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Utility maximization

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Abstract

The aim of this thesis is to study portfolio optimisation problems combining a risk free asset with risky assets. A single period model with finite probability space is first considered. The convex duality approach is used to find the optimal trading strategy that maximises the final wealth.

A time-continuous model is then assumed, where the stock prices are driven by stochastic differential equations. The Dynamic Programming Principle and some of the related results are stated and used to solve the classical Merton's problem where we seek to maximize the expected utility of consumption under an infinite time horizon assuming a deterministic interest rate and deterministic stock volatilities.

These assumptions are then relaxed. A Hobson and Rogers stochastic volatility model is considered, where the volatility is expressed as an exponentially-weighted mean of historic log-prices. The HJB equation is used to derive a non linear ODE of the value function which can be linearised by a change of variable. Then assuming a finite time horizon T, Feynman-Kac theorem provides a solution of the ODE, and by taking the limit $T \to \infty$ and considering a transversality condition, the solution of the ODE can be written as the expectation of an Ito process. Finally, a numerical example is provided to illustrate the results, where the expectation is approximated by Monte Carlo simulations and the optimal consumption and investment strategies and the effects of varying the parameters on these optimums are analysed.

Finally, a consumption-investment problem is considered under a non deterministic interest rate driven by a Vasicek model. The ODE is derived using the HJB equation, which is again non linear and can't be solved analytically. The subsolution-supersolution method developed by Fleming and Pang is used to find an upper and lower bound of the optimal solution.

Keywords: Optimal consumption-investment, HJB equation, Dynamic Programming Principle, Stochastic volatility, Vasicek Interest Rate Model.

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" Gratitude is the memory of the heart. "
Hans Christian Andersen

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The notations below will be used in this thesis:

a.s	Almost surely
$^\top$	Transpose
$^{-1}$	Inverse
tr	Trace
∇	Gradient
D^2	Hessian matrix
$\mathbb{E}$	Expected values
Ω	Probability space
$\mathbb{P}$	Probability measure
$\mathbb{Q}$	Risk-neutral probability measure
$\mathbb{R}$	Real numbers
$\mathbb{R}^n$	Real vectors of dimension n
$\mathscr{M}_{n,d}(\mathbb{R})$	Real matrices of dimensions $n * d$
$u.v$	Scalar product of u and v
V_x	First order partial derivative of V
V_{xy}	Second order partial derivative of V

Table 1: Notations

1 Introduction

1.1 Literature review

" Don't gamble; take all your savings and buy some good stock and hold it till it goes up, then sell it. If it don't go up, don't buy it. " - Will Rogers

The objective of every investor is to gain the highest possible return and take the lowest risk. He can build a portfolio by considering two types of assets: a risk-free asset that returns a fixed low rate and consequently doesn't imply any risk and a risky asset that gives a preeminent expected return against a higher risk. Depending on his preferences, the investor will either invest all of his wealth in one of these assets or allocate it between the two assets and have a trade-off between high return and riskiness. But how can he assess the performance of a given strategy?

The first approach that has been used is the expected value of the payoff, until Nicholas Bernoulli introduced the "St. Petersburg Paradox" in 1713. Suppose a gambler can enter the following game: a fair toss coin is tossed until a head appears and the gambler gets 2^n where n is the number of times the coin was flipped. The outcome of this game is $Y = 2^n$ with probability $P(\underbrace{TTTT\cdots}_{n-1 \text{ times}}H) = 2^{-n}$.

The question now is: how much this game is worth ? According to the expected value approach, the gambler can pay :

$$\mathbb{E}(Y) = \sum_{n=1}^{\infty} \frac{1}{2^n} * 2^n = \infty$$

This is absolutely not a reasonable price of the game. In 1738, Nicholas's cousin Daniel Bernoulli came up with a solution that revolutionized the world of finance. He suggested to alter the nominal amount and replace it by the utility of this amount, and proposed to take a logarithmic utility function $U(x) = log(x)$:

$$\mathbb{E}(U(Y)) = \sum_{n=1}^{\infty} \frac{1}{2^n} * U(2^n) < \infty$$

Arrow [2] suggested to take a bounded utility function and De Buffon [10] argued that some sufficiently improbable outcomes are "morally impossible" and should be ignored.

Bernoulli's idea gave birth to the Marginalist Revolution in 1817, and a growth in interest about utility maximization raised since then.

Some researchers were interested in the single period model. In this setting, Markowitz [28] and Tobin [52] showed the *"Efficient Set Theorem"* stating that if the returns are normally distributed and the utility function is concave, then the problem of maximising the expected utility is equivalent to minimizing the standard deviation of the returns for every concave utility function and maximizing the expected returns for any standard deviation of the returns. If the returns are also independent, the theorem holds for the quadratic utility function.

Fama [22] showed an extension of this theorem for a wider class of distributions: symmetric stable Paretian returns [26]. This class of stable distributions was first introduced by Lévy [38].

Elton and Guber [20] proved a necessary condition for expected utility maximization under log-normal distribution: the optimal portfolio should lie on an exterior boundary in mean variance space for all utility function U.
They have also found [19] that maximizing the expected utility is equivalent to maximising the geometric mean in the particular case of logarithmic utility.

The statistical properties of the efficient frontier have been investigated by Jobson [35], Broadie [9], Chopra and Ziemba [15], Best and Ding [5] and MacLean and Weldon [39]. They studied distribution tests and confidence sets to assess the assumption of independence and normal distribution of log-returns.

The continuous time modeling has seen a grown interest after the introduction of stochastic calculus. Merton has formulated and solved the optimal consumption-investment model [42] under log-normally distributed returns with deterministic volatility and interest rate and a Constant Relative Risk Aversion (CRRA) utility function using the Dynamic Programming approach. It was developed by Bellman [3] in 1975 and had been used in Optimal Control theory.

However, the assumptions made in Merton's problem aren't realistic. For instance, interest rates aren't deterministic in real life and should be modeled by a stochastic differential equation. The main interest rate models that have been introduced in the literature are Vasicek [54], CIR [16] and Hull–White [32] models.

Some papers studied utility maximisation under stochastic interest rates. In the finite time horizon case, Chang and Chang [13] derived a closed-form solution in a complete market under a Hyperbolic Absolute Risk Aversion (HARA) utility function using the Legendre transform. Dong [34] derived the optimal solution of the consumption-investment problem under CRRA utility function for a market that consists of a risk-free asset where the interest rate is driven by a Vasicek model, a zero-coupon bond and a risky stock that follows a log-normal model. Korn and Kraft [13] and Grasselli [14] used the CIR interest rate dynamics to derive the optimal solution under HARA utility.

For the infinite horizon time case, Fleming and Pang [24] developed a subsolution-supersolution method and provided upper and lower bounds for the value function under a CRRA utility function. The drawback of this method is that some of the supersolutions/subsolutions are not explicitly derived but they are rather given in a parametric form. Trybula [53] used Fleming and Pang's approach to derive explicit bounds in the case of a consumption problem without investment. We use the same approach as Fleming and Pang to derive a numerical approximation of the optimal solution under a Vasicek interest rate model.

The volatility is a very important parameter in pricing and hedging financial securities. It has been proved that it is not constant (see [6] and [50]) and models based on deterministic volatility don't match the implied market distributions. Several models have been suggested to model the randomness that drives the volatility processes.

Cox and Ross [17], Geske [27]), Rubinstein [49] and Bensoussan et al [4] suggested to model it by a diffusion process that is correlated with stock prices and/or with a firm's debt. Johnson and Shanno [36], Scott [50], Hull and White ([33] and [31]) and Wiggins [55] described the randomness of the volatility by a Brownian motion independent of the price process.

In 1982, Engle [21] introduced ARCH processes and in 1986, Bollerslev [7] introduced GARCH processes to model asset volatilities. Theses models capture the stylized facts that log-returns volatility is clustered and highly persistent. These proporties are also captured by Hobson and Rogers [29] model. They introduced in 1998 an endogenous model of volatility that is expressed as an exponentially-weighted average of the moments of historic log-prices. This model had the advantage of being observable and also consistent with the 'smiles' and 'skews' implied by the market.

Kraft [37] addressed the optimization problem under a Heston stochastic volatility model and the power utility function. Chacko and Viceira [12] investigated the case of a dynamic hedging portfolio and derived closed-form formulas of optimal consumption and investment strategies for an infinite horizon. Fleming and Hernandez-Hernandez [23] and Fouque, Papanicolaou and Sircar [25] were also interested in the optimization problem with stochastic volatility.

In this paper, we model the volatitlity process by Hobson and Rogers [29] model, and write the solution of the HJB equation as an expectation using Feynman-Kac theorem and by imposing a transversality condition ([1] and [18], [42]), then this expectation is computed using Monte Carlo simulations. This is a powerful method that provides high level results. This idea has been suggested by Ravi in his paper [40].

Other researchers combined the stochastic volatility model together with stochastic interest rates. For example, Chang and Rong [14] used a CIR interest rate model and Heston's stochastic volatility model to derive a closed-form solution of the expected utility under a finite time horizon for both power utility and logarithmic utility.

Noh and Kim [44] addressed the case of a Vasicek asset price volatility model and a stochastic interest model such that the mean return $\mu(r_t)$ and volatility $\sigma(r_t)$ satisfy some conditions using HARA utility and logarithmic utility. They characterized the optimum by a supersolution and subsolution.

Finally, there is a plenty of variants of Merton's problem that take into account transaction costs, taxes, random stopping time, etc. These problems have been briefly discussed by Rogers [48], and some researchers were interested in these problems and tackled them in a more detailed manner. We refer to the recent paper of Hobson, Tse and Zhu's [30] for the multi-asset investment and consumption problem with transaction costs.

2 A convex optimisation method

2.1 Model specifications

In this first chapter, we consider a single period model with initial time $t = 0$ and final time $t = 1$, and finite probability space $\Omega = \{\omega_1, ..., \omega_K\}$. P is the historical probability measure and Q is the risk-neutral measure.

Let $B = \{B_0, B_1\}$ denotes the bank account process, such that $B_0 = 1$, and $B_1 = 1 + r$ where r is the interest rate.

We consider N securities, and denote the price process of the i-th security by $S_i = (S_i(0), S_i(1))$. The time $t = 0$ price $S_i(0)$ is a deterministic positive number and $S_i(1)$ is a non-negative random variable representing the time $t = 1$ price.

We also consider the discounted price process $S_i^*(t) := \dfrac{S_i(t)}{B_t}$ for $t \in \{0, 1\}$.

We denote by the vector $H = (H_1, ..., H_N)$ the trading strategy, where H_i is the number of units held of the i-th security, and $\mathbb{H}$ the set of all trading strategies.

Finally, we denote the wealth process of the portfolio by $V = (V_0, V_1)$, its value at time $t \in \{0, 1\}$ is $V_t = B_t + \sum_{n=1}^{N} H_n S_n(t)$.

2.2 Optimisation problem

The problem consists of finding the optimal strategy, hence the need of quantifying the performance of a given strategy H. One of the performance measures used is the expected utility.

Let $u : \mathbb{R} \times \Omega \to \mathbb{R}$ be a function such that $w \to u(w, \Omega)$ is differentiable, concave, and strictly increasing for each $\omega \in \Omega$. This function represents the utility of the wealth ω.

The optimisation problem is formulated as:

$$\begin{array}{ll} \underset{H \in \mathbb{H}}{\text{maximize}} & \mathbb{E}u\left(V_1(H_1, \cdots, H_N)\right) \\ \text{subject to} & V_0 = v \end{array}$$

Why this is a convex optimisation problem?

The function u is assumed to be a concave function and the expectation is linear and increasing, so:

$$
\begin{aligned}
u(\lambda V+(1-\lambda)W) &\geq \lambda u(V)+(1-\lambda)u(W) && \text{(By concavity of u)}\\
\mathbb{E}u(\lambda V+(1-\lambda)W) &\geq \mathbb{E}(\lambda u(V)+(1-\lambda)u(W)) && \text{(Expectation is increasing)}\\
\mathbb{E}u(\lambda V+(1-\lambda)W) &\geq \mathbb{E}(\lambda u(V))+(1-\lambda)\mathbb{E}(u(W)) && \text{(By linearity of expectation)}
\end{aligned}
$$

Thus the objective function is concave and the set of constraints is clearly convex.

The objective function can be written as:

$$
\begin{aligned}
\mathbb{E}u\left(V_1\right) &= \sum_{\omega\in\Omega} \mathbb{P}(\omega)u\left(V_1(\omega),\omega\right)\\
&= \sum_{\omega\in\Omega} \mathbb{P}(\omega)u\left(B_1\left\{v+H_1\Delta S_1^*+\cdots+H_N\Delta S_N^*\right\},\omega\right)
\end{aligned}
$$

We compute the gradient as follows:

$$
\begin{aligned}
\nabla(V_1) &= \left(\sum_{\omega\in\Omega} \mathbb{P}(\omega)u'\left(B_1(\omega)\left\{v+H_1\Delta S_1^*(\omega)+\cdots+H_N\Delta S_N^*(\omega)\right\},\omega\right)B_1(\omega)\Delta S_n^*(\omega)\right)_{1\leq n\leq N}\\
&= \left(\mathbb{E}\left[B_1u'\left(V_1\right)\Delta S_n^*\right]\right)_{1\leq n\leq N}
\end{aligned}
$$

So that the first order condition can be written as:

$$
\sum_{\omega\in\Omega} \mathbb{P}(\omega)u'\left(B_1(\omega)\left\{v+H_1\Delta S_1^*(\omega)+\cdots+H_N\Delta S_N^*(\omega)\right\},\omega\right)B_1(\omega)\Delta S_n^*(\omega)=0, \quad n=1,\ldots,N
$$

This system of N equations is not always easy to solve, this is why we will introduce the risk neutral approach.

2.3 Risk neutral approach

First, let's get some intuition about this approach.

Let's define $Q_u(\omega) := \mathbb{P}(\omega)u'(V_1(\omega),\omega)B_1(\omega)$. If Q_u is proportional to the risk-neutral probability measure $\mathbb{Q}$, then the first order condition is equivalent to:

$$
\mathbb{E}^{\mathbb{Q}}\left(\Delta S_n^*\right)=0, \quad n=1,\ldots,N \tag{1}
$$

By the property of the risk neutral probability measure, S_n^* is a $\mathbb{Q}$-martingale, i.e $\mathbb{E}^{\mathbb{Q}}\left(\Delta S_n^*\right)=0$, for $n=1,\ldots,N$, which is the same as the system (1). Hence the link between the risk neutral measure and our optimal investment problem. We will look at more details in the following sections.

Recall the objective function of the optimisation problem:

$$(H_1, \cdots, H_N) \mapsto \mathbb{E}u\left(V_1(H_1, \cdots, H_N)\right),$$

which is the decomposition of the two functions:

$$\begin{array}{lllll} f & : & \mathbb{W}_v & \to & \mathbb{R} \\ & & V_1 & \mapsto & \mathbb{E}u(V_1) \end{array} \quad \text{and} \quad \begin{array}{lllll} V_1 & : & \mathbb{H} & \to & \mathbb{W}_v \\ & & (H_1, \cdots, H_N) & \mapsto & B_1(v + \sum H_n \Delta S_n^*) \end{array}$$

Where $\mathbb{W}_v$ is the set of attainable wealths.

The function V_1 maps the trading strategies into the real random variables representing the final wealth, and the function f maps these wealths to the real expected utility which we aim to maximise.

The approach is the following [46]:

1. Find the optimal attainable wealth V_1^* which maximises the objective function.
2. Replicate the wealth V_1^* and find the corresponding hedging vector $H = (H_1, \cdots, H_N)$.

2.3.1 Complete models

Recall that a model is complete if every payoff V_1 is replicable, and that there exists a unique risk-neutral probability measure, under which the time 0 fair value of the claim V_1 is $V_0 = \mathbb{E}^{\mathbb{Q}}(V_1)$.

Thus, under the assumption of complete market, a wealth V_1 is attainable iff the discounted final wealth verifies $v = \mathbb{E}^{\mathbb{Q}}(V_1^*)$.

The set of attainable wealths can be then written as:

$$\mathbb{W}_v = \left\{ V \quad \text{r.v} : \quad \mathbb{E}^{\mathbb{Q}}\left(\frac{V}{B_1}\right) = v \right\}$$

.

Step 1:

According to the above approach, one needs to solve the following optimisation problem:

$$\begin{array}{ll} \text{maximize} & \mathbb{E}u(V) \\ \text{subject to} & \mathbb{E}^{\mathbb{Q}}(V/B_1) = v \end{array}$$

To do so, we introduce the Lagrange multiplier:

$$L(V,\lambda) = (V) - \lambda\left(\mathbb{E}^{\mathbb{Q}}\frac{V}{B_1}\right)$$

and the density $L = \dfrac{\mathbb{Q}}{\mathbb{P}}$, so that for any random variable Y, one has:

$$\mathbb{E}^{\mathbb{Q}}(Y) = \mathbb{E}^{\mathbb{P}}(LY)$$

and

$$\begin{aligned} L(V,\lambda) &= \mathbb{E}\left[u(V) - \lambda\frac{LV}{B_1}\right] \\ &= \sum_{\omega\in\Omega}\mathbb{P}(\omega)\left[u(V(\omega)) - \lambda\frac{L(\omega)V(\omega)}{B_1(\omega)}\right] \end{aligned}$$

So now we are interested in the following problem:

$$\begin{array}{ll} \text{maximize} & L(V,\lambda) \\ \text{subject to} & \mathbb{E}^{\mathbb{Q}}(V/B_1) = v \end{array}$$

The necessary conditions can be written as:

$$u'(V^*(\omega)) = \lambda\frac{L(\omega)}{B_1(\omega)} \quad \text{, for all } \omega\in\Omega$$

i.e

$$V^*(\omega) = u'^{-1}\left(\lambda\frac{L(\omega)}{B_1(\omega)}\right) \quad \text{, for all } \omega\in\Omega \tag{2}$$

where u'^{-1} denotes the inverse function of u'.

Solving the equation 2 would allow to have the expression of the optimal wealth as a function of λ, then λ is determined such that the constraint is not violated:

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_1}u'^{-1}\left(\lambda\frac{L}{B_1}\right)\right] = v$$

Step 2:

In this step, we find the trading strategy $H = (H_1,\cdots,H_N)$ to replicate the wealth V of equation 2. Thus, we need to solve a system of K equations:

$$B_1(\omega_i)\left(v + \sum_{n=1}^{N}H_n\Delta S_n^*(\omega_i)\right) = V(\omega_i) \quad i\in\{1,\cdots,K\}$$

3 A DPP approach

In this chapter, we will consider an other approach to solve the control problems based on the Dynamic Programming Principle (DPP) that we will introduce later.

3.1 Model specifications

We consider in this chapter a continuous-time setting.

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ a filtered probability space and $W = (W^1, \cdots, W^d)$ a d-dimensional Brownian motion.

We consider a financial market with two types of assets: risk-free asset representing the bank account, and a vector of risky assets $S_t = (S_t^1, \cdots, S_t^n)$ which are the solutions of the following SDEs:

$$dB_t = r_t B_t\, dt, \qquad dS_t = \mu(t, S_t)\, dt + \sigma(t, S_t)\, dW_t, \tag{3}$$

where $\mu(t, S_t) \in \mathbb{R}^n$ and $\sigma(t, S_t) \in \mathscr{M}_{n*d}(\mathbb{R})$.

The components of the second equation are:

$$dS_t^i = \mu_i(t, S_t)\, dt + \sum_{j=1}^{d} \sigma_{ij}(t, S_t)\, dW_t^j, \quad 1 \leq i \leq n$$

In the next sections, we consider the following drifts $\mu_i(t, S_t)$ and volatilities $\sigma_{ij}(t, S_t)$ functions:

$$\mu_i(t, S_t) = \mu_i S_t^i, \qquad \sigma_{ij}(t, S_t) = \sigma_{ij} S_t^i,$$

where μ_i and σ_{ij} are constants.

We denote by X_t and Y_t the money invested in the bank account and risky assets respectively, and ω_t the total wealth. We have then:

$$\begin{aligned} \omega_t &= X_t + Y_t, \\ Y_t &= n_t \,.\, S_t, \end{aligned}$$

where n is a n-dimensional previsible process, its component n_t^i represents the number of shares of the i-th asset hold in the portfolio at time t and $(u.v)$ denotes the scalar product of u and v.

Assuming that the stocks S deliver a n-dimensional adapted vector of dividends δ , and that the agent makes an endowment of a process e and consumes at the rate process c, we can write the dynamics of X and Y as:

$$\begin{array}{rcl} dX_t & = & r_t X_t\,dt + e_t\,dt - c_t\,dt, \\ dY_t & = & n_t \,.\, (dS_t + \delta_t\,dt), \end{array}$$

so that the final wealth has the dynamics:

$$\begin{aligned} d\omega_t &= dX_t + dY_t \\ &= r_t \quad X_t \quad dt + e_t\,dt - c_t\,dt + n_t \,.\, (dS_t + \delta_t\,dt) \\ &= r_t \quad (\omega_t - n_t.S_t) \quad dt + e_t\,dt - c_t\,dt + n_t \,.\, (dS_t + \delta_t\,dt) \end{aligned}$$

The processes e, δ, S and r are assumed given. We will take $e = 0$ and $\delta = 0$, i.e there is not an endowment of money or dividends:

$$d\omega_t = r_t \omega_t\,dt + \theta_t \,.\, (\sigma\,dW_t + (\mu - r_t)\,dt) - c_t\,dt \tag{4}$$

where $\theta_t := n_t.S_t$ is the vector of each asset worth.

The equation (4) that describes the dynamics of the wealth process can be written as:

$$d\omega_t = f(\omega(t), \nu(t)) \tag{5}$$

given that: $\omega(t_0) = \omega_0$ and such that :

$$\nu(t) = (c(t), \theta(t))$$

Theorem 1 (Carathèodory)
Suppose that:

1. $f(\cdot,\cdot)$ *is continuous,*
2. $f(\cdot,\nu)$ *satisfies the Lipschitz condition, i.e. there exists a constant* $L_f > 0$ *such that:*
$$|f(z,\nu) - f(z',\nu)| \le L_f |z - z'|$$
for all $z, z' \in \mathbb{R}^d$ *and* $\nu \in A$,
3. $f(z, \nu(t))$ *is measurable with respect to* t.

Then, there is a unique absolutely continuous function $y : [t_0, T] \to \mathbb{R}^d$ *that satisfies*

$$y(s) = z_0 + \int_{t_0}^{s} f(y(\tau), \nu(\tau))\,d\tau$$

Proof 1
The proof can be found in [41].

The theorem 1 guarantees, under some conditions, the existence and uniqueness of the solution of equation (5) and that the solution has the same properties as the function y.

Finally, the agent has to control the processes c and θ in a way such that her wealth ω is always non-negative. Thus, the couple (θ, c) is constrained to live in the set of admissible controls $\mathscr{A} := \{ \quad (\theta, c) \quad | \quad \omega \geq 0 \quad \text{a.s.} \}$.

3.2 Structure of optimal control problems

In an optimal control problem, we aim at finding the control processes (c and n in section (3.1) for example). To do so, one needs to pay particular attention to the four following elements [51]:

- **State process $\mathscr{Z}$(.)**: This process describes the state and hypotheses of the problem. In the setting of the previous section, the wealth of the investor is the state process and it is characterized by equation (4):

$$\mathscr{Z}(.) := \omega(.)$$

  Any other information about the parameters of the problem should be taken into account in the state process.

- **Control process ν(.)**: This process is the one that the investor can control in order to optimize his objective. It is represented by the processes (θ, c) in the previous section:

$$\nu(.) := \begin{bmatrix} \theta(.) \\ c(.) \end{bmatrix}$$

- **Set of admissible controls $\mathscr{A}$**: The control process $\nu(.)$ has to respect certain constraints in order that the strategy can be admissible. We call the set of such control processes the admissible controls:

$$\mathscr{A} := \{ \quad (\theta, c) \quad | \quad \omega \geq 0 \quad \text{a.s.} \}$$

- **Objective function**: The aim of the agent is to achieve a certain optimal criterion. To assess the performance of a given control process, one needs to set an optimisation measure. In the majority of optimal control problems, this function is taken as the expectation of a certain utility function of the control process and the investor seeks to maximise that quantity:

$$J(\nu(\cdot)) = \mathbb{E}\left[\int_0^T u(t, \nu(t))dt + G(T, \omega_T) \mid \mathscr{Z}_0 = z \right]$$

where T is the time horizon, it can be finite or infinite. The function u will mostly be taken as:

$$u(t,x) = e^{-\rho t} U(x)$$

where U is the Constant-Relative-Risk-Aversion (CRRA) function $U(x) := \dfrac{x^{1-R}}{1-R}$ s.t $R > 0$ and $R \neq 1$.

The discount factor $e^{-\rho t}$ guarantees the convergence of the integral.

We can notice that the problem suggests a feedback process: the control changes if there is a change in the state process. Thus, we will use the Dynamic Programming Principle (DPP) since it leads to solutions in a feedback format [41].

The aim of the agent is to attain the supremum of the above functional. The value function is then defined as:

$$V(z) := \sup_{\nu \in \mathscr{A}} J(\nu(\cdot))$$

Some control problems aim to minimise a cost function:

$$V(z) := \inf_{\nu \in \mathscr{A}} J(\nu(\cdot))$$

Then instead of solving the problem only at the initial time $t = 0$, we will be interested in finding the optimal strategy at every time t, i.e:

$$V(t,z) = \sup_{\nu \in \mathscr{A}_z} \mathbb{E}\left[\int_t^T e^{-\rho s} \frac{c_s^{1-R}}{1-R} ds \mid Z(t) = z\right]$$

where $\mathscr{A}_z$ denotes the set of admissible control processes of a state process Z such that $Z_t = z$.

3.3 Dynamic Programming Principle (DPP)

The Dynamic Programming Principle (DPP) is a sine qua non theorem in tackling optimal investment problems.

Theorem 2 (Dynamic Programming Principle)
For any stopping time $\tau \geq t$

$$V(t,z) = \sup_{\nu \in \mathcal{A}_{t,z}} \mathbb{E}\left[\int_t^\tau u(s,\nu(s))ds + V\left(\tau, Z_{t,z}^{\nu}(\tau)\right) \mid \mathcal{F}_t\right]$$

where $Z_{t,z}^{\nu}(\tau)$ is the value of the state process starting at time t with value z and controlled by the process ν evaluated at time τ.

3.4 Hamilton Jacobi Bellman Equation (HJB Equation)

Theorem 3 (The Davis-Varaiya Martingale Principle of Optimal Control (MPOC) [48]) *Assume that:*

1. *There exists a function* $V : [0,T] \times \mathbb{R}^+ \to \mathbb{R}$ *which is* $C^{1,2}$*, such that:*
$$V(T,\cdot) = u(T,\cdot),$$
2. *For any* $(n,c) \in \mathscr{A}(w_0)$*:*
$$Y_t \equiv V(t,w_t) + \int_0^t u(s,c_s)\,ds \text{ is a supermartingale,} \tag{6}$$
3. *For some* $(n^*,c^*) \in \mathscr{A}(w_0)$ *the process* Y *is a martingale.*

Then (n^*,c^*) *is optimal, and the value of the problem starting from initial wealth* w_0 *is*

$$V(0,w_0) = \sup_{(n,c)\in\mathscr{A}(w_0)} \mathbb{E}\left[\int_0^T u(t,c_t)\,dt + u(T,w_T)\right]$$

By equation (6), and by applying Ito's formula, we obtain:

$$\begin{aligned} dY_t &= V_t dt + V_w dw + \frac{1}{2}V_{ww}(dw)^2 + u(t,c)dt \\ &= V_w\theta\cdot\sigma dW + \left\{u(t,c) + V_t + V_w(r_t w + \theta\cdot(\mu - r_t) - c) + \frac{1}{2}\left|\sigma^T\theta\right|^2 V_{ww}\right\}dt \end{aligned}$$

by assuming the dynamics of ω in equation (3).

Then, by integrating from t to $t+h$:

$$Y_{t+h} - Y_t = \underbrace{\int_t^{t+h} V_w\theta\cdot\sigma dW}_{\text{Local martingale}} + \int_t^{t+h}\left\{u(s,c) + V_s + V_w(r_s w + \theta\cdot(\mu - r_s) - c) + \frac{1}{2}\left|\sigma^T\theta\right|^2 V_{ww}\right\}ds$$

So,

$$Y_t = E\left[Y_{t+h} - \int_t^{t+h}\left\{u(s,c) + V_s + V_w(r_s w + \theta\cdot(\mu - r_s) - c) + \frac{1}{2}\left|\sigma^T\theta\right|^2 V_{ww}\right\}ds \mid \mathscr{F}_t\right]$$

By setting

$$\mathcal{L}^\nu v := \mu(t,x,\nu)\cdot\nabla v + \frac{1}{2}\operatorname{tr} a(t,x,\nu)D^2 v$$

where,

$$a(t,x,\nu) := \sigma(t,x,\nu)\sigma(t,x,\nu)^t \quad \text{and} \quad \operatorname{tr} a := \sum_{i=1}^d a_{ii}$$

and in view of the DPP,

$$\sup_{\nu \in \mathcal{A}_{t,x}} E\left[-\int_t^{t+h}\left(\frac{\partial}{\partial t}v + \mathcal{L}^{\nu(s)}v + L\right)ds\right] = 0$$

Assuming that μ, a, L are continuous, and dividing the above equation by h and letting h go to zero, we obtain

$$-\frac{\partial}{\partial t}v(t,x) + H\left(x,t,\nabla v(t,x), D^2 v(t,x)\right) = 0$$

where

$$H(\omega, \nabla V(\omega), D^2 V(\omega)) := \sup_{(\theta,c)\in\mathscr{A}}\left[\mu(\omega,\theta,c)\,\nabla V(\omega) + \frac{1}{2}\mathrm{tr}(\sigma\sigma^T(\omega,\theta,c)D^2V(\omega)) + U(c)\right] \tag{7}$$

3.5 Verification theorem

Let $\beta > 0$ and $f : \mathbb{R}^n \times A \to \mathbb{R}$ a measurable function. For $x \in \mathbb{R}^n$, we denote by $\mathcal{A}(x)$ the subset of controls α in $\mathcal{A}_0$ such that:

$$\mathbb{E}\left[\int_0^\infty e^{-\beta s}|f(X_s^x, \alpha_s)|\,ds\right] < \infty$$

The gain function is defined as:

$$J(x,\alpha) = \mathbb{E}\left[\int_0^\infty e^{-\beta s} f(X_s^x, \alpha_s)\,ds\right]$$

for all $x \in \mathbb{R}^n$ and $\alpha \in \mathcal{A}(x)$.

The associated value function is:

$$v(x) = \sup_{\alpha\in\mathcal{A}(x)} J(x,\alpha)$$

Theorem 4 (Infinite horizon [11])
Let $w \in C^2(\mathbb{R}^n)$, and satisfies a quadratic growth condition, i.e. there exists a constant C such that

$$|w(x)| \le C\left(1+|x|^2\right), \quad \forall x \in \mathbb{R}^n$$

i- Suppose that

$$\beta w(x) - \sup_{a\in A}[\mathcal{L}^a w(x) + f(x,a)] \ge 0, \quad x \in \mathbb{R}^n$$
$$\limsup_{T\to\infty} e^{-\beta T}\mathbb{E}[w(X_T^x)] \ge 0, \quad \forall x \in \mathbb{R}^n, \forall \alpha \in \mathcal{A}(x)$$

Then $w \ge v$ on $\mathbb{R}^n$.

ii- Suppose further that:

- *For all $x \in \mathbb{R}^n$, there exists a measurable function $\hat{\alpha}(x), x \in \mathbb{R}^n$, valued in A such that*

$$\begin{aligned}\beta w(x) - \sup_{a \in A} [\mathcal{L}^a w(x) + f(x,a)] &= \beta w(x) - \mathcal{L}^{\hat{\alpha}(x)} w(x) - f(x, \hat{\alpha}(x)) \\ &= 0\end{aligned}$$

- *The SDE*

$$dX_s = b(X_s, \hat{\alpha}(X_s))\, ds + \sigma(X_s, \hat{\alpha}(X_s))\, dW_s$$

  admits a unique solution, denoted by $\hat{X}_s^x$, given an initial condition $X_0 = x$, satisfying

$$\liminf_{T \to \infty} e^{-\beta T} E\left[w\left(\hat{X}_T^x\right)\right] \leq 0$$

- *The process $\left\{\hat{\alpha}\left(\hat{X}_s^x\right), s \geq 0\right\}$, lies in $\mathcal{A}(x)$.*

Then:

$$w(x) = v(x), \quad \forall x \in \mathbb{R}^n$$

and $\hat{\alpha}$ is an optimal Markovian control

4 Merton's problem

4.1 Problem formulation

As mentioned in section 3.2, we need to specify the following elements for the optimal control problem:

- **State process:**

  The model is driven by the differential equations of the wealth:

$$d\omega_t \quad = \quad r\omega_t\, dt + \theta_t.(\sigma\, dW_t + (\mu - r)\, dt) - c_t\, dt, \tag{8}$$

  where $r, \mu, \sigma, \beta, \bar{r}, \sigma_r$ are constants and the correlation parameter η lies in $[-1, 1]$. The state process is in this case $Z_t := \omega_t$. It is the unique solution of the following equation:

$$dZ_t \quad = rZ_t + \theta_t.(\mu - r) - c_t\, dt + \theta_t.\sigma\, dW_t \tag{9}$$

  where W_t is a standard Brownian motion.

  Thus, (Z_t) is a diffusion process with drift and volatility:

  $\mu(\omega, r, \theta, c) := r\omega_t + \theta_t.(\mu - r) - c_t$ and $\sigma(\omega, r, \theta, c) := \theta_t.\sigma$ respectively.

- **Control process:**

  The state process is controlled by the consumption process c_t and the money invested in the risky assets θ_t, ν is then $\nu_t := \begin{bmatrix} c_t \\ \theta_t \end{bmatrix}$.

- **Set of admissible controls $\mathscr{A}_{\omega_0}$:**

  The admissible control process for a state process ω which starts at ω_0 at time 0 is:

  $$\mathscr{A}_{\omega_0} := \{ \quad (\theta, c) \quad | \quad \left(\pi := \frac{\theta}{\omega}, c\right) \text{ bounded, adapted ; } \omega \geq 0 \quad \text{ a.s. } \}$$

  The reason behind taking the constraint of bounded processes is the following: If $\pi := \dfrac{\theta}{\omega}$ takes a negative value that is too large in absolute value, then the agent needs to borrow a very big amount. This situation is not realistic: the agent can't borrow as much as she wants.

- **Objective functional:**

  The time horizon is infinite in this problem. The objective functional is the expected discounted utility derived from consumption:

  $$J(Z(\cdot), \nu(\cdot)) = E\left[\int_0^\infty e^{-\rho t} U(c(t)) dt \mid w_0 = w_0\right]$$

  where the utility function is the CRRA function $U(\omega) := \dfrac{\omega^{1-R}}{1-R}$.

The value function is the supremum of the objective functional:

$$V(t, \omega) = \sup_{(\theta, c) \in \mathscr{A}_\omega} \mathbb{E}\left[\int_t^\infty e^{-\rho s} \frac{c_s^{1-R}}{1-R} ds \mid w(t) = w\right]$$

4.2 Time-homogeneous property:

This problem is time-homogeneous: By changing the variable inside the integral $s \mapsto s - t$, we can deduce that:

$$V(t, \omega) = e^{-\rho t} V(0, \omega)$$

This is why we will focus on $V(0, \omega)$ and we will write $W(\omega)$ to denote $V(0, \omega)$:

$$V(t, \omega) = e^{-\rho t} V(\omega)$$

4.3 Scaling property

Let's consider an admissible control process $\hat{\nu}_t := \begin{bmatrix} \hat{c}_t \\ \hat{\theta}_t \end{bmatrix} \in \mathscr{A}_{\hat{\omega}_0}$. That means that the wealth process $\hat{\omega}_t$ controlled by these processes and driven by the dynamics:

$$d\hat{\omega}_t = r\hat{\omega}_t + \hat{\theta}_t.(\mu - r) - \hat{c}_t\,dt + \hat{\theta}_t.\sigma\,dW_t$$

is a.s positive, and starts at ω_0.

Now let's consider $\lambda > 0$, the control process $\nu'_t := \begin{bmatrix} c'_t \\ \theta'_t \end{bmatrix} := \lambda \begin{bmatrix} \hat{c}_t \\ \hat{\theta}_t \end{bmatrix}$ and the wealth process $\omega'_t = \lambda\hat{\omega}_t$. It verifies the SDE:

$$\begin{aligned} d\omega'_t &= r\lambda\hat{\omega}_t + \lambda\hat{\theta}_t.(\mu - r) - \lambda\hat{c}_t\,dt + \lambda\hat{\theta}_t.\sigma\,dW_t \\ &= \lambda\,d\hat{\omega}_t \end{aligned}$$

Thus the wealth process ω' is also a.s positive.
Furthermore, the processes $\pi' := \dfrac{\theta'}{\omega'}$ and c' are bounded and adapted, so ν' is an admissible control process of a wealth that starts from $\lambda\hat{\omega}_0$.

So we can conclude that $\lambda\nu \in \mathscr{A}_{\lambda\omega_0}$ is equivalent to $\nu \in \mathscr{A}_{\omega_0}$.

We can also notice that, for all $\omega > 0$:

$$\begin{aligned} V(\lambda\omega) &= \sup_{(\theta,c)\in\mathscr{A}_{\lambda\omega}} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\frac{c_t^{1-R}}{1-R}dt \mid \omega(0) = \lambda\omega\right] \\ &= \sup_{(\lambda\theta,\lambda c)\in\mathscr{A}_{\lambda\omega}} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\frac{(\lambda c_t)^{1-R}}{1-R}dt \mid \omega(0) = \lambda\omega\right] \\ &= \sup_{(\theta,c)\in\mathscr{A}_{\omega}} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\frac{(\lambda c_t)^{1-R}}{1-R}dt \mid \omega(0) = \omega\right] \\ &= \lambda^{1-R}V(\omega) \end{aligned}$$

By taking $\omega = 1$, we obtain $V(\lambda) = \lambda^{1-R}V(1)$, which means that V is proportional to the utility function U.

Now, we need to write down the HJB equation and try to solve it. If we find a solution, then most likely it will be the solution to our optimisation problem.

4.4 HJB equation

The associated HJB equation is given by:

$$\rho V(\omega) - H(\omega, \nabla V(\omega), D^2 V(\omega)) = 0, \qquad \omega \in \mathbb{R} \tag{10}$$

where:

$$H(\omega, \nabla V(\omega), D^2 V(\omega)) := \sup_{(\theta,c)\in\mathscr{A}} \left[\mu(\omega,\theta,c)\, \nabla V(\omega) + \frac{1}{2}\mathrm{tr}(\sigma\sigma^T(\omega,\theta,c) D^2 V(\omega)) + U(c) \right] \tag{11}$$

First, we compute the quantities:

$$\mu(\omega,k,c)^T \nabla V(\omega) \;=\; (\omega r + \theta.(\mu - r) - c) V_\omega$$

and

$$\begin{aligned} \mathrm{tr}\left(\sigma\sigma^T(\omega,\theta,c) D^2 V(\omega)\right) &= \mathrm{tr}\left(\theta.\sigma(\theta.\sigma)^\top V_{\omega\omega}\right) \\ &= \theta.\sigma(\theta.\sigma)^\top V_{\omega\omega} \end{aligned}$$

By substituting these expressions in the equation (11), we obtain:

$$\begin{aligned} H(\omega, \nabla V(\omega), D^2 V(\omega)) &= \sup_{(\theta,c)\in\mathscr{A}} [(\omega r + \theta.(\mu - r) - c)\, V_\omega + \tfrac{1}{2}\theta.\sigma(\theta.\sigma)^\top V_{\omega\omega} + U(c)] \\ &= \omega r V_\omega + \sup_c [-c V_\omega + U(c)] + \sup_\theta \left[\theta.(\mu - r)\, V_\omega + \tfrac{1}{2}\theta.\sigma(\theta.\sigma)^\top V_{\omega\omega}\right] \end{aligned}$$

4.5 Optimal control parameters

This step consists of solving the optimisation problems:

$$\sup_{c\geq 0} [-c V_\omega + U(c)] \quad \text{and} \quad \sup_{\theta \geq 0} \left[\theta.(\mu - r)\, V_\omega + \frac{1}{2}\theta.\sigma(\theta.\sigma)^\top V_{\omega\omega}\right]$$

By the time-homogeneous and scaling properties (see sections 4.2 and 4.3), the value function can be written as:

$$V(t,\omega) = e^{-\rho t} V(\omega) = e^{-\rho t} \gamma_M^{-R} \frac{\omega^{1-R}}{1-R}$$

- **Optimal c^*:**

  Since $R > 0$, the function $c \mapsto -c V_\omega + \dfrac{c^{1-R}}{1-R}$ is concave and thus it admits a supremum.
  The first order condition can be written as:

  $$\partial_c(-c V_\omega + \frac{c^{1-R}}{1-R}) = -\omega^{-R}\gamma_M^{-R} + c^{-R} = 0$$

  i.e. the optimal proportion of wealth consumed is:

  $$q^* = \frac{c^*}{\omega} = \gamma_M$$

- **Optimal** θ^***:**

  The function $\omega \mapsto \theta.(\mu - r)V_\omega + \frac{1}{2}\theta.\sigma(\theta.\sigma)^\top V_{\omega\omega}$ is quadratic. Since $V_{\omega\omega} < 0$ (the value function should be concave), we can deduce that it has a maximum. The gradient of this function is:

  $$\nabla_\theta(\theta.(\mu - r)V_\omega + \frac{1}{2}\theta.\sigma(\theta.\sigma)^\top V_{\omega\omega}) = (\mu - r)^\top V_\omega + \theta^\top \sigma\sigma^\top V_{\omega\omega}$$

  So that the first order condition can be written as:

  $$\theta^{*\top}\sigma\sigma^\top V_{\omega\omega} = -(\mu - r)^\top V_\omega$$

  i.e

  $$\theta^* = -\frac{V_\omega}{V_{\omega\omega}}(\sigma\sigma^\top)^{-1}(\mu - r)$$

  i.e

  $$\theta^* = \omega R^{-1}(\sigma\sigma^\top)^{-1}(\mu - r)$$

  And the vector of optimal proportions of wealth invested in risky assets is:

  $$\pi_M = R^{-1}(\sigma\sigma^\top)^{-1}(\mu - r)$$

4.6 Value function

Let's plug the optimal values in the equation (10):

$$e^{-\rho t}\left[\frac{R}{1-R}(\gamma_M w)^{1-R} - \rho\gamma_M^{-R}u(w) + rw\gamma_M^{-R}w^{-R} + \frac{1}{2}\gamma_M^{-R}w^{1-R}|\kappa|^2/R\right] = 0$$
$$\frac{e^{-\rho t}w^{1-R}\gamma_M^{-R}}{1-R}\left[R\gamma_M - \rho - (R-1)\left(r + \frac{1}{2}|\kappa|^2/R\right)\right] = 0$$

where

$$\kappa \equiv \sigma^{-1}(\mu - r)$$

So:

$$\gamma_M = \frac{\rho + (R-1)\left(r + \frac{1}{2}|\kappa|^2/R\right)}{R}$$

Thus the value function is:

$$V(t,w) = e^{-\rho t}\gamma_M^{-R}\frac{\omega^{1-R}}{1-R}$$

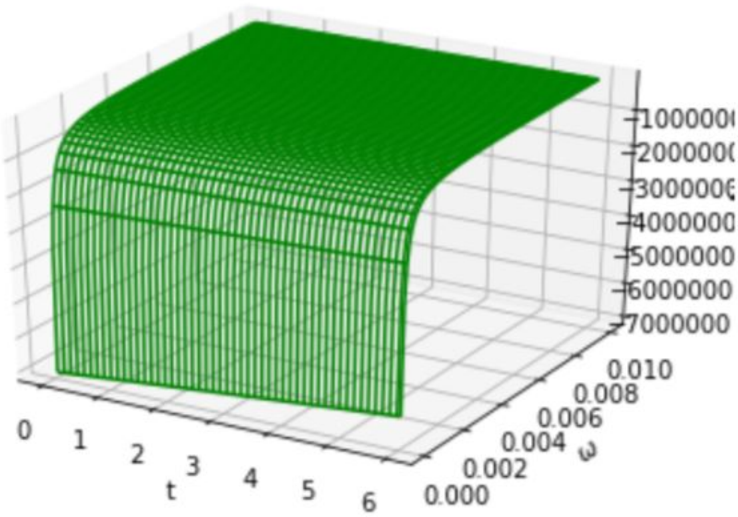

Figure 1: The value function of Merton's problem as a function of time t and wealth ω_t

4.7 Analysis of optimal values

Both control processes c^* and θ^* are proportional to the wealth.

The optimal controls suggest that the agent should allocate a constant proportion π_i^* of wealth in each asset i. This constant is inversely proportional to the volatility of the asset: the greater is the volatility, the higher is the risk and the lower is the optimal proportion of the asset.

We can also notice that it's proportional to the excess return of stocks over bonds adjusted by the risk: this is called the sharpe ratio; it measures the profits associated with risk-taking investment in a given stock.

If for an asset i, $\mu_i = r$, which means that the expected return is equal to the risk-free rate, the allocation to that risky asset i is null. This can be expected since investing that money in the risk-free asset yields to the same return without taking any risk.

5 Stochastic Volatility

In Merton's problem, a Black and Scholes model of stocks dynamics is assumed. Although this model has proven a high tractability, the stocks log-returns prices are supposed normally distributed with a constant variance and independent increments. This is not consistent with the market implied distributions, and the deterministic volatility needs to be altered by a Stochastic Volatility (SV) model.

Different approaches have been suggested for the dynamics of volatility process:

- Level dependent volatility models:

$$dS_t = rS_t dt + \sigma(S_t) dW_t$$

  Cox and Ross [17], Geske [27], Rubinstein [49] and Bensoussan et al [4] suggested and derived prices formulas for the Constant Elasticity of Variance (CEV) model where $\sigma(S_t) = \sigma S_t^{-(1-\alpha)}$ such that $0 < \alpha < 1$. This model, has the same drawback as Black and Scholes: the volatility is deterministic.

- Stochastic volatility models:

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t$$

  - Stein-Stein:

$$d\sigma_t = \kappa(\theta - \sigma_t) dt + \tilde{\sigma} dB_t$$

    this is the same as Vasicek model for interest rates (see section 6 for parameters interpretation).

  - Hull-White:

$$\sigma_t = \sqrt{v_t} \quad ; \quad dv_t = \mu v_t dt + \tilde{\sigma} v_t dB_t$$

    where the parameters μ and $\tilde{\sigma}$ are constants.

  - Heston:

$$\sigma_t = \sqrt{v_t} \quad ; \quad dv_t = \kappa(\theta - v_t) dt + \tilde{\sigma}\sqrt{v_t} dB_t$$

    where $\kappa, \theta, \tilde{\sigma}$ are positive constants.

  - Scott: $\sigma_t = \sqrt{v_t}$ and variance process v_t satisfies

$$v_t = \exp(y_t) \quad ; \quad dy_t = k(\ln(\theta) - y_t) dt + \epsilon dB_t$$

  where B and W are correlated Brownian motions with correlation coefficient ρ constant.

- Exponentially weighted average of the historic log-price:

$$dS_t = \mu S_t\,dt + \sigma_t S_t\,dW_t$$

Hobson and Rogers [29] suggested to model the randomness in the stochastic volatility by the previous log-prices, so that no new Brownian motion is introduced.

$$\sigma_t = f(Y_t) \quad ; Y_t = \int_{-\infty}^{t} \lambda e^{\lambda(s-t)}(X_s - X_t)\,ds \quad ; \quad X_t = \ln(S_t) \tag{12}$$

In this variant of Merton's problem, we model the volatility by the Exponentially weighted average of the historic log-price model.

Y_t can be expressed as:

$$\begin{aligned} Y_t &= \int_{-\infty}^{t} \lambda e^{\lambda(s-t)} X_s\,ds - X_t \int_{-\infty}^{t} \lambda e^{\lambda(s-t)}\,ds \\ &= e^{-\lambda t}\int_{-\infty}^{t} \lambda e^{\lambda s} X_s\,ds - X_t \end{aligned} \tag{13}$$

By setting A_t and H_t as $A_t := \int_{-\infty}^{t} \lambda e^{\lambda s} X_s\,ds$ and $H_t := e^{-\lambda t} A_t$, we have then:

$$\begin{aligned} dA_t = \lambda e^{\lambda t} X_t \quad \text{and} \quad dH_t &= -\lambda e^{-\lambda t} A_t\,dt + e^{-\lambda t}\,dA_t \quad \text{(By Ito's formula)} \\ &= -\lambda e^{-\lambda t} A_t\,dt + \lambda X_t\,dt \\ &= -\lambda\left(e^{-\lambda t}\int_{-\infty}^{t} \lambda e^{\lambda s} X_s\,ds - X_t\right)dt \\ &= -\lambda Y_t\,dt \end{aligned}$$

So that we have:

$$\begin{aligned} dY_t &= dH_t - dX_t \\ &= -\lambda Y_t\,dt - dX_t \end{aligned} \tag{14}$$

By Ito's lemma applied to X_t, we obtain:

$$\begin{aligned} dX_t &= \frac{dS_t}{S_t} - \frac{1}{2}\frac{dS_t^2}{S_t^2} \\ &= \left(\mu - \frac{\sigma_t^2}{2}\right)dt + \sigma_t\,dW_t \end{aligned}$$

By substituting this expression in the equation (14), we obtain:

$$dY_t + \lambda Y_t\,dt = -\left\{\left(\mu - \frac{1}{2}f(Y_t)^2\right)dt + f(Y_t)\,dW_t\right\} \tag{15}$$

5.1 Problem formulation

- **State process:**

The model is driven by two differential equations:

$$\begin{aligned} d\omega_t &= r\omega_t dt + \theta_t [f(Y_t)dW_t + (\mu - r)dt] - c_t dt, \\ dY_t &= -\lambda Y_t dt - \left(\mu - \frac{1}{2} f(Y_t)^2\right) dt - f(Y_t) dW_t \end{aligned} \tag{16}$$

where μ, r, λ are constants.

We set the state process to be the vector $Z_t := \begin{pmatrix} \omega_t \\ Y_t \end{pmatrix}$. It is the unique solution of the following equation:

$$dZ_t = \begin{bmatrix} r\omega_t + \theta_t(\mu - r) - c_t \\ -\lambda Y_t - \mu + \dfrac{1}{2} f(Y_t)^2 \end{bmatrix} dt + \begin{bmatrix} \theta_t f(Y_t) & 0 \\ -f(Y_t) & 0 \end{bmatrix} \begin{bmatrix} dB_t^1 \\ dB_t^2 \end{bmatrix} \tag{17}$$

where $\left(B_t^1, B_t^2\right)$ is a two-dimensional standard Brownian motion.

Thus, (Z_t) is a diffusion process with drift and volatility matrix:

$\mu(\omega, Y, \theta, c) := \begin{bmatrix} \omega r + \theta(\mu - r) - c \\ -\lambda Y - \mu + \frac{1}{2} f(Y)^2 \end{bmatrix}$ and $\sigma(\omega, Y, \theta, c) := \begin{bmatrix} \theta f(Y) & 0 \\ -f(Y) & 0 \end{bmatrix}$ respectively.

- **Control process:**

The control process is set to be the vector $\nu_t := \begin{bmatrix} c_t \\ \theta_t \end{bmatrix}$.

- **Objective functional:**

The objective functional is the expected discounted utility derived from consumption:

$$V(t, \omega, y) = \sup_{(\theta, c) \in \mathscr{A}_{(\omega, y)}} E\left[\int_t^\infty e^{-\rho s} U(c(s)) ds \mid \omega_t = \omega, Y_t = y\right]$$

where the utility function is the CRRA function $U(\omega) := \dfrac{\omega^{1-R}}{1-R}$.

5.2 HJB equation

The associated HJB equation is given by:

$$\rho V(\omega,Y) - H(\omega,Y,\nabla V(\omega,Y),D^2V(\omega,Y)) = 0, \qquad \omega, Y \in \mathbb{R} \tag{18}$$

where:

$$\begin{aligned} H(\omega,Y,\nabla V(\omega,Y),D^2V(\omega,Y)) \ := \ \sup_{\theta\geq 0, c\geq 0} \ & [\mu(\omega,Y,\theta,c)\,\nabla V(\omega,Y) + \\ & +\tfrac{1}{2}\mathrm{tr}(\sigma\sigma^T(\omega,Y,\theta,c)D^2V(\omega,Y)) + U(c) \end{aligned} \tag{19}$$

Let's compute:

$$\begin{aligned} \mu(\omega,Y,k,c)^T\nabla V(\omega,Y) &= \begin{bmatrix} \omega r + \theta\,(\mu - r) - c \\ -\lambda Y - \mu + \frac{1}{2}f(Y)^2 \end{bmatrix}^T \begin{bmatrix} V_\omega \\ V_Y \end{bmatrix} \\ &= (\omega r + \theta\,(\mu - r) - c)\,V_\omega + \left(-\lambda Y - \mu + \tfrac{1}{2}f(Y)^2\right)V_Y \end{aligned}$$

and

$$\begin{aligned} \mathrm{tr}\left(\sigma\sigma^T(\omega,Y,\theta,c)D^2V(\omega,Y)\right) &= \mathrm{tr}\left(\begin{bmatrix} \theta f(Y) & 0 \\ f(Y) & 0 \end{bmatrix}\begin{bmatrix} \theta f(Y) & 0 \\ f(Y) & 0 \end{bmatrix}^T \begin{bmatrix} V_{\omega\omega} & V_{\omega Y} \\ V_{\omega Y} & V_{YY} \end{bmatrix}\right) \\ &= f(Y)^2\left(\theta^2 V_{\omega\omega} - 2\theta V_{\omega Y} + V_{YY}\right) \end{aligned}$$

By substituting these expressions in the equation (19), we obtain:

$$\begin{aligned} H(\omega,Y,\nabla V(\omega,r),D^2V(\omega,Y)) &= \sup_{\theta\geq 0,c\geq 0}[(\omega r + \theta\,(\mu - r) - c)\,V_\omega + \left(-\lambda Y - \mu + \frac{1}{2}f(Y)^2\right)V_Y + \\ &\quad +\tfrac{1}{2}f(Y)^2\left(\theta^2 V_{\omega\omega} - 2\theta V_{\omega Y} + V_{YY}\right) + U(c)] \\ &= \omega r V_\omega + \left(-\lambda Y - \mu + \frac{1}{2}f(Y)^2\right)V_Y + \tfrac{1}{2}f(Y)^2 V_{YY} + \\ &\quad + \sup_{\theta\geq 0}\left[\theta\,(\mu - r)\,V_\omega + \tfrac{1}{2}f(Y)^2\theta^2 V_{\omega\omega} - f(Y)^2\theta V_{\omega Y}\right] \\ &\quad + \sup_{c\geq 0}\,[-cV_\omega + U(c)] \end{aligned} \tag{20}$$

By the same arguments as in sections (4.2) and 4.3, the value function can be written as:

$$V(\omega,Y) = U(\omega)g(Y) = \frac{\omega^{1-R}}{1-R}g(Y)$$

5.3 Optimal control parameters

Here we derive the optimal values of consumption and the wealth invested in the stock by solving the optimisation problems:

$$\sup_{c\geq 0}[-cV_\omega + U(c)] \quad \text{and} \quad \sup_{\theta\geq 0}\left[\theta(\mu-r)V_\omega + \frac{1}{2}f(Y)^2\theta^2 V_{\omega\omega} - f(Y)^2\theta V_{\omega Y}\right]$$

- **Optimal c^*:**

  By the same assumption as the previous section, the first order condition can be written as:

$$\partial_c(-cV_\omega + U(c)) = -\omega^{-R}g(Y) + c^{-R} = 0$$

  i.e. the optimal proportion of wealth consumed is:

$$q^* = \frac{c^*}{\omega} = (g(Y))^{-\frac{1}{R}}$$

- **Optimal θ^*:**

  The function $\omega \mapsto \theta(\mu-r)V_\omega + \frac{1}{2}f(Y)^2\theta^2 V_{\omega\omega} - f(Y)^2\theta V_{\omega Y}$ is polynomial of second degree. Assuming $V_{\omega\omega} < 0$, we can deduce that it has a maximum attained at:

$$\theta^* = \frac{-f(Y)^2\omega^{-R}g'(Y) + (\mu-r)g(Y)\omega^{-R}}{Rf(Y)^2 g(Y)\omega^{-R-1}}$$

  And the optimal proportion of wealth invested in the risky asset is:

$$\begin{aligned}\pi^* &= \frac{-f(Y)^2 g'(Y) + (\mu-r)g(Y)}{Rf(Y)^2 g(Y)} \\ &= \frac{1}{R}\left[\frac{\mu-r}{f(Y)^2} - \frac{1}{g}g'\right]\end{aligned}$$

We can notice that the optimal consumption and investment proportions of wealth don't depend on th wealth ω.

By substituting c^* and θ^* in (30), we obtain the second order ODE:

$$Rg^{1-1/R} - \rho g + r(1-R)g + (1-R)\frac{\left\{(\mu-r)g - f^2 g'\right\}^2}{2f^2 Rg} + \frac{1}{2}f^2 g'' - (\lambda Y + \mu - \frac{1}{2}f^2)g' = 0 \quad (21)$$

5.4 Solving the ODE

By reordering the terms of the equation (21), we obtain:

$$\begin{aligned}\left[r(1-R)-\rho+\frac{(\mu-r)^2(1-R)}{2f^2R}\right]g+\left[-\frac{(\mu-r)(1-R)}{R}-\mu+\frac{1}{2}f^2-\lambda Y\right]\frac{\partial g}{\partial Y}\\ +(1-R)\frac{f^2}{2R}\frac{1}{g}\left(\frac{\partial g}{\partial Y}\right)^2+\frac{1}{2}f^2\frac{\partial^2 g}{\partial Y^2}+Rg^{1-1/R} &= 0\end{aligned} \tag{22}$$

This equation is not linear, to linearise it [43] we introduce the function h such that: $g(Y)=h(Y)^R$.
The derivatives of g are:

$$\begin{aligned}\frac{\partial g}{\partial Y} &= Rh(Y)^{R-1}\frac{\partial h}{\partial Y}\\ \frac{\partial^2 g}{\partial Y^2} &= Rh(Y)^{R-2}\left[(R-1)\left(\frac{\partial h}{\partial Y}\right)^2+h(Y)\frac{\partial^2 h}{\partial Y^2}\right]\end{aligned}$$

So that:

$$\begin{aligned}\frac{1}{g}\left(\frac{\partial g}{\partial Y}\right)^2 &= R^2h(Y)^{-R}\left(\frac{\partial h}{\partial Y}\right)^2\\ \frac{\partial^2 g}{\partial Y^2} &= Rh(Y)^{R-2}\left[(R-1)\left(\frac{\partial h}{\partial Y}\right)^2+h(Y)\frac{\partial^2 h}{\partial Y^2}\right]\end{aligned}$$

Thus

$$(1-R)\frac{f^2}{2R}\frac{1}{g}\left(\frac{\partial g}{\partial Y}\right)^2+\frac{1}{2}f^2\frac{\partial^2 g}{\partial Y^2} = \frac{1}{2}f^2Rh(Y)^{R-1}\frac{\partial^2 h}{\partial Y^2}$$

Plugging this formula into equation (22), we obtain the following linear second-order ODE:

$$\begin{aligned}\left[r\frac{1-R}{R}-\frac{\rho}{R}+\frac{(\mu-r)^2(1-R)}{2f^2R^2}\right]h+\left[-(\mu-r)\frac{1-R}{R}-\mu+\frac{1}{2}f^2-\lambda Y\right]\frac{\partial h}{\partial Y}\\ +\frac{1}{2}f^2\frac{\partial^2 h}{\partial Y^2}+1 &= 0\end{aligned} \tag{23}$$

This equation doesn't admit a closed-form solution in general.
Inspired by the paper [40], we will find a solution of the ODE by considering first a finite time horizon T using a boundary condition.

By considering: $h(Y,t):=e^{-\frac{\rho}{R}t}h(Y)$, the equation (23) has the form:

$$\frac{\partial h}{\partial t}(Y,t)+\hat{\mu}(Y,t)\frac{\partial h}{\partial Y}(Y,t)+\frac{1}{2}\hat{\sigma}^2(Y,t)\frac{\partial^2 h}{\partial Y^2}(Y,t)-\hat{\phi}(Y,t)h(Y,t)+\hat{F}(Y,t)=0$$

defined for all Y and $t \in [0,T]$, with terminal condition

$$h(Y,T) = \psi(Y)$$

where:

$\hat{\mu}(Y,t)$	$-(\mu - r)\dfrac{1-R}{R} - \mu + \dfrac{1}{2}f^2(Y) - \lambda Y$
$\hat{\sigma}^2(\hat{Y},t)$	$f^2(Y)$
$\hat{\phi}(Y,t)$	$-r(1-R) - \dfrac{(\mu-r)^2(1-R)}{2f^2(Y)R}$
$\hat{F}(Y,t)$	$e^{-\frac{\rho}{R}t}$

Table 2: Coefficients of PDE of h

The Feynman-Kac theorem states that the solution can be written as a conditional expectation:

$$h(Y,t) = \mathbb{E}\left[\int_t^T e^{-\int_t^r \hat{\phi}(\hat{Y}_\tau,\tau)d\tau}\hat{F}\left(\hat{Y}_r,r\right)dr + e^{-\int_t^T \hat{\phi}(\hat{Y}_\tau,\tau)d\tau}\psi\left(\hat{Y}_T\right) \mid \hat{Y}_t = Y\right] \tag{24}$$

where $\hat{Y}$ is an Itô process verifying the equation:

$$d\hat{Y} = \hat{\mu}(\hat{Y},t)dt + \hat{\sigma}(\hat{Y},t)dW \tag{25}$$

such that W is a Brownian motion and the initial condition for $\hat{Y}(t)$ is $\hat{Y}(t) = Y$.

Adding the *transversality condition* ([1] and [18], [42]):

$$\lim_{T\to\infty} \mathbb{E}\left[e^{-\int_t^T \hat{\phi}(\hat{Y}_\tau,\tau)d\tau}\psi\left(\hat{Y}_T\right) \mid \mathcal{F}_t\right] = 0,$$

and taking the limit $T \to \infty$ in equation (24), we obtain :

$$h(Y,t) = \mathbb{E}\left[\int_t^\infty e^{-\int_t^s \hat{\phi}(\hat{Y}_\tau,\tau)d\tau}e^{-\frac{\rho}{R}s}ds \mid \hat{Y}_t = Y\right]$$

To find the value of this function, we will use Monte Carlo (MC) simulation.

5.5 Numerical solution

In this section, we will implement numerically the optimal solution of the consumption-investment problem assuming a Hobson-Rogers SV model.

The stock dynamics are:

$$dS_t = \mu S_t\,dt + \sigma_t S_t\,dW_t$$

where

$$\sigma_t = f(Y_t) \quad ; Y_t = \int_{-\infty}^{t} \lambda e^{\lambda(s-t)}(X_s - X_t)\,ds \quad ; \quad X_t = \ln(S_t)$$

Recall that the optimal controls are:

$$q^* = \frac{c^*}{\omega} = (g(Y))^{-\frac{1}{R}}$$

and

$$\pi^* = \frac{1}{R}\left[\frac{\mu - r}{f(Y)^2} - \frac{1}{g}g'\right]$$

and the value function is:

$$V(t,\omega,y) = e^{-\rho t}U(\omega)g(y) = e^{-\rho t}\frac{\omega^{1-R}}{1-R}g(y)$$

where

$$g(y) = h(y)^R$$

$$h(y) = \mathbb{E}\left[\int_0^\infty \exp\left(-\int_0^s \hat{\phi}\left(\hat{Y}_\tau,\tau\right)d\tau\right)\exp\left(-\frac{\rho}{R}s\right)\mathrm{d}s \mid \hat{Y}_0 = y\right]$$

$$d\hat{Y} = \hat{\mu}(\hat{Y},t)dt + \hat{\sigma}(\hat{Y},t)dW$$

The functions $\hat{\mu}(\hat{Y},t)$, $\hat{\sigma}(\hat{Y},t)$ and $\hat{\phi}(\hat{Y},t)$ are given in table (2).

To find q^*, π^* and $V(t,\omega,y)$, we will compute $h(y)$ using N_s MC.
To derive π^*, we need to compute an approximation of $g'(Y)$ by the finite-difference method:

$$\frac{\partial g(y)}{\partial y} \approx \frac{g(y+\Delta) - g(y)}{\Delta}$$

Steps:

We can summarize the steps of the approximation as the following:

1. Discretize the time onto a grid of N_t equally-spaced values : $0 = t_0 \leq \cdots \leq t_N$ for a large value of t_N;

2. Simulate N_s paths of the process Y starting at y : $(Y_{t_i})_{0\leq i\leq t_N}$ using MC simulations:

$$Y_{t_{i+1}} = Y_{t_i} + \left[-(\mu-r)\frac{1-R}{R} - \mu + \frac{1}{2}f^2(Y_{t_i}) - \lambda Y_{t_i}\right]\frac{T_N}{N} + \frac{\sigma(1+Y_{t_i})}{\sqrt{2+Y_{t_i}^2}}\sqrt{\frac{T_N}{N}}Z_{i+1}$$

where Z_{i+1} are independent standard normal variables.

3. Approximate $\int_0^\infty \exp\left(-\int_0^s \phi(Y_\tau,\tau)\,d\tau\right)\exp\left(-\frac{\rho}{R}s\right)ds$ by Riemann sums:

$$\tilde{h} = \sum_{i=0}^{N_t}\exp\left(-\sum_{j=0}^{i}\phi(Y_{t_j})\frac{t_N}{N_t}\right)\exp\left(-\frac{\rho}{R}t_i\right)\frac{t_N}{N_t};$$

4. Take the average of these values and obtain $h(y)$ then $g(y)$:

$$\hat{h}(y) = \frac{1}{N_s}\sum_{i=1}^{N_s}\tilde{h}_i \quad ; \quad \hat{g}(y) = \hat{h}(y)^R$$

5. Compute approximations of $\frac{\partial g(y)}{\partial y}$:

$$\hat{g'}(y) = \frac{\hat{g}(y+\Delta)-\hat{g}(y)}{\Delta};$$

6. Compute π^*, q^* and V.

For this numerical part, we will consider the following parameters:

Parameter	**Value**
μ	0.15
r	0.05
R	2
λ	0.1
ρ	0.02
f(y)	$\sigma\frac{1+y^2}{2+y^2}$
σ	0.35
t_N	25
N_s	1000
N_t	10000
Δ	10^{-1}

Table 3: Parameter values of the numerical solution of SV model

Plot of f:

In Hobson and Rogers stochastic volatility model, the volatility function is assumed to be Lipshitz. The choice made for f guarantees this condition. Furthermore, f is an even function.

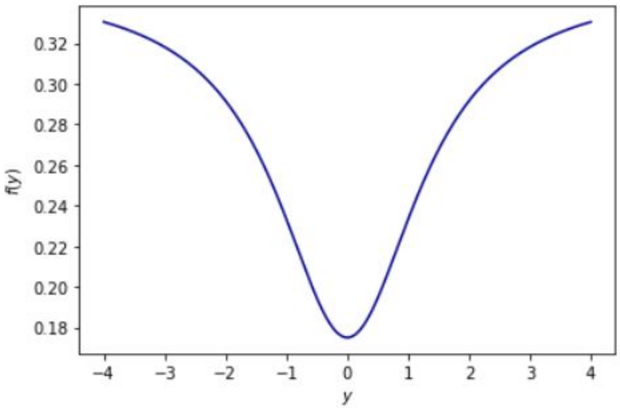

Figure 2: Function $f : y \mapsto \sigma \dfrac{1+y^2}{2+y^2}$

Computation of g(5):

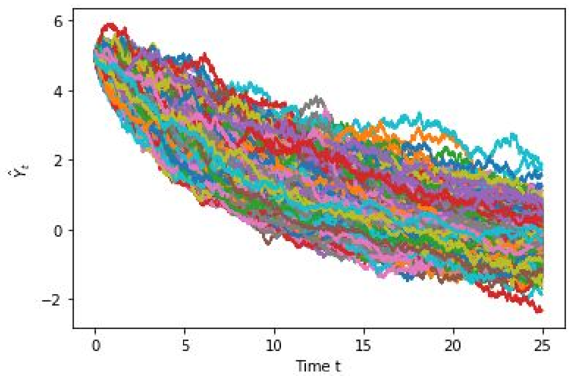

Figure 3: N_s Path simulations of $\hat{Y}_t$ with $Y_0 = 5$ and $N_s = 1000$

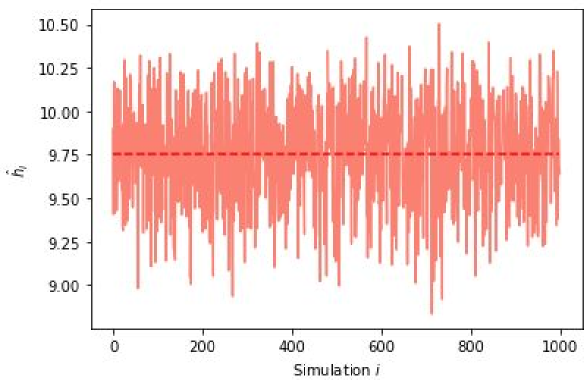

Figure 4: N_s simulations of $\hat{h}(5)$ with $N_s = 1000$

The figure (3) represents N_s MC paths simulations of the process $\hat{Y}_t$ with the dynamics given by equation (25). The figure (4) shows the simulated values of $\hat{h}(5)$. These values fluctuate around the mean with a small variance, which makes the algorithm stable and reliable. We obtain $\hat{h}(5) \approx 9.75$ and $\hat{g}(5) \approx 95$.

Plots and analysis of g, g', q^* and π^*

Let's plot the functions g, g' and the optimal consumption and investment proportions q^* and π^*:

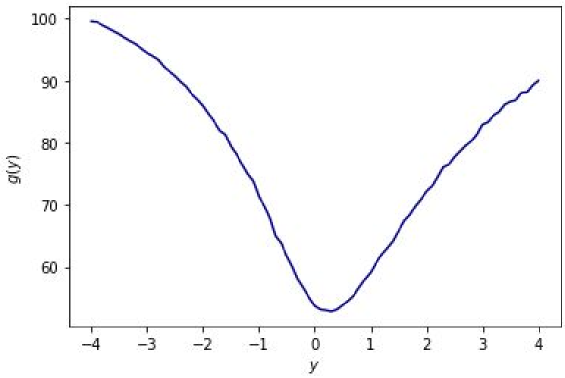

Figure 5: The numerical solution of the function $y \mapsto \hat{g}(y)$

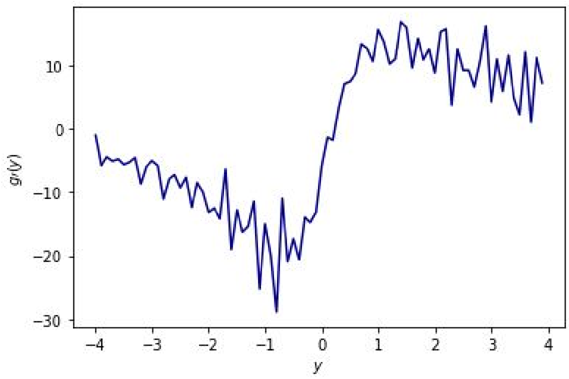

Figure 6: The numerical derivative of the function $y \mapsto \hat{g}'(y)$

We can notice that the function g has the same shape as the function f. The plot of g is smooth and the one of g' is less smoother, this might be due to the step Δ used to approximate the derivative.

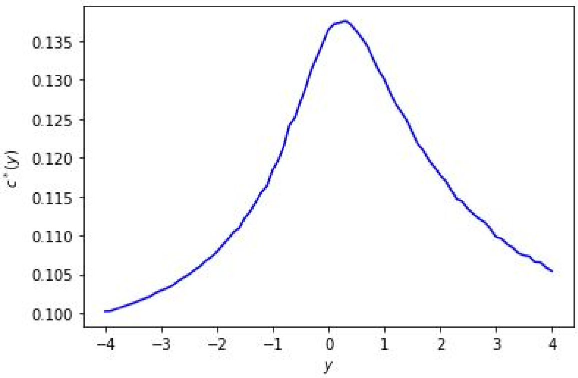

Figure 7: The optimal consumption proportion of wealth c^* as a function of y

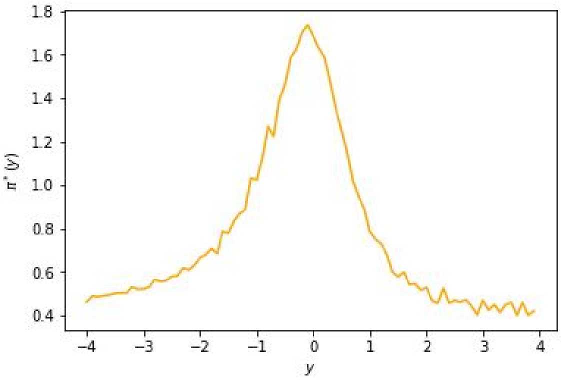

Figure 8: The optimal investment proportion of wealth π^* as a function of y

We can see in figure 8 that the investment rate is surprisingly greater than 1 for some values of the offset. This might be due to the computational errors.

The consumption rate in figure 7 is high for low offsets, and it gets lower for higher offsets y. This behaviour is similar to Merton's problem, where the consumption proportion is inversely proportional to the volatility when $R > 1$ which corresponds to a risk averse investor.

The same behaviour can be seen for the investment proportion. In both models, the investor invests less in stocks with high volatility.

Analysis of parameters effect:

In this section, we will analyze the effects of varying the parameters of risk aversion R, volatility σ of the function f and the discounting rate of past information λ on the consumption and investment proportions.

Effect of risk aversion parameter R:

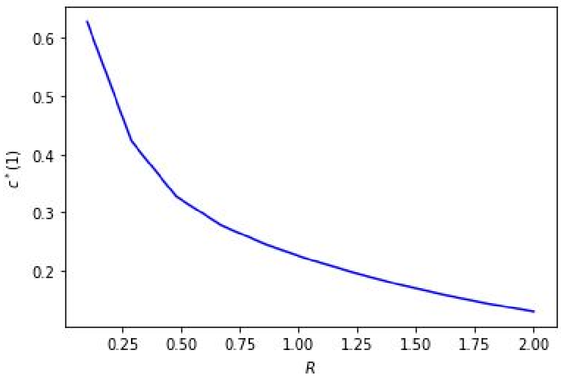

Figure 9: Effect of varying R on the optimal consumption proportion of wealth c^* evaluated at $y = 1$

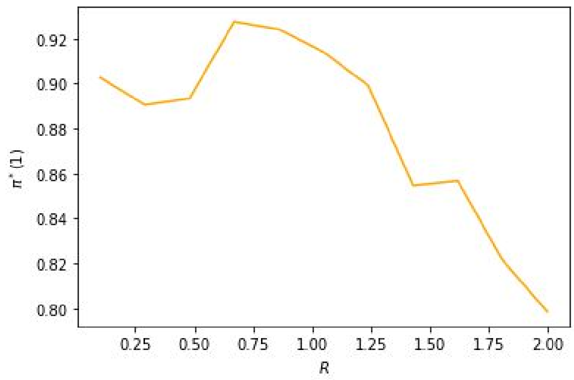

Figure 10: Effect of varying R on the optimal investment proportion of wealth π^* evaluated at $y = 1$

The figures 9 and 10 show that the optimal consumption and investment proportions decrease as the factor R increases, i.e as the investor becomes more risk-averse. This can be guessed intuitively as a risk-averse investor would take less risk, and thus would consume and invest less in the risky asset.

Effect of the volatility parameter σ:

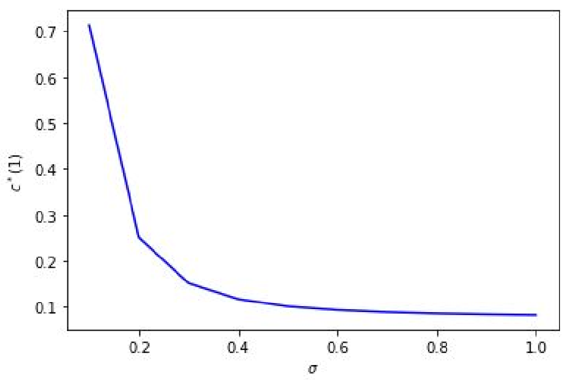

Figure 11: Effect of varying σ on the optimal consumption proportion of wealth c^* evaluated at $y = 1$

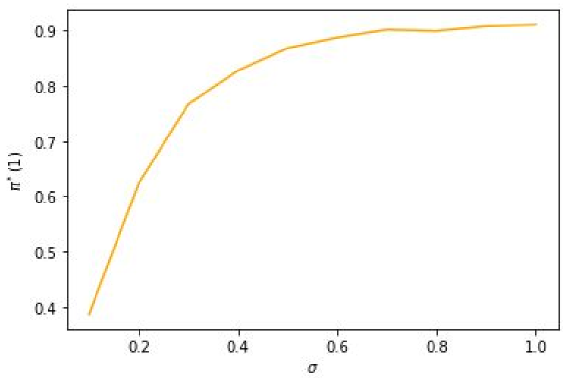

Figure 12: Effect of varying σ on the optimal investment proportion of wealth π^* evaluated at $y = 1$

The plots 11 and 12 suggest that the consumption should be lower and the investment should be higher as the parameter σ increases. The first part is as expected, but the second part concerning the investment proportion is surprising since intuitively, the higher is a volatility parameter the less should the investor take risk and invest in the risky asset. The parameter σ can be interpreted as the limit of the volatility of the asset prices as $y \to \infty$.

Effect of the discounting rate of past information parameter λ:

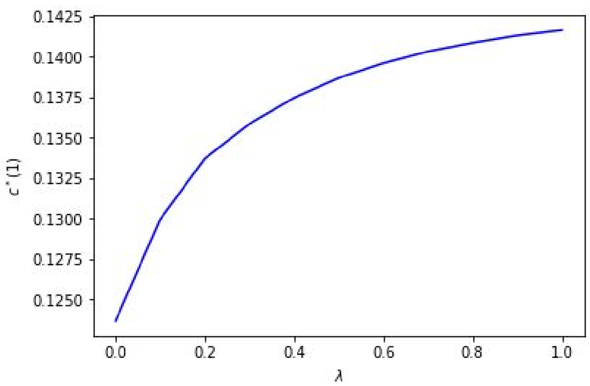

Figure 13: Effect of varying λ on the optimal consumption proportion of wealth c^* evaluated at $y = 1$

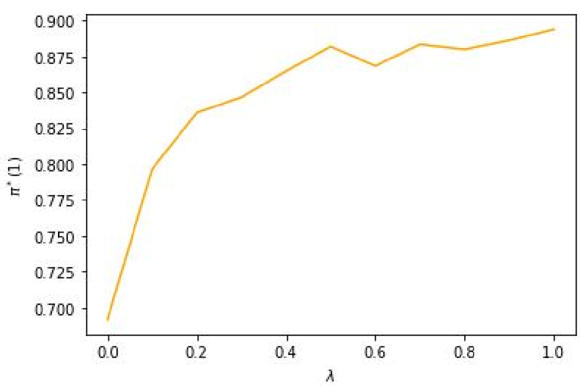

Figure 14: Effect of varying λ on the optimal investment proportion of wealth π^* evaluated at $y = 1$

As λ increases, the investor relies more on the past stock prices. The optimal proportions of the figures 13 and 14 suggest that the investor should consume and invest more in the risky asset, this result might be surprising at first sight since the investor won't be relying on recent information; but at the same time, this might be expected as for such stochastic volatility model, the investor believes that the volatility is persistent and clustered. Thus, there is a trade-off between relying on recent information and taking historic past values into account. We can therefore expect these plots to be increasing for small values of λ and then decreasing for very high values.

6 Stochastic interest rate

In the classical Merton's problem, the agent invests in a risky asset and a risk-free asset assuming a deterministic fixed interest rate r. In this section, we relax this assumption and consider the following stochastic differential equation for interest rate :

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t$$

where W_t is a standard Brownian Motion.

The choice of $\mu(t, r_t)$ and $\sigma(t, r_t)$ differs from a model to another. The main interest rate models [8] are:

- Vasicek (1977) [54]:

$$dr_t = k(\theta - r_t)dt + \sigma dW_t, \quad \alpha = (k, \theta, \sigma)$$

- Cox-Ingersoll-Ross (CIR, 1985) [16]:

$$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t, \quad \alpha = (k, \theta, \sigma), \quad 2k\theta > \sigma^2$$

- Dothan / Rendleman and Bartter [47]:

$$dr_t = ar_t dt + \sigma r_t dW_t, \left(r_t = x_0 e^{(a - \frac{1}{2}\sigma^2)t + \sigma W_t}, \alpha = (a, \sigma)\right)$$

- Exponential Vasicek:

$$r_t = \exp(z_t), \quad dz_t = k(\theta - z_t)dt + \sigma dW_t, \quad \alpha = (k, \theta, \sigma)$$

Each model has its advantages and drawbacks.

The Vasicek model assumes a linear equation that can be solved explicitly which makes it tractable. The expectation of r_t converges to θ when $t \to \infty$, which means that the process r_t is mean reverting. However, this model allows r_t to take negative values with positive probability. This assumption used to be a drawback before the financial crisis of 2008, but is legitimate since then.

The CIR model assumes a mean reverting non negative process r_t and is usually closer to market implied distributions than Vasicek model, but r_t follows a Chi-squared distribution under this model and thus it is less tractable.

The third one is a Black and Scholes equation. Although r_t has a log-normal distribution and is tractable, this model is not mean reverting; in fact, the expectation of r_t can converge to ∞ if $a > 0$ which is not realistic since interest rates are controlled by the Central Bank.

Finally, the fourth equation models a log-normal, non negative and mean reverting process but is not tractable.

We will assume a Vasicek model for interest rate r_t to derive the formulas. In this model, the parameter θ can be interpreted as the long term mean, k as the speed of convergence to the long term mean and σ is the volatility. The variance $var(r_t)$ converges to $\frac{\sigma^2}{2k}$ as $t \mapsto \infty$, which means that increasing the speed k or decreasing the volatility σ leads to decrease the uncertainty and hinders the variance to explode.

6.1 Problem formulation

- **State process:**

  The model is driven by two differential equations:

$$
\begin{aligned}
d\omega_t &= r_t\omega_t\,dt + \theta_t\,(\sigma\,dW_t + (\mu - r_t)\,dt) - c_t\,dt, \\
dr_t &= \beta\,(\bar{r} - r_t)\,dt + \sigma_r\,dB_t, \\
d\langle W, B\rangle_t &= \eta\,dt,
\end{aligned}
\tag{26}
$$

  where $\mu, \sigma, \beta, \bar{r}, \sigma_r$ are constants and the correlation parameter η lies in $[-1, 1]$.

  We set the state process to be the vector $Z_t := \begin{bmatrix} \omega_t \\ r_t \end{bmatrix}$. It is the unique solution of the following equation:

$$
dZ_t = \begin{bmatrix} r_t\omega_t + \theta_t\,(\mu - r_t) - c_t \\ \beta\,(\bar{r} - r_t) \end{bmatrix} dt + \begin{bmatrix} \sigma\theta_t & 0 \\ \sigma_r\eta & \sigma_r\sqrt{1-\eta^2} \end{bmatrix} \begin{bmatrix} dB_t^1 \\ dB_t^2 \end{bmatrix}
\tag{27}
$$

  where $\left(B_t^1, B_t^2\right)$ is a two-dimensional standard Brownian motion.

  Thus, (Z_t) is a diffusion process with drift and volatility matrix:

  $\mu(\omega, r, \theta, c) := \begin{bmatrix} \omega r + \theta\,(\mu - r) - c \\ \beta\,(\bar{r} - r) \end{bmatrix}$ and $\sigma(\omega, r, \theta, c) := \begin{bmatrix} \sigma\theta & 0 \\ \sigma_r\eta & \sigma_r\sqrt{1-\eta^2} \end{bmatrix}$ respectively.

- **Control process:**

  The control processes are the same as in Merton's problem: the consumption process c_t and θ_t representing the value of the holding asset, ν is thus set to be the vector $\nu_t := \begin{bmatrix} c_t \\ \theta_t \end{bmatrix}$.

- **Objective functional:**

  The objective functional is the expected discounted utility derived from consumption:

$$J(Z(\cdot),\nu(\cdot)) = \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c(t))dt \mid \omega(0)=\omega, r(0)=r\right]$$

  where the utility function is the CRRA function $U(\omega) := \dfrac{\omega^{1-R}}{1-R}$.

  Our goal is to find the supremum of the above functional at every time t:

$$V(t,\omega,r) = \sup_{(\theta,c)\in\mathscr{A}_{(t,\omega,r)}} \mathbb{E}\left[\int_t^\infty e^{-\rho s} U(c_s)\, ds \mid \omega(t)=\omega, r(t)=r\right]$$

Again the scaling property gives the following decomposition of the value function:

$$V(t,\omega,r) = e^{-\rho t} U(\omega) f(r)$$

6.2 HJB equation

The associated HJB equation for this infinite horizon problem is given by:

$$\rho V(\omega,r) - H(\omega,r,\nabla V(\omega,r), D^2 V(\omega,r)) = 0, \qquad \omega, r \in \mathbb{R} \tag{28}$$

where:

$$H(\omega,r,\nabla V(\omega,r),D^2V(\omega,r)) := \sup_{\theta\geq 0, c\geq 0}\left[\mu(\omega,r,\theta,c)\,\nabla V(\omega,r) + \frac{1}{2}\mathrm{tr}(\sigma\sigma^T(\omega,r,\theta,c)D^2V(\omega,r)) + U(c)\right]$$

(29)

First, we compute the quantities:

$$\begin{aligned} \mu(\omega,r,k,c)^T \nabla V(\omega,r) &= \begin{bmatrix} \omega r + \theta(\mu - r) - c \\ \beta(\bar{r} - r) \end{bmatrix}^T \begin{bmatrix} V_\omega \\ V_r \end{bmatrix} \\ &= (\omega r + \theta(\mu - r) - c)\, V_\omega + \beta(\bar{r} - r)\, V_r \end{aligned}$$

and

$$\begin{aligned}\operatorname{tr}\left(\sigma\sigma^T(\omega,r,\theta,c)D^2V(\omega,r)\right) &= \operatorname{tr}\left(\begin{bmatrix}\sigma\theta & 0\\ \sigma_r\eta & \sigma_r\sqrt{1-\eta^2}\end{bmatrix}\begin{bmatrix}\sigma\theta & 0\\ \sigma_r\eta & \sigma_r\sqrt{1-\eta^2}\end{bmatrix}^T\begin{bmatrix}V_{\omega\omega} & V_{\omega r}\\ V_{\omega r} & V_{rr}\end{bmatrix}\right)\\ &= \sigma^2\theta^2V_{\omega\omega}+2\sigma\sigma_r\eta\theta V_{\omega r}+\sigma_r^2V_{rr}\end{aligned}$$

By substituting these expressions in the equation (6.2), we obtain:

$$\begin{aligned}H(\omega,r,\nabla V(\omega,r),D^2V(\omega,r)) &= \sup_{\theta\geq 0,c\geq 0}[(\omega r+\theta(\mu-r)-c)V_\omega+\beta(\bar{r}-r)V_r+\\ &\quad +\tfrac{1}{2}\sigma^2\theta^2V_{\omega\omega}+\sigma\sigma_r\eta\theta V_{\omega r}+\tfrac{1}{2}\sigma_r^2V_{rr}+U(c)]\\ &= \omega rV_\omega+\beta(\bar{r}-r)V_r+\tfrac{1}{2}\sigma_r^2V_{rr}+\sup_{c\geq 0}[-cV_\omega+U(c)]\\ &\quad +\sup_{\theta\geq 0}\left[\theta(\mu-r)V_\omega+\tfrac{1}{2}\sigma^2\theta^2V_{\omega\omega}+\sigma\sigma_r\eta\theta V_{\omega r}\right]\end{aligned} \tag{30}$$

where:

$$V(\omega,r)=U(\omega)f(r)=\frac{\omega^{1-R}}{1-R}f(r)$$

6.3 Optimal control parameters

Here we derive the optimal values of consumption and the wealth invested in the stock by solving the optimisation problems:

$$\sup_{c\geq 0}[-cV_\omega+U(c)] \quad \text{and} \quad \sup_{\theta\geq 0}\left[\theta(\mu-r)V_\omega+\frac{1}{2}\sigma^2\theta^2V_{\omega\omega}+\sigma\sigma_r\eta\theta V_{\omega r}\right]$$

- **Optimal c^*:**

Since $R>0$, the function $c\mapsto -cV_\omega+\dfrac{c^{1-R}}{1-R}$ is concave and thus it admits a supremum.
The first order condition can be written as:

$$\partial_c(-cV_\omega+U(c))=-\omega^{-R}f(r)+c^{-R}=0$$

i.e. the optimal proportion of wealth consumed is:

$$q^*=\frac{c^*}{\omega}=(f(r))^{-\frac{1}{R}}$$

- **Optimal θ^*:**

The function $\omega \mapsto \theta(\mu - r) V_\omega + \frac{1}{2}\sigma^2\theta^2 V_{\omega\omega} + \sigma\sigma_r\eta\theta V_{\omega r}$ is polynomial of second degree. Assuming $V_{\omega\omega} < 0$, we can deduce that it has a maximum attained at:

$$\theta^* = \frac{\sigma\sigma_r\eta\omega^{-R} f'(r) + (\mu - r) f(r)\omega^{-R}}{R\sigma^2 f(r)\omega^{-R-1}}$$

And the optimal proportion of wealth invested in the risky asset is:

$$s^* = \frac{\theta^*}{\omega} = \frac{\sigma\sigma_r\eta f'(r) + (\mu - r) f(r)}{R\sigma^2 f(r)}$$

By substituting c^* and θ^* in (30), we obtain the second order ODE:

$$Rf^{1-1/R} - \rho f + r(1-R)f + (1-R)\frac{\{(\mu - r)f + \sigma\sigma_r\eta f'\}^2}{2\sigma^2 Rf} + \frac{1}{2}\sigma_r^2 f'' + \beta(\bar{r} - r)f' = 0 \quad (31)$$

6.4 Solving the ODE

The ODE (31) doesn't admit a closed-form solution.

However, we can notice that this ODE looks like the ODE (21). The main difference is the term of $\sigma\sigma_r\eta f'$. In fact, in the previous problem, we managed to linearise the ODE and write the solution as an expectation using Feynman Kac theorem. We can have the same shape of ODE if we take $\eta = 1$ and $\sigma = \sigma_r$, which means that the interest rate and the stock prices are driven by the same source of randomness and they also have the same volatility. These assumptions don't reflect the real dynamics of stock prices and interest rates.

If $\eta = 0$, i.e the stock prices and interest rates are driven by independent Brownian motions, then the ODE becomes:

$$Rf^{1-1/R} - \rho f + r(1-R)f + (1-R)\frac{(\mu - r)^2}{2\sigma^2 R} f + \frac{1}{2}\sigma_r^2 f'' + \beta(\bar{r} - r)f' = 0$$

Subsolution and supersolution method:

By considering

$$Z(r) := \ln f(r)$$

Flemming and Pang [24], [45] showed under some conditions the existence of a subsolution $\underline{Z}$ and supersolution $\bar{Z}$ such that:

$$\underline{Z}(r) \leqslant Z(r) \leqslant \bar{Z}(r), \quad \forall r \in \mathbf{R}$$

We state their findings adapted to our framework as the following:

Theorem 5 (Case $R > 1$ [24])
Define

$$a_1 := \frac{-2(1-R)}{3\sigma^2 R^2}, \quad a_2 := \mu - \sigma^2 R$$

Then there exists a constant $\bar{a}_3 > 0$ such that for any $a_3 \geq \bar{a}_3$

$$\underline{Z}(r) := -R \ln\left(a_1 (r - a_2)^2 + a_3\right)$$

is a subsolution.

Define

$$b_1 := \frac{R-1}{2\sigma^2 R^2}, \quad b_2 := \mu - \sigma^2 R$$

$$b_3 := b_1 \frac{2\sigma_r^2 \left[\frac{3}{2} - (1-R) + (1-R)\eta^2\right] - 2\eta\sigma^3\sigma_r R(1-R) + \left|\beta(\bar{r} - b_2)\right|}{2\beta + \left|\beta(\bar{r} - b_2)\right|}$$

If

$$\begin{aligned}\rho \geq \; & \mu(1-R) + R\left[2\beta\left|\beta(\bar{r} - b_2)\right| - \tfrac{\sigma^2(1-R)}{2}\right] + \\ & -\frac{2(1-R)\sigma_r^2\left[\frac{3}{2} - (1-R) + (1-R)\eta^2\right] - 2\eta\sigma^3\sigma_r(1-R)^2 R + (1-R)\left|\beta(\bar{r} - b_2)\right|}{2\sigma^2 R\left[2\beta + \left|\beta(\bar{r} - b_2)\right|\right]}\end{aligned}$$

Then

$$Z(r) := \log\left[\left(b_1 (r - b_2)^2 + b_2\right)^{-R}\right]$$

is a supersolution .

Theorem 6 (Case $R < 1$ [24])
Assume

$$\rho > (1-R)\mu - \frac{\sigma^2}{2} R(1-R)$$

Define K_1 as

$$K_1 := \log \tilde{K}_1$$

where

$$\tilde{K}_1^{-\frac{1}{R}} = \frac{1}{R}\left[\rho - \mu(1-R) + \frac{\sigma^2}{2} R(1-R)\right]$$

Then, for any $K_2 \leq K_1$, K_2 is a subsolution.

Define:

$$R_1 := \frac{\sigma_r^2 - 2\beta\eta\sigma\sigma_r}{\sigma^2\beta^2 + \sigma_r^2 - 2\beta\eta\sigma\sigma_r}$$

If

$$\max\{1, R_1\} < R < 1$$

In addition, define

$$\mu_1 := -2\sigma^2\left[1 + \frac{\gamma\eta^2}{R}\right]$$
$$\mu_2 := 2\beta + \frac{2(1-R)\eta\sigma_r}{\sigma R}$$
$$\mu_3 := -\frac{1-R}{2\sigma^2 R}$$

Let a^+, a^- be the real roots of $\mu_1 a^2 + \mu_2 a + \mu_3 = 0$, $0 < a^- < a^+$.

Then for any $a_1 \in (a^-, a^+)$, there exist constants $a_2 > K_1$ and $C_1(a_1)$, where $C_1(\cdot)$ is given by:

$$C_1(a_1) := \frac{4\lambda_1(a_1)\lambda_3(a_1) - \lambda_2^2(a_1)}{4\lambda_1(a_1)}$$
$$\lambda_1(a_1) := \mu_1 a_1^2 + \mu_2 a_1 + \mu_3$$
$$\lambda_2(a_1) := -\left[2\beta\bar{r} + \frac{2(1-R)\eta\sigma_r}{\sigma R}\right]a_1 + \frac{\mu(1-R)}{\sigma_1^2 R)} - (1-R)$$
$$\lambda_3(a_1) := -a_1\sigma_r^2 - \frac{(1-R)\mu^2}{2\sigma^2 R}$$

such that

$$\bar{Z}(r) \equiv a_1 r^2 + a_2$$

is a supersolution if $\beta > -C_1(a_1)$.

6.5 Numerical example

For this numerical part, we will consider the following parameters:

Parameter	Value
μ	0.15
$\bar{r}$	0.04828
R	2
η	0.45
ρ	0.02
σ	0.35
σ_r	0.01
β	0.2

Table 4: Parameter values of the numerical solution of stochastic interest rate model

In this case $R > 1$, the supersolution and subsolution are given by theorem (5).

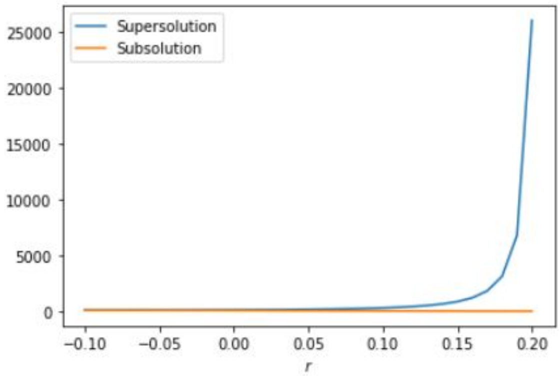

Figure 15: Supersolution $\bar{f}(r)$ and subsolution $\underline{f}(r)$ of the function f

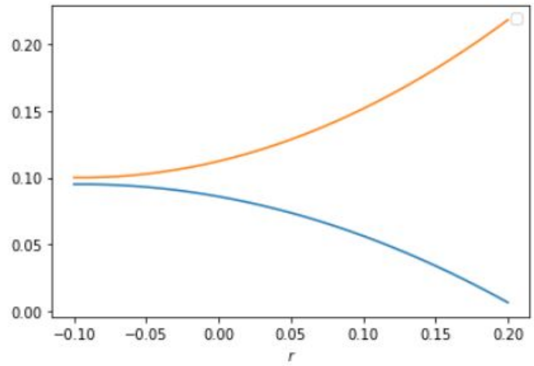

Figure 16: Supersolution $\bar{q}^*(r)$ and subsolution $\underline{q}^*(r)$ of the consumption proportion $q^*(r)$

The figure 15 shows the evolution of the supersolution and subsolution of the function f. The function f is comprised between these two functions. As the consumption proportion in this model is $q^* = (f(r))^{-\frac{1}{R}}$, we can find upper and lower bounds for the optimal consumption proportion given by the figure 16.

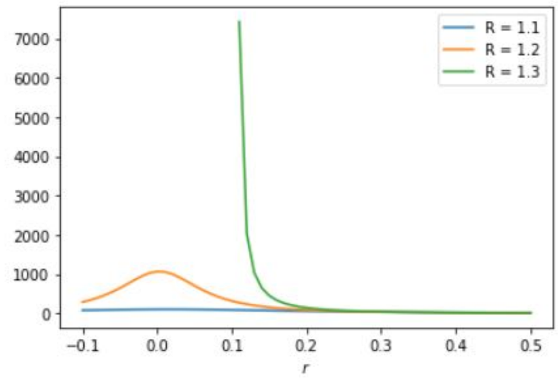

Figure 17: Effect of varying the risk aversion rate R on the supersolution $\bar{f}(r)$ of the function f

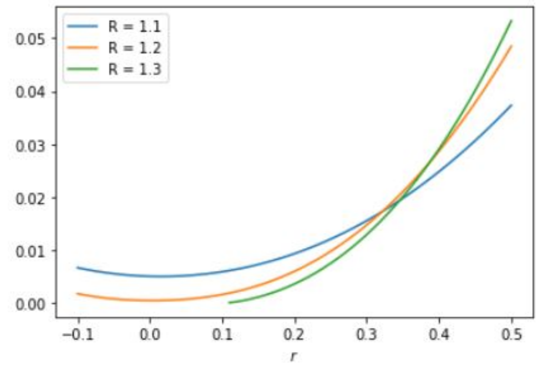

Figure 18: Effect of varying the risk aversion rate R on the subsolution $\underline{q}^*(r)$ of the consumption proportion $q^*(r)$

The figure 17 shows the effect of varying the parameter R on the supersolution and subsolution. We can notice that the the supersolution attains a supremum value that goes to ∞ when R is increasing. This results in a big upper bound and gives poor information about the optimal value. As for the figure 18, it shows that the shape of the consumption proportion is steadier. The other parameters $\bar{r}$, β, η and ρ don't have a big effect on the shape of the supersolution/subsolution.

The drawback of this method is that we don't have any information about the investment proportion π^*.

7 Conclusion

In this thesis, we studied utility maximisation problems. First, we assumed a discrete-time single period model. This mathematically simple setting is an unrealistic representation of the financial market, but is an important milestone in the theory of finance.

We tackled after that the consumption-investment problem under a continuous time model, where we considered a financial market consisting of a risk-free asset and risky assets and the objective is to maximize the expected Constant-Relative-Risk-Aversion utility of consumption under an infinite time horizon. The Dynamic Programming Principle and the HJB equation were used to derive a colsed-form solution of Merton's problem: The investor should allocate constant proportions of wealth to the risky assets and each fraction should be proportional to the Sharpe Ratio of the asset which is equal to the expected excess return over interest rate adjusted by the volatility.

In Merton's problem, several assumptions are made, for instance the stock prices volatilities and interest rates are deterministic. We investigated the consistency of these assumptions with the market implied distributions, then we modified the model to take into account the randomness of these two parameters.

First, we assumed a stochastic volatility model where the dynamics of stock price volatilities are described by Hobson and Rogers stochastic volatility expressed as an exponentially-weighted mean of historic log-prices. This model captures the volatility clustering and persistence and the Leverage Effect observed in the market and is consistent with the volatility smile and skewness.

Writing down the HJB equation leads to a non linear ODE. Using a change of variable, the equation is reduced to a linear ODE that can be solved using Feynman-Kac theorem and expressed as an expectation of an Ito Process. Using MC simulations, we simulated the process paths and computed an estimation of the expectation and the optimal investment and consumption proportions. This numerical method is powerful and gives high-standard approximations.

We found some similarities with Merton's problem: the investor consumes and invests less in stocks when the offset is high. We have also found that the more the investor is risk-averse the less should be the consumption and investment proportions. Furthermore, the greater is the discounting rate of past information λ (but not too large at the same time), the more the investor is comfortable to consume and invest.

We considered then a model where the interest rate follows a Vasicek model. There is no closed-form formula of the optimums under the infinite time horizon. We used the subsolution-supersolution method discussed by Fleming and Pang to find upper and lower bounds of the solution. This method doesn't provide a characterization of the optimal investment proportion, but gives an idea about the range of the value function and the consumption fraction.

Rogers discussed a numerical method in his book [48] to solve this problem. He used the policy improvement method to approximate the value function. This method results in a non-steady algorithm that doesn't converge for several values of parameters. Furthermore, this approach doesn't have a probabilistic interpretation.

Finally, in this thesis we relaxed only two of the non realistic assumptions in Merton's problem. We didn't take into account neither transaction costs nor taxes nor different utility functions. We also assumed a continuous trading and didn't impose any consumption constraints. For a more realistic and accurate portfolio optimisation, one can use the Reinforcement Learning approach where an agent is trained and is learning from a dynamic environment and aims to maximize its reward.

References

[1] Andrew Ang and Jun Liu. "Risk, return, and dividends". In: *Journal of Financial Economics* 85.1 (2007), pp. 1–38.

[2] Kenneth J Arrow. *Essays in the theory of risk-bearing*. Tech. rep. 1970.

[3] Richard E Bellman and Stuart E Dreyfus. *Applied dynamic programming*. Princeton university press, 2015.

[4] Alain Bensoussan, Michel Crouhy, and Dan Galai. "Stochastic equity volatility and the capital structure of the firm". In: *Philosophical Transactions of the Royal Society of London. Series A: Physical and Engineering Sciences* 347.1684 (1994), pp. 531–541.

[5] MJ Best and B Ding. "On the continuity of the minimum in parametric quadratic programs". In: *Journal of Optimization Theory and Applications* 86.1 (1995), pp. 245–250.

[6] Robert C Blattberg and Nicholas J Gonedes. "A comparison of the stable and student distributions as statistical models for stock prices". In: *The journal of business* 47.2 (1974), pp. 244–280.

[7] Tim Bollerslev. "Generalized autoregressive conditional heteroskedasticity". In: *Journal of econometrics* 31.3 (1986), pp. 307–327.

[8] Damiano Brigo and Fabio Mercurio. *Interest rate models-theory and practice: with smile, inflation and credit*. Springer Science & Business Media, 2007.

[9] Mark Broadie. "Computing efficient frontiers using estimated parameters". In: *Annals of operations research* 45.1 (1993), pp. 21–58.

[10] G Buffon. "Essai d'arithmétique morale". In: *Supplémenta l'Histoire naturelle* 4 (1777), p. 1777.

[11] Álvaro Cartea, Sebastian Jaimungal, and José Penalva. *Algorithmic and high-frequency trading*. Cambridge University Press, 2015.

[12] George Chacko and Luis M Viceira. "Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets". In: *The Review of Financial Studies* 18.4 (2005), pp. 1369–1402.

[13] Hao Chang and Kai Chang. "Optimal consumption–investment strategy under the Vasicek model: HARA utility and Legendre transform". In: *Insurance: Mathematics and Economics* 72 (2017), pp. 215–227.

[14] Hao Chang and Xi-min Rong. "An investment and consumption problem with CIR interest rate and stochastic volatility". In: *Abstract and Applied Analysis*. Vol. 2013. Hindawi. 2013.

[15] Vijay K Chopra and William T Ziemba. "The effect of errors in means, variances, and covariances on optimal portfolio choice". In: *Handbook of the Fundamentals of Financial Decision Making: Part I*. World Scientific, 2013, pp. 365–373.

[16] John C Cox, Jonathan E Ingersoll Jr, and Stephen A Ross. "A theory of the term structure of interest rates". In: *Theory of valuation*. World Scientific, 2005, pp. 129–164.

[17] John C Cox and Stephen A Ross. "The valuation of options for alternative stochastic processes". In: *Journal of financial economics* 3.1-2 (1976), pp. 145–166.

[18] Darrell Duffie. *Dynamic asset pricing theory*. Princeton University Press, 2010.

[19] Edwin J Elton and Martin J Gruber. "On the optimality of some multiperiod portfolio selection criteria". In: *The Journal of Business* 47.2 (1974), pp. 231–243.

[20] Edwin J Elton and Martin J Gruber. "Portfolio theory when investment relatives are lognormally distributed". In: *The Journal of Finance* 29.4 (1974), pp. 1265–1273.

[21] Robert F Engle. "Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation". In: *Econometrica: Journal of the Econometric Society* (1982), pp. 987–1007.

[22] Eugene F Fama and James D MacBeth. "Risk, return, and equilibrium: Empirical tests". In: *Journal of political economy* 81.3 (1973), pp. 607–636.

[23] Wendell H Fleming and Daniel Hernández-Hernández. "An optimal consumption model with stochastic volatility". In: *Finance and Stochastics* 7.2 (2003), pp. 245–262.

[24] Wendell H Fleming and Tao Pang. "An application of stochastic control theory to financial economics". In: *SIAM Journal on Control and Optimization* 43.2 (2004), pp. 502–531.

[25] Jean-Pierre Fouque, George Papanicolaou, and K Ronnie Sircar. *Derivatives in financial markets with stochastic volatility*. Cambridge University Press, 2000.

[26] Andrea Gamba. "Portfolio analysis with symmetric stable paretian returns". In: *Current topics in quantitative finance*. Springer, 1999, pp. 48–69.

[27] Robert Geske. "The valuation of compound options". In: *Journal of financial economics* 7.1 (1979), pp. 63–81.

[28] Markowitz H. "Portfolio Selection". In: *The Journal of Finance* (1952).

[29] David G Hobson and Leonard CG Rogers. "Complete models with stochastic volatility". In: *Mathematical Finance* 8.1 (1998), pp. 27–48.

[30] David Hobson, SL Alex, and Yeqi Zhu. "A multi-asset investment and consumption problem with transaction costs". In: *Finance and Stochastics* 23.3 (2019), pp. 641–676.

[31] John Hull. "An analysis of the bias in option pricing caused by a stochastic volatility". In: *Advances in futures and options research* 3 (1988), pp. 29–61.

[32] John Hull and Alan White. "The general Hull–White model and supercalibration". In: *Financial Analysts Journal* 57.6 (2001), pp. 34–43.

[33] John Hull and Alan White. “The pricing of options on assets with stochastic volatilities”. In: *The journal of finance* 42.2 (1987), pp. 281–300.

[34] Dong Jiuying. “Optimal Investment Consumption Model with Vasicek Interest Rate”. In: *2007 Chinese Control Conference*. IEEE. 2007, pp. 391–394.

[35] JD Jobson. “Confidence regions for the mean-variance efficient set: an alternative approach to estimation risk”. In: *Review of Quantitative Finance and Accounting* 1.3 (1991), p. 235.

[36] Herb Johnson and David Shanno. “Option pricing when the variance is changing”. In: *Journal of financial and quantitative analysis* (1987), pp. 143–151.

[37] Holger Kraft*. “Optimal portfolios and Heston’s stochastic volatility model: an explicit solution for power utility”. In: *Quantitative Finance* 5.3 (2005), pp. 303–313.

[38] Paul Lévy. “Calcul des probabilités”. In: (1925).

[39] Leonard C Maclean and K Laurence Weldon. “Estimating multivariate random effects without replication”. In: *Communications in Statistics-Theory and Methods* 25.7 (1996), pp. 1447–1469.

[40] Ravi Malhotra. “Portfolio Management in the Presence of Stochastic Volatility”. PhD thesis. University of London, 2008.

[41] Falcone Maurizio. “Optimal Control and the Dynamic Programming Principle”. In: Jan. 2014, pp. 1–8. ISBN: 978-1-4471-5102-9. DOI: 10.1007/978-1-4471-5102-9_209-1.

[42] Robert C Merton. “Lifetime portfolio selection under uncertainty: The continuous-time case”. In: *The review of Economics and Statistics* (1969), pp. 247–257.

[43] Marek Musiela and Thaleia Zariphopoulou. “An example of indifference prices under exponential preferences”. In: *Finance and Stochastics* 8.2 (2004), pp. 229–239.

[44] Eun-Jung Noh and Jeong-Hoon Kim. “An optimal portfolio model with stochastic volatility and stochastic interest rate”. In: *Journal of Mathematical Analysis and Applications* 375.2 (2011), pp. 510–522.

[45] Tao Pang. “Portfolio optimization models on infinite-time horizon”. In: *Journal of optimization theory and applications* 122.3 (2004), pp. 573–597.

[46] Stanley Pliska. *Introduction to mathematical finance*. Blackwell publishers Oxford, 1997.

[47] Richard J Rendleman. “The pricing of options on debt securities”. In: *Journal of Financial and Quantitative Analysis* (1980), pp. 11–24.

[48] Leonard CG Rogers. *Optimal investment*. Vol. 1007. Springer, 2013.

[49] Mark Rubinstein. “Displaced diffusion option pricing”. In: *The Journal of Finance* 38.1 (1983), pp. 213–217.

[50] Louis O Scott. “Option pricing when the variance changes randomly: Theory, estimation, and an application”. In: *Journal of Financial and Quantitative analysis* (1987), pp. 419–438.

[51] H Mete Soner. *Stochastic optimal control in finance*. Scuola normale superiore, 2004.

[52] James Tobin. "Liquidity preference as behavior towards risk". In: *The review of economic studies* 25.2 (1958), pp. 65–86.

[53] Jakub Trybuła. "Optimal consumption problem in the Vasicek model". In: *Opuscula Mathematica* 35.4 (2015), pp. 547–560.

[54] Oldrich Vasicek. "An equilibrium characterization of the term structure". In: *Journal of financial economics* 5.2 (1977), pp. 177–188.

[55] James B Wiggins. "Option values under stochastic volatility: Theory and empirical estimates". In: *Journal of financial economics* 19.2 (1987), pp. 351–372.

Acronyms