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Use of Kernel Methods for Dynamic Hedging Incomplete Markets

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Declaration

The work contained in this thesis is my own work unless otherwise stated.

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Abstract

Hedging, an indispensable part of risk management, is a strategy that tries to limit risks in financial assets. Typical hedging techniques involve taking opposite positions in derivatives that correspond to an existing position. When making predictions of future value movements of financial derivatives, the model assumption of the underlying asset needs to be imposed, leading to a non-portable and model-specified hedging strategy. In this paper, by drawing inspiration from Black-Scholes discrete-time hedging and Q-learning framework, a model-free hedging strategy is proposed and computed by Dynamically Controlled Kernel Estimation (DCKE), a start-of-the-art framework for pricing and hedging. The newly proposed hedging strategy is a function of hedging assets value and the value of the financial derivatives to be hedged with no constraint of the model assumptions of them. As a result, it also works in the incomplete market where the underlying asset is not tradable and the financial derivative needs to be hedged by a partially correlated asset. Moreover, as it does not constrain the target derivative product, it can be used to hedge not only options but also xVA in the incomplete market.

Contents

List of Figures

List of Tables

Introduction

This paper aims to propose a model-free methodology based on kernel estimation to dynamically hedge the incomplete market where the perfect hedging is not possible. In the financial industry, hedging is an indispensable part of risk management. It is a way of risk management strategy that aims at reducing the risk of loss caused by contingent future price movement. By holding the opposite position in the related assets, the potential losses in investment can be offset to some extent. In the complete and arbitrage-free market, referring to the celebrated Black-Scholes-Merton model, there exists a unique risk-neutral measure under which options have a unique no-arbitrage price, and all the risks could be perfectly hedged by continuously trading the underlying asset and the risk-free bank account in a self-financing way. The unique price is computed by the discounted payoff under the risk-neutral measure, and there is no risk of mismatch between the hedging portfolio value and the true option value. The position of the underlying asset could be determined by the first-order derivative of the option value to the underlying asset value, namely the delta-hedge.

However, the perfect hedge is not operable in the real world, and there are several reasons for that.

- The perfect hedge is only possible in the complete market, where all risks are directly linked to tradable assets, and the real market is far from being complete. In the real world, a hedging strategy can only be operated at a discrete-time. The removal of the continuous-trading assumption would expose the trader to the risk of hedging error.
- In the incomplete market setting, pricing and hedging an option is no longer a no-arbitrage problem, instead, it is a trade-off between the expected utility and the extra risks needed to be taken. The attitude towards risk exerts an impact on the hedging strategy through a risk-averse parameter λ. And hedging strategy does not totally depend on the determination of option prices. The maximization of expected utility/ the minimization of local variance replaces the role of the no-arbitrage principle.
- Even if the perfect hedge is attainable, it is not desirable. The reduction in risk provided by hedging also typically results in a reduction in potential profits. Even if the perfect hedge is possible, the potential profit would be eliminated as well.

As a result, the perfect hedging is neither attainable nor desirable in the incomplete market. And this leads to the problem we try to solve: how can we effectively pricing and hedging in the incomplete market?

Looking closer, the crucial part for the hedger to construct an effective hedging strategy is making precise estimation about the future value change with the information available at present, which necessitates the use of conditional expectation. In this paper, we employ the dynamically controlled kernel estimation (DCKE) to compute the conditional expectation. By using that, the option pricing and delta hedge can be computed at the same time. In the industry, Least Square Monte Carlo (LSMC) is commonly used to compute conditional expectation. DCKE outperforms existing methods in terms of convergence speed and the estimation in delta and "tails". On top of traditional kernel estimation, control variates are added to DCKE.

In this paper, we aim to propose an effective and model-free method for pricing and hedging in the incomplete market. This paper is structured as follows. Chapter 1 starts from the economic set-ups of the incomplete market and an optimal hedging strategy in terms of variance minimization is proposed under the Q-learning framework, following by the introduction of two methodologies that help compute the hedging strategy by giving an estimation of conditional expectation. The fitting results and features of them are discussed. Chapter 2 applies the outstanding DCKE in the incomplete market. We first assume the underlying asset follows the Geometric Brownian motion (GBM) and Ornstein–Uhlenbeck (OU) process, and the results are compared with the closed-form formula to illustrate that our results are valid without the restriction of model choices. Followed by justifying the model-free property of the newly proposed hedging strategy, more complex models are applied. Copulas, a data-driven model which does not have a closed-form hedging solution is considered, and the corresponding hedging strategy is computed. In Chapter 3, we switch the focus toward CVA and aim to find a hedging strategy for CVA in the incomplete market. We employ the default probability approach to evaluate counterparty credit risk and compute the newly proposed strategy under simple model assumptions, the result of which is then compared with the benchmark provided by closed-form solutions for specified models.

Chapter 1

Set-ups and Methodologies

We will first set up a simple incomplete market model based on simple options writing on an underlying asset but hedging with another partially correlated tradable asset. Taking the risk averse parameter into consideration, a general hedging strategy is derived with the help of Q-learning framework. And then a commonly used method in the industry, Least Squares Monte Carlo (LSM), is introduced along with the brief analysis of its fitting results and drawbacks. After that, dynamically controlled kernel estimation, a brand new methodology that outperforms LSM in terms of convergence speed and fitting at tails, is discussed.

1.1 Set-ups and Q-equation in the incomplete market

Suppose there is an option writing on a non-tradable underlying asset S, and there exists another tradable asset H which is partially correlated to the asset S. To hedge the option, a hedge portfolio composed of a tradable asset H and a risk-free bank account B is set up. At time t, the value of the hedge portfolio is:

$$\Pi_t = x_t H_t + B_t$$

, where x_t is the position taken in the asset H to hedge the option at time t.
The aim is to match the value of hedge portfolio to the value of option as close as possible. At maturity, these two values could be set identically. In addition, at maturity, the position in the option would be closed, as the option would either be exercised or not. And the position in the asset H should be closed along with the option (i.e. $x_T = 0$), altering the contingent value of portfolio into certain wealth. Thus we have the boundary condition at maturity T

$$\Pi_T = B_T = G_T(S_T) \tag{1.1.1}$$

, where $G_T(\cdot)$ is the payoff function.
Before maturity, the portfolio value Π_t is not measurable, but can be computed backward recursively using the restriction of hedging portfolio being self-financing. From time t to time $t+1$, the bank account would earn the interest at the risk-free interest rate r, changing from B_t to $e^{r\Delta t}B_t$. With the position in the asset H unchanged, the value of portfolio Π_{t+1} right before the next re-balancing is then $x_t H_{t+1} + e^{r\Delta t}B_t$. And the portfolio value at time $t+1$ right after the re-balancing is $B_{t+1} + x_{t+1}H_{t+1}$. As the hedging strategy is self-financing, equating these two values, we have

$$e^{r\Delta t}B_t + x_t H_{t+1} = B_{t+1} + x_{t+1}H_{t+1}$$

Re-arranging the equation and combine with the terminal condition B_T, the bank account at time t can be computed backward recursively:

$$B_t = e^{-r\Delta t}[B_{t+1} + x_{t+1}H_{t+1} - x_t H_{t+1}] \text{ , } t = T-1, \ldots, 0$$

In order to compute the hedging portfolio value Π_t at $t < T$, add $x_t H_t$ to both sides and get the following recursive equation for Π_t,

$$\begin{aligned}\implies \Pi_t &= e^{-r\Delta t}[B_{t+1} + x_{t+1}H_{t+1} - x_t H_{t+1} + e^{r\Delta t} x_t H_t] \\ &= e^{-r\Delta t}[\Pi_{t+1} - x_t(H_{t+1} - e^{r\Delta t} H_t)] \\ &=: e^{-r\Delta t}[\Pi_{t+1} - x_t \Delta H_t] \text{ , where } \Delta H_t = H_{t+1} - e^{r\Delta t} H_t \end{aligned} \tag{1.1.2}$$

A recursive equation of portfolio value Π_t is gotten as above. Combined with the boundary condition (1.1.1), the portfolio value could be computed step by step backward.

To find the optimal hedging strategy, we need to think about the purpose of hedging. Implementing a hedging strategy is to reduce the future risks the traders are expose to. The hedging strategy $\{x_t\}_{t=0}^T$ that we want is the one minimizes the variance of hedging portfolio value Π_t. Employing the recursive relationship of portfolio value, we get the following result. We first compute the conditional variance of variance

$$\begin{aligned} Var(\Pi_t|\mathscr{F}_t) &= Var(e^{-r\Delta t}[\Pi_{t+1} - x_t \Delta H_t]|\mathscr{F}_t) \\ &= Var(\Pi_{t+1}|\mathscr{F}_t) + x_t^2 Var(\Delta H_t|\mathscr{F}_t) - 2x_t Cov(\Pi_{t+1}, \Delta H_t|\mathscr{F}_t) \end{aligned} \tag{1.1.3}$$

Computing the first order derivative of (1.1.3), the optimal hedge would be obtained as follows.

$$\begin{aligned} x_t^* &= \arg\min Var(\Pi_t|\mathscr{F}_t) \\ &= \arg\min (Var(\Pi_{t+1}|\mathscr{F}_t) + x_t^2 Var(\Delta H_t|\mathscr{F}_t) - 2x_t Cov(\Pi_{t+1}, \Delta H_t|\mathscr{F}_t)) \\ &= \frac{Cov(\Pi_{t+1}, \Delta H_t|\mathscr{F}_t)}{Var(\Delta H_t|\mathscr{F}_t)} \text{ , } t = T-1, \ldots, 0 \end{aligned} \tag{1.1.4}$$

We first define the fair price of option C_t as the expected value of hedging portfolio: $\hat{C}_t = E[\Pi_t|\mathscr{F}_t]$. It is worth noting that, in the incomplete market, pricing an option is no longer a no-arbitrage problem, but a trade-off between the expected return and the extra risk needs to be taken. As a result, the fair price is not same as the ask price. Extra risk premium needs to be added on the top of the fair price to compensate for the extra risk taken. With risk averse parameter λ, we can set the risk premium to be the risk averse parameter times the discounted variance of portfolio value in the future, and then the ask price becomes

$$C_t^{\text{ask}} = E_t[\Pi_t + \lambda \sum_{t'=t}^{T} e^{-r(t'-t)} Var(\Pi_{t'}|\mathscr{F}_{t'})|\mathscr{F}_t]$$

The object moves from minimizing the hedging error to minimizing the ask price, a more general hedging is set up. Notably, this hedge can be used for both speculating and hedging, instead of only aims at hedging. Finding the minimum ask price C_t^{ask} is same as finding the maximum of its negative. Maximizing $V_t := -C_t^{\text{ask}}$ then lies in the reinforcement learning area, therefore we state the problem using the language of Q-learning.

We now re-write the action value function to find the recursive relationship:

$$\begin{aligned} V_t &= E_t[-\Pi_t - \lambda \sum_{t'=t}^{T} Var(\Pi_{t'}|H_{t'} = h', x_{t'} = x')|H_t = h, x_t = x] \\ &= E_t[-\Pi_t - \lambda Var(\Pi_t|H_t = h, x_t = x) - \lambda \sum_{t'=t+1}^{T} Var(\Pi_{t'}|H_{t'} = h', x_{t'} = x')|H_t = h, x_t = x] \\ &= E_t[-\Pi_t - \lambda Var(\Pi_t|H_t = h, x_t = x) + e^{-r\Delta t}(V_{t+1} + E_{t+1}[\Pi_{t+1}|\mathscr{F}_{t+1}])|H_t = h, x_t = x] \end{aligned}$$

Replacing equation (1.1.2) back into the derivation and re-arranging the equation, we obtain the Bellman equation:

$$V_t = E_t[R(x_t, H_t, H_{t+1}) + e^{-r\Delta t} V_{t+1}|H_t = h, x_t = x] \text{ , } t = T-1, \ldots, 0 \tag{1.1.5}$$

, where the one-step reward is defined as

$$\begin{aligned} R(x_t, H_t, H_{t+1}) &= e^{-r\Delta t} x_t \Delta H_t - \lambda Var(\Pi_t|H_t = h, x_t = x) \\ &= e^{-r\Delta t} x_t \Delta H_t - \lambda e^{-2r\Delta t} E_t[\hat{\Pi}_{t+1}^2 - 2x_t \Delta \hat{H}_t + x_t^2 (\hat{H}_t)^2|H_t = h, x_t = x] \end{aligned}$$

, where $\hat{\Pi}_{t+1} := \Pi_{t+1} - \mathbb{E}[\Pi_{t+1}]$ and $\Delta\hat{H}_t := \Delta H_t - \mathbb{E}[\Delta H_t]$.

The action value function, also known as the Q function, defines the value of taking action x in state h under a policy π, denoted by $Q_\pi(h, x)$. Mathematically, $x_t = \pi(h_t, t)$. x_t is the action taken at time t, whereas $\pi(h_t, t)$ is the function mapping the state ($H_t = h, t = t$) to the action x_t.

We define the function $Q_t = \mathbb{E}[V_t|\mathscr{F}_t] = E_t[-\Pi_t - \lambda \sum_{t'=t}^{T} Var(\Pi_{t'}|\mathscr{F}_{t'})|\mathscr{F}_t]$ and aim to find the maximum value of it, setting up a value maximization problem.

The object function Q_t could be regarded as the expected value of the total reward over any and all successive steps, starting from the current state, proposed by Halperin (2019)[1].

By decomposing the value function into two parts, namely the immediate reward plus the discounted future values, the complex optimisation problem would be break into simple, recursive sub-problems to find optimal solution:

$$Q_t^* = \mathbb{E}_t[R_t + \gamma \max_{x_{t+1}} Q_{t+1}^* | H_t = h, x_t = x]$$

, with the terminal condition, $R_T = -\lambda Var(\Pi_T)$ and $Q_T = -\Pi_T - \lambda Var(\Pi_T)$.
Putting it back to the value function Q_t,

$$Q_t = E_t[e^{-r\Delta t} x_t \Delta H_t - \lambda e^{-2r\Delta t} E_t[\hat{\Pi}_{t+1}^2 - 2x_t \Delta\hat{H}_t + x_t^2 (\hat{H}_t)^2 | \mathscr{F}_t] + e^{-r\Delta t} Q_{t+1} | \mathscr{F}_t] \tag{1.1.6}$$

Observing the value function Q, it can be regarded as a quadratic function of position x_t. Computing the first order derivative and setting it to be zero, we can then get the optimal hedging strategy

$$x_t^* = \frac{E_t[\Delta\hat{H}_t \hat{\Pi}_{t+1} + \frac{1}{2\lambda e^{-r\Delta t}} \Delta H_t | \mathscr{F}_t]}{E_t[(\Delta\hat{H}_t)^2 | \mathscr{F}_t]} \tag{1.1.7}$$

The optimal hedging strategy presented above is a critical result for this paper. Note that during the derivation, the only condition used is that the hedging strategy is self-financing. As a result, the hedging strategy proposed can be used to hedging any product with contingent future value, such as option and CVA, which would be discussed in Chapter 1 and Chapter 2 respectively. To hedge a product, just set the product value being hedged equal to the expectation of hedging portfolio value at each time step.

When hedging an option, taking the limit as $\Delta t \to 0$, the optimal policy around time t could be obtained

$$\lim_{\Delta t \to 0} x_t^* = \frac{\partial \hat{C}_t}{\partial H_t} + \frac{\mu' - r}{2\lambda \sigma'^2} \frac{1}{H_t}$$

Note that if $\mu' = r$ or the risk averse parameter $\lambda \to \infty$, it converges to the local-risk minimization delta given by equation 1.1.4. Furthermore, when the partial correlation $\rho \to 1$, underlying asset and tradable asset being perfect correlated, it converges to the Black-Scholes delta.

Plugging equation 1.1.7 back to equation 1.1.6, a recursive formula for the optimal action-value function is obtained:

$$Q_t^* = \gamma \mathbb{E}_t[Q_{t+1}^* - \gamma\lambda \hat{\Pi}_{t+1}^2 + \gamma\lambda x^* 2 (\Delta\hat{H}_t)^2] \text{ , for } t = 0, \ldots, T-1$$

1.2 Methodologies

In this section, two methodologies of computing conditional expectation are discussed, namely Least Squares Monte Carlo (LSM) and Dynamically Controlled Kernel Estimation (DCKE). The regression enables us to estimate the pricing and hedging strategy along with the evolution of asset price, which provides new information with time passes by. The regression help avoid the necessity of nested Monte Carlo, leading to higher computational efficiency. LSM is a commonly used methodology for estimating price and hedging strategy conditional on the current asset price, the algorithm of which is introduced in the following sections. The analysis of its fitting results and shortcomings is presented as well. According to Kienitz (2021)[2], DCKE manages to address the drawbacks of LSM, proves a patent and efficient methodology. It is a combination of kernel regression, control variates and gaussian process regression. Each of them would be introduced, and this section is finished with the algorithm of DCKE.

1.2.1 Least Squares Monte Carlo

The Least Squares Monte Carlo approach, as per Carriere (1996)[3], Longstaff and Schwartz (2001)[4], is commonly used in the industry to do the pricing and hedging for financial derivatives by working out the conditional expectations. Least Squares Monte Carlo is an approximate dynamic programming approach used to price and manage options with early and multiple exercise opportunities. The underlying asset price evolves with time, which provides new information, leading the changes of option price and hedging strategy. LSM provides a convenient and efficient way to do the pricing and hedging at intermediate time step, avoiding the repeatedly simulation at each (i.e. avoiding the use of nested Monte Carlo simulation).

1.2.1.1 Least Squares regression

In the scenario of option pricing, there are two common applications of the regression: function approximation and variance minimization.

To define the setting for the function approximation, referring to Grau (2008)[5], there are several assumptions needed:

Assumption 1.2.1. A data set $(\mathbf{X}, \mathbf{y})$ is provided, $\mathbf{X} \in \mathcal{R}^{n,s}$,$\mathbf{y} \in \mathcal{R}^n$.

Assumption 1.2.2. The rows $\mathbf{x}^i \in \mathcal{R}^s$ are independent and identially distributed realizations of a random vector with a probability density function $p(\mathbf{x})$, which is non-zero everywhere on the cube $\mathcal{D} := [\mathbf{x}_{min}, \mathbf{x}_{max}]$, and zero outside.

Assumption 1.2.3. The provided values of $\mathbf{y}$ are noisy observations of $f(\mathbf{x}^i)$:

$$y^i = f(\mathbf{x}^i) + \epsilon^i \text{ , for } i = 1, \ldots, n$$

, where ϵ^i is random error with mean zero (i.e. $\mathbf{E}[\epsilon^i] = 0$) and is independent of $\mathbf{x}^i$.

Assumption 1.2.4. the function $f : \mathcal{R}^s \to \mathcal{R}$ has a representation $f(\mathbf{x}) = \sum_{j=1}^{\infty} a_j b_j(\mathbf{x})$, $\mathbf{x} \in \mathcal{R}^s$, where $b_j \in \mathcal{B}, j = 1, \ldots, \infty$ are bounded basis functions b_j: $\mathcal{R}^s \to \mathcal{R}$ of a vector space $\mathcal{B} \subset \mathcal{C}^1$ with $||b_j(\mathbf{x})||_\infty = c_j < \infty$, $\exists \mathbf{x} \in \mathcal{D} : |b_j(\mathbf{x})| > 0$, for $j = 1, \ldots, \infty$.

Let the assumptions above to be satisfied. The local basis approximation $\tilde{f}^m$ of the function f induced by the set of samples $(\mathbf{X}, \mathbf{y})$, $\mathbf{X} \in \mathcal{R}^{n,s}$,$\mathbf{y} \in \mathcal{R}^n$ with function space $\mathcal{B}^m$ spanned by the vasis function $b_i, \ldots, b_m \in \mathcal{B}$, is given by

$$\tilde{f}^m = \sum_{j=1}^{m} \tilde{a}_j^n b_j(\mathbf{x})$$

, where $\tilde{f}^m \in \mathcal{B}^m \in \mathcal{B}$, and $\tilde{a}_j^n \in \mathcal{R}$ for $j = 1, \ldots, m$.

And the coefficient vector $\tilde{\mathbf{a}}^m = A(\mathbf{X}, \mathbf{y})$, $\tilde{\mathbf{a}}^m = (\tilde{a}_1^n, \ldots, \tilde{a}_m^n)^T$ with a suitable function $A(\mathbf{X}, \mathbf{y})$ *iff*

$$\forall \epsilon \; \exists N(\epsilon) : \left\| \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} - \begin{pmatrix} \tilde{a}_1^n \\ \vdots \\ \tilde{a}_m^n \end{pmatrix} \right\| < \epsilon, \quad \forall n \geq N(\epsilon)$$

We then move to the question of how to find the function $A(\mathbf{X}, \mathbf{y})$, which determines the coefficient vector $\tilde{\mathbf{a}}^m$ given the noisy observations $\mathbf{y} \in \mathcal{R}^n$ and the basis functions of interest $b_j(\mathbf{x}) \in \mathcal{B}^m$, thus giving the local basis approximation. The result we provide by the following theorem.

Theorem 1.2.5. *Let the assumptions stated above be satisfied. The local basis approximation $\tilde{f}^m(\cdot)$ of function $f(\cdot)$ based on a set $(\mathbf{X}, \mathbf{y})$ of n noisy observations is given by*

$$\tilde{f}^m = \sum_{j=1}^{m} \tilde{a}_j^n b_j(\mathbf{x})$$

, where

$$\begin{aligned} \tilde{\mathbf{a}}^m = A(\mathbf{X}, \mathbf{y}) &= \arg\min_{\tilde{\mathbf{a}}^m} ||\mathbf{B}(\mathbf{X})^T \tilde{\mathbf{a}}^m - \mathbf{y}||_2 \\ &= (\mathbf{B}(\mathbf{X})^T \mathbf{B}(\mathbf{X}))^{-1} \mathbf{B}(\mathbf{X})^T \mathbf{y} \end{aligned} \tag{1.2.1}$$

, with

$$\mathbf{B}(\mathbf{X}) := \begin{pmatrix} b_1(\mathbf{x}^1) & \dots & b_m(\mathbf{x}^1) \\ \vdots & \ddots & \vdots \\ b_1(\mathbf{x}^n) & \dots & b_m(\mathbf{x}^n) \end{pmatrix}$$

Then $\lim_{m,n\to\infty} \tilde{f}^m(\mathbf{x}) = \mathbf{E}[\mathbf{y}|\mathbf{x}]$*, an unbiased estimator of the observations* $\mathbf{y}$ *given* $\mathbf{x}$*.*

1.2.1.2 Demonstration of LSM on a European call option

Here we explain the algorithm for LSM using a European call option (Algorithm 1). The details of path generation would be explained later in section 2.1.3:

Algorithm 1: LSM for hedging and pricing a European call option.

Data: strike price K; risk-free interest rate r; number of time steps T; length of time step Δt; number of paths n; number of samples u; price evolution sequence S_{t_i} with each length n for $i = 0, \dots, T$; option price sequence V_{t_i} with each length n for $i = 0, \dots, T$

Result: mesh points of spot price sequence M_{t_i}; estimation of option price sequence $\hat{V}_{t_i}$; delta hedge position $\hat{x}_{t_i}$ with each length n for $i = 0, \dots, T$

Set arrays: $V^j_{t_T} = (S^j_{t_T} - K)^+$ for $j = 1, \dots, n$; $\hat{V}_{t_i}$; $\hat{x}_{t_i}$;

for *at each time step* t_i *for* $i = T-1, \dots, 0$ *backward* **do**

- draw u samples from S_{t_i}: $M^j_{t_i}$ for $j = 1, \dots, u$
- get a_{ik} by fitting the regression using exact simulation: $\sum_k a_{ik} b_k(S^j_{t_i}) \approx e^{-r\Delta t} V^j_{t+1}$
- put sample into fitted regression model: $\hat{V}^j_{t_i} = \sum_k a_{ik} b_i(M^j_{t_i})$
- $\hat{V}^j_{t_i} = 1D - interpolation(M^j_{t_i}, \hat{V}^j_{t_i}; S^j_{t_i})$
- $V^j_{t_i} = e^{-r\Delta t} V^j_{t_{i+1}}$

hedging strategy $\hat{x}^j_{t_i} = \frac{\partial \sum_k a_{ik} b_i(M^j_{t_i})}{\partial M^j_{t_i}}$

Consider a European call option with parameters stated in Table 1.1, we use the Least Squares Monte Carlo with Legrendre basis function to do the pricing and hedging at intermediate time steps using the algorithm 1. The prices of option given by LSM and Black-Scholes formula at time $t = 0.25$ are shown in Figure 1.1, and the delta hedges of which are shown in Figure 1.2.

General Features	Values
Initial underlying stock price S_{t_0}	100
Strike price K	100
Risk-free rate r	0.01
Volatility for underlying stock σ	0.3
Maturity time T	1 year
Time step t_i	[0, 0.25, 0.5, 0.75, 1.0]
Number of path n	10000
Basis function $b(\mathbf{x})$	Legrendre
Polinomial degree k	10

Table 1.1: European call option parameters for LSM

In Figure 1.1, the resulting option value obtained by LSM is compared with the option value given by Black-Scholes pricing formula. It can be seen that the fitting error increases along with the increase of underlying asset price, suggesting that LSM gives a better fitting in the "out-of-money" case than in the "in-the-money" case. And the similar tail deviation can be observed in Figure 1.2, the large discrepancies lead to the motivation of finding a more efficient and accurate methodology. In the next section, a start-of-the-art methodology would be introduced, which manages to fix the drawbacks of LSM approach.

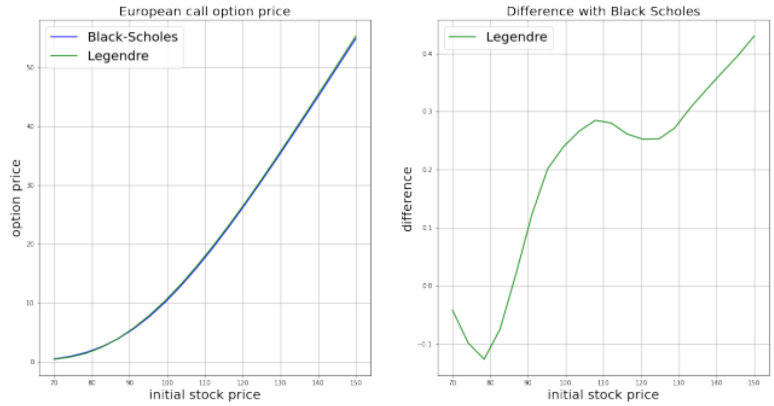


Figure 1.1: Pricing a European call option with LSM at $t = 0.25$

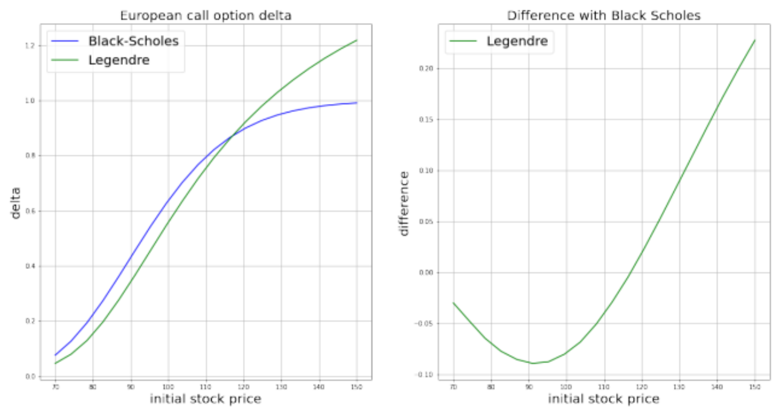


Figure 1.2: Delta hedge of a European call option with LSM at $t = 0.25$

1.2.2 Dynamically Controlled Kernel Estimation

Observing the optimal hedging strategy we get in equation (1.1.7), the key to computing the hedging strategy is to efficiently estimate the conditional expectation. Luckily, the dynamically controlled kernel estimation is there to help. It outperforms the existing methods such as Least Squares Monte Carlo in terms of convergence speed and fitting at the "tails". Dynamically controlled kernel estimation is a methodology developed based on the kernel regression, control variates and Gaussian Process regression. Each of them would be briefly discussed as follows.

1.2.2.1 Kernel regression

In statistics, the kernel estimation is a non-parametric way to estimate the probability density function of a random variable. The expectation of some variable Y conditional on the random variable X could be denoted as

$$\mathbb{E}[Y|X] = \mathbf{m}(X;Y)$$

,where $\mathbf{m}$ represents the kernel estimator.

Before stating what is the kernel estimation, the notion of local regression needs to be clarified. The local regression is a combination of moving average and polynomial regression, breaking the global regression into many polynomial regressions with smaller data set. The idea of it is to perform a weighted polynomial regression, in other words, to perform a regression around a point of interest using only training data that are "local" to that point, where the weights are determined by the kernel function of choice.

With given degree p, a polynomial regression is performed against y and $(x, x^2, \ldots, x^p)$. The Taylor polynomial of an α-th differentiable function f at point x is

$$T(f;x;p)(a) = \sum_{|\alpha|\leq p} \frac{\partial^\alpha f(x)}{\alpha!}(a-x)^\alpha$$

The coefficients of local regression are computed by minimizing the cost function

$$J_x(\beta) = \sum_{i=1}^{N}(T(f;x;p)(x_i) - y_i)^2\omega_i$$

, where the weights ω_i are usually chosen to be the exponential kernel $K(\mathbf{x}_i) = exp(-\frac{||\mathbf{x}_i||^2}{2})$.

One of the commonly-used kernel estimator is Nadaraya-Watson Regression, as per Nadaraya (1964)[6] and Watson(1964)[7], and that is the case of local regressions with degree $p = 0$. The Nadaraya-Watson (NW) estimator $\mathbf{m}$ is

$$\hat{\mathbf{m}}^{NW}(\mathbf{x};\mathbf{y}) = \frac{\sum_{i=1}^{n}(K_h(\mathbf{x}-\mathbf{x}_i)\mathbf{y}_i)}{\sum_{i=1}^{n}K_h(\mathbf{x}-\mathbf{x}_i)} \tag{1.2.2}$$

, where K_h is the scaled kernel function with bandwidth parameter h of choice: $K_h(\mathbf{x}) = h^{-d}K(\frac{\mathbf{x}}{d})$, where d is the dimension of $\mathbf{x}$.

Another popular kernel estimators is Locally Linear estimator, which is the case of local regressions with degree $p = 1$. The locally linear kernel weighted estimator (LL), as per Fan (1992)[8], is given by

$$\hat{\mathbf{m}}^{LL}(\mathbf{x};\mathbf{y}) = \frac{1}{n}\frac{\hat{s}_2(\mathbf{x};h) - \hat{s}_1(\mathbf{x};h)(\mathbf{x}_i - \mathbf{x})\mathbf{y}_i}{\hat{s}_2(\mathbf{x};h)\hat{s}_0(\mathbf{x};h) - \hat{s}_1(\mathbf{x};h)^2}K_h(\mathbf{x}-\mathbf{x}_i) \tag{1.2.3}$$

, where $\hat{s}_j(\mathbf{x};h) = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{x}_i - \mathbf{x})^j K_h(\mathbf{x}-\mathbf{x}_i)$, for $j = 0, 1, 2$.

The bandwidth of the kernel estimation has a strong impact on the results, the choice of which is a trade-off between variance and bias. A popular choice of fixed bandwidth parameter for univariate Gaussian distributed data is Silverman's rule of thumb, as per Silverman (1987)[9].

$$h^* = 0.9 * \min(\hat{\sigma}, \frac{IQR}{1.34}) * n^{\frac{1}{5}} \tag{1.2.4}$$

, where $\hat{\sigma}$ represents the standard deviation of the samples, IQR is the interquantile $(Q_{0.75} - Q_{0.25})$ range with sample size of n.

For the multivariate case, assuming a normal product kernel and a true normal density with $\mathbf{\Sigma} = \mathbf{I}_d$, and the bandwidth parameter $h_k^* = \hat{\sigma}_k n^{-\frac{1}{d+4}}$ for the multivariate version.

It is worth noting that the fixed bandwidth may not be ideal as the optimal bandwidth may differ in different regions of the domain. So we employ the variable kernel instead, which would help deal with the unreasonable smoothness of the distribution.

The optimal hedging strategy $\{x_t\}_{t=0}^{T}$ obtained in the equation (1.1.7) is a ratio of two conditional expectations. And then the Nadaraya-Watson estimator and Locally Linear estimator can be of use to help compute the hedging strategy by giving estimation of conditional expectations.

1.2.2.2 Control variates

When we conduct a regression, the estimation error follows a Normal distribution with zero mean and a non-zero variance. For Monte Carlo simulation, the expectation of random variable Y is estimated by the sample mean of the realisations $y_1, \ldots, y_N$ drawn.

$$\hat{\mu}_Y = \mathbb{E}[Y] = \frac{1}{N}\sum_{i=1}^{N} y_i$$

And the error $\mu_Y - \hat{\mu}_Y$ approximately follows a Normal distribution $N(0, \frac{\sigma^2}{N})$.

If we can manage to reduce the variance of error, the quality of estimation would be improved. The idea of variance reduction is to replace the variable Y with another variable Y^* with same mean but smaller variance: $\mathbb{E}[Y^*] = \mathbb{E}[Y]$ and $Var[Y^*] < Var[Y]$. As a result, the estimator $\mu^* = \mathbb{E}[Y^*]$ would convergent to same mean, but the smaller variance would lead to faster convergence speed.

Control variates is a standard technique to reduce variance, and it is fist applied in pricing financial derivatives by Boyle (1977)[10]. With a control variate Z, the Y^* defined above would be found as $Y^* := Y + \beta(Z - \mu_Z)$, where μ_Z is the mean of variable Z. We then check if the variance is reduced.

$$\mathbb{E}[Y^*] = \mathbb{E}[Y] + \beta(\mathbb{E}[Z] - \mu_Z) = \mathbb{E}[Y] = \mu_Y$$
$$Var(Y^*) = Var(Y) + \beta^2 Var(Z) + 2\beta Cov(Y, Z)$$

The coefficient β^* could be computed by minimizing the variance of Y^*.

$$\beta^* = \arg\min Var(Y^*) = -\frac{Cov(Y, Z)}{Var(Z)}$$

$$\Longrightarrow Var(Y^*) = Var(Y) + \frac{Cov^2(Y, Z)}{Var(Z)} - 2\frac{Cov^2(Y, Z)}{Var(Z)} = (1 - \rho^2_{Y,Z})Var(Y) \leq Var(Y)$$

As is shown above, the control variate does help reduce the variance. And it also works in the case of conditional expectation, with any x, the estimator $\hat{\mu}_Y = \mathbb{E}[Y|X = x]$ could be replaced by $\hat{\mu}^*_Y = \mathbb{E}[Y^*|X = x]$ where

$$Y^* = Y + \beta(Z - \mu_Z(x)) \text{ , with coefficient } \beta^* = -\frac{Cov(Y, Z|X = x)}{Var(Z|X = x)}$$

Moreover, compare the optimal hedging obtained in equation (1.1.4) and the optimal coefficient β^*,

$$x^*_t = \frac{\mathbb{E}_t[\Delta\hat{H}_t\hat{\Pi}_{t+1} + \frac{1}{2\lambda e^{-r\Delta t}}\Delta H_t|\mathscr{F}_t]}{\mathbb{E}_t[(\Delta\hat{H}_t)^2|\mathscr{F}_t]} = \frac{\mathbb{E}[(\Delta H_t - \mathbb{E}[\Delta H_t])(\Pi_{t+1} - \mathbb{E}[\Pi_{t+1}]) + \frac{1}{2\lambda e^{-r\Delta t}}\Delta H_t|\mathscr{F}_t]}{\mathbb{E}[(\Delta H_t - \mathbb{E}[\Delta H_t])^2|\mathscr{F}_t]}$$
$$\beta^* = -\frac{Cov(Y, Z|X = x)}{Var(Z|X = x)} = -\frac{\mathbb{E}[(Y - \mathbb{E}[Y])(Z - \mathbb{E}[Z])|X = x]}{\mathbb{E}[(Z - \mathbb{E}[Z])^2|X = x]}$$

It can be seen that these two values are tightly related if we choose the underlying tradable asset H to be the control variate, and the risk averse parameter $\lambda \to \infty$, making the second term of optimal hedging numerator tends to zero.

In addition, the control variate can be extended to multiple controls, which contributes to improve the estimation with multiple underlying assets. Suppose the mean $\mathbb{E}[\mathbf{X}]$ is known for $\mathbf{X} = (X^1, \ldots, X^z)^T$, the regressand Y could then be replaced by

$$Y_{CV}(\boldsymbol{\beta}) = Y + \beta_1(X^1 - \mathbb{E}[X^1]) + \cdots + \beta_z(X^z - \mathbb{E}[X^z]) = Y + \boldsymbol{\beta}^T(\mathbf{X} - \mathbb{E}[\mathbf{X}])$$

, with the coefficients $\boldsymbol{\beta} \in \mathbf{R}^z$.

Minimizing the variance of Y_{CV}, the optimal coefficients are computed by

$$\boldsymbol{\beta}^* = \Sigma_X^{-1}\Sigma_{XY}$$

, where Σ_X with dimensionality $z \times z$ and Σ_{XY} with dimensionality $z \times 1$ are the variance matrix for regressor $\mathbf{X}$ and the covariance matrix of $\mathbf{X}$ and Y respectively. After replacing Y by Y_{CV}, the prediction variance reduces:

$$Var(Y) = \sigma^2_Y \text{ , and } Var(Y_{CV}) = (1 - R^2)\sigma^2_Y$$

, with R^2 being the squared correlation coefficient between X and Y:

$$R^2 = (\frac{Cov(\mathbf{X}, Y)}{\sqrt{Var(\mathbf{X})Var(Y)}})^2 = \frac{\Sigma^T_{XY}\Sigma^{-1}_X\Sigma_{XY}}{\sigma^2_Y}$$

Another convenient and popular choice of coefficients $\boldsymbol{\beta}$ is $\beta^*_i = -b_i$,for $i = 1, \ldots, z$ where the b_i's are the solution to the linear regression, as per Haugh (2010)[11]:

$$Y = c + b_1X^1 + \cdots + b_zX^z + \epsilon$$

It is worth noting that the intercept term c cannot be ignored when computing the least square solution.

1.2.2.3 Gaussian process regression

Gaussian progress regression (GPR) is a non-parametric method of regression. It lies in the area of Bayesian inference and works well in the case of small data set. The supervised machine learning and Gaussian process regression give the estimation of parameters in different forms. To be specific, the supervised machine learning learns the exact value of parameters, but the distribution of parameters would be obtained by performing a Gaussian process regression. Thus the uncertainty of predictions could be measured, such as by evaluating the 95% confidence interval. In terms of calculation, consider a linear regression

$$y_i = \beta x_i + \epsilon_i$$

, where the error $\epsilon_i \sim N(0, \sigma_n^2)$. Regard the coefficient as a random variable, and assume a prior distribution for the coefficient $\beta \sim N(0, \sigma_p^2)$,

$$p(y|x, \beta) \sim N(\mu, \sigma_n^2)$$

, with $\mu = \beta x$.

Employing the Bayes' theorem we can obtain that

$$p(\beta|x, y) = \frac{p(y|x, \beta)p(\beta)}{p(y|x)} \propto exp(-\frac{1}{2}(\beta - \bar{\beta})^2 A)$$

, where $\bar{\beta} = \frac{(\frac{x^2}{\sigma_n^2} + \sigma_p^{-1})^{-1} xy}{\sigma_n^2}$ and $A = \frac{x^2}{\sigma_n^2} + \sigma_p^{-1}$, leading to the posterior distribution

$$p(\beta|x, y) \sim N(\bar{\beta}, A^{-1}).$$

Finally, the predictive distribution could be obtained as

$$p(y^*|x^*, x, y) \sim N(\frac{A^{-1} x x^* y}{\sigma_n^2}, x^{*2} A^{-1}).$$

Rather than claiming the value function relates to some specific models, a Gaussian process can represent the objective function obliquely, but rigorously by selecting different prior distributions and kernel functions, thus removes the restriction of models. And it can be extended to multi-dimensional case.

For any point $\mathbf{x} \in \mathbf{R}^d$, a Gaussian process is a random process assigning a random variable $f(\mathbf{x})$ to the point, where the joint distribution of a finite number of the variables is itself Gaussian.

$$\boldsymbol{p(f|\mathrm{X}) = N(f|\mu, \mathrm{K})}$$

, where$f = (f(\mathbf{x_1}, \ldots, \mathbf{x}_N))$, $\boldsymbol{\mu} = (m(\mathbf{x}_1), \ldots, m(\mathbf{x}_N))$ and the kernel matrix $K_{ij} = k(\mathbf{x}_i, \mathbf{x}_j)$. A popular choice of kernel function k is radial basis kernel function (RBF), which measures the distance of a pair of points:

$$k(\mathbf{x}_i, \mathbf{x}_j) = \sigma_f^2 exp(-\frac{1}{2l^2}(\mathbf{x}_i - \mathbf{x}_j)^T(\mathbf{x}_i - \mathbf{x}_j))$$

, with σ_f^2 expresses the vertical variation of the function, and l is the scaling parameter, controlling the smoothness of the function. By employing radial kernel function, it makes sense logically that the points nearby have larger impact of resulting estimation compared to points which are far from the local point. As the Gaussian Processes are so flexible that the mean function can be assumed to take zero everywhere.

Given a number of pairs of realisations $(\mathbf{X}, \mathbf{y})$, the multi-dimensional posterior distribution can help give predictions of new inputs $\mathbf{X}^*$:

$$\boldsymbol{p(f^*|\mathrm{X}^*, \mathrm{X}, y) = N(f^*|\mu^*, \Sigma^*)}$$

with the joint of known observation and predictions follows a multivariate Gaussian distribution:

$$\begin{pmatrix} \mathbf{y} \\ f^* \end{pmatrix} \sim N\left(\mathbf{0}, \begin{pmatrix} \mathbf{K}^y & \mathbf{K}^* \\ \mathbf{K}^* & \mathbf{K}^{**} \end{pmatrix}\right)$$

, with $\mathbf{K}^y = k(\mathbf{X}, \mathbf{X}) + \sigma_y^2 \mathbf{I}$, $\mathbf{K}^* = k(\mathbf{X}, \mathbf{X}^*)$ and $\mathbf{K}^{**} = k(\mathbf{X}^*, \mathbf{X}^*)$. And the parameters of predictive posterior distribution could be computed as

$$\boldsymbol{\mu}^* = \mathbf{K}^{*T} \mathbf{K}^{y-1} \mathbf{y} \text{ , and } \boldsymbol{\Sigma}^* = \mathbf{K}^{**} - \mathbf{K}^{*T} \mathbf{K}^{y-1} \mathbf{K}^*.$$

1.2.2.4 Dynamically Controlled Kernel Estimation

Combining the kernel regression, control variate and Gaussian process regression, we get the dynamically controlled kernel estimation, proposed by Lee (2020)[12]. The algorithm is stated as follows.

- Perform a local regression to get the estimation of conditional expectation $\tilde{y}_i = \mathbb{E}[Y|X = x_i]$.
- Compute control variate using local regression $\tilde{z}_i = \mathbb{E}[Z|X = x_i]$ and coefficient β^*.
- Perform a Gaussian process regression of x_i against $\bar{y}_i = \tilde{y}_i + \beta^*(\tilde{z}_i - \mu_Z(x_i))$.

The newly proposed DCKE algorithm is proved to outperform the traditional LSM particularly in the estimation of deltas and the 'tails' in examples of the Black-Scholes and the Heston model, and also provides more precise intermediate-value predictions than both Grau's and Potters' methods. Here we demonstrate how this method could be applied in the case of hedging and pricing a European call option in the following section.

Chapter 2

Vanilla Option Hedging in the Incomplete Market

Following the economic set-ups and methodology introduction in Chapter 1, several simple models for asset price evolution are considered. We now hedge the incomplete markets in some simple cases with the algorithm stated in previous sections. Here we focus on the incompleteness in terms of the partial correlation between the underlying asset and the tradable asset. The known closed-form formula provide the benchmark for the optimal hedging strategy computed by our algorithm, thus help validate that the newly proposed optimal hedging strategy is model-free. As a result, the valid hedging strategy for more complex models without closed-form result could be computed.

2.1 Black-Scholes-Merton model

2.1.1 GBM process and pathwise derivative

Referring to the celebrated Black-Scholes-Merton model, [13] under the physical measure P, the value evolution process of the underlying asset could be modeled as

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2.1.1}$$

$$\implies \quad S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \tag{2.1.2}$$

, where μ depicts the drift and σ measures the volatility. And W_t is a Wiener process.

And we assume the stochastic differential equation (SDE) for the tradable asset H is

$$\begin{aligned} dH_t &= \mu' H_t dt + \sigma' H_t dW'_t \\ \implies \quad H_t &= H_0 e^{(\mu' - \frac{1}{2}\sigma'^2)t + \sigma' W'_t} \end{aligned} \tag{2.1.3}$$

, where $dW_t dW'_t = \rho dt$, depicting the partial correlation between S_t and H_t with $\rho \in (-1, 1)$. The partial correlation would expose the hedger to the risk of hedging error.

Black-Scholes Call option delta could be represented as a pathwise derivative of the option value with respect to the tradable asset price. Under Geometric Brownian motion, as the option value C_{t_i} is differentiable with respect to the asset price H_{t_i}, the valid interchange between expectation and derivative empowers the computation of the kernelised pathwise derivative estimation as follows.

Plugging $dW_t = \rho dt/dW_{t'}$ back to (2.1.1), the SDE of underlying asset S_t can be represented as

$$dS_t = \mu S_t dt + \sigma S_t \rho dt/dW_{t'} \tag{2.1.4}$$

When the drifts of S and H being equal, the derivative could be computed as

$$\begin{aligned}\frac{dS_t}{dH_t} &= \frac{\mu S_t dt + \sigma S_t \rho dt/dW'_t}{\mu H_t dt + \sigma' H_t dW'_t} \\ &= \frac{\mu S_t dt dW'_t + \sigma S_t \rho dt}{\mu H_t dt dW'_t + \sigma' H_t (dW'_t)^2} \\ &= \frac{\rho \sigma S_t dt}{\sigma' H_t (dW'_t)^2} = \frac{\rho \sigma S_t}{\sigma' H_t} \end{aligned} \tag{2.1.5}$$

, which enables us to compute the pathwise derivative:

$$\begin{aligned}\frac{dC_{t_i}}{dH_{t_i}} = \mathbb{E}[\frac{dG^j_{t_i}}{dH_{t_i}}] &= \mathbb{E}[\frac{dG^j_{t_i}}{dS^j_{t_T}} \frac{dS^j_{t_T}}{dS^j_{t_i}} \frac{dS^j_{t_i}}{dH^j_{t_i}}] \\ &= \mathbb{E}[e^{-r(t_T - t_i)} \mathbf{1}_{\{S^j_{t_T} > K\}} \frac{S^j_{t_T}}{S^j_{t_i}} \frac{\rho \sigma S^j_{t_i}}{\sigma' H^j_{t_i}}] \\ &= \mathbb{E}[e^{-r(t_T - t_i)} \mathbf{1}_{\{S^j_{t_T} > K\}} \frac{\rho \sigma S^j_{t_T}}{\sigma' H^j_{t_i}}] \end{aligned} \tag{2.1.6}$$

, where $G^j_{t_i} := e^{-r(t_T - t_i)}(S^j_{t_T} - K)^+$ represents the discounted payoff. Applying pathwise derivative, we get the delta at each meshed point k for $k = 1, 2, \dots, u$ of underlying at time step t_i:

$$\hat{x}^k_{t_i} = \hat{\mathbf{m}}_h(M_k, H_{t_i}, S_{t_i}; e^{-r(t_T - t_i)} \mathbf{1}_{\{S^j_{t_T} > K\}} \frac{\rho \sigma S_{t_T}}{\sigma' H_{t_i}}) \tag{2.1.7}$$

, with kernel estimation $\hat{\mathbf{m}}_h$ defined in the equation (1.2.2).

2.1.2 Analytical optimal policy

Referring to Basak and Chabakauri (2011)[14], the closed-form formulae for hedging an untradable stock with a partial correlated stock is derived. And the linear PDE was derived as per Windcliff(2006)[15]. Here we mimic the algorithms to derive the one for hedging a European call option using a partially correlated asset, and employ it as a benchmark for the newly proposed optimal hedging strategy.

Proposition 2.1.1 (Optimal hedge under BSM model)**.** *Suppose there is a European call option writing on an untradable underlying asset S with the strike price K. Setting up a hedging portfolio Π with a partial correlated asset H and risk-free bank account B, when the price evolution follows Black-Scholes-Merton model, the optimal hedging strategy $\{x^*_t\}^T_{t=0}$ is give by*

$$x^*_t = \frac{\rho\sigma}{\sigma'} \frac{S_t}{H_t} \frac{\partial \mathbb{E}^*_t[e^{-r(T-t)} G(S_T)]}{\partial S_t} \tag{2.1.8}$$

, where the payoff function $G(S_T) = (S_T - K)^+$.

Proof. Suppose we take a position of shorting one unit of European call option and the fair value is denoted by V_t. Thus the hedging error is $Y_t := \Pi_t - V_t$.

$$Y = xH + B - V$$

Assume $B = V - xH$ at time t, the change of hedging error over the time period $[t, t + dt]$ could be derived as follows.

$$\begin{aligned} dY &= -[\dot{V} + \mu S V' + \frac{\sigma^2 S^2}{2} V'']dt - \sigma S V' dW + r(V - xH)dt + x(\mu' H dt + \sigma' H dW') \\ &= -[\dot{V} + \mu S V' + \frac{\sigma^2 S^2}{2} V'' - r(V - xH) - x\mu' H]dt - \sigma S V' dW + x\sigma' H dW' \end{aligned}$$

, where $\dot{V} = \frac{\partial V}{\partial t}$, $V' = \frac{\partial V}{\partial S}$ and $V'' = \frac{\partial^2 V}{\partial S^2}$.

The variance of dY is

$$Var(dY) = \mathbb{E}[(-\sigma S V' dW + x\sigma' H dW')^2] = [x^2 \sigma'^2 H^2 + \sigma^2 S^2 V'^2 - 2\sigma S V' x \sigma' H \rho]dt$$

Employing the strategy of variance-minimization, the optimal hedging could be found as

$$x_t^* = \arg\min Var(dY) = \frac{\rho\sigma}{\sigma'}\frac{S_t}{H_t}V' \tag{2.1.9}$$

Conduct Monte Carlo simulation, $V_T = \mathbb{E}_T[(S_T - K)^+]$, and then the partial derivative $V' = \frac{\partial \mathbb{E}_t^*[e^{-r(T-t)}(S_T-K)^+]}{\partial S_t}$, substituting it back to the equation (2.1.9), the result is derived as required. □

From proposition 3.2.1, we expect the optimal hedging policy converges to the Black-Scholes delta hedge when $\rho = 1$ and volatilities for two assets are the same, which simplify our case back to the complete market where there exist no arbitrage opportunity. Furthermore, the analytical optimal hedging strategy also agrees with the expectation of pathwise derivative proposed by equation (2.1.5) under GBM, where the payoff function is differentiable with respect to the asset price.

It is worth noting that the optimal hedge proposed by equation (1.1.7) converges to closed-form optimal hedge given by equation (3.2.3) when the time lag $\Delta t \to 0$ and the risk averse parameter $\lambda \to \infty$, derived as follows.

The fair price of option was defined as the expectation of hedging portfolio value at time t:

$$\hat{C}_t = \mathbb{E}[\Pi_t|\mathscr{F}_t]$$

By Tower property, the optimal hedge could be expressed as follows when risk averse parameter $\lambda \to \infty$,

$$\begin{aligned}\lim_{\lambda\to\infty} x_t^* &= \lim_{\lambda\to\infty} \frac{E_t[\Delta\hat{H}_t\hat{C}_{t+1} + \frac{1}{2\lambda e^{-r\Delta t}}\Delta H_t|\mathscr{F}_t]}{E_t[(\Delta\hat{H}_t)^2|\mathscr{F}_t]} \\ &\to \frac{E_t[\Delta\hat{H}_t\hat{C}_{t+1}|\mathscr{F}_t]}{E_t[(\Delta\hat{H}_t|\mathscr{F}_t]} \\ &= \frac{Cov(\Delta H_t, C_{t+1}|\mathscr{F}_t)}{Var(\Delta H_t|\mathscr{F}_t)}\end{aligned} \tag{2.1.10}$$

Applying first-order Taylor expansion,

$$\hat{C}_{t_{i+1}} = C_{t_i} + \frac{\partial C_{t_i}}{\partial H_{t_i}}\Delta H_{t_i} + \mathbf{O}(\Delta t_i) \tag{2.1.11}$$

Plugging equation (2.1.11) and equation (2.1.5), which depicts the derivative of underlying asset price S_{t_i} with respect to the tradable asset price H_{t_i} under Black-Scholes-Merton model, back into equation (2.1.10), we get

$$\lim_{\Delta t_i\to 0} x_t^* = \frac{\partial C_{t_i}}{\partial H_{t_i}} = \frac{\partial S_{t_i}}{\partial H_{t_i}}V' = \frac{\rho\sigma}{\sigma'}\frac{S_t}{H_t}V' \tag{2.1.12}$$

, which agrees with the optimal hedge given by proposition (3.2.1), meaning the newly proposed optimal hedge converges to continuous-time closed-form optimal hedge when time step tends to zero.

2.1.3 Path generation

Following GBM, the underlying asset price evolution process could be derived by solving the stochastic differential equation (SDE) as follows,

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad \Longrightarrow \quad S_t = S_0 exp((\mu - \frac{\sigma^2}{2})t + \sigma W_t)$$

$$dH_t = \mu' H_t dt + \sigma' H_t dW_t' \quad \Longrightarrow \quad H_t = H_0 exp((\mu' - \frac{\sigma'^2}{2})t + \sigma' W_t')$$

Here we generate samples at several discrete time steps $t_i, i = 1, \ldots, T$ with n samples at each time step $S_{t_i}^j$ for $j = 1, \ldots, n$.

$$S_{t_{i+1}}^j = S_{t_i}^j exp((\mu - \frac{\sigma^2}{2})(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}Z_j) \tag{2.1.13}$$

$$H^j_{t_{i+1}} = H^j_{t_i} exp((\mu' - \frac{\sigma'^2}{2})(t_{i+1} - t_i) + \sigma' \sqrt{t_{i+1} - t_i} Z'_j) \quad (2.1.14)$$

, where $\begin{pmatrix} Z \\ Z' \end{pmatrix}$ are drawn from a Multi-Normal distribution $N(\mu, \mathbf{\Sigma})$ with $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\mathbf{\Sigma} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. The asset price trajectories S_{t_i} and H_{t_i} can be generated by the equations (2.1.13) and (2.1.14) respectively from time 0 through time T starting at the current price S_{t_0} and H_{t_0} with parameters correlation, drifts and volatilities given.

2.1.4 Practical experiment

As stated before, when the drift of asset value evolution equals the risk-free rate and the risk averse parameter being large enough, the optimal hedge converges to the Black-Scholes delta hedge. Here we set the drift equals the risk-free rate, and focus on the impact of risk averse parameter λ and correlation ρ. Considering a European call option with parameter settings in Table 2.1, we employ the DCKE to compute the conditional expectations, thus giving the hedging strategy at an intermediate time step with algorithm 2 stated. In real world, we should never expect the volatilities of two asset to be identical, and the correlation would not be identical for different pairs of assets. Thus, we expect the optimal hedging strategy, computed by the model-free method we aim to propose, to perfectly converge to the closed-form delta hedge for discrete Black-Scholes model without the constraint of correlation ρ and volatilities σ and σ'.

General Features	Values
Initial underlying stock price S_{t_0}	100
Initial tradable stock price H_{t_0}	100
Strike price K	100
Risk-free rate r	0.01
Drift for underlying stock μ	0.01
Drift for tradable stock μ'	0.01
Volatility for underlying stock σ	[0.2, 0.25, 0.3]
Volatility for tradable stock σ'	[0.4, 0.35, 0.3]
Correlation ρ	[0.5, 0.7, 0.9]
Risk averse λ	10^7
Maturity time T	1 year
Time step t_i	[0, 0.9, 1.0]
Number of path n	1000000

Table 2.1: European call option parameters for DCKE

Here we use the meshed points among 1% and 99% percentiles of the tradable asset at the intermediate time step, and employ the discounted payoff approach to compute the pricing and hedging strategy. The bandwidth at each meshed point are scaled by a ratio of the maximal kernel size to the local kernel size under the fixed kernel, with a constant specifies the maximum value capping the bandwidth from below, the value of which relates to the number of simulated paths, and help control the smoothness at the tails.

As the underlying asset price S_t and the tradable asset price H_t both exert influence on the hedging strategy, it is reasonable to perform a two-dimensional kernel regression in the case of hedging a European option using partial correlated asset, the process of which are stated in algorithm 2.

Implementing the algorithm 2, the results with partial correlation $\rho = 0.9$ and risk averse parameter $\lambda = 1$ are shown in the figures 2.1. As is shown in the figures, the optimal hedge we derived from minimizing the variance perfectly converge to the kernelised pathwise derivative, also the analytical optimal hedging strategy for GBM process. Under high underlying asset price along with low tradable asset price, the deviation of two deltas becomes noticeable, suggesting the algorithm suffers from the edge noise.

We aim to propose a model-free methodology, in other words, we expect these two deltas converges without the constraint of underlying model and parameters. To be specific, the kernelised pathwise delta and variance minimizing optimal hedging strategy should also converge under any

Algorithm 2: 2-Dim DCKE of hedging and pricing a European call option.

Data: strike price K; risk-free interest rate r; number of time steps T; length of time step Δt; number of paths n; number of meshed samples u; price evolution sequence $(P_{t_i} = (S_{t_i}, H_{t_i}))$ with each length n for $i = 0, \dots, T$; price evolution sequence $(P^*_{t_i} = (S^*_{t_i}, H^*_{t_i}))$ with each length n^* for $i = 0, \dots, T$; partial correlation ρ; hyperparameter input for bandwidth adjustment of price h_p; hyperparameter input for bandwidth adjustment of delta h_d; risk averse parameter λ; cap of variable kernel ratio C

Result: mesh points of spot price sequence $M_{t_i} = (M_{t_i,S}, M_{t_i,H})$; coefficient for control variates $\hat{\beta}_p$ for pricing and $\hat{\beta}_d$ for hedging; option fair price sequence $C^{(\text{fair})}_{t_i}$; kernel estimation of delta x'_{t_i}; optimal hedge position x_{t_i} derived by variance minimisation with each length n for $i = 0, \dots, T$

Set arrays: $C^{j,(\text{fair})}_{t_T} = (S^j_{t_T} - K)^+$; $C^{j,(\text{ask})}_{t_T} = C^{j,(\text{fair})}_{t_T} + \lambda Var(C^{(\text{fair})}_{t_T})$ for $j = 1, \dots, n$; $\hat{C}^{(\text{fair})}_{t_i}$; $\hat{C}^{(\text{ask})}_{t_i}$; $\hat{x}_{t_i}$;

M_{t_i}: samples between 5% - 95% percentiles of generated spot prices

$M^*_{t_i}$: samples between 5% - 95% percentiles of generated spot prices

$\Sigma = cov(P_{t_1}, P_{t_2})$ with $LL^T = \Sigma^{-1}$

$L' = Ln^{-\frac{1}{d+4}}$

$L'_p = L' * h_p$

$L'_d = L' * h_d$

$h = 0.9\min(\sqrt{Var(S_{t_T})}, \frac{IQR(S_{t_T})}{1.34})$;

for *time steps from* $i = T-1$ *to* $i = 0$ *backward* **do**

  for *each mesh point* $k = 1, \dots, u$ **do**

    $K^k_p = \sum_{j=1}^n e^{-\sum_{v=1}^d (P_{t_2} - M^k) L'^2_p}$

  $K^k_{\text{ratio}} = \min(\frac{K^*}{K^k_{t_i}}, C)$

  variable kernel $h_{\text{new}} = hK^k_{\text{ratio}}$

  $\hat{C}^{k,(\text{fair})}_{t_{i+1}} = \hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; C^{k,(\text{fair})}_{t_{i+1}})$ (kernel estimator $\mathbf{m}$ defined in equation 1.2.2)

  $\hat{\Delta H}^k_{t_{i+1}} = \hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; \Delta H^k_{t_{i+1}})$

  $\hat{\beta}_p = \frac{\hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; (\Pi_{t_{i+1}} - \hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; \Pi_{t_{i+1}}))(\Delta H^k_{t_{i+1}} - \hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; \Delta H^k_{t_{i+1}})))}{\hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; (\Delta H^k_{t_{i+1}} - \hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; \Delta H^k_{t_{i+1}}))^2)}$

  $\hat{x}^k_{t_i} = \frac{\hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; (\Pi_{t_{i+1}} - \hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; \Pi_{t_{i+1}}))(\Delta H^k_{t_{i+1}} - \hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; \Delta H^k_{t_{i+1}})) + \frac{1}{2\lambda\gamma}\Delta H_{t_i})}{\hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; (H^k_{t_{i+1}} - \hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; \Delta H^k_{t_{i+1}}))^2)}$

  $\hat{x}'^k_{t_i} = \hat{\mathbf{m}}_h(M_k, H_{t_i}, S_{t_i}; e^{-r(t_T - t_i)} \mathbf{1}_{\{S^j_{t_T} > K\}} \frac{\rho\sigma S_{t_T}}{\sigma' H_{t_i}})$

  $\hat{\beta}_d = \frac{\hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; (\frac{dG_{t_T}}{dM^k_{t_i,H}} - \hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; (\frac{dG_{t_T}}{dM^k_{t_i,H}}))(\Delta H^k_{t_{i+1}} - \hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; \Delta H^k_{t_{i+1}})))}{\hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; (\Delta H^k_{t_{i+1}} - \hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; \Delta H^k_{t_{i+1}}))^2)}$

  $\hat{C}^{*(\text{fair})}_{t_i} = e^{-r\Delta t} GPR(RBF, M^*_{t_i}, \hat{C}^{(\text{fair})}_{t_i} - \hat{\beta}_p(\hat{H}_{t_{i+1}} - e^{r\Delta t} M_{t_i,H}))$

  $\hat{x}^*_{t_i} = GPR(RBF, M^*_{t_i}, \hat{x}_{t_i} - \hat{\beta}_d(\hat{H}_{t_{i+1}} - e^{r\Delta t} M_{t_i,H}))$

  $\hat{x}'^*_{t_i} = GPR(RBF, M^*_{t_i}, \hat{x}'_{t_i} - \hat{\beta}_d(\hat{H}_{t_{i+1}} - e^{r\Delta t} M_{t_i,H}))$

Compute the optimal hyperparameter by minimising validation error $h^*_p, h^*_d = \arg\min_{h_p, h_d \sim U(0.01,5)} ||\hat{x}^*_{t_i} - \hat{x}'^*_{t_i}||$ via *hyperopt*

Repeat the process above with hyperparameter h^*_P and h^*_d.

Result: $\hat{C}^{(\text{fair})}_{t_i}$; $\hat{x}^*_{t_i}$; $\hat{x}'^*_{t_i}$

.

correlation ρ and with different volatilities σ and σ', which are the two cases we present in figure 2.2 and 2.4 respectively.

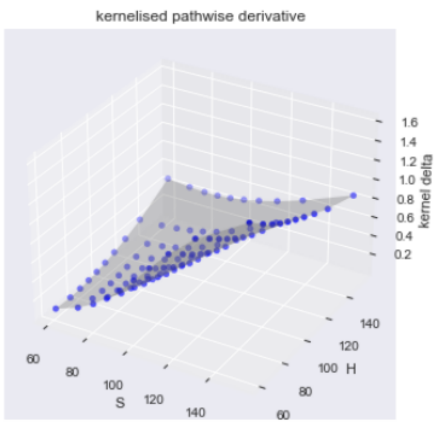
First keeping the risk averse parameter λ to be large and fixed, the optimal pricing and hedging results are shown in figure 2.2. As is shown in the figures, the hedging strategy is sensitive to the correlation ρ. With the increase of correlation, the holding position of tradable asset in the hedging portfolio increases. The quality of hedging improves when the correlation between two assets increases, and converges to the perfect hedge when $\rho \to 1$. And there exists a deviation at the right tail in the Q-Q plot (right). To explore the error, we present the deviation in 3-dimensional figures (figure 2.3). It can be seen that the deviation is large on the edge.

Turning to the locally linear (LL) estimator may help address the edge noise. As per Hardle(1990)[16] and Fan (1992)[8], both estimators are consistent, and the bias is shown in Table 2.2. It can be seen from the table, with the same variance, the bias of locally linear estimator is smaller than that of Nadaraya-Watson estimator. The additional bias term of NW estimator contributes to the edge noise because the denominator tends to disappear at the boundary.

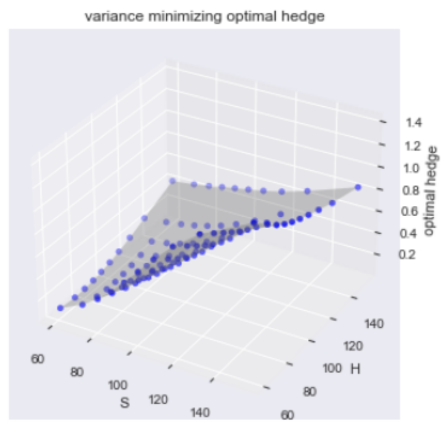
Estimator	Nadaraya-Watson	Locally Linear
Bias	$h^2(0.5m_h''(x) + \frac{m_h'(x)f'(x)}{f(x)})\int z^2 k_h(z)dz$	$h^2(0.5m_h''(x))\int z^2 k_h(z)dz$
Variance	$\frac{\sigma^2(x)}{f(x)nh}\int k_h^2(x)dz$	$\frac{\sigma^2(x)}{f(x)nh}\int k_h^2(x)dz$

Table 2.2: bias and variance of NW and LL estimators up to $\mathbb{O}(h^2)$

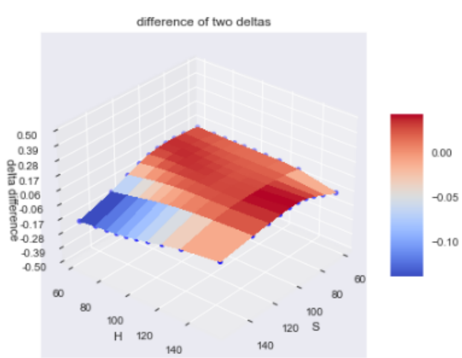
Then also keeping the correlation fixed, the pricing and hedging are computed with different volatilities for underlying asset S and tradable asset H, shown in figure 2.4. The holding position of tradable asset in the hedging portfolio increases with the increase of underlying asset volatility, and decrease with the rise of tradable asset volatility, which agrees with the closed-form formula for optimal hedging proposed by proposition 3.2.1.



(a) Kernelised pathwise derivative



(b) Variance minimization optimal hedge



(c) Difference between two deltas

(d) Q-Q plot of pathwise derivative against optimal hedge

Figure 2.1: Result with risk averse parameter $\lambda = 10^7$ and partial correlation $\rho = 0.9$

2.2 Ornstein–Uhlenbeck process

Ornstein–Uhlenbeck (OU) process, named after Leonard Ornstein and George Eugene Uhlenbeck (1930)[17], is a commonly-used stochastic process with application financial mathematics, proposed by Leonard Ornstein and George Eugene Uhlenbeck, which manages to depict the trend that the asset price has a tendency of the walk to move back towards a central location. When the asset price is further away from the center, the attraction becomes greater.

2.2.1 OU process and pathwise derivative

Referring to the OU process, at time t, the spot price of underlying asset S follows the following stochastic differential equation,

$$dS_t = \theta_s(\mu_s - S_t)dt + \sigma_s dW_t \tag{2.2.1}$$

, where θ_s is mean reversion speed, μ_s is the mean reversion level, and σ_s represents the volatility of price evolution. It is worth noting that, the main drawback of of OU process is S_t may take negative values, but for a wide range of relevant parameter values, the probability of getting negative is very small. Similarly, the price evolution of tradable asset H follows

$$dH_t = \theta_h(\mu_h - H_t)dt + \sigma_h dW'_t$$

, with $dW_t dW'_t = \rho dt$, depicting the partial correlation between two asset prices. The derivative of S_t with respect to H_t could be computed using the same derivation for GBM:

$$\frac{dS_t}{dH_t} = \frac{\rho\sigma_s}{\sigma_h} \tag{2.2.2}$$

Take integral to the both sides of equation 2.2.1:

$$S_t = S_0 e^{-\theta_s t} + \mu_s(1 - e^{-\theta_s t}) + \sigma_s \int_0^t e^{-\theta_s(t-s)} dW_s$$

Then employing the Ito's isometry, the closed form formula could be derived as follows.

$$S_t = S_0 e^{-\theta_s t} + \mu(1 - e^{-\theta_s t}) + \frac{\sigma_s}{2\theta_s} e^{-\theta_s t} W_{e^{2\theta_s t}-1} \tag{2.2.3}$$

, and similar result could be computed for tradable asset H:

$$H_t = H_0 e^{-\theta_h t} + \mu(1 - e^{-\theta_h t}) + \frac{\sigma_h}{2\theta_h} e^{-\theta_h t} W'_{e^{2\theta_h t}-1}$$

From equation 2.2.3, the recursive relationship of asset price could be expressed as follows.

$$S_{t_i} = S_{t_{i-1}} + \theta_s(\mu_s - S_{t_{i-1}})\Delta t + \sigma_s W_{\Delta t}$$

, with $\Delta t = t_i - t_{i-1}$ being the time step. Thus the derivative of S_T with respect to S_t could be derived out.

$$\begin{aligned}
&\frac{\partial S_{t_i}}{\partial S_0} = \frac{\partial S_{t_{i-1}}}{\partial S_0} - \theta_s \frac{\partial S_{t_{i-1}}}{\partial S_0}\Delta t + \mathbf{O}(\Delta t^2) \approx (1 - \theta_s \Delta t)\frac{\partial S_{t_{i-1}}}{\partial S_0} \\
\Longrightarrow &\frac{\partial S_{t_i}}{\partial S_{t_{i-1}}} = (1 - \theta_s \Delta t) \\
\Longrightarrow &\frac{\partial S_T}{\partial S_t} = \frac{\partial S_T}{\partial S_{T-1}} \frac{\partial S_{T-1}}{\partial S_{T-2}} \cdots \frac{\partial S_{t+1}}{\partial S_t} = (1 - \theta_s \Delta t)^n
\end{aligned}$$

, with the number of time step $n = \frac{T-t}{\Delta t}$. When the time step $\Delta t \to 0$, the number of time step $n \to \infty$, leading the derivative to

$$\frac{\partial S_T}{\partial S_t} \to e^{-\theta_s(T-t)} \tag{2.2.4}$$

Putting equation 2.2.2 and equation 2.2.4 together, the pathwise derivative of a European call option value C_t with respect to the tradable asset H_t could be computed,

$$\begin{aligned} \frac{dC_t}{dH_t} = \frac{d\mathbf{E}[e^{-r(T-t)}(S_T-K)^+]}{dH_t} &= \mathbf{E}[\frac{de^{-r(T-t)}(S_T-K)^+}{dH_t}] \\ &= \mathbf{E}[e^{-r(T-t)}\frac{d(S_T-K)^+}{dS_T}\frac{dS_T}{dS_t}\frac{dS_t}{dH_t}] \\ &= \mathbf{E}[e^{-(r+\theta_s)(T-t)}\mathbf{1}_{\{S_T>K\}}\frac{\rho\sigma_s}{\sigma_h}] \end{aligned} \tag{2.2.5}$$

, which agrees with the time homogeneous property of OU process and can provide benchmark for the hedging strategy proposed by equation 1.1.7.

2.2.2 Analytical optimal strategy

Using the same algorithm as for proposition 3.2.1, we derive the closed-form optimal hedging strategy for OU process as follow.

Proposition 2.2.1 (Optimal hedge under OU process)**.** *Suppose there is a European call option writing on an untradable underlying asset S with the strike price K. Setting up a hedging portfolio Π with a partial correlated asset H and risk-free bank account B, when the asset price evolution follows an OU process, the optimal hedging strategy $\{x_t^*\}_{t=0}^T$ is give by*

$$x_t^* = \frac{\rho\sigma_s}{\sigma_h}\frac{\partial\mathbb{E}_t^*[e^{-r(T-t)}G(S_T)]}{\partial S_t} \tag{2.2.6}$$

, where the payoff function $G(S_T) = (S_T - K)^+$.

Proof. As stated before, we set up a hedging portfolio with value Π_t composed by a tradable asset H_t and risk-free bank account B_t, to hedge one-unit short position of a European call option written on the underlying asset S_t, the fair value of which at time t is denoted by V_t

Thus the hedging error $Y_t := \Pi_t - V_t = xH_t + B_t - V_t$. Applying Ito's lemma,

$$\begin{aligned} dY &= -[\dot{V} + V'\theta_s(\mu_s - S) + \frac{1}{2}\sigma_s^2]dt - \sigma_s V' dW_t + x(\theta_h(\mu_h - H_t)dt + \sigma_h dW_t') + r(V - xH)dt \\ &= [\dot{V} + V'\theta_s(\mu_s - S) + \frac{1}{2}\sigma_s^2 - r(V - xH) + x(\theta_h(\mu_h - H_t)]dt - \sigma_s V' dW_t + x\sigma_h dW_t' \end{aligned}$$

The variance of hedging error

$$Var(dY) = \mathbb{E}[(-\sigma_s V' dW_t + x\sigma_h dW_t')^2] = \mathbb{E}[(\sigma_s^2 V'^2 + x^2\sigma_h^2 - 2x\sigma_s\sigma_h V')dt]$$

It can be seen that the the variance of hedging error is a quadratic function of the holding position in the hedging portfolio. Compute the first derivative and set it to be zero,

$$\frac{\partial Var(dY)}{\partial x} = 2\sigma_h^2 x - 2\sigma_s\sigma_h V' := 0 \tag{2.2.7}$$

Under the variance minimization criterion, the optimal hedging strategy is given by

$$x_t^* = \arg\min Var(Y_t) = \frac{\rho\sigma_s}{\sigma_h}V' \tag{2.2.8}$$

□

Similar to the finding under BSM model, the newly proposed hedging strategy given by equation (1.1.7) converges to the closed-form optimal hedging strategy stated by proposition (2.2.1) under OU process. Again plugging equation (2.1.11) and equation (2.2.2) back to the newly proposed optimal hedging strategy in the case $\lambda \to \infty$ and time lag $\Delta t_i \to 0$,

$$\lim_{\Delta t_i \to 0} x_{t_i}^* = \frac{\partial C_{t_i}}{\partial H_{t_i}} = \frac{\rho\sigma_s}{\sigma_h}V' \tag{2.2.9}$$

, showing the convergence of optimal hedging strategy under OU process when $\lambda \to \infty$ and time lag $\Delta t_i \to 0$.

Moreover, the newly proposed optimal hedging only converges to the analytical optimal hedging strategy stated above when the risk averse parameter $\lambda \to \infty$. When λ does not tend to infinity, the hedging proves to be more expensive. Employing the variance minimization criteria to find the optimal hedging, the target function we want to minimize is:

$$Var(\Pi_t|\mathscr{F}_t)$$

Taking the investors' attitude toward risk into consideration, the target function we want to minimize is instead

$$\mathbb{E}[\Pi_t + \lambda \sum_{t'=t}^{T} e^{-r(t'-t)} Var(\Pi_{t'}|\mathscr{F}_{t'})|\mathscr{F}_t] \tag{2.2.10}$$

Decomposing the target into sub-minimizing problem at each time step. The optimal hedging strategy is found by

$$x_t^* = \arg\min \mathbb{E}[\Pi_t + \lambda Var(\Pi_t)|\mathscr{F}_t] \tag{2.2.11}$$

Mimic the derivation of proposition 2.2.1, the optimal hedging strategy with risk averse parameter is then

$$x_t^* = \frac{\rho\sigma_s V' + \frac{1}{2\lambda\sigma_h}[\theta_h(\mu_h - H) + rH]}{\sigma_h} \tag{2.2.12}$$

Compare the optimal hedging strategy with and without risk averse parameter under OU processed:

- with λ: $x_t^* = \frac{\rho\sigma_s V' + \frac{1}{2\lambda\sigma_h}[\theta_h(\mu_h - H) + rH]}{\sigma_h}$
- without λ: $\frac{\rho\sigma_s}{\sigma_h} V'$

When λ does not tends to infinity, the price is no longer a risk-free price, but a trade-off between the expected return and the extra risks need to be taken. It can be seen that when the investors are willing to have higher expectation of portfolio value at the cost of taking extra risk, the holding position of tradable asset in the hedging portfolio becomes higher.

2.2.3 Path generation

To generate sample paths, here we employ the Euler Scheme, a simple and robust scheme of generating samples out of a stochastic differential equation (SDE) or ordinary differential equation (ODE). Given a stochastic differential equation, Euler-Maruyama method (also known as Euler Scheme) is a method for the approximate numerical solution. By partitioning the interval [0, T] into N equal subintervals, it provide the approximation to the true solution random variable by a Markov chain. The method is explained in the case of OU process as follows.

We apply the Euler Scheme to generate paths using the discretized equation to the SDE of Ornstein-Uhlenbeck process :

$$S_{t_i} = S_{t_{i-1}} + \theta_s(\mu_s - S_{t_{i-1}})\Delta t + \sigma_s W_{\Delta t}$$

$$H_{t_i} = H_{t_{i-1}} + \theta_h(\mu_h - H_{t_{i-1}})\Delta t + \sigma_h W'_{\Delta t}$$

The sample generating process is stated as follows:

- Decide a number of n to get grid points: $0 = t_1 < t_2 < \cdots < t_{n-1} < t_n = t_T$, and time step $\Delta t = t_{i+1} - t_i$
- Drawing $(n-1) \times m$ pairs of independent samples (Z_i^s, Z_i^h) for $i = 1, \ldots, n$ from a Multi-Normal distribution $N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}\right)$, where m represents the number of paths, and then reshape the list of samples into a matrix with dimensional $(n-1) \times m$.

- For $i = 1, \dots, n$ and $j = 1, \dots, m$, compute

$$S_i^j = S_{t_{i-1}}^j + \theta_s(\mu_s - S_{t_{i-1}}^j)\Delta t + \sigma_s\sqrt{\Delta t}Z_{i-1,j}^s$$

$$H_i^j = H_{t_{i-1}}^j + \theta_h(\mu_h - H_{t_{i-1}}^j)\Delta t + \sigma_h\sqrt{\Delta t}Z_{i-1,j}^h$$

And matrix of underlying and tradable asset price evolution could then be obtained.

2.2.4 Practical experiment

Here we keep the same parameters as in Table 2.1 and add extra parameters for OU process as stated in Table 2.3.

General Features	Values
Strike price K	105
Mean reversion speed θ_s and θ_h	[0.1, 0.1]
Mean reversion level μ_s and μ_h	[150, 150]
Correlation ρ	[0.3, 0.5, 0.7]

Table 2.3: European call option extra parameters for OU process

As stated before, we expect our algorithm performs well under arbitrary correlation ρ and also in the case with difference between volatilities between σ_s and σ_h. Figure 2.5 shows the resulting optimal hedging strategy (left) and the Q-Q plots of closed-form benchmark, proposed by equation 2.2.3, against the optimal hedging strategy given by equation 1.1.7 under different correlation between underlying asset and tradable asset $\rho \in \{0.3, 0.5, 0.7\}$. According to the results shown, the holding position of tradable asset in the hedging portfolio would rise along with the increase of correlation between two assets. And two deltas match perfectly under each level of correlation, suggesting that our algorithm works well under OU process.

We manage to illustrate that the newly proposed optimal hedging strategy works in the cases of GBM and OU process mathematically and practically, suggesting that the optimal hedging proposed by equation (1.1.7) is model-free. We then can apply more complex models in following sections, which have no closed-form formula to provide benchmarks.

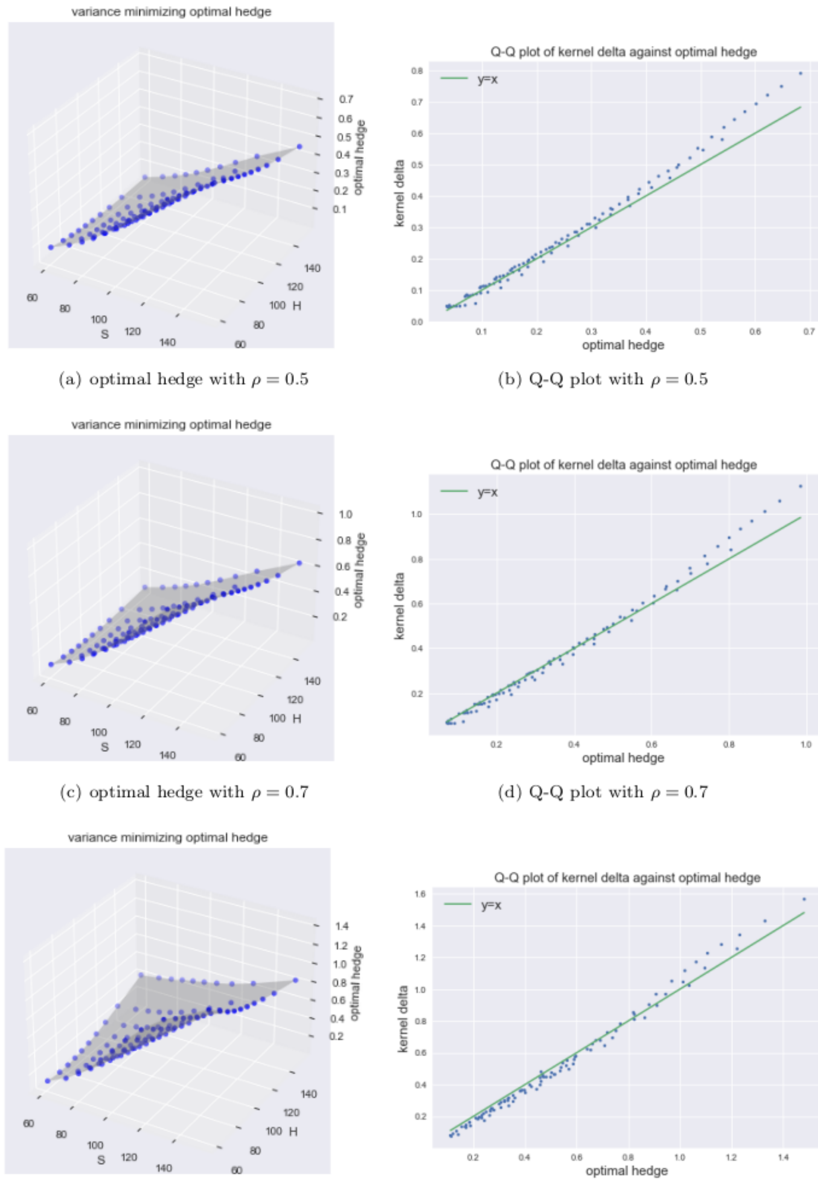
2.3 Copulas

A copula of d-dimension is a joint cumulative distribution function on the uni hypercube $[0,1]^d$ with uniform marginal distributions. It provides a method of building multivariate distributions by separately choosing the marginal distributions and the dependence structure.

Here we use the copula to learn the dependence structure of historical daily return of the pair of assets, namely underlying asset and tradable asset, and then use the fitted model to generate synthetic predictions of future daily returns, thus obtain a matrix of asset price evolution as before.

- Fit copulas with historical daily return of Apple and S&P500 from 01/01/2000 to today.
- Using fitted model to generate predicted daily returns with number of $(n-1) \times N$, where n represents the time steps in total, whereas N is the number of paths.
- Set initial stock price $P_0 = [\text{AAPL}_0, \text{SP500}_0]$ be the close price of Apple and S&P500 today, and $P_i = P_{i-1} \times (1 + returns)$, for $i = 1, \dots, n$.

By doing the steps stated above, the matrix of asset price evolution is then obtained. Inputting the price matrix and implementing algorithm 2, the optimal hedging strategy is presented in figure 2.6, where a similar shape as before is observed.



(a) optimal hedge with $\rho = 0.5$

(b) Q-Q plot with $\rho = 0.5$

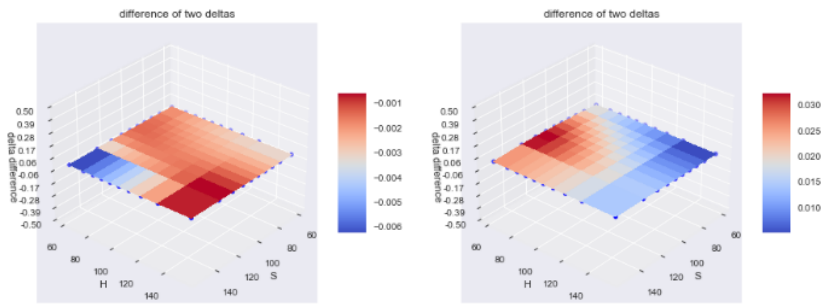
(c) optimal hedge with $\rho = 0.7$

(d) Q-Q plot with $\rho = 0.7$

(e) optimal hedge with $\rho = 0.9$

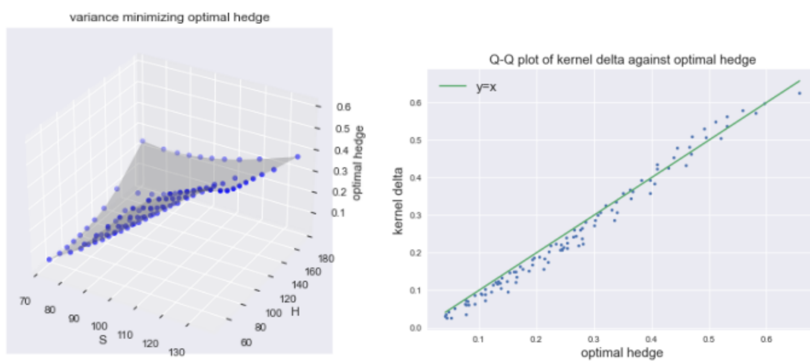
(f) Q-Q plot with $\rho = 0.9$

Figure 2.2: Comparison of two deltas under different correlation $\rho \in \{0.5, 0.7, 0.9\}$

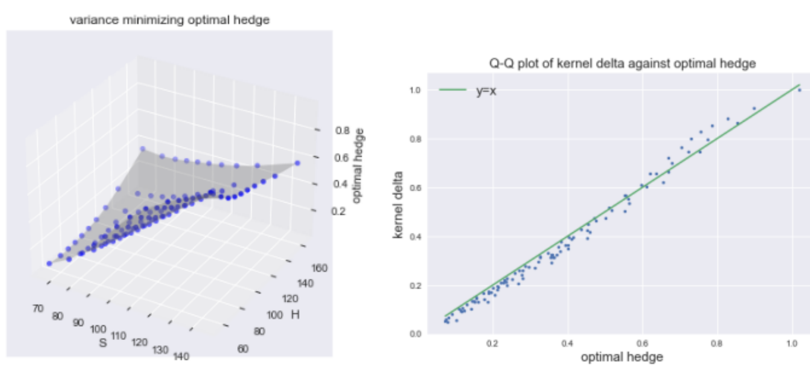


(a) difference between two deltas with $\rho = 0.5$ (b) difference between two deltas with $\rho = 0.7$

Figure 2.3: Difference between two deltas under different correlation ρ presented in 3-dim figures

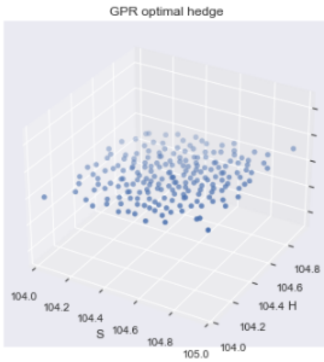


(a) optimal hedge with $\sigma = 0.2, \sigma' = 0.4$ (b) Q-Q plot with $\sigma = 0.2, \sigma' = 0.4$



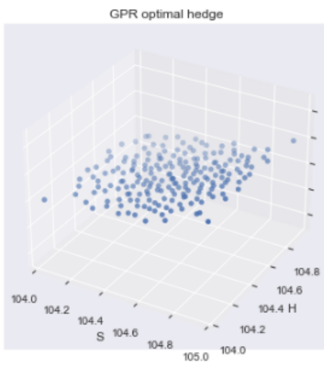
(c) optimal hedge with $\sigma = 0.25, \sigma' = 0.35$ (d) Q-Q plot with $\sigma = 0.25, \sigma' = 0.35$

Figure 2.4: Comparison of two deltas under different pairs of volatilities σ and σ'

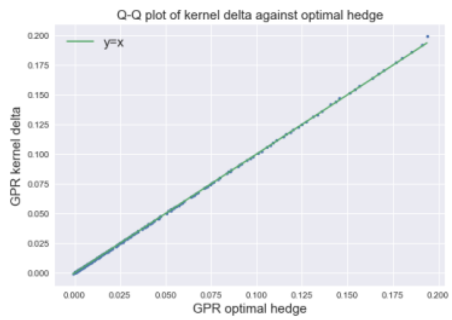


(a) optimal hedge with $\rho = 0.3$

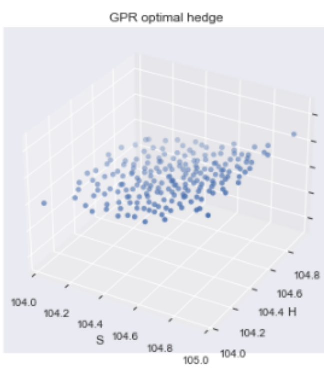
(b) difference between two deltas with $\rho = 0.3$



(c) optimal hedge with $\rho = 0.5$



(d) difference between two deltas with $\rho = 0.5$



(e) optimal hedge with $\rho = 0.7$

(f) difference between two deltas with $\rho = 0.7$

Figure 2.5: Result with risk averse parameter $\lambda = 10^7$ and partial correlation $\rho \in \{0.3, 0.5, 0.7\}$ under OU process

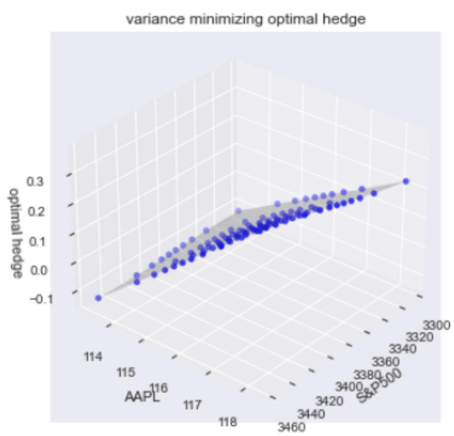


Figure 2.6: Optimal hedging strategy using Copulas of AAPL and SP500

Chapter 3

CVA Hedging in the Incomplete Market

Credit Valuation Adjustment (CVA) is a change to the market value of derivative instruments to account for counterparty credit risk. It represents the discount to the standard derivative value that a buyer would offer after taking into account the possibility of a counterparty's default. CVA is the most widely known of the valuation adjustments, collectively known as xVA. Putting it another way, it can be seen as a price that an investor would pay to hedge the counterparty credit risk of a derivative instrument. It gives the expectation of future loss caused by the default of counterparty, discounted to the current time under risk-neutral measure. We are interested in hedging CVA in the incomplete market. This chapter aims to find a model-free hedging strategy for a CVA, written on one underlying asset but is hedged with another partially correlated asset. Followed by introducing CVA and its hedging, the model-free hedging strategy is derived. Adding model assumptions for underlying asset price evolution and default intensity, the newly proposed hedging strategy is compared with analytical optimal hedging under each scenario to validate the model-free property of newly proposed hedging strategy.

3.1 Credit Valuation Adjustment and CVA Hedging

When an entity enters an OTC contract with a counterparty, the fact that the counterparty may default before the final settlement of contract exposes the entity to the counterparty default risk. As such, the counterparty might not respect its payment obligations.

Definition 3.1.1. (Counterparty credit risk) Referring to Basel II[18], the counterparty credit risk is defined as the risk that the counterparty to a transaction could default before the final settlement of the transaction's cash flows. An economic loss would occur if the transactions or portfolio of transactions with the counterparty has a positive economic value at the time of default.

Pricing of counterparty credit risk leads to the notion of Credit Valuation Adjustment (CVA). According to the party concerned, it can be divided into two categories: unilateral CVA and bilateral CVA, which assumes only counterparty can default and considers the bilateral nature of counterparty credit risk respectively. In this chapter, we focus on unilateral CVA and assumes one side of transaction parties is defaultable whereas the other is default-free. There are two ways of valuating risk: the default time approach (DTA) and the default probability approach (DPA). The DTA valuates the default time explicitly. It is an intuitive way but the lack of complete information about the entity's default time makes this approach unlikely to implement. DPA focuses on the probability distribution of default time instead of the default time itself, and sometimes leads to simple closed-form solutions. Dividing the time period into very small time intervals, we can assume that the counterparty could only default at the end of each small period.

Mathematically, under default probability approach, CVA is formulated as

$$\mathrm{CVA}_t = (1 - Rec)\mathbb{E}[\int_t^T D(t,s)V_s^+ \delta_\tau(ds)] \tag{3.1.1}$$

, where Rec is the client recovery rate, $D(0,t_i)$ represents the discount factor from time t_i back to 0 , $V_t^+ := \max(0, V_t)$ and $\delta_\tau(dt)$ denotes the probability of default time $\tau \in (t, t+dt)$.

Denote the default intensity by γ_t. For the counterparty to default in the time interval $(t, t+dt)$, it has to first survive the period $(0,t)$ with probability $e^{-\int_0^t \gamma_s ds}$ and then default in $(t, t+dt)$ with probability $\gamma_t dt$. Thus $\delta_\tau(dt) = e^{-\int_0^t \gamma_s ds}\gamma_t dt$, as a result of which,

$$\text{CVA}_t = (1 - Rec)\mathbb{E}[\int_t^T D(t,s)V_s^+ e^{-\int_t^s \gamma_u du}\gamma_s ds] \tag{3.1.2}$$

, with boundary condition

$$\text{CVA}_T = \lim_{t \to T} \text{CVA}_t \to 0 \tag{3.1.3}$$

As CVA measures the future default risk, it shrinks to 0 at maturity along with the close-out of the underlying financial derivative.

Note that when the underlying derivative is a European call option, the discounted option value

$$D(t,t_i)V_{t_i}^+ = e^{-r(t_i - t)}(E^{-r(T-t_i)}V_T) = V_t \text{ , for } t_i \in (t,T) \tag{3.1.4}$$

Plugging equation (3.1.4) back to equation (3.1.2) and switching to the discrete scenario, for intermediate time step $t = 0, \ldots, T$, CVA is formulated as

$$\begin{aligned}\text{CVA}_t &= (1 - Rec)\sum_{i=t+1}^{N} D(t,t_i)V_{t_i} e^{-\sum_{l=t}^{l<i-1}\gamma_i \Delta t}\gamma_{i-1}\Delta t \\ &= (1 - Rec)V_t \sum_{i=t+1}^{N} e^{-\sum_{l=t}^{l<i-1}\gamma_i \Delta t}\gamma_{i-1}\Delta t \end{aligned} \tag{3.1.5}$$

Re-structure equation (3.1.5) and the recursive relationship of CVA could be derived as follows.

$$\begin{aligned}\text{CVA}_t &= (1 - Rec)\sum_{i=t+1}^{N} D(t,t_i)V_{t_i} e^{-\sum_{l=t}^{l<i-1}\gamma_i \Delta t}\gamma_{i-1}\Delta t \\ &= (1 - Rec)V_t \sum_{i=t+1}^{N} e^{-\sum_{l=t}^{l<i-1}\gamma_i \Delta t}\gamma_{i-1}\Delta t \\ &= (1 - Rec)\Delta t e^{-r\Delta t}V_{t+1}[\gamma_t + e^{-\gamma_t \Delta t}\sum_{i=t+2}^{N} e^{-\sum_{l=t+1}^{l<i-1}\gamma_i \Delta t}\gamma_{i-1}] \\ &= (1 - Rec)V_t\gamma_t\Delta t + e^{-(r+\gamma_t)\Delta t}\text{CVA}_{t+1} \end{aligned} \tag{3.1.6}$$

And we can compute the value of CVA backward from $\text{CVA}_T = 0$.

CVA is a credit hybrid option on the contingent exposure of a derivative contract or a portfolio of derivative contracts. Like other options products, CVA can be hedged via dynamic hedging by delta hedging with the underlying derivative (and/or option on the underlying derivative) and credit default swap (CDS).

Assume there is a CVA for a European call option writing on an underlying asset S_t, but is hedged with another partially correlated asset H_t and risk-free bank account B_t. Setting up the hedging portfolio:

$$\Pi_t = x_t H_t + B_t$$

, where x_t represents the holding position of asset H_t in the hedging portfolio at time t.

At time T, the holding position x_T should be zero as we want to alter the contingent value of portfolio into certain wealth at maturity, eliminating the risk. Thus the boundary condition is then

$$B_T = \Pi_T = \text{CVA}_T = 0$$

As stated in section 1.1, the optimal hedging strategy proposed by equation (1.1.7):

$$x_t^* = \frac{E_t[\Delta\hat{H}_t\hat{\Pi}_{t+1} + \frac{1}{2\lambda e^{-r\Delta t}}\Delta H_t|\mathscr{F}_t]}{E_t[(\Delta\hat{H}_t)^2|\mathscr{F}_t]}$$

still stands with a little adjustment, namely calibrating the expectation of hedging strategy to the value of CVA at each time step $t_i \in (t_0, \ldots, t_T)$

$$\text{CVA}_t = \mathbb{E}[\Pi_t|\mathscr{F}_t] \text{ ,} \textit{with boundary condition } \Pi_T = 0$$

3.2 CVA Hedging with Black-Scholes-Merton Assets and OU Default Intensity

In this section, we add model assumptions to CVA setting, and compare the closed-form optimal hedging strategy with the newly proposed hedging strategy to verifies the robustness of the latter. Assume the financial derivative to which we want to do the credit valuation adjust is a European call option written on a stock S_t, and is hedged with a partial correlated stock H_t. And the default intensity of counterparty is γ_t. Suppose asset price diffusion obeys Black-Scholes-Merton model and the default intensity follows an OU process under risk-neutral measure:

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t \\ dH_t &= rH_t dt + \sigma' H_t dW'_t \\ d\gamma_t &= \theta_\gamma(\mu_\gamma - \gamma_t)dt + \sigma_\gamma dW_t^\gamma \end{aligned} \tag{3.2.1}$$

, with $dW_t dW'_t = \rho dt$ to depict the partial correlation between underlying asset and hedging asset.

To find the hedging strategy, the sensitivities of CVA with respect to its risk factor (e.g. stock price diffusion and the volatile of default intensity) need to be computed. Suppose the underlying derivative is a European call option whose price is denoted by V_t. The terminal value is given by the payoff function of European call option: $V_T = (S_T - K)^+$. Adding the model assumptions into equation (3.1.5), assume default intensity γ_t is independent of underlying option value V_t and CVA_t is differentiable with respect to H_t, then the pathwise delta of CVA_t with respect to tradable asset H_t is

$$\begin{aligned} \Delta_t = \frac{\partial \mathbb{E}[\text{CVA}_t]}{\partial H_t} &= \mathbb{E}[\frac{\partial\ \text{CVA}_t}{\partial H_t}] \\ &= \mathbb{E}[\frac{\partial}{\partial H_t}(1-Rec)V_t \sum_{i=t+1}^{N} e^{-\sum_{l=t}^{l<i-1}\gamma_i \Delta t}\gamma_{i-1}\Delta t] \\ &= \mathbb{E}[(1-Rec)\sum_{i=t+1}^{N} e^{-\sum_{l=t}^{l<i-1}\gamma_i \Delta t}\gamma_{i-1}\Delta t \frac{\partial V_t}{\partial S_t}\frac{\partial S_t}{\partial H_t}] \\ &= \mathbb{E}[(1-Rec)\sum_{i=t+1}^{N} e^{-\sum_{l=t}^{l<i-1}\gamma_i \Delta t}\gamma_{i-1}\Delta t \frac{\partial e^{-r(T-t)}V_T}{\partial S_T}\frac{\partial S_T}{\partial S_t}\frac{\partial S_t}{\partial H_t}] \\ &= \mathbb{E}[(1-Rec)\sum_{i=t+1}^{N} e^{-\sum_{l=t}^{l<i-1}\gamma_i \Delta t}\gamma_{i-1}\Delta t e^{-r(T-t)}\mathbf{1}_{\{S_T>K\}}\frac{\rho\sigma S_T}{\sigma' H_t}] \end{aligned} \tag{3.2.2}$$

Crépey managed to compute CVA using Gaussian process regression (2019)[19], and we aim to achieve the same goal and take a step further, hedging the CVA, with DCKE.

3.2.1 Path Generation

We first assume the underlying asset price follows the celebrated Black-Scholes-Merton model and the default intensity obeys an OU process. To compute the value of CVA, the sample paths of stock price evolution, option value and default intensity are required and needed to be combined. The algorithm of path generation is declared in algorithm 3.

3.2.2 Analytical Optimal Strategy

In this section, we aim to find the optimal hedging strategy under Black-Scholes model assumption, and use it to provide benchmark for newly proposed hedging strategy.

Proposition 3.2.1 (Optimal hedge for CVA when asset price follows BSM model)**.** *Suppose there is a CVA of a European call option writing on an untradable underlying asset S with the strike price K. Assume the default intensity is independent of underlying option value. Setting up a hedging portfolio Π with a partial correlated asset H and risk-free bank account B, when the price evolution follows Black-Scholes-Merton model, the optimal hedging strategy $\{x_t^*\}_{t=0}^T$ is give by*

$$x_t^* = (1-Rec)\sum_{i=t+1}^{N} e^{-\sum_{l=t}^{l<i-1}\gamma_i \Delta t}\gamma_{i-1}\Delta t \frac{\rho\sigma}{\sigma'}\frac{S_t}{H_t}\frac{\partial \mathbb{E}_t^*[e^{-r(T-t)}G(S_T)]}{\partial S_t} \tag{3.2.3}$$

Algorithm 3: Monte Carlo sample paths generation for CVA.

Data: underlying option strike price K; risk-free interest rate r; number of time steps T; length of time step Δt; initial stock price $(P_{t_0} = (S_{t_0}, H_{t_0}))$; volatilities for two asset σ_s, σ_h; partial correlation between underlying asset and tradable asset ρ; initial default intensity γ_0; mean reverting speed θ; mean reverting level μ, volatility of default density σ_γ; number of paths n;

Output: Stock price sequence $P_{t_i} = (S_{t_i}, H_{t_i})$ for $i = 0, \dots, T$ with each length n; option value V_{t_i} for $i = 0, \dots, T$ with each length n; default intensity γ_{t_i} for $i = 0, \dots, T-1$ with each length n; value of credit valuation adjustment CVA_{t_i} for $i = 0, \dots, T$ with each length n

set vector initial stock price $P^j_{t_0} = (S^j_{t_1}, H^j_{t_0})$ for $j = 1, \dots, n$.
set vector initial default density $\gamma^j_{t_0} = \gamma_{t_0}$ for for $j = 1, \dots, n$.
for *time steps from* $i = 1$ *to* $i = T$ **do**

  Draw random numbers $\begin{pmatrix} Z^j \\ Z'^j \end{pmatrix}$ from a Multi-Normal distribution $N(\mu, \mathbf{\Sigma})$ with $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\mathbf{\Sigma} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ for $j = 1, \dots, n$;
  Underlying stock price $S^j_{t_i} = S^j_{t_{i-1}} exp((r - \frac{\sigma_s^2}{2})\Delta t + \sigma_s \sqrt{\Delta t} Z^j)$ for $j = 1, \dots, n$;
  Tradable stock price $H^j_{t_i} = H^j_{t_{i-1}} exp((r - \frac{\sigma_h^2}{2})\Delta t + \sigma_h \sqrt{\Delta t} Z'^j)$ for $j = 1, \dots, n$;
  $P^j_{t_i} = (S^j_{t_i}, H^j_{t_i})$ for $j = 1, \dots, n$.

for *time steps from* $i = 1$ *to* $i = T-1$ **do**

  Draw random numbers Z^j from Gaussian distribution $N(0, 1)$ for $j = 1, \dots, n$;
  $\gamma^j_{t_i} = \gamma^j_{t_{i-1}} + \theta_\gamma(\mu - \gamma^j_{t_{i-1}})\Delta t + \sigma_\gamma \sqrt{\Delta t} Z^j$.

Set option value at maturity $V^j_{t_T} = (S_{t_T} - K)^+$ for $j = 1, \dots, n$.
Set CVA at maturity $\text{CVA}_{t_T} = 0$ for $j = 1, \dots, n$.
for *time steps from* $i = T-1$ *to* $i = 0$ *backward* **do**

  Option value $V^j_{t_i} = e^{-r\Delta t} V^j_{t_{i+1}}$ for $j = 1, \dots, n$;
  $\text{CVA}^j_{t_i} = (1 - Rec)V_{t_i}\gamma_{t_i}\Delta t + e^{-(r+\gamma_{t_i}\Delta t}\text{CVA}_{t_{i+1}}$ for $j = 1, \dots, n$.

Result: $\gamma^j_{t_i}$ for $i = 0, \dots, T-1$ and $j = 1, \dots, n$;
$P^j_{t_i}$;$V^j_{t_i}$; $\text{CVA}^j_{t_i}$ for $i = 0, \dots, T$ and $j = 1, \dots, n$

, where the payoff function $G(S_T) = (S_T - K)^+$.

Proof. Define $f(\gamma_t, t) := (1 - Rec) \sum_{i=t+1}^{N} e^{-\sum_{l=t}^{l \le i-1} \gamma_l \Delta t} \gamma_{i-1} \Delta t$, then the credit valuation adjustment becomes $\text{CVA}_t = f(\gamma_t, t)V_t$.

Suppose we take a position of shorting one unit of CVA for a European call option whose fair value is denoted by V_t. Thus the hedging error is $Y_t := \Pi_t - \text{CVA}_t$.

$$Y = xH + B - \text{CVA}$$

Assume $B = \text{CVA} - xH$ at time t, the change of hedging error over the time period $[t, t+dt]$ could be derived as follows.

$$\begin{aligned} dY &= -[f(\gamma_t, t)(\dot{V} + \mu SV') + \frac{\sigma^2 S^2}{2} V'' f(\gamma_t, t)^2]dt - \sigma SV' f(\gamma_t, t)dW + r(V - xH)dt + x(\mu' H dt + \sigma' H dW') \\ &= -[f(\gamma_t, t)(\dot{V} + \mu SV') + \frac{\sigma^2 S^2}{2} V'' f(\gamma_t, t)^2 - r(V - xH) - x\mu' H]dt - \sigma SV' f(\gamma_t, t)dW + x\sigma' H dW' \end{aligned}$$

, where $\dot{V} = \frac{\partial V}{\partial t}$, $V' = \frac{\partial V}{\partial S}$ and $V'' = \frac{\partial^2 V}{\partial S^2}$.
Referring to Ito isometry, the variance of dY is

$$\begin{aligned} Var(dY) &= \mathbb{E}[(-\sigma SV' f(\gamma_t, t)dW + x\sigma' H dW')^2] \\ &= [x^2 \sigma'^2 H^2 + \sigma^2 S^2 V'^2 f(\gamma_t, t)^2 - 2\sigma SV' x\sigma' H \rho f(\gamma_t, t)]dt \end{aligned}$$

Observing the variance of hedging error derived above, it can be seen as a quadratic function of x_t. Employing the strategy of variance-minimization, the optimal hedging could be found as

$$x_t^* = \arg\min Var(dY) = f(\gamma_t, t)\frac{\rho\sigma}{\sigma'}\frac{S_t}{H_t}V' \tag{3.2.4}$$

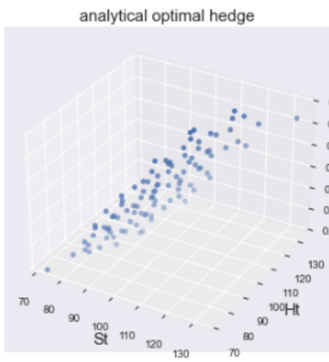
Conduct Monte Carlo simulation, $V_T = \mathbb{E}_T[(S_T - K)^+]$, and then the partial derivative $V' = \frac{\partial \mathbb{E}_t^*[e^{-r(T-t)}(S_T-K)^+]}{\partial S_t}$, substituting V' and $f(\gamma_t, t)$ back to the equation (3.2.4), the result is derived as required. □

3.2.3 Practical Experiment

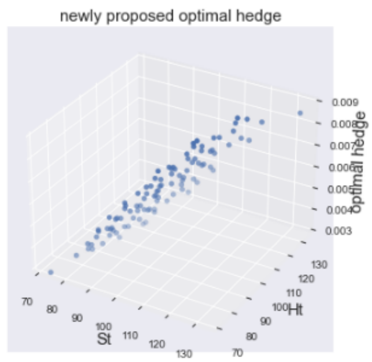
Following the parameters in Table 3.1, same as investigating the hedging for a European call option, a 2-dimensional DCKE is implemented to find the hedging strategy for CVA at each intermediate time step as stated in algorithm 4. Both the newly proposed hedging strategy and analytical optimal hedging strategy under model assumptions specified before are computed and then compared. The two hedging results at intermediate time step $t = 0.4$ are shown in figure 3.1 (panel (a) and (b) respectively), where similar shapes are observed. To get a better understanding of the performance, the Q-Q plot of newly proposed hedging against the analytical optimal hedging is presented in 3.1 panel (c), showing a good consistence of two hedging strategies. In figure 3.2, Q-Q plots of two strategies at all intermediate time steps are shown, by observing which, it is reasonable for us to conclude that the algorithm works well during the whole process.

General Features	Values
Initial stock price (S_{t_0}, H_{t_0})	(100,100)
Strike price K	100
Risk-free rate r	0.01
Volatilities for stocks (σ,σ')	(0.3,0.3)
Correlation ρ	0.7
Risk averse λ	10^7
Initial default intensity γ_0	0.05
Mean reverting speed for default intensity θ_γ	1
Mean reverting level for default intensity μ_γ	0.05
Volatility for default intensity σ_γ	0.01
Maturity time T	1 year
Time step t_i	[0, 0.2, 0.4, 0.6, 0.8, 1.0]
Number of path n	20000

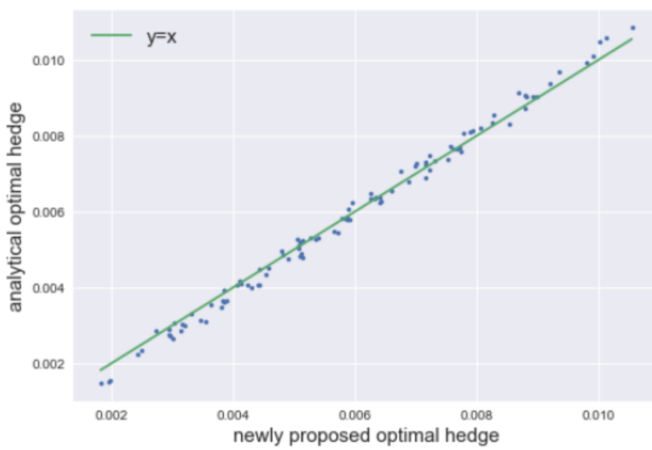
Table 3.1: CVA parameters for DCKE



(a) newly proposed optimal hedge at t=0.4



(b) analytical optimal hedge at t=0.4



(c) Q-Q plot of kernel delta against optimal hedge at t=0.4

Figure 3.1: Results of CVA hedging at time $t = 0.4$ using DCKE

Algorithm 4: 2-Dim DCKE of hedging CVA.

Data: underlying option strike price K; risk-free interest rate r; number of time steps T; length of time step Δt; number of paths n; number of meshed samples u; price evolution sequence $(P_{t_i} = (S_{t_i}, H_{t_i}))$ with each length n for $i = 0, \dots, T$; price evolution sequence $(P^*_{t_i} = (S^*_{t_i}, H^*_{t_i}))$ with each length n^* for $i = 0, \dots, T$; partial correlation ρ; hyperparameter input for bandwidth adjustment of price h_p; hyperparameter input for bandwidth adjustment of delta h_d; risk averse parameter λ; cap of variable kernel ratio C; default intensity γ

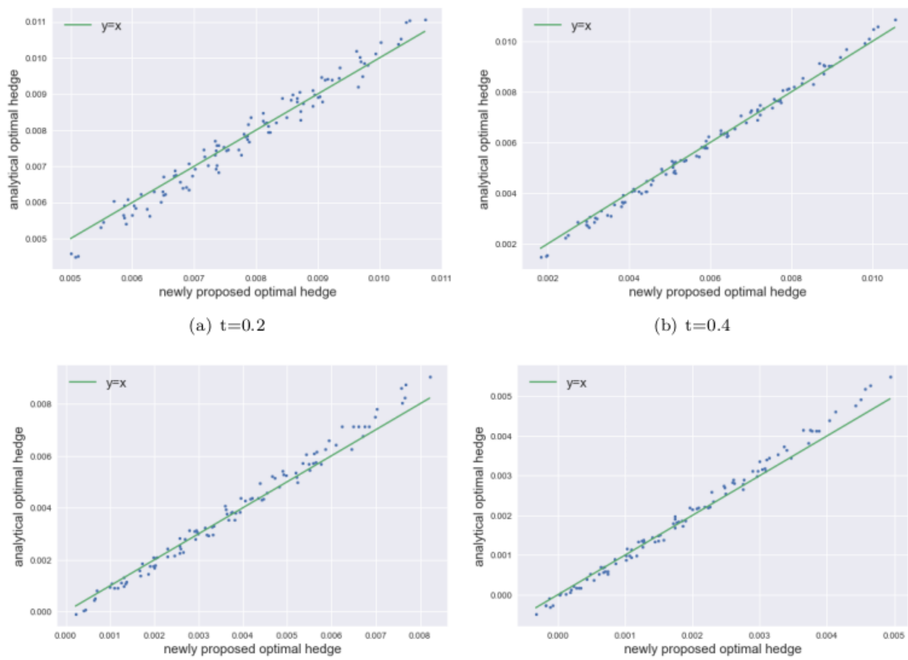
Output: mesh points of spot price sequence $M_{t_i} = (M_{t_i,S}, M_{t_i,H})$; coefficient for control variates $\hat{\beta}_p$ for pricing and $\hat{\beta}_d$ for hedging; option value sequence V_{t_i}; CVA sequence CVA_{t_i}; kernel estimation of delta x'_{t_i}; optimal hedge position x_{t_i} derived by variance minimisation with each length n for $i = 0, \dots, T$

Set arrays: $V^j_{t_T} = (S^j_{t_T} - K)^+$; $\mathrm{CVA}^j_{t_T} = 0$ $\hat{V}^j_{t_i}$; $\hat{\mathrm{CVA}}^j_{t_i}$; $\hat{x}_{t_i}$; $\hat{x}'_{t_i}$;
M_{t_i}: samples between 5% - 95% percentiles of generated spot prices
$M^*_{t_i}$: samples between 5% - 95% percentiles of generated spot prices
$\Sigma = cov(P_{t_1}, P_{t_2})$ with $LL^T = \Sigma^{-1}$
$L' = Ln^{-\frac{1}{d+4}}$
$L'_p = L' * h_p$
$L'_d = L' * h_d$

$h = 0.9 \min(\sqrt{Var(S_{t_T})}, \frac{IQR(S_{t_T})}{1.34})$;
for *time steps from* $i = T-1$ *to* $i = 0$ *backward* **do**
  for *each mesh point* $k = 1, \dots, u$ **do**
    $K^k_p = \sum_{j=1}^n e^{-\sum_{v=1}^d (P_{t_2} - M^k) {L'_p}^2}$
  $K^k_{\text{ratio}} = \min(\frac{K^*}{K^k_{t_i}}, C)$
  variable kernel $h_{\text{new}} = hK^k_{\text{ratio}}$
  $\hat{\mathrm{CVA}}_{t_{i+1}} = \hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; \mathrm{CVA}_{t_{i+1}})$ (kernel estimator $\mathbf{m}$ defined in equation 1.2.2)
  $\hat{\Delta H}^k_{t_{i+1}} = \hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; \Delta H^k_{t_{i+1}})$
  $\Pi_{t_{i+1}} = \mathrm{CVA}_{t_{i+1}}$
  $\hat{\beta}_p = \frac{\hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; (\Pi_{t_{i+1}} - \hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; \Pi_{t_{i+1}}))(\Delta H^k_{t_{i+1}} - \hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; \Delta H^k_{t_{i+1}})))}{\hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; (\Delta H^k_{t_{i+1}} - \hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; \Delta H^k_{t_{i+1}}))^2)}$
  $\hat{x}^k_{t_i} = \frac{\hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; (\Pi_{t_{i+1}} - \hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; \Pi_{t_{i+1}}))(\Delta H^k_{t_{i+1}} - \hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; \Delta H^k_{t_{i+1}})) + \frac{1}{2\lambda\gamma}\Delta H_{t_i})}{\hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; (H^k_{t_{i+1}} - \hat{\mathbf{m}}_{h_{\text{new}}}(M^k_{t_i}, S_{t_i}; \Delta H^k_{t_{i+1}}))^2)}$
  $x'^k_{t_i} = (1 - Rec)(t_{i+1} - t_i) \sum_{i=t+1}^N \gamma_{i-1} e^{-\sum_{l=t}^{i-2} \gamma_l (t_{i+1} - t_i)} e^{-r(t_T - t_i)} \mathbf{1}_{\{S^j_{t_T} > K\}} \frac{\rho\sigma S_{t_T}}{\sigma' H_{t_i}}$
  $\hat{x}'^k_{t_i} = \hat{\mathbf{m}}_h(M_k, H_{t_i}, S_{t_i}; x'^k_{t_i})$
  $\hat{\beta}_d = \frac{Cov(\hat{x}'^k_{t_i}, \Delta H_{t_i} | \mathscr{F}_t))}{Var(\Delta H_{t_i} | \mathscr{F}_t)}$
  $\hat{\mathrm{CVA}}_{t_i} = e^{-r\Delta t} GPR(RBF, M^*_{t_i}, \hat{\mathrm{CVA}}_{t_i} - \hat{\beta}_p(\hat{H}_{t_{i+1}} - e^{r\Delta t} M_{t_i,H}))$
  $\hat{x}^*_{t_i} = GPR(RBF, M^*_{t_i}, \hat{x}_{t_i} - \hat{\beta}_d(\hat{H}_{t_{i+1}} - e^{r\Delta t} M_{t_i,H}))$
  $\hat{x}'^*_{t_i} = GPR(RBF, M^*_{t_i}, \hat{x}'_{t_i} - \hat{\beta}_d(\hat{H}_{t_{i+1}} - e^{r\Delta t} M_{t_i,H}))$
  $V_{t_i} = e^{-r(t_{i+1} - t_i)} V_{t_{i+1}}$
  $\mathrm{CVA}_t = (1 - Rec) V_{t_i} \gamma_{t_i} (t_{i+1} - t_i) + e^{-(r + \gamma_{t_i})(t_{i+1} - t_i)} \mathrm{CVA}_{t_{i+1}}$

Compute the optimal hyperparameter by minimising validation error
$h^*_p, h^*_d = \arg\min_{h_p, h_d \sim U(0.01,5)} ||\hat{x}^*_{t_i} - \hat{x}'^*_{t_i}||$ via *hyperopt*
Repeat the process above with hyperparameter h^*_P and h^*_d.

Result: $\hat{\mathrm{CVA}}_{t_i}$; $\hat{x}^*_{t_i}$; $\hat{x}'^*_{t_i}$



(a) t=0.2

(b) t=0.4

(c) t=0.6

(d) t=0.8

Figure 3.2: Q-Q plots of kernel delta against optimal hedge for CVA at intermediate time steps

Conclusion

In this paper, by drawing inspiration from Q-learning and Black-Scholes discrete-time hedging, we propose a model-free way of finding the optimal hedging strategy in a general incomplete market where a hedger aims at reducing the risk of a non-tradable asset on a contingent claim. When deriving the newly proposed hedging strategy, we employ the criterion of variance minimization and refer to the framework of Q-learning, breaking the complex optimization problem into simple, recursive sub-problems to find the optimal solution. The only condition used during the derivation is that the hedging portfolio is self-financing, no assumption of assets' price evolution or the assumption of target financial derivatives' value evolution entered into the derivation. As a result, the optimal hedging strategy derived can be applied to any scenario without the constraint of underlying model assumptions and can be applied to hedge various financial derivatives, such as options and CVA.

The newly proposed hedging strategy is a ratio of two conditional expectations. Two methodologies are then introduced to compute the conditional expectation: Least Squares Monte Carlo (LSM) and Dynamically Controlled Kernel Estimation (DCKE). LSM is the standard method for computing conditional expectations in the industry currently. After comparing the pricing and hedging strategy of a European call option with the closed-form result provided by Black-Scholes pricing and delta-hedge, it can be seen that the large fitting error appears with a high underlying asset price. LSM gives a better fitting in the "out-of-money" case than in the "in-the-money" case. Also, the derivation around tails can be observed. To overcome these drawbacks, we turned to DCKE, a brand-new algorithm for conditional expectation computation, which outperforms the existing methods in terms of convergence speed and the fitting at "tails". It is a combination of kernel regression, control variates, and Gaussian process regression.

With DCKE, we computed the newly proposed hedging strategy and then compared it with the model-specified analytical optimal hedging strategies for simple models. For DCKE, we can choose the degree of kernel regression. When the degree $p = 0$, the kernel estimator is also known as Nadaraya-Watson (NW) regression. It gives a good fit but deviates around tails to some extent. And the fitting is further improved when the degree of kernel estimation increases. When the degree $p = 1$, it is known as the Locally Linear (LL) regression. Both of these two estimators are biased. Compared with NW regression, it provides smaller bias under the same level of variance of fitting errors, helping improve the fitting further. We then stick to the choice of Locally Linear regression. The model-free property is verified in the case of the Black-Scholes-Merton model and Ornstein–Uhlenbeck process respectively. The consistency of them suggests our algorithm is generic, and it works under various models. What makes this algorithm valuable is that it can be applied to more complex models and other non-parametric models, whose closed-form formula for optimal hedging is hard to compute analytically or does not exist. Suppose there is an option written on Apple stock price and is hedged with S&P500. We choose Copulas to depict the dependence and correlation of two datasets. We calibrate the Copulas using the historical daily returns of S&P500 and Apple and then use it to generate future predictions. The hedging strategy is computed and a similar shape of the hedging strategy is observed.

We then switch the research towards CVA and focus on unilateral CVA. It is a change to the market value of derivative instruments to account for counterpart credit risk. There are two measures to evaluate risks: the default time approach (DTA) and the default probability (intensity) approach (DPA). Due to the lack of information on enterprises' default time, the DTA is unlikely to be implemented. We employ the default probability approach to evaluate risk. Imposing simple model assumptions: the assets price diffusion follows a Geometric Brownian motion whereas the default intensity obeys an OU process, the analytical optimal hedging strategy is computed. And the newly proposed hedging strategy is also computed with DCKE. These two strategies agree with each other at each intermediate time step. The accordance of them again illustrates that the newly proposed hedging strategy not only can be used to hedge options, but also CVA. Combining the hedging for risk-free option and CVA, the credit value adjusted option can therefore be hedged.

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Instructor

PAGE 1

PAGE 2

PAGE 3

PAGE 4

PAGE 5

PAGE 6

PAGE 7

PAGE 8

PAGE 9

PAGE 10

PAGE 11

PAGE 12

PAGE 13

PAGE 14

PAGE 15

PAGE 16

PAGE 17

PAGE 18

PAGE 19

PAGE 20

PAGE 21

PAGE 22

PAGE 23

PAGE 24

PAGE 25

PAGE 26

PAGE 27

PAGE 28

PAGE 29

PAGE 30

PAGE 31

PAGE 32

PAGE 33

PAGE 34

PAGE 35

PAGE 36

PAGE 37

PAGE 38

PAGE 39

PAGE 40

PAGE 41

PAGE 42

PAGE 43

PAGE 44

PAGE 45