + C, \lambda_2 - C) \in \mathcal{P}^*(\mathbb{P}^1, \mathbb{P}^2)$ representing constant shifts in European payoffs, so this infimum is not unique[32]. From equations above We could deduce that the static super-replicatwon strategy containing Vanilla payoff $\lambda_1(s_1)$ and $\lambda_2(s_2)$ with market prices $\mathbb{E}^{\mathbb{P}^1}[\lambda_1(S_1)]$ and $\mathbb{E}^{\mathbb{P}^2}[\lambda_1(S_2)]$ generates a portfolio $\lambda_1(s_1) + \lambda_2(s_2)$ whose value is greater than or equal to the payoff $c(s_1, s_2)$ at maturity. We can interpret (2.2.1) as the robust super-replication price of c within the framework of mathematical finance.

2.2.2 Optimal transport formulation

The equation (2.2.1) could be transformed by adding a Kuhn-Tucker multiplier, which is a positive measure on $\mathbb{R}_+^2$, to the inequality (2.2.2)[32]:

Theorem 2.2.3.

$$MK_2 = \sup_{\mathcal{P}^*(\mathbb{P}^1, \mathbb{P}^2)} \mathbb{E}^{\mathbb{P}}[c(S_1, S_2)], \tag{2.2.3}$$

where $\mathcal{P}^*(\mathbb{P}^1, \mathbb{P}^2) = \{\mathbb{P} : S_1 \sim \mathbb{P}^1, \mathbb{P} : S_2 \sim \mathbb{P}^2\}$.

We call this expression the primal problem. When (2.2.3) is equal to (2.2.1), this equation is known as the Kantorovich duality. Keep in mind that the OT problems are usually written with an 'inf' rather than a 'sup', while this optimisation MK_2 is to maximise the cost function $E^{\mathbb{P}}[c(S_1, S_2)]$ over the convex set of joint measures with $\mathbb{P}^1$ and $\mathbb{P}^2$. If the cost function is continuous and the set $\mathcal{P}^*(\mathbb{P}^1, \mathbb{P}^2)$, then the supremum over $\mathcal{P}^*(\mathbb{P}^1, \mathbb{P}^2)$ can be obtained by Prokhorov's theorem. This proves the existence of a $\mathbb{P}^* \in \mathcal{P}^*(\mathbb{P}^1, \mathbb{P}^2)$(may not be unique) such that $MK_2 = \mathbb{E}^{\mathbb{P}^*}[c(S_1, S_2)]$[32].

Moreover, the following proposition will show that the infimum in Monge-Kantorovich formulation can be attained by a pair (λ, λ^*) of bounded continuous c-concave functions.

Proposition 2.2.4.

$$MK_2 = \inf_{\lambda \in C_b} \mathbb{E}^{\mathbb{P}^1}[\lambda^*(S_1)] + \mathbb{E}^{\mathbb{P}^2}[\lambda(S_2)], \tag{2.2.4}$$

where $\lambda^*(s_1) \equiv \sup_{s_2 \in \mathbb{R}_+}\{c(s_1, s_2) - \lambda(s_2)\}$ *is the c-concave transform of* λ.

2.3 Martingale optimal transport

In this section we will discuss the martingale optimal transport(MOT). Firstly, some basics will be introduced.

Let us consider one asset with payoff $c(s_1, s_2)$ evaluated at two dates $t_1 < t_2$. We suppose the Vanilla options of all strikes whose maturities are t_1 and t_2 can be traded, so the distribution of S_1 and $S_2 (S_1 \equiv S_{t_1}, S_2 \equiv S_{t_2})$ can my implied. Then we define the model-independent value upper bound, which is consistent with t_1 and t_2 Vanilla options, to be the OT problem in a martingale version. We construct the dual problem:

Definition 2.3.1.

$$\widehat{MK_2} \equiv \inf_{\mathcal{M}^*(\mathbb{P}^1, \mathbb{P}^2)} \mathbb{E}^{\mathbb{P}^1}[\lambda_1(S_1)] + \mathbb{E}^{\mathbb{P}^2}[\lambda_2(S_2)], \tag{2.3.1}$$

where $\mathcal{M}^*(\mathbb{P}^1, \mathbb{P}^2)$ is the set of function $\lambda_1 \in L^1(\mathbb{P}^1), \lambda_2 \in L^1(\mathbb{P}^2)$, and H is a bounded continuous function on $\mathbb{R}_+$ such that

$$\lambda_1(s_1) + \lambda_2(s_2) + H(s_1) \cdot (s_2 - s_1) \geq c(s_1, s_2), \forall (s_1, s_2) \in \mathbb{R}_+^2. \tag{2.3.2}$$

This represents a semi-static hedging strategy containing European Call options with payoffs λ_1 and λ_2 and a delta strategy at time t_1, it generates a Profit & Loss(P&L) at time t_2 with zero cost. A delta-hedging $H_0(S_0) \cdot (s_1 - S_0)$ can be performed but since it could be incorporated into $\lambda_1(s_1)$, so it is not included[32]. We could have also added any intermediate delta-hedging term $H_i(S_0, ..., s_{t_i}) \cdot (s_{t_{i+1}} - s_{t_i})$ with $0 < t_i < t_{i+1} \leq t_2$, but the optimal solution is obtained when $H_i = 0$. A corollary later will support this.

Compared $\widehat{MK_2}$ to the OT MK_2 in equation (2.2.1), we can deduce that $\widehat{MK_2} \leq MK_2$ because of the existence of the function H.

2.3.1 MOT formulation

Similar to what we have done in the previous section, the MOT duality (2.3.1) can be formulated as below: the primal problem.

Definition 2.3.2.

$$\widehat{MK_2}^* = \sup_{\mathbb{P} \in \mathcal{M}(\mathbb{P}^1, \mathbb{P}^2)} \mathbb{E}^{\mathbb{P}}[c(S_1, S_2)], \tag{2.3.3}$$

where $\mathcal{M}(\mathbb{P}^1, \mathbb{P}^2) = \{\mathbb{P} : \mathbb{E}^{\mathbb{P}}[S_2|S_1] = S_1, S_1 \sim \mathbb{P}^1, \mathbb{P} : S_2 \sim \mathbb{P}^2\}$ is the set of discrete martingale measures on $\mathbb{R}^2_+$ with $\mathbb{P}^1$ and $\mathbb{P}^2$.

The set $\mathcal{M}(\mathbb{P}^1, \mathbb{P}^2)$ is convex and weakly compact, so $\widehat{MK_2}^*$ can be achieved at the extremal point[32]. The proposition below will show the sufficient and necessary condition such that this set is not empty.

Proposition 2.3.3. *The set $\mathcal{M}(\mathbb{P}^1, \mathbb{P}^2)$ is non-empty if and only if the mean of either $\mathbb{P}^1$ or $\mathbb{P}^2$ is S_0 and $\mathbb{P}^1 \leq \mathbb{P}^2$ are in convex order[14].*

Definition 2.3.4. $\mathbb{P}^1 \leq \mathbb{P}^2$ are in convex order if and only if

$$\mathbb{E}^{\mathbb{P}^1}[(S_1 - K)^+] \leq \mathbb{E}^{\mathbb{P}^2}[(S_2 - K)^+], \quad \forall K \in \mathbb{R}_+.$$

In addition, we can reconstruct $\mathcal{M}(\mathbb{P}^1, \mathbb{P}^2)$ by its extremal points, according to the Krein-Milman theorem. We now present the following theorem characterizing extremal points:

Theorem 2.3.5. *For $\mathbb{P} \in \mathcal{M}(\mathbb{P}^1, \mathbb{P}^2)$, the properties below are equivalent[19]:*

- $\mathbb{P} \in Ext(\mathcal{M}(\mathbb{P}^1, \mathbb{P}^2))$*, which is the set of extremal points.*
- *The set of all functions $c \in L^1(\mathbb{P})$ is dense in $L^1(\mathbb{P})$, where*

$$c(s_1, s_2) = \lambda_1(s_1) + \lambda_2(s_2) + H_1(s_1)(s_2 - s_1), \quad \mathbb{P} - a.s.$$

  for some $\lambda_1 \in L^1(\mathbb{P}^1), \lambda_2 \in L^2(\mathbb{P}^2)$ and $H \in L^0(\mathbb{P}^1)$

Based on this theorem we find that a payoff $c \in L^1()\mathbb{P}$ could be roughly replicated in the semi-static sense. Thus, there exists some point $\mathbb{P} \in Ext(\mathcal{M}(\mathbb{P}^1, \mathbb{P}^2))$ in a robust, complete model where all payoffs $c \in L^1(\mathbb{P})$ are obtainable by the semi-static hedging.

Now we introduce another important theorem.

Theorem 2.3.6. *Suppose that $\mathbb{P}^1 \leq \mathbb{P}^2$ are probability measures in convex order on $\mathbb{R}_+$ with first moment S_0, and Assumption 2.2.1 holds, then $\widehat{MK_2}^* = \widehat{MK_2}$.*

The theorem tells us if the assumptions above hold, then there is no duality gap. With our knowledge of duality so far, we introduce the following corollary to explain why we do not include any intermediate delta-hedging in Definition 2.3.1.

Corollary 2.3.7. *Define*

$$\widehat{MK_2}^{3/2} \equiv \inf_{\mathcal{M}^*_{3/2}(\mathbb{P}^1,\mathbb{P}^2)} \mathbb{E}^{\mathbb{P}^1}[\lambda_1(S_1)] + \mathbb{E}^{\mathbb{P}^2}[\lambda_2(S_2)],$$

where $\mathcal{M}^*_{3/2}(\mathbb{P}^1, \mathbb{P}^2)$ is the set of functions $\lambda_1 \in L^1(\mathbb{P}^1), \lambda_2 \in L^1(\mathbb{P}^2)$, and $H(resp. H_{3/2})$ is bounded continuous function on $\mathbb{R}_+(resp. \mathbb{R}^2_+)$ such that $\forall(s_1, s_{3/2}, s_2) \in \mathbb{R}^3_+$. And

$$\lambda_1(s_1) + \lambda_2(s_2) + H(s_1)(s_{3/2} - s_1) + H_{3/2}(s_1, s_{3/2})(s_2 - s_{3/2}) \geq c(s_1, s_2), \quad \forall(s_1, s_2).$$

Then we have

$$\widehat{MK_2}^{3/2} = \widehat{MK_2}.$$

This corollary shows that there is no need for the delta hedging at intermediate dates since the optimal value for $\widehat{MK_2}^{3/2}$ and $\widehat{MK_2}$ are the same.

2.3.2 Connection with Hamilton-Jacobi-Bellman equation

We will show that the minimisation in MOT can be restricted to the class of concave envelope. For a function g, its concave envelope is the smallest concave function greater than or equal to g.

Proposition 2.3.8. *For probability measures $\mathbb{P}^1 \leq \mathbb{P}^2$ in convex order on $\mathbb{R}_+$ with mean S_0, and Assumption 2.2.1 holds. We have*

$$\widehat{MK_2} = \inf_{\lambda \in L^1(\mathbb{P}^2)} \mathbb{E}^{\mathbb{P}^1}[(c(S_1, \cdot) - \lambda(\cdot))^{**}(S_1)] + \mathbb{E}^{\mathbb{P}^2}[\lambda(S_2)], \tag{2.3.4}$$

where λ^{**} denotes its concave envelope[25].

By using the formulation (2.3.4), we may connect the martingale Monge-Kantorovich formulation with the solution of Hamilton-Jacobi-Bellman equation:

Corollary 2.3.9.

$$\widehat{MK_2} = \inf_{u(1,\cdot) \in L^1(\mathbb{P}^2)} \mathbb{E}^{\mathbb{P}^1}[u(0, S_1, S_1)] + \mathbb{E}^{\mathbb{P}^2}[u(1, S_2)],$$

where

$$u(0, s_1, s_2) \equiv \sup_{\sigma \in [0,\infty]} \mathbb{E}^{\mathbb{P}}[c(s_1, S_T) - u(1, S_T) | S_0 = s],$$

and $dS_t = \sigma_t dB_t$. B is a Brownian motion and σ is an adapted(w.r.t the filtration of B) unbounded control process.

If s_1 is fixed, $u(0, s_1, s)$ represents a value function of a stochastic control problem containing the maximisation of the expectation of $c(s_1, S_T) - u(1, S_T)$ over all controls σ. In the view of mathematical finance, the SDE of S_t represents an unbounded uncertain volatility model[32].

If we assume that the dual (2.3.1) can be obtained by $(\lambda^*_1, \lambda^*_2, H^*)$, then the primal (2.3.3) tells that the payoff could be perfectly dynamically replicated under $\mathbb{P}^* \in \mathcal{M}(\mathbb{P}^1, \mathbb{P}^2)$, i.e.

$$\lambda^*_1(s_1) + \lambda^*_2(s_2) + H^*(s_1)(s_2 - s_1) = c(s_1, s_2), \quad \mathbb{P}^* - a.s. \tag{2.3.5}$$

for the optimal martingale measure $\mathbb{P}^*$. Equation (2.3.5) holds because

$$\mathbb{E}^{\mathbb{P}^*}[\lambda^*_1(S_1) + \lambda^*_2(S_2) + H^*(S_1)(S_2 - S_1) - c(S_1, S_2)] =$$
$$\mathbb{E}^{\mathbb{P}^1}[\lambda^*_1(S_1)] + \mathbb{E}^{\mathbb{P}^2}[\lambda^*_2(S_2)] - \mathbb{E}^{\mathbb{P}^*}[c(S_1, S_2)] = 0.$$

2.3.3 The discrete martingale Fréchet-Hoeffding solution

In this section, we will demonstrate the explicit solution to $\widehat{MK_2}$ under the martingale Spence-Mirrlees condition $\partial_{s_1 s_2 s_2} c > 0$, which gives a martingale measure similar to Fréchet-Hoeffding solution. Under this condition, the optimal measure will be independent of payoff and only depend on $\mathbb{P}^1$ and $\mathbb{P}^2$. Then the infimum in the dual problem could be obtained.

Explicit solution for the primal

The extremal probability measure $\mathbb{P}^*$ coincides with the unique left-monotone martingale transference[24]:

Definition 2.3.10. $\mathbb{P} \in \mathcal{M}(\mathbb{P}^1, \mathbb{P}^2)$ is left-monotone if we can find a Borel set $\Gamma \subset \mathbb{R} \times \mathbb{R}$ such that $\mathbb{P}[(X, Y) \in \Gamma] = 1$. And $\forall (x, y_1), (x, y_2), (x', y') \in \Gamma$ with $x < x'$, then $y' \notin (y_1, y_2)$.

For simplicity, we assume that $\delta F \equiv F_2 - F_1$ has a unique maximum m. In real market, distribution implied by Vanilla option also satisfy this condition[32]. Now we will characterise $\mathbb{P}^*$ in terms of ODEs by using the primal formulation, and we denote F_1 and F_2 to be the cumulative distribution functions of $\mathbb{P}^1$ and $\mathbb{P}^2$.

We define $\mathbb{P}^* \in \mathcal{M}(\mathbb{P}^1, \mathbb{P}^2)$ as

$$\mathbb{P}^*(ds_1,\ ds_2) = \mathbb{P}^1(ds_1)(q(s_1)\delta_{T_u(s_1)}(ds_2) + (1 - q(s_1))\delta_{T_d(s_1)}(ds_2))\ ,$$
$$q(x) = \frac{x - T_d(x)}{T_u(x) - T_d(x)}$$

with the maps $T_d(x) \le x \le T_u(x)$, T_u is increasing and T_d is decreasing. They are defined by

$$\begin{aligned} T_u(x) &= T_d(x) = x,\ x \le m \\ T_u(x) &= F_2^{-1}(F_1(x) + \delta F(T_d(x)) \end{aligned} \tag{2.3.6}$$

and

$$T_d'(x) = -\frac{T_u(x) - x}{T_u(x) - T_d(x)} \frac{F_1'(x)}{F_2'(T_d(x)) - F_1'(T_d(x))} \tag{2.3.7}$$

We need to solve the first-order OED (2.3.7) with the initial condition $T_d(m) = m$ to find $T_d(x)$ for $x \ge m$. Also, $T_d(x)$ can be expressed as the unique solution $t \in \mathbb{R}_+$ of

$$-\int_t^m (g(x,\zeta) - \zeta) d\delta F(\zeta) + \int_m^x (g(\zeta, m) - \zeta) dF_1(\zeta) = 0, \tag{2.3.8}$$

where $g(x, \zeta) \equiv F_2^{-1}(F_1(x) + \delta F(\zeta))$ and $t \le m \le x$.

Explicit solution for the dual

We first define a triple of dual variables $(\lambda_1^*, \lambda_2^*, H^*)$ with a smooth function c.

The dynamic hedging component H^* is defined by

$$H^{*'}(s_1) = \frac{c_{s_1}(s_1, T_u(s_1)) - c_{s_1}(s_1, T_d(s_1))}{T_u(s_1) - T_d(s_1)},\ \forall s_1 \ge m \tag{2.3.9}$$

The payoff function λ_2^* is defined by

$$\begin{aligned} \lambda_2^{*'}(s_2) &= c_{s_2}(T_u^{-1}(s_2),\ s_2) - H^* \circ T_u^{-1}(s_2)\ ,\ \forall s_2 \ge m \\ &= c_{s_2}(T_d^{-1}(s_2),\ s_2) - H^* \circ T_d^{-1}(s_2)\ ,\ \forall s_2 < m. \end{aligned} \tag{2.3.10}$$

And λ_1^* is

$$\begin{aligned} \lambda_1^*(s_1) &= \mathrm{E}^{\mathbb{P}^*}[c(S_1,\ S_2) - \lambda_2^*(S_2) | S_1 = s_1] \\ &= q(s_1)(c(s_1,\ .) - \lambda_2^*) \times T_u(s_1) + (1 - q(s_1))(c(s_1,\ .) - \lambda_2^*) \times T_d(s_1)\ ,\ \forall s_1 \in \mathbb{R}_+ \end{aligned} \tag{2.3.11}$$

With the theorem below we are able to obtain the solution for the dual.

Theorem 2.3.11. *For probability measures $\mathbb{P}^1 \leq \mathbb{P}^2$ on $\mathbb{R}_+$ with first moment S_0, and the assumption that $\delta F \equiv F_2 - F_1$ has a unique maximum m holds. And we further suppose that $\lambda_1^{*+} \in L^1(\mathbb{P}^1)$,$\lambda_2^{*+} \in L^1(\mathbb{P}^2)$, and the partial derivative of $c_{s_1 s_2 s_2}$ exists with $c_{s_1 s_2 s_2} > 0$ on $\mathbb{R}_+^2$. Then*

- $(\lambda_1^*,\ \lambda_2^*,\ H^*) \in \mathcal{M}^*(\mathbb{P}^1,\ \mathbb{P}^2)$.
- *the strong duality holds for the martingale transport problem, $\mathbb{P}^*$ is a solution of $\widehat{MK}_2^*$ and $(\lambda_1^*, \lambda_2^*, H^*)$ is a solution of $\widehat{MK}_2$:*

$$\mathbb{E}^{\mathbb{P}^*}[c(S_1, S_2)] = \widehat{MK}_2^* = \widehat{MK}_2 = \mathbb{E}^{\mathbb{P}^1}[\lambda_1^*(S_1)] + \mathbb{E}^{\mathbb{P}^2}[\lambda_2^*(S_2)].$$

2.4 Methods for numerical implementation

In this section, we will explore how to find the optimal solutions to the MOT primal and dual problems, and what computational techniques should be utilised. For simplicity, we will consider the case of one asset evaluated at two dates $t_1 < t_2$ with the corresponding Vanilla options. And Assumption 2.2.1 holds for the remaining part of this chapter.

2.4.1 $\mathcal{U}$-quantization

Before introducing the method of solving the MOT primal problem numerically, we discuss some foundations related to this method. This is because we need to reduce the primal problem to a linear program by quantising measures. Baker[8] presents various methods for the quantisation of measures, for example, the $L2$ quantisation. However, it has been proved that the $L2$ quantisation does not preserve the convex order[8]. By Proposition 2.3.3 we know that the set $\mathcal{M}(\mathbb{P}^1, \mathbb{P}^2)$ will be empty if $\mathbb{P}^1 \leq \mathbb{P}^2$ are not in convex order. Therefore, Baker[8] introduces another method called $\mathcal{U}$-quantisation.

Definition 2.4.1. For a probability measure with distribution function $F(x)$, its quantile function is

$$F^{-1}(p) = \inf\{x \in \mathbb{R} : p \leq F(x)\}$$

Definition 2.4.2. For an integer n, define a probability measure $\mu \in \mathcal{P}(\mathbb{R})$ with distribution function $F(u) = \int_{-\infty}^{u} d\mu(x)$. Then the $\mathcal{U}$-quantization of μ is

$$U(a_1, ..., a_n) = \frac{1}{n}\sum_{i=1}^{n} \delta_{a_i}, \qquad \text{where} \quad a_i = n\int_{\frac{i-1}{n}}^{\frac{i}{n}} F^{-1}(u)du, \tag{2.4.1}$$

and δ_x is the Dirac point mass at x. And its distribution function F^* is

$$F^* = \begin{cases} 0 & , x \leq a_1 \\ \frac{i}{n} & , x \in [a_i, a_{i+1}) \\ 1, & x \geq a_n. \end{cases}$$

Some useful properties of $\mathcal{U}$-quantisation that convince us to use this method for numerical implementations will be shown below.

Theorem 2.4.3. *($\mathcal{U}$-quantization preserves the mean of a measure) Define a probability measure μ with distribution function F, and $U(a_1, ..., a_n)$ to be its $\mathcal{U}$-quantization. Then μ and $U(a_1, ..., a_n)$ have the same mean.*

Theorem 2.4.4. *($\mathcal{U}$-quantization preserves the convex order) For $\mu, \nu \in \mathcal{P}(\mathbb{R})$ with $\mathcal{U}$-quantizations $U(a_1, ..., a_n)$ and $U(b_1, ..., b_n)$. If $\mu \leq \nu$ are in convex order, then $U(a_1, ..., a_n) \leq U(b_1, ..., b_n)$ are in convex order.*

As we discussed before, there is no duality gap if Proposition 2.3.3 and Theorem 2.3.6 hold, so that $\widehat{MK_2}$ can be attained. Therefore, the theorems above assure us that the duality gap will not appear if we utilise the $\mathcal{U}$-quantisation, since the mean and convex order of original measures will not be changed.

2.4.2 Solving the MOT primal problem

Now we will discuss how to solve the MOT primal by using the $\mathcal{U}$-quantisation. Firstly, we will need the implied probability measures $\mathbb{P}^1$ and $\mathbb{P}^2$ from the prices of Vanillas with a range of strikes K quoted in the market. However, not all options with desired strikes are traded in reality. Thus we will use interpolation methods to build a complete spectrum of strikes with the available market information. The details of this step will not be presented here.

We emphasise that the implied probability measures $\mathbb{P}^1$ and $\mathbb{P}^2$ can be obtained by differentiating the market values of Calls twice with respect to strikes K, as Breeden and Litzenberger[11] claim. So after acquiring Call prices with desired strikes, we could approximate the partial derivative of $C(t_i, K)$ by differencing it over small step sizes and repeat the same process to get the second-order derivative of $C(t_i, K)$, thereby obtaining the marginals.

We quantilize $\mathbb{P}^1$ and $\mathbb{P}^2$ with distribution function F and G, for fixed integers n and m respectively:

$$\mathbb{P}^1(a_1, ..., a_n) = \frac{1}{n}\sum_{i=1}^{n} \delta_{a_i}, \qquad \text{where} \quad a_i = n\int_{\frac{i-1}{n}}^{\frac{i}{n}} F^{-1}(u)du,$$

and

$$\mathbb{P}^2(b_1, ..., b_m) = \frac{1}{m}\sum_{j=1}^{m} \delta_{b_j}, \qquad \text{where} \quad b_j = m\int_{\frac{j-1}{m}}^{\frac{j}{m}} G^{-1}(u)du,$$

Also, we discretise the price of underlying such that asset takes n possible values $(x_i)_{1\leq i\leq n}$ at time t_1 and takes m possible values $(y_j)_{1\leq j\leq n}$ at time t_2.

According to Guo and Obłój[12], we could transfer the MOT primal (2.3.3) into the following linear programming problem:

$$U(K) = \max_{(p_{i,j})_{1\leq i\leq n, 1\leq j\leq m} \in \mathbb{R}_+^{n,m}} \sum_{i=1}^{n}\sum_{j=1}^{m} p_{i,j}c(x_i,\ y_j) \tag{2.4.2}$$

subject to:

$$\sum_{j=1}^{m} p_{i,j} = a_i,\ i = 1,\ ...,\ n$$

$$\sum_{i=1}^{n} p_{i,j} = b_j,\ j = 1,\ ...\ m$$

$$\sum_{j=1}^{m} p_{i,j}b_j = a_i x_i,\ i = 1,\ ...,\ n$$

For the lower bound:

$$L(K) = \min_{(p_{i,j})_{1\leq i\leq n, 1\leq j\leq m} \in \mathbb{R}_+^{n,m}} \sum_{i=1}^{n}\sum_{j=1}^{m} p_{i,j}c(x_i,\ y_j), \tag{2.4.3}$$

subject to the same constraints above.

We summarise the steps of algorithm below:

- Interpolate t_1- and t_2 Vanillas available in the market to derive options prices with desired strikes.
- Use (2.1.3) to calculate the second-order derivative of market values of Call options w.r.t the strike K, to obtain the implied measures $\mathbb{P}^1$ and $\mathbb{P}^2$.
- Discretise the asset prices s_1 at time t_1 and s_2 at time t_2 on a two-dimensional grid.
- Conduct $\mathcal{U}$-quantization on $\mathbb{P}^1$ and $\mathbb{P}^2$, to obtain $a_i's$ and $b_j's$.
- Use the simplex algorithm to solve the linear programs (2.4.2) and (2.4.3).

2.4.3 Solving the MOT dual problem

According to Pierre Henry-Labordère[32], we could solve the MOT dual problem by linear programming.

We transfer $\mathbb{E}^{\mathbb{P}^1}[\lambda_1(S_1)]$ and $\mathbb{E}^{\mathbb{P}^2}[\lambda_2(S_2)]$ into a sum of weighted prices of Call options in the market and the value v of other components in the portfolio. There are N Calls whose strikes ranging from $[\alpha_1\%, \alpha_2\%] \times S_0$ maturing at time t_1 and t_2 respectively.

$$\mathbb{E}^{\mathbb{P}^1}[\lambda_1(S_1)] + \mathbb{E}^{\mathbb{P}^2}[\lambda_2(S_2)] \approx v + \sum_{j=1}^{N} \omega_1^j C(t_1,\ K_1^j) + \sum_{j=1}^{N} \omega_2^j C(t_2,\ K_2^j)\ ,$$

where $C(t_i,\ K)$ is the market value of a Call of maturity t_i and strike K.

Then the upper bound of the portfolio $\widehat{MK_2}$ becomes

$$U(K) = \min_{v,(\omega_1^j),(\omega_2^j),H(\cdot)} \quad v + \sum_{j=1}^{N} \omega_1^j C(t_1,\ K_1^j) + \sum_{j=1}^{N} \omega_2^j C(t_2,\ K_2^j) \tag{2.4.4}$$

subject to:

$$v + \sum_{j=1}^{N} \omega_1^j (s_1 - K_j)^+ + \sum_{j=1}^{N} \omega_2^j (s_2 - K_j)^+ + H(s_1)(s_2 - s_1) \geq c(s_1, s_2), \quad \forall (s_1, s_2) \in \mathbb{R}_+^2.$$

In addition, we could also discretise the price of underlying s_1 with n possible values $(x_i)_{1\leq i\leq n}$ and s_2 with m possible values $(y_k)_{1\leq k\leq m}$ at time t_1 and t_2 respectively. Then the linear program (2.4.4) will have $n \times m$ constraints:

$$v + \sum_{j=1}^{N} \omega_1^j (x_i - K_j)^+ + \sum_{j=1}^{N} \omega_2^j (y_j - K_j)^+ + H(x_i)(y_j - x_i) \geq c(x_i, y_j), \quad \forall (x_i, y_j) \in \mathbb{R}_+^2,$$

for $1 \leq i \leq n, 1 \leq j \leq m$.

Thus, we have reduced the MOT dual problem for the upper bound into a standard linear program $U(K)$, which could be numerically solved via optimisation algorithm.

As for the lower bound, we simply need to replace min by max in (2.4.4), and reverse the inequality sign in constraints.

$$L(K) = \max_{v,(\omega_1^j),(\omega_2^j),H(\cdot)} \quad v + \sum_{j=1}^{N} \omega_1^j C(t_1,\ K_1^j) + \sum_{j=1}^{N} \omega_2^j C(t_2,\ K_2^j) \tag{2.4.5}$$

subject to:

$$v + \sum_{j=1}^{N} \omega_1^j (x_i - K_j)^+ + \sum_{j=1}^{N} \omega_2^j (y_j - K_j)^+ + H(x_i)(y_j - x_i) \leq c(x_i, y_j), \quad \forall (x_i, y_j) \in \mathbb{R}_+^2,$$

for $1 \leq i \leq n, 1 \leq j \leq m$.

We summarise the steps of algorithm below:

- Determine the number and range of strikes of Vanillas, and collect the market prices $C(t_i, K)$.
- Discretise the asset prices s_1 at time t_1 and s_2 at time t_2 on a two-dimensional grids to build the constraints.
- Solve the linear program.

2.5 Numerical implementation

2.5.1 MOT primal

In this section, we will focus on how to conduct the $\mathcal{U}$-quantisation by utilising the market information. The reason that we will not show the full implementation is that in practice the option prices are influenced by various factors such as intrinsic value, time value and the methods of pricing in reality. The assumption that marginals we obtain by differentiating the Call prices with respect to strikes are in a martingale measure may easily be violated, so the mean of $\mathcal{U}$-quantisation points in $\mathbb{P}^1(a_1, ..., a_n)$ and $\mathbb{P}^2(b_1, ..., b_n)$ may not be the same. Besides the above reasons, the choice of Call prices due to bid-ask spread, different interpolation methods and numerical accuracy may also influence the measure we intend to acquire. Therefore, the probability measures built up by solely relying on Call prices in the market might not be 'perfect' enough to solve the linear program (2.4.2). So the consistency of observed Vanilla prices is a crucial step, which is hardly met in practice and is one of the greatest challenges in the algorithm. Still, we will present the results of $\mathcal{U}$-quantisation.

Firstly, we collect the data of Call options on the stock Amazon "AMZN" with strikes ranging from \$2950 to \$3500 on 21 August 2020. The expiration dates of two options are t_1 : 28 August and t_2 : 4 September 2020 respectively, which means the maturities are 7 days and 14 days. Not all options with strikes on $[2950, 3500]$ are available in the market, so we conduct the interpolation on option prices and corresponding strikes by the spline method. We will show only the case of the first option with maturity $T = 7$ days, the similar results of the other option can be found in the code link.

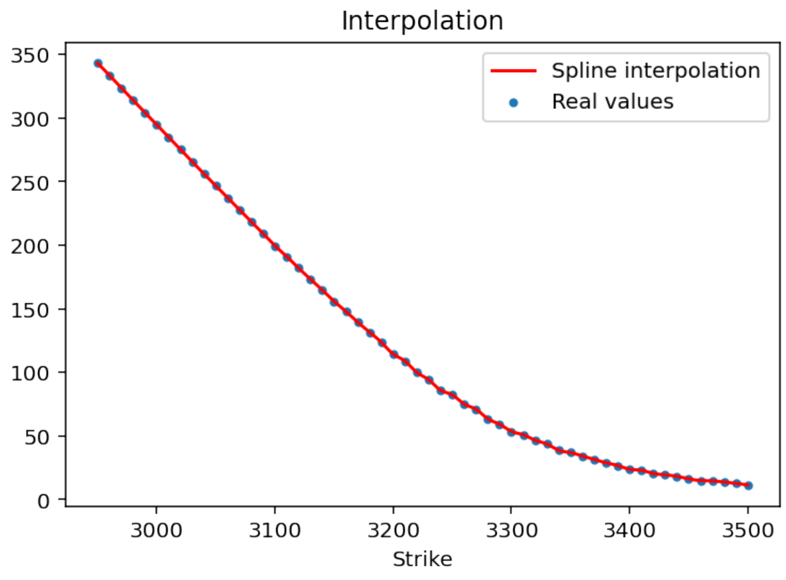

Figure 2.1: Interpolation on available Call prices with strikes.

After doing this, we are able to utilise the desired Call prices with strikes depending on the number of points we set in the interpolation.

Then we compute the implied marginals of stock from the second derivative of Call prices w.r.t. strikes. Due to the fluctuation of Call prices, some values of $\partial_K^2 C^i(K)$ become negative, but the probability should be non-negative. So we also smoothen the curve for Call prices in the process of interpolation. Also, in order to prevent obtaining unrealistic Call prices, we do not interpolate any prices whose strikes are not in the range $[2950, 3500]$ supported by the real market. Then we standardise the probability measure we obtain since we assume the sum of all marginals should be equal to 1.

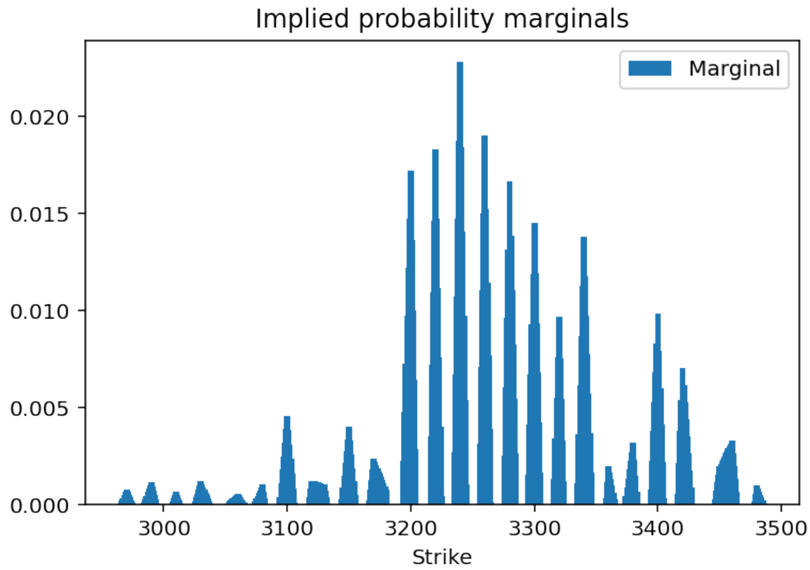

Figure 2.2: Implied probability marginals after interpolation.

Subsequently, we build the CDF and quantile function by equation (2.4.1) to $\mathcal{U}$ quantise the measures $\mathbb{P}^1$ and $\mathbb{P}^2$ from the implied probability marginals, with $n = 10$. The results are as follows:

$\mathcal{U}$-quantization	a_i's and b_i's
$\mathbb{P}^1(a_1, ..., a_n)$	2961, 3084, 3137, 3212, 3254, 3311, 3339, 3368, 3401, 3447
$\mathbb{P}^2(b_1, ..., b_n)$	3002, 3139, 3171, 3209, 3236, 3243, 3280, 3320, 3363, 3403

Table 2.1: $\mathcal{U}$-quantization

This means, for example, we predict that the stock s_1 on the date t_1: 28 August 2020 will have the following probability distribution function:

$$F^*(x) = \begin{cases} 0 & , x \leq a_1 \\ \frac{i}{10} & , x \in [a_i, a_{i+1}) \\ 1, & x \geq a_{10}, \end{cases}$$

where a_i's are listed above.

2.5.2 MOT dual

In this section, we will implement a numerical example for the MOT dual problem.

We will use the options on the stock Amazon in the previous section to find the value bounds for a Call option expiring at t_1, along with a hedging strategy containing one t_1-Vanilla, one t_2-Vanilla and some risk-free bounds V, and the delta at t_1. The market prices of t_1- and t_2-Vanillas are \$108.85 and \$106.05, strikes are \$3210 and \$3260 respectively. After interpolating Call prices and discretising the asset prices on a two-dimensional grid of 100 values, we solve the linear programs (2.4.4) and (2.4.5) to obtain the value bounds of this portfolio. As we can see in the figure below, except the lower bound reduces to zero for large strikes, the real market values indeed lie with the bounds we implement.

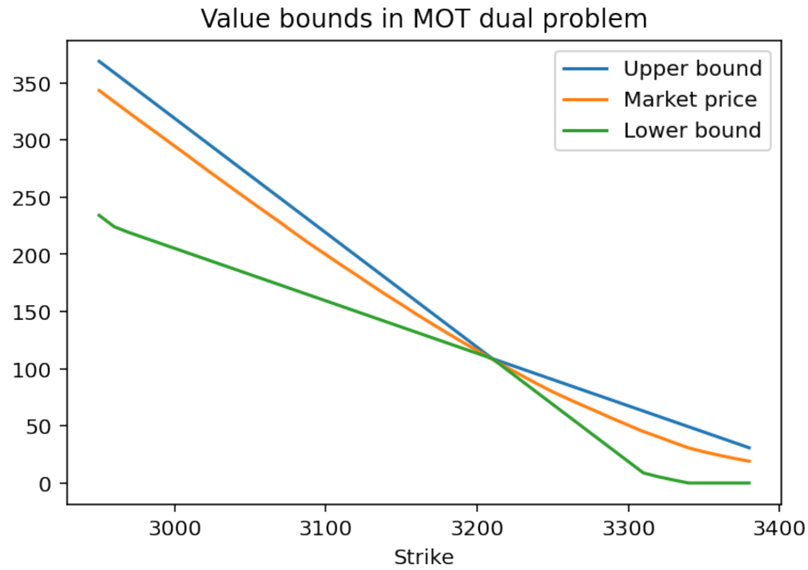

Figure 2.3: Value bounds produced by linear programs in MOT dual.

2.5.3 Comparison between UVM and MOT

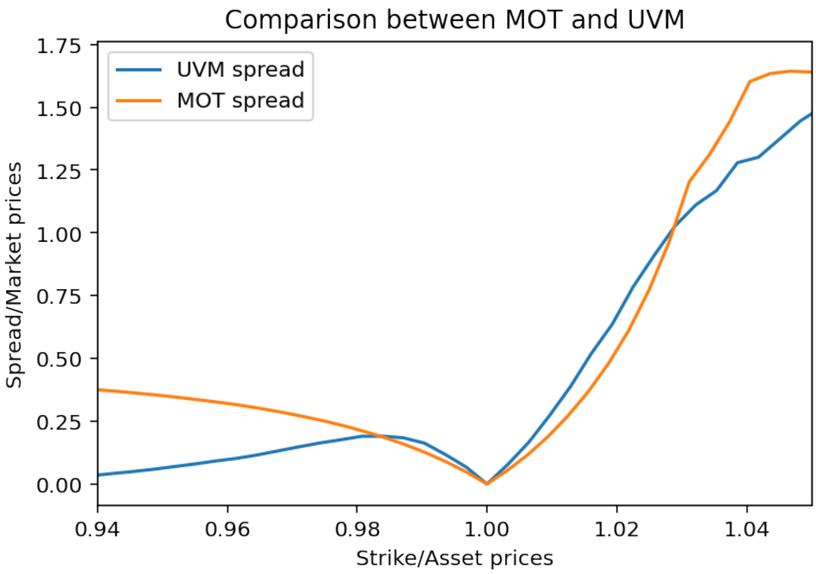

Figure 2.4: Comparison between UVM and MOT in numerical implementations

We also present the spreads of value bounds obtained by using λ-UVM and MOT in previous sections. In order to compare the performances in their numerical implementations more straightforwardly, we divide strikes by the initial asset price and value spreads by corresponding market prices of the derivative to standardise the x- and y-axis. Note that we are trying to observe some potential differences between these two methods; hence not all characteristics may appear in this numerical comparison. We find in Figure 2.4 that, when strikes are close to the initial asset price, the value bounds of MOT are slightly narrower, but there is no much difference between two value spreads in the most of these cases. Furthermore, the UVM works well when the strike is much smaller than S_0; its value spread presents a trend converging to zero. The value bound of UVM is also narrower when K is slightly large than S_0. So the performance of UVM is moderately better than that of MOT. In general, the UVM requires more assumptions that the asset price process behaves under some specific models, and its volatility band needs to be calibrated. So the accuracy of UVM depends on several factors, and any violation of these assumptions may compromise the model. In our implementations, we chose the trinomial model to simulate the price process. In fact, the usage of other models may also affect the numerical results. As a whole, the UVM performs well since the value bounds we obtained in the last section is relatively narrow and it based on more assumptions. When it comes to MOT, just like the problem we discussed in Section 2.5.1 for the primal, the points in $\mathcal{U}$-quantisations of probability measures with the same mean are hard to obtain from the market due to various reasons, and the marginals obtained may not be in a martingale measure. Also, it is difficult to ensure the implied probability marginal obtained are in a martingale measure, and the interpolation we used may also cause the uncertainty. For the MOT dual, the linear programming form of it can be easily implemented and solved, although different methods for solving linear programming might influence the value bounds. In summary, we may obtain better value bounds for derivatives by using the method based on more assumptions. However, these assumptions might be violated in the real world. Therefore, it is essential to consider carefully about the trade-off between the robustness and model assumptions in order to produce practical value bounds when the model risk is acceptably low at the same time.

2.6 Conclusion

In this chapter, we first discussed the optimal transport problems, which subsequently leads us to a similar problem with a martingale constraint, namely MOT. We deduced the primal and dual problems for MOT with important properties and presented the methods for their numerical implementations, which allows us to reduce the MOT problem into a linear program.

Finally, we demonstrated how to conduct the $\mathcal{U}$-quantisation and presented the value bounds for a derivative with hedging strategy in the MOT dual problem. Then the comparison between MOT and UVM in terms of their numerical performances and basic properties was conducted.

Chapter 3

Robust pricing for path-independent options with alternative methods

3.1 Introduction

In this chapter, alternative methods of pricing path-dependent options, such as Digital, Lookback and Barrier options, will be discussed. Our objective here is to deduce the optimal upper and lower bounds which are in line with the market information. For the sake of simplicity, we will suppose that the riskless interest rate is zero. Furthermore, we will assume the case that all Calls and Puts with the desired strikes are available on the market, which can be implemented through interpolation methods similar to the previous chapter.

The technique is similar to what we used in the first chapter. We studied the uncertain volatility model, where the calibration of the volatility band and the assumption of underlying's behaviours are required. In contrast, in this chapter, we only need the Vanilla options market prices to generate value bounds for path-dependent options. The main idea is to use super-replicating strategies to produce the value bounds, then to examine whether those bounds are optimal by utilising the assumption that asset prices process is a martingale consistent with the market information under the equivalent martingale measure with zero interest rate. Also, we need to deduce the probability marginals for Vanilla at certain points.

We will introduce basic definitions and properties of the Digital, Lookback and Barrier options, then the optimal value bounds in a discrete sense. Finally, some numerical results will be presented.

We assume the Call option can be traded at prices $C(K)$ with all possible strikes K and maturity date $T = 1$ year. We further assume $S_0 = 1$, by defining the EMM $\mathbb{Q}$ and the probability law p of S_T, we have the mean of p to be 1 due to the martingale property of S. Then we know the Call price is

$$C(K) = E^{\mathbb{Q}}[(S_T - K)^+] = \int_0^\infty (x - K)p(dx), \quad \forall K \geq 0 \tag{3.1.1}$$

Note that we have $C'(K) = -p([K, \infty))$. And if the function of Call price w.r.t the asset price $x \to C(x)$ is twice differentiable, then we also have $p(dx) = C''(x)dx$. The details can be found in [11]. So the Call price is

$$C(K) = \int_0^\infty (x - K)C''(x)dx, \quad \forall K \geq 0.$$

3.2 Lookback options

We define the price for the underlying asset to be $(S_t)_{t\geq 0}$ with $S_0 = 1$, and the payoff of the Lookback option with maturity T to be

$$\sup_{0\leq t\leq T} S_t.$$

We suppose the continuous updating of the maximum. The upper and lower bound will still be valid even if the asset prices are observed in discrete time. Since in a discrete sense, the payoff can only be shrunk for the upper bound. Setting T to be one of the updating points leads to the soundness of the lower bound.

Before we present the value bounds for the Lookback option, we need the following assumptions[10]:

Assumption 3.2.1. (Continuous market) We can find a family of Call options with maturity T, the continuum of possible strike prices exists for each of them. Moreover, we are able to long Calls in arbitrary amounts, including fractions, and short them at the same price, which is independent of the quantity.

Assumption 3.2.2. (Regularity of Call prices) The price of a Call option $C(K)$ is a decreasing, convex function of K with $C(0) = 1$ for $K \geq 0$, and $C(K)$ reduces to zero as K goes to infinity. If we sell a European Call before its maturity, then its value will always be the intrinsic value of the corresponding American Call option. So at time $t < T$ we can sell a European Call with strike K for at least $(S_t - K)^+$.

We firstly define the barycentre function b_v

$$b_v(x) = \frac{1}{v((x,\infty))}\int_{(x,\infty)} yv(dy).$$

For a random variable X with continuous distribution v, then we define the Hardy-Littlewood transform of v to be v^*, which is the law of $b_v(X)$. More specifically, $b_v(k)$ is the point on the x-coordinate where the tangent at K to the function $C_v(K) = \int_{\mathbb{R}}(y-K)^+ v(dk)$ intersects the x-axis, and $v^*((b_v(K),\infty))$ is the modulus of the slope of this tangent. More details about the barycentre function can be found in [10]. Then we show the value bounds for the Lookback option.

Proposition 3.2.3. *Under the above assumptions, the upper bound for the price of the Lookback option is*

$$U = \int xp^*(dx),$$

and the lower bound is

$$L = C(1) + 1.$$

Proof. Here we will only prove the lower bound, the proof of the upper bound can be found in [10]. It is obvious that

$$\sup_{0\leq t\leq T} S_t \geq (S_T - 1)^+ + 1, \qquad S_0 = 1, \tag{3.2.1}$$

which can be interpreted as the payoff of the Lookback option is greater than the payoff of a Call option with strike 1 plus 1. So we are able to pay $L = C(1) + 1$ for the Lookback option without any risk.

We consider a price process containing a single jump at time $T/2$ with $S_0 = 1$. The jump follows the law p corresponding to C(K) in (3.1.1). Thus, the process is a martingale since

$$E^{\mathbb{Q}}[S_T - 0] = \int_0^\infty (x-0)p(dx) = C(0) = S_0 = 1.$$

By taking the expectation on both sides, the inequality (3.2.1) becomes equality. So $C(1) + 1$ is the largest lower bound for this Lookback option. □

Note that U denotes the lowest price at which a risk-averse dubious investor is willing to sell the Lookback option; L denotes the highest price that he/she is willing to pay. The arbitrage-free condition tells that the price of the Lookback option has to lie within the interval $[L, U]$.

3.3 Digital options

Now let us consider the Digital option, whose payoff is one unit if the price of an underlying asset crosses a constant Barrier B before the maturity $T = 1$, and the payoff is zero otherwise. Suppose the initial price $S_0 < B$, and we denote H_B to be the first time of the asset price reaching the Barrier, so $H_B = \inf\{t : S_t \geq B\}$ and the option has the payoff $\mathbb{1}_{\{H_B \leq 1\}}$. We preclude the extreme case that $\mathbb{Q}(S_T \leq B) = 1$ which leads to the Call price $C(B) = 0$, so $C(B) > 0$.

Now we construct a model-independent lower bound on the value of the Digital option. Intuitively, we know that at maturity T the asset price may be lower than the Barrier B even if it crosses B at any time before T, so $\{S_1 \geq B\} \subseteq \{H_B \leq 1\}$. We take the expectation on both sides of the following inequality containing the Digital's payoff[13]

$$\mathbb{1}_{\{S_1 \geq B\}} \leq \mathbb{1}_{\{H_B \leq 1\}},$$

we have

$$-C'(B) = p([B, \infty)) = \mathbb{Q}(S_1 \geq B) \leq \mathbb{Q}(H_B \leq 1) = \mathbb{E}[\mathbb{1}_{\{H_B \leq 1\}}].$$

Proposition 3.3.1. *The largest lower bound L on the value of a Digital option is $-C'(B)$, where $C = C(K)$ is the price of a Call option with strike K.*

$L = -C'(B)$ is the largest lower bound because if the asset price process S is 1 for $t < T$ and it jumps to value above the Barrier B at time T, the above inequalities become equalities.

Now we work on the model-independent upper bound of the Digital. We consider a super-replicating strategy for the one-touch Digital option whose arbitrage-free price is $D(B)$: long $(B - K)$ unit of a Call option with strike K and short $(B - K)$ forward contract of the underlying asset whenever the price of underlying crosses the Barrier. So we have

$$\mathbb{1}_{\{H_B \leq 1\}} \leq \frac{(S_T - K)^+}{B - K} + \frac{S_{H_B} - S_T}{B - K} \times \mathbb{1}_{\{H_B \leq 1\}} \tag{3.3.1}$$

Let us first check the inequality (3.3.1). If the underlying does not reach the Barrier, then the left-hand side term reduces to zero and the right-hand side term is non-negative. In the other case, the value on the left becomes one and $S_{H_B} \geq B$. If we replace the part $(S_T - K)^+$ by $(S_T - K)$ to make the right-hand side value even smaller, the inequality holds.

We take the expectation and optimise the amount of the Call to obtain

$$D(B) \leq \inf_{K<B} \frac{C(K)}{B - K} + 0,$$

where the zero term on the right-hand side means the zero cost of a forward.

Proposition 3.3.2. *The smallest upper bound U on the value of a Digital option is* $\inf_{K<B} \frac{C(K)}{B-K}$.

we also examine whether the upper bound U is optimal[9]. Recall that C is differentiable and $C(B) > 0$ so the infimum $\inf_{K<B} \frac{C(K)}{B-K}$ can be obtained at some point $a(p, B) < B$ and $-C'(a) = \frac{C(a)}{B-a}$. We also have $\mathbb{Q}(S_T > a) = p([a, \infty]) = -C'(a)$. Let $\gamma \in (0, S_0)$ solve

$$\frac{S_0 - \gamma}{B - \gamma} = \frac{C(a)}{B - a}.$$

We also build S to be a constant except at time $T/2$ and T. $S_{T/2}$ takes value B with probability $\frac{C(a)}{B-a}$ and value γ otherwise. Furthermore, restrict the law p of S_T to $\{S_T > a\}$ on $S_{T/2=B}$ and the law p of S_T to $\{S_T \leq a\}$ on $S_{T/2=a}$. So $\{S_{T/2} = B\} = \{S_T > a\}$ and we obtain

$$S_T \mathbb{1}_{\{S_{T/2}=B\}}] = C(a) + a \cdot P(S_T > a) = C(a) + a\frac{C(a)}{B - a} = B\frac{C(a)}{B - a} = B \cdot P(S_T > a) = B \cdot P(S_{T/2} = B),$$

and $\{S_{T/2} = \gamma\} = \{S_T \leq a\}$. So

$$E[S_T \mathbb{1}_{\{S_{T/2}=\gamma\}}] = E[S_T(1 - \mathbb{1}_{\{S_T>a\}})] = S_0 - \frac{BC(a)}{B-a},$$

which implies $E(S_T) = S_0$ and $E[S_T \mathbb{1}_{\{S_{T/2}=\gamma\}}]$ is equal to

$$S_0 - B\frac{S_0-\gamma}{B-\gamma} = \frac{\gamma(B-S_0)}{B-\gamma} = \gamma(1 - \frac{S_0-\gamma}{B-\gamma}) = \gamma P(S_{T/2} = \gamma)\ .$$

Then we find that $E[S_T|\mathbb{F}_{T/2}] = S_{T/2}$, which means $E[S_{T/2}] = E[S_T]$. And also S is a martingale since $E[S_{T/2}|\mathbb{F}_0] = E[S_{T/2}] = S_0 = E[S_t]$. With the law p of S_T, $\{H_B \leq 1\} = \{S_{T/2} = B\} = \{S_T > a\}$ and $\mathbb{Q}(S_T > a) = -C'(a) = \frac{C(a)}{B-a}$, we conclude $P(D) = U$ in this case and thus U is the optimal upper bound.

Proposition 3.3.3. *For $B < S_0$ the optimal value bound for this Digital option, with payoff $\mathbb{1}_{\{H_B\leq 1\}}$ and $H_B = \inf\{t : S_t \leq B < S_0\}$, is[13]*

$$\left[1 + C'(B), \frac{P(d)}{d-B}\right],$$

where d is the point where $\inf_{K>B} \frac{P(K)}{K-B}$ is attained, and $P(K)$ is the price of the Put option on the same underlying with strike K.

Note that the upper and lower bounds we obtain are fully model-free. None assumption of underlying asset's behaviour is needed since the dynamics of S is built up by the market information, i.e. Call option prices on the market.

3.4 Barrier options

In the previous section, we have developed the optimal upper and lower bounds for the price of Digital options. Let us now derive the value bounds for another path-dependent derivative, namely single Barrier options. We will particularly focus on the upper bounds for up-and-in and up-and-out Calls and Puts since the prices of down-and-in and down-and-out Barrier options could be driven by 'up' Barriers[13], and the lower bounds can be obtained by utilising the 'in-and-out' parity.

Similar to the technique we used in the previous section, we aim to obtain the optimal value bounds for Barrier options via the construction of a super-replicating strategy which bounds the payoff. Then the optimal results can be proven by introducing the martingale property such that the bounds can be actually obtained. In this section, only proofs of optimal value bounds for certain Barrier options will be presented, proofs of rest of value bounds could be found in detail in [13].

3.4.1 Upper bounds of Barrier Call options

Up-and-in

We keep H_B to be the same as the previous section. The payoff of the up-and-in Call Barrier with strike K and maturity T = 1 is the same as that of a Call if the price of an underlying asset crosses the Barrier B before T, and is zero otherwise, i.e.

$$(S_T - K)^+ \mathbb{1}_{\{H_B \leq T\}}.$$

Note that if $K \geq B$, then the payoff is simply the payoff of a Call. So we will consider the case $B > K$. Recall that $a(p, B)$ is the point where the infimum $\inf_{K<B} \frac{C(K)}{B-K}$ can be reached.

Proposition 3.4.1. *For $K < B$, the optimal upper bound for the price of the up-and-in Barrier Call option is*

$$\begin{cases} C(K) & , \text{if } a(p,B) \le K \\ \frac{B-K}{B-a} \cdot C(a) & , \text{otherwise.} \end{cases}$$

Proof. We construct a super-replicating strategy: long $(B-K)/(B-\beta)$ unit of Call option with strike β and short $(\beta - K)/(B-\beta)$ unit of forward contract of the underlying asset whenever the underlying crosses the Barrier. So we have

$$(S_1 - K)^+ \mathbb{1}_{\{H_B \le 1\}} \le \frac{(B-K)}{(B-\beta)}(S_1 - \beta)^+ + \frac{(\beta - K)}{(B-\beta)}(B - S_1)\mathbb{1}_{\{H_B \le 1\}}, \quad \forall \beta \in (K, B). \quad (3.4.1)$$

This is because if $(S_1 \le K)$ or $(H_B > 1)$, then the left-hand side term reduces to zero and the right-hand side term is positive. Also, if $(S_1 > K)$ and $(H_B \le 1)$, then (3.4.1) becomes the equality for $(S_1 \ge \beta)$. For $K < S_1 < \beta$ we then have the strict inequality.

We denote the price of this Barrier option to be $C^{Barrier}(B)$, and know the cost of a forward contract is zero. By taking expectations we obtain

$$C^{Barrier}(B) \le \frac{(B-K)}{B-\beta} \cdot C(\beta) + 0.$$

When we examined the upper bound in Proposition 3.3.2, we know the infimum of the first term on the right-hand side above can be attained at $\beta \equiv a(p, B) \vee K$. According to Brown et al.[13], we could build the price process S, which is a martingale, then we have

$$(S_1 > a) \subseteq (H_B \le 1) \subseteq (S_1 \ge a) \ ,$$

If $a < K$ then $(S_1 - K)^+ \mathbb{1}_{\{H_B \le 1\}} = (S_1 - K)^+$ and the equality holds in (3.4.1). If $a \ge K$ then $(S_1 - a)^+ = 0$ on $(H_B > 1)$ and

$$\frac{(B-K)}{(B-a)}(S_1 - a)^+ + \frac{(a-K)}{(B-a)}(B - S_1) = (S_1 - K)$$

on $(H_B \le 1)$. So we obtain the equality for $\beta = a$ and for all possible values of S_1 in (3.4.1). Therefore we have shown that there is a price process for which the expected payoff of the option equals the value bound we have advanced. □

Up-and-out

The payoff of the up-and-out Barrier Call option is

$$(S_T - K)^+ \mathbb{1}_{\{H_B \ge T\}}.$$

For $K \ge B$ the option will be knocked out for a positive payoff, so option always has zero value.

Proposition 3.4.2. *For $K < B$, the optimal upper bound for the price of the up-and-out Barrier Call option is*

$$C(K) - C(B) - (B-K)p([B, \ \infty)) = C(K) - C(B) + (B-K)C'(B)$$

And here we only consider the case that the price process of the underlying is discrete.

3.4.2 Upper bounds of Barrier Put options

Up-and-in

In this section, the optimal upper bounds of Barrier Put options will be presented. The payoff of an up-and-in Put option with strike K and maturity T = 1 is

$$(K - S_T)^+ \mathbb{1}_{\{H_B \leq T\}}.$$

We will use the fact that the supremum $\sup_{K<B} \frac{C(B)-P(K)}{B-K}$ can be attained at point $\alpha(p, B)$ [13].

Proposition 3.4.3. *For $K < B$, the optimal upper bound for the price of the up-and-in Barrier Put option is*

$$\begin{cases} P(K) & , \text{if } \alpha(p,B) > K \\ \frac{K-\alpha}{B-\alpha} \cdot C(B) + \frac{B-K}{B-\alpha} \cdot P(\alpha) & , \text{otherwise}, \end{cases}$$

where $P(K)$ is the price of the Put option with strike K. For $B \leq K$, the optimal upper bound is

$$C(K) + \frac{K-B}{B-a} \cdot C(a).$$

Proof. For $K < B$, we build a super-replicating strategy: long $\frac{K-\beta}{B-\beta}$ unit of a Call option with strike B and $\frac{B-K}{B-\beta}$ unit of a Put option with strike β, also short $\frac{\beta-K}{B-\beta}$ unit of a forward contract of the underlying asset whenever the underlying crosses the Barrier. So we have

$$(K - S_1)^+ \mathbb{1}_{\{H_B \leq 1\}} \leq \frac{(K-\beta)}{(B-\beta)}(S_1 - B)^+ + \frac{(B-K)}{(B-\beta)}(\beta - S_1)^+ + \frac{(K-\beta)}{(B-\beta)}(B - S_1)\mathbb{1}_{\{H_B \leq 1\}}. \quad (3.4.2)$$

By taking the expectation on both sides, the construction of martingale S and choice of $\beta = \alpha \vee K$, we obtain the desired result, including equality.

For $B \leq K$, we transfer (3.4.2) into the following inequality[13]

$$(K - S_1)^+ \mathbb{1}_{\{H_B \leq 1\}} \leq (S_1 - K)^+ + \frac{(K-B)}{(B-\beta)}(S_1 - \beta)^+ + \frac{(K-\beta)}{(B-\beta)}(B - S_1)\mathbb{1}_{\{H_B \leq 1\}}. \quad (3.4.3)$$

The optimal choice of β is $\beta = \alpha$. By constructing a martingale S as we did in the proof of Proposition 3.3.2, we can show the equality in (3.4.3). □

Up-and-out

The payoff of an up-and-out Put option is

$$(K - S_T)^+ \mathbb{1}_{\{H_B > T\}}.$$

Proposition 3.4.4. *For $K < B$, the optimal upper bound for the price of the up-and-in Barrier Put option is $P(K)$. For $B \leq K$, the bound is*

$$P(B) + (K - B)(1 + C'(B)).$$

3.4.3 Lower bounds of Barrier options

Let us recall the 'in-and-out' parity.

$$(S_T - K)^+ \mathbb{1}_{\{H_B \leq T\}} = (S_T - K)^+ - (S_T - K)^+ \mathbb{1}_{\{H_B > T\}} \quad (3.4.4)$$

If we substitute (3.4.4) into the left-hand side of (3.4.1), we obtain

$$(S_1 - K)^+ - (S_1 - K)^+ \mathbb{1}_{\{H_B > 1\}} \leq \frac{(B-K)}{(B-\beta)}(S_1 - \beta)^+ + \frac{(\beta-K)}{(B-\beta)}(B - S_1)\mathbb{1}_{\{H_B \leq 1\}}$$

By taking the expectation on both sides and the knowledge of minimum point of $\frac{B-K}{B-\beta} \cdot C(\beta)$, we obtain

$$C(K) - \frac{B-K}{B-a} \cdot C(a) \leq \mathbb{E}[(S_1 - K)^+ \mathbb{1}_{\{H_B \leq 1\}}]$$

Thus, we deduce the optimal lower bound for the up-and-out Barrier Call option. A similar technique can also obtain the bound for the up-and-in.

Proposition 3.4.5. *For $K < B$, the optimal lower bound for the price of the up-and-in Barrier Call option is*

$$C(B) - (B-K)C'(B).$$

For $B \leq K$, the lower bound is $C(K)$.

Note that the up-and-out Call becomes worthless when $K \geq B$. Furthermore, if $a \leq K < B$, then the lower bound we deduced becomes negative. So the lower bound in these two cases is 0.

Proposition 3.4.6. *For $K < a < B$, the optimal lower bound for the price of the up-and-out Barrier Call option is*

$$C(K) - \frac{B-K}{B-a} \cdot C(a)$$

Similarly, to derive optimal lower bounds for Barrier Puts, we use the following 'in-and-out' parity:

$$(K - S_T)^+ \mathbb{1}_{\{H_B \leq T\}} = (K - S_T)^+ - (K - S_T)^+ \mathbb{1}_{\{H_B > T\}}$$

Proposition 3.4.7. *The optimal lower bound for the price of the up-and-in Barrier Put option is[13]*

$$\begin{cases} 0 & \text{, if } K < B, \\ P(K) - P(B) - (K-B)(1 + C'(B)) & \text{, if } B \leq K. \end{cases}$$

The optimal lower bound for the price of the up-and-out Barrier Put option for $K < B$ is

$$\begin{cases} 0 & \text{, if } K < \alpha, \\ P(K) - \frac{K-\alpha}{B-\alpha} \cdot C(B) - \frac{B-K}{B-\alpha} \cdot P(\alpha) & \text{, otherwise.} \end{cases}$$

And when $B \leq K$, the lower bound is

$$K - S_0 - \frac{K-B}{B-a} \cdot C(a).$$

3.4.4 Optimal bounds for other kind of Barrier options

In addition, we will summarise the optimal value bounds for other kinds of Barrier options in discrete sense, according to Brown et al.[13].

Lemma 3.4.8. *For $B < S_0$ and $H_B = \inf\{t : S_t \leq B < S_0\}$, we define $d(p, B)$ and $\delta(p, B)$ to be the points where the infimum $inf_{K>B} \frac{P(K)}{K-B}$ and $inf_{K>B} \frac{P(B)-C(K)}{K-B}$ can be obtained respectively, then we have*

$$\mathbb{P}(H_B \leq 1) \leq \frac{P(d)}{d-B},$$

and

$$\mathbb{P}(H_B \leq 1) \geq \mathbb{P}(S_1 \leq B) + \frac{P(B) - C(\delta)}{\delta - B}.$$

Proposition 3.4.9. *The optimal upper and lower value bounds for 'down-and-in' and 'down-and-out' Barrier Call and Put options are as follows:*

(i) The optimal value bound for 'down-and-in' Call option for $K \leq B$ is

$$\left[C(K) - C(B) + (B-K)\cdot C'(B), P(K) + \frac{B-K}{d-B}\cdot P(d)\right].$$

For $B < K$ the bound is

$$\left[0, \frac{(\delta \vee K) - K}{(\delta \vee K) - B}\cdot P(B) + \frac{K-B}{(\delta \vee K) - B}\cdot C(\delta \vee K)\right].$$

(ii) The optimal value bound for 'down-and-out' Call option is: for $K < B$,

$$\left[S_0 - K - \frac{B-K}{d-B}\cdot P(d), C(B) - (B-K)\cdot C'(B)\right],$$

for $B \leq K$

$$\left[C(K) - \frac{(\delta \vee K) - K}{(\delta \vee K) - B}\cdot P(B) - \frac{K-B}{(\delta \vee K) - B}\cdot C(\delta \vee K), C(K)\right].$$

(iii) For $B < K$, the optimal value bound for 'down-and-in' Put option is

$$\left[P(B) + (K-B)(1 + C'(B)), \frac{K-B}{(d \wedge K) - B}\cdot P(d \wedge K)\right].$$

For $K \leq B$, the price of 'down-and-in' simply becomes the price of a put.

(iv) For $B < K$, the optimal value bound for 'down-and-out' Put option is

$$\left[P(K) - \frac{K-B}{(d \wedge K) - B}\cdot P(d \wedge K), P(K) - P(B) - (K-B)(1 + C'(B))\right].$$

For $K \leq B$, the price of 'down-and-in' reduces to zero.

3.5 Numerical Implementations

3.5.1 Introduction

In this section, we will numerically implement the value bounds for some path-dependent options. We initialise the condition that the underlying asset price $S_0 = 2.5$ and the maturity of options $T = 1$ year, with zero interest rate and no dividend.

We will first construct an arbitrage-free SVI volatility surface introduced by Gatheral and Jacquier[17], then compute the Black-Scholes Call and Put prices, which we assume to be the market prices, in order to obtain the value bounds for those options based on theoretical results in the previous sections. Then we will use Monte-Carlo simulation to compute the values of options under Black-Scholes model, and CEV(constant elasticity of variance) model as Browns et al. did in [13], thereby examining whether values actually lie within the bounds obtained.

3.5.2 Barrier options

Black-Scholes Model

Since the interest rate is assumed to be zero, we will first use the following Black-Scholes model to simulate the Barrier option prices.

$$dS_t = \sigma(K)S_t dt \tag{3.5.1}$$

Before computing the optimal value bounds, we need the volatilities in (3.5.1) and use them to simulate Call and Put prices with different strikes. We follow Gatheral and Jacquier[alt.01], who introduced the SVI volatility surface, to generate arbitrage-free volatilities which exhibit similar behaviours as the market, as shown below. We assume these values are true volatilities of the prices of the underlying asset with different strikes in the market.

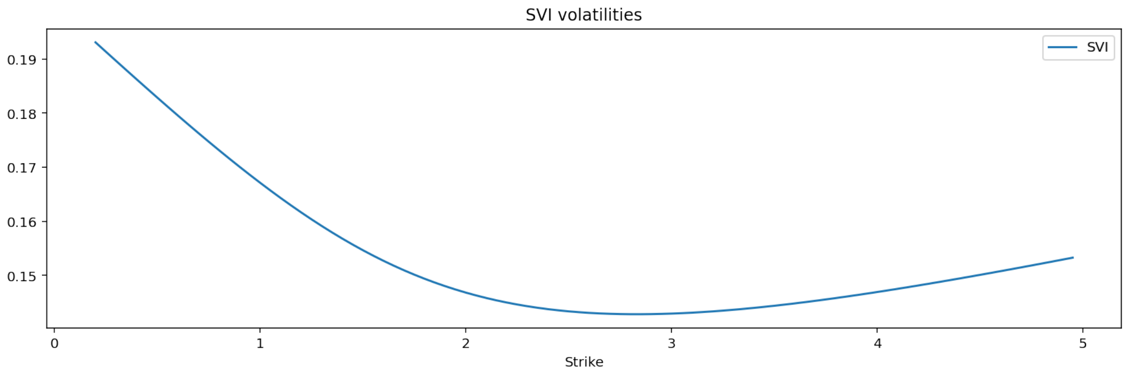

Figure 3.1: SVI arbitrage-free volatilities.

Then we use the Black-Scholes analytical solutions to compute the Call and Put option prices with the corresponding strikes and volatilities we obtained, which are assumed to be real market prices. Note that the purpose of this numerical implementation is to examine whether the optimal bounds we derived in previous sections are valid under some models. So even if the market prices are not 'real', the effectiveness of value bound can still be proven as long as the environment where we are working represents the similar behaviours as the real market and is rational.

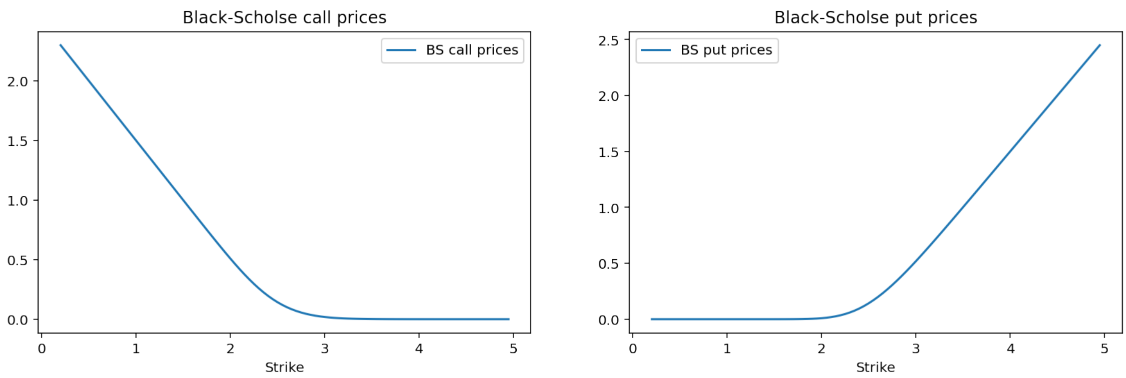

Figure 3.2: Black-Scholes Call and Put option prices.

Now we use the Monte-Carlo simulation to generate the 'up-and-in' and 'up-and-out' Barrier Call and Put options respectively. The time of simulations is $M = 100000$, and the time step is $1/250$ year. We define the initial underlying asset to be $S_0 = 2.5$ and Barrier $B = 2.6$. At the same time we calculate $C'(B)$ by central difference method, a and α by solving $\arg\min_{K<B} \frac{C(K)}{B-K}$ and $\arg\max_{K<B} \frac{C(B)-P(K)}{B-K}$ respectively. Then we use the formulas in Section 3.4 to compute the theoretical optimal upper and lower bounds for the values of those Barrier options. The results are shown in Figure 3.3.

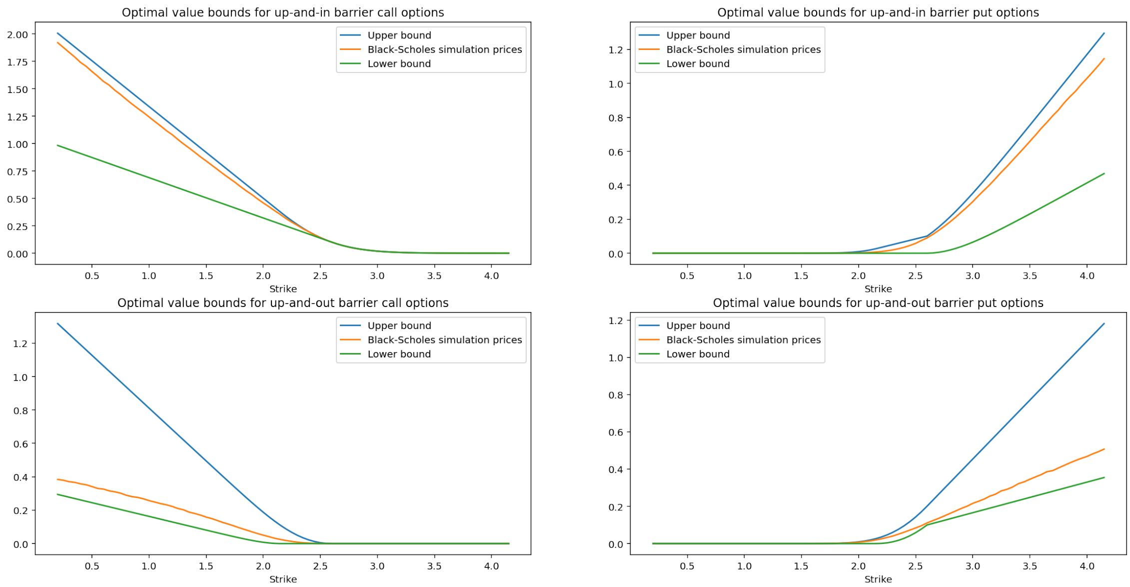

Figure 3.3: Optimal value bounds for 'up-and-in' and 'up-and-out' Calls and Puts under BS model.

In Figure 3.3 we observe that the prices simulated under the Black-Scholes model lie perfectly well within the theoretical value bounds. We notice that the upper and lower bounds coincide when $K > B$ for 'up-and-in' Call, this is because the option reduces to a simple Call for $K > B$. Moreover, the 'up-and-out' Call is knocked out when $K > B$. Other seemingly coincidences of value bounds for two Barrier Puts result from small values of these bounds, which are close to zero. We also consider the case that the volatility that we use to price options is not consistent with the market information. For example, we assume that the volatility is always a constant $\sigma = 0.2$ and never changes, and use it price all Barrier options above with different strikes, then check the connection between the optimal bounds and the simulation results.

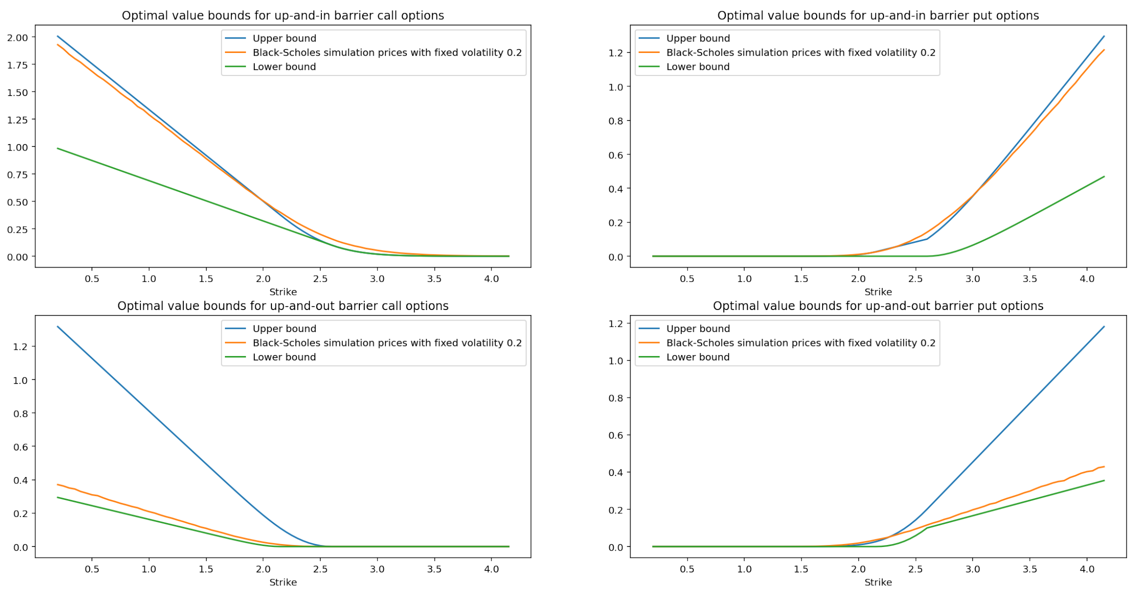

Figure 3.4: Optimal value bounds for 'up-and-in' and 'up-and-out' Calls and Puts under BS model with constant volatility $\sigma = 0.2$.

As we can see, the prices simulated with constant volatility more or less cross the theoretical optimal value bounds. This means, for instance, some arbitrage opportunities appear when the simulated price is above the optimal upper bound because we overprice the option. So volatilities we use should always be consistent with the market information.

CEV model

We now use another model which was introduced by [16], called Constant elasticity of variance model(CEV):

$$dS_t = \sigma(K)\sqrt{S_t}dt,$$

where the elasticity is chosen to be -1. Similarly, we use this model to simulate the prices of four Barrier options above, then compare these value with the corresponding theoretical optimal value bounds. The results are summarised in Figure 3.5.

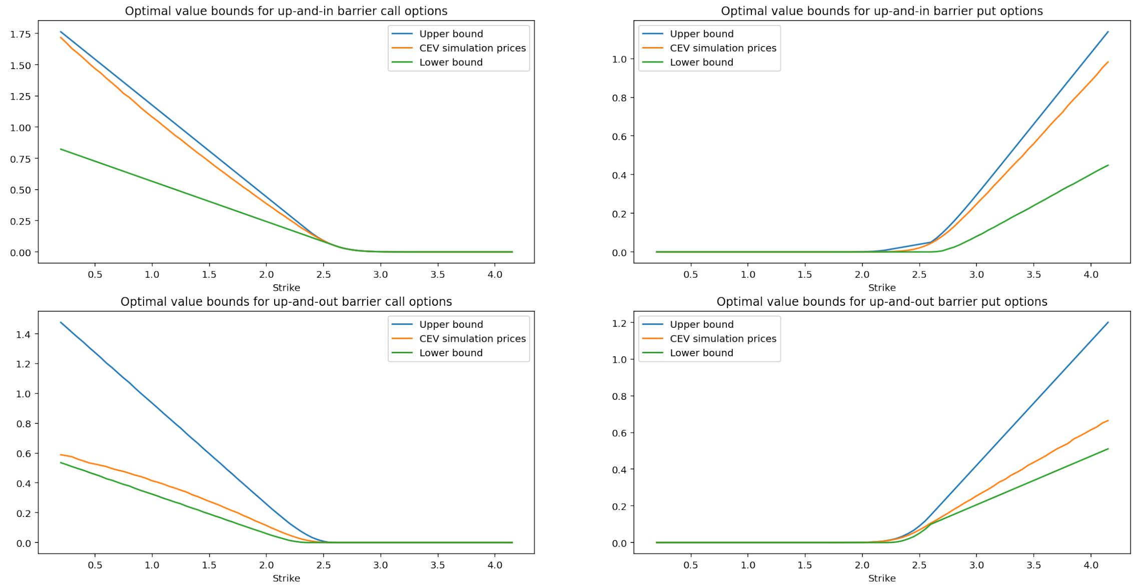

Figure 3.5: Optimal value bounds for 'up-and-in' and 'up-and-out' Calls and Puts under CEV model.

Optimal value bounds for other Barrier options

Moreover, we also simulate the prices of 'down-and-in' and 'down-and-out' Barrier Call and Put options respectively under the BS model, and compute their optimal value bounds according to Proposition 3.4.9, for the sake of completeness. The Barrier we set for those options is $B = 2.4$.

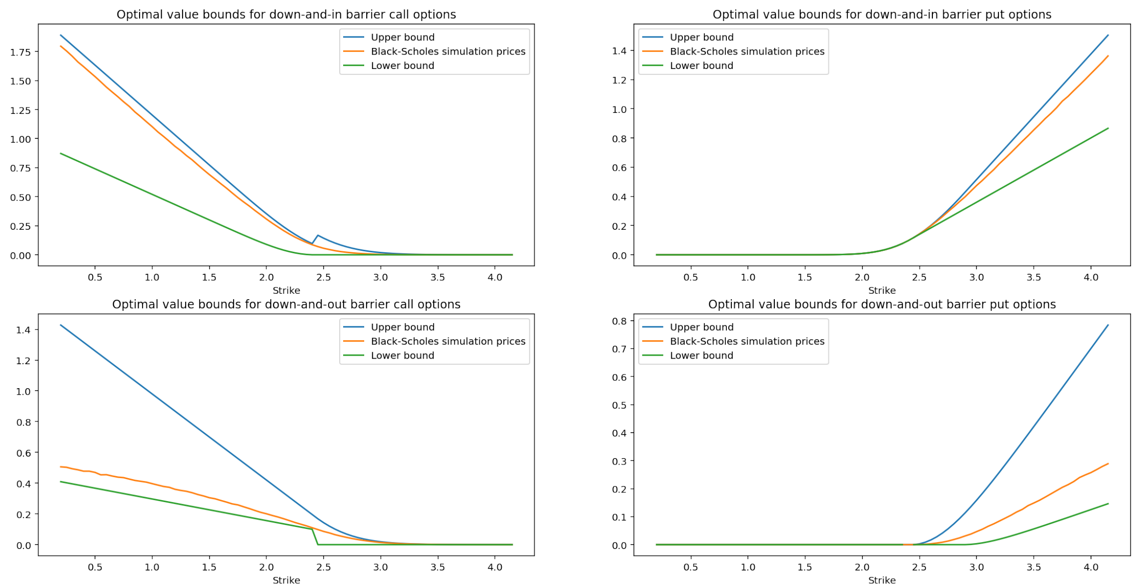

Figure 3.6: Optimal value bounds for 'down-and-in' and 'down-and-out' Calls and Puts under BS model.

It is obvious to see the prices lie within the bound. The reason for the coincidence of upper and lower bounds for 'down-and-in' Barrier Put option is that for $K < B$, the Barrier option reduces to a simple Put option.

3.5.3 Digital option

We also compute the optimal upper and lower bounds for Digital options in two cases that $2.5 = S_0 < B = 2.6$ and $S_0 > B = 2.4$, according to Section 3.3. The prices of options are again simulated by the Black-Scholes model. As we can see below, the prices of Digital options lie within the theoretical optimal value bounds.

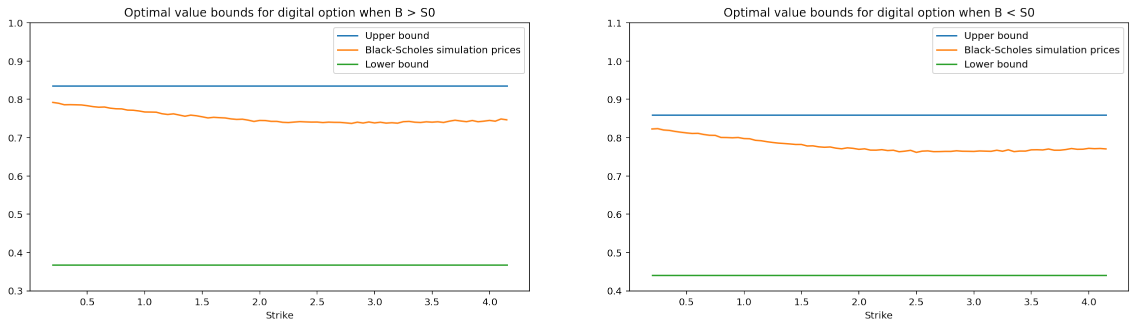

Figure 3.7: Optimal value bounds for Digital options under BS model for $S_0 > B$ and $S_0 < B$.

3.6 Conclusion

In this chapter, we demonstrated the methods of robust pricing for Lookback, Digital and Barrier path-dependent options. By utilising the Vanilla prices in the market, we are able to produce the optimal value bounds for these options. Then, some of these theoretical value bounds were verified under the Black-Scholes and CEV models by numerical implementations.

Also, the consequence of using constant volatility that is independent of market information was presented in an example. Therefore we generated volatilities based on SSVI to examine this model-indenpendent method.

Conclusion

In this thesis, we studied three methods of robust pricing. Although a few assumptions were still proposed, these methods have significantly reduced the dependence of obtained value bounds on deterministic models, thereby reducing the model risk. We first explored how to derive optimal upper and lower bounds for derivatives by assuming a volatility band and utilising BSB equation. Then we introduced the Lagrangian UVM to find the optimal combination of hedging instruments for the best possible value bounds. The properties and theorems of optimal transport problems and martingale optimal transport were discussed, so we were able to reduce the MOT into linear programs: the primal and the dual problem. Then we studied the methods of solving LP to obtain optimal value bounds. Finally, the robust approaches to pricing path-dependent options such as the Lookback, Digital and Barrier options have also been studied.

The only inputs we needed to generate robust value bounds were market Vanilla prices since all of these methods are exogenous. In our numerical implementations, the prices of the actual derivatives indeed lied within the value bounds generated by these three methods. In the comparison between numerical performances of UVM and MOT, we found that the methods based on more assumptions may produce a better/narrower value bounds. In contrast with MOT, which needs fewer assumptions, UVM requires that the actual volatility lies within the predetermined volatility band and the discounted underlying price process is a true martingale under some EMMs. So we should carefully balance the robustness and model assumptions when pricing derivatives, since narrower value bounds are based on the assumptions that may be violated and not correctly describe the real world, while too wide bounds are without any practical use.

The additional study could be put on the calibration of the volatility band in UVM since the width of the band indeed influence the final results. The MOT primal problem is difficult to solve since the quantisation points of implied probability measures obtained from the market may not have the same mean and the implied probability marginals may not be in a martingale measure. So possible alternative numerical methods of solving the MOT dual worth investigating. Moreover, the numerical methods discussed in this thesis are mostly in the discrete sense, so further research on their continuous version and the corresponding effects may be beneficial to the field of robust option pricing.

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