Approximation Error in Dependence Iteration for Default Modelling

by

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Declaration Form

The work contained in this thesis is my own work unless otherwise stated.

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Abstract

We examine the differences between two default time simulation procedures: (a) single simulation of multivariate default times throughout the whole time frame, the correct procedure and (b) iterated simulation which means we break down the simulation in the first procedure into N consecutive and identical time steps. The second approach is an approximation that is often used in the industry for logistic / IT reasons. To calculate default probability, we utilise copula function as it could capture dependence structure among entities with arbitrary margins. With regard to our study, default time of an entity is defined as a random variable with exponential distribution as in the intensity model. Likewise, we divide the problems into two big cases: all survival case, where we calculate the probability that all names survive at terminal time and mixed survival case, where we compute the probability that some names default and other survive. We then focus on 2 and 3-dimensional copula for each case.

According to Brigo and Chourdakis [16], computing default probability with either procedure 1 or 2 is indistinguishable if the copula has self-chaining characteristic. Indeed, the idea of self-chaining copula originally comes from Multivariate Lack of Memory (MLOM) property which can be found in Marshall-Olkin (or multivariate exponential) distribution. In this thesis, we expand the results in [16] by analysing the error pattern as a function of Kendall's tau rank correlation in the bivariate case. We further numerically examine the error in trivariate case and mixed survival/default states. We also compute the error analytically when N goes to infinity for a few copulas.

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Contents

List of Figures

List of Tables

1 Introduction

1.1 Background

In the last decade, uncertainty problems of the financial sector have been dramatically increasing. A considerable amount of giant companies suffered great losses or, even worse, went bankrupt. The phenomenon started in late 2007 when the global financial crisis hit most of the countries and has not fully recovered until now. The collapse of Lehman Brother, the fourth-largest United States investment bank, in September 2008 is one of the most tragic stories in the financial industry. A related point to be considered is a significant rise of sensitivity level of the industry with respect to political and social issues. Take the recent case where Britain voted to leave the European Union as an example; several industries have already experienced financial effects although the new regulations have not been fully settled. This sensitivity issue has a great impact on credit exposure levels, which is considered as a trigger of systemic risk. Learning from the past events, investment banks and financial service companies are now dedicating a huge effort to improve their risk management system, especially in credit risk area.

According to McNeil, Frey and Embrechts [58, page 3] the term credit risk means the possibility that we will not receive the payments which are previously settled in specific agreement due to borrower's default. As we can see from the definition, the word default has a significant role in credit risk area. Brigo, Morini and Pallavicini [20, page 47] describe default as an event where a company could not fulfil the promised payments to its counter-party. Modelling and simulating default probability of several dependent entities are challenging tasks even for an expert quantitative analyst. Referring to Brigo and Mercurio in [19], there are three types of default probability models:

- Firm Value models: At first, Merton [59] constructed such models based on a stochastic process of the firms value in 1974. Whenever the value of that firm drops such that the firm could not pay its liability at maturity time, default does occur. In the next 2 years, Black and Cox [10] improved these models in order to make them more applicable to the real industry. Development of the latter models was inspired by barrier options pricing models. Nevertheless, some experts think that these approaches are less natural compared with the following models.

- Intensity models: the second models, which will be used in this thesis, were developed to overcome some shortages in firm value models. By adapting intensity models, default event does not relate to default-free market data. Moreover, a default time is interpreted as Poisson process' initial jump time with constant, deterministic time-varying, or stochastic intensity. This topic will be discussed more precisely in the following section.

- Hybrid models: a combination of the first two approaches.

Everyone might agree that financial disaster in 2008 was undoubtedly caused by simple yet powerful mathematical model which is known as the Gaussian copula. In fact, the disaster would not have happened if people had carefully looked at the model assumptions. Frequent misuse of mathematical models in the financial industry becomes one of our motivation to do further research in the related area. In spite of its shortage, the copula framework is still very popular among analysts due to its ability to describe the dependence structure of several random variables with arbitrary marginal distributions. Besides the copula function, there is also a single number that measures dependency level among several random variables which is usually called as rank correlation. Kendall's tau, one of the most famous examples of rank correlation, is used as a benchmark to determine parameter values of our default model.

We are interested in analysing consistency between 2 different simulation procedures of several famous copulas. In the first (or correct) procedure, default probability within the specified time frame, from 0 to final time T, is calculated directly in single copula simulation. Whereas in the second (or approximation) procedure, default probability is computed iteratively. This means the time interval is broken down into N identical sub-intervals. We then simulate local copula function along all consecutive sub-intervals leading to the terminal time T. In particular, this topic is divided into two main cases: (1) analysing and comparing iteration error for the case where all companies survive at final time T, (2) studying iteration error for the case in which some companies default and the rests survive. Additionally, the effect of iterating 2-dimensional copula on its dependency structure is also examined in this study.

In order to solve those problems, Multivariate Lack of Memory (MLOM) property is introduced. It leads us to several more complex mathematical terms such as self-chaining copula, Marshall-Olkin distribution function, markovian survival indicator, and so on. This study has been first aimed by Brigo and Chourdakis [16] which is then extended to a more challenging problem by Brigo, Mai, Scherer in [18]. As stated by the latter paper, there are two main reasons why the second simulation approach is more convenient than the other:

- Basel III, which is issued to resolve vagueness in liquidity risk area, is a set of banking regulations published in June 2011 by the Basel Committee on Banking Supervision. However, these regulations do not have legal force since they are only designed as standard banking guidelines. According to BIS consultative documents in [9], Basel III proposes that "Banks trading book exposures be assigned to a small number of liquidity horizon categories. [10 days, 1 month, 3 months, 6 months, 1 year][...]". In order to implement this guideline, a financial institution needs to develop a proper multivariate default simulation with the shorter time steps as proposed in Basel III.

- In Mathematical Finance theory, Brownian Motion (BM) is the most fundamental stochastic processes describing asset movements. Since asset movements are random processes, it is common that the values are updated progressively according to the predetermined time steps; consider stock prices evolutions in Binomial tree method as an example. On the other hand, the default times in intensity model are usually treated as random variables which are static and usually simulated once during the whole time frame. The problem may arise if we want to simulate default probabilities of assets which are driven by BM. Therefore, approximation procedure is preferable in the real industry.

1.2 Literature Review

In this part of the paper, we introduce and discuss several basic mathematical concepts needed for the rest of the paper. From the very basic idea such as copula to the more sophisticated theory like stochastic survival indicator. All of these materials are summarised from the past related literature to integrate the main problems we try to figure out.

While discussing mathematical concepts, notations are crucial things that could not be ignored. Therefore, we would like to introduce them in short. Generally, we are working with probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a set of all feasible outcomes, $\mathcal{F}$ is the set of events, and $\mathbb{P}$ is a probability whose value lies inside $[0, 1]$ interval. Moreover, a random variable τ_i taking values in $\mathcal{T} = \mathbb{R}_+$ is denoted as default time of company i. Another important notation taken from [16] is survival function. One-dimensional survival function at deterministic point t is defined as

$$S(t) = \mathbb{P}(\tau > t) = 1 - F(t), \tag{1.1}$$

where F is cumulative distribution function of τ.

1.2.1 Intensity Models

We refer to [8, page 221] and [20, page 65] as our general references. It has been stated that there are three ways to model default events. The first one, Firm Value model, was first introduced by Merton [59] in 1974. Default event is described as "an event where the value of debt is larger than equity at terminal time", see [59, page 9]. Hence, default time is predictable. This seems to be an appropriate primary model when modelling default time at first until we need to deal with short-term credit spreads.

In the next 2 years from the time Merton proposed his model, Black and Cox [10] announced their version of Firm Value model. The main difference of their set-up and the former is a consideration of earlier default time. Brigo, Morini, Pallavicini [20, page 51-53] shows that these two models result in relatively different outcomes. Although the latter closely resembles empirical data, this

model still has some shortcomings, especially if Credit Default Swap (CDS) calibration comes to our concern.

In order to overcome those problems, Intensity model was first mentioned at the beginning of nineties. This model, which sometimes known as Reduced Form model, has been widely used and developed since then. Jarrow and Turnbull [44], the first two experts who introduced Intensity models, describe this set-up as a modification of Merton's Firm Value model embedded with more sophisticated interest rates dynamics and Poisson process. Inspired by the previous works of some experts in this area, Lando [43] was then generalised the model by considering credit spread effect. Instead of Poisson process, Cox process is utilised in his model so that random form of intensities can be incorporated. Moreover, Lando's set-up is classified as "affine term structure" model which is related to Duffie and Singleton [31] framework. Even though affine models are popular among credit rating agencies due to its simplicity, Jarrow and Yu [45] shows that the aforementioned models are not accurate for some common cases. Recently, plenty of scholars have been doing a lot of researches in hybrid models. Broadly speaking, such models link hazard rate in Intensity models to firm's value of equity, see Duffie and Lando [30] as well as Fadugba and Edogbanya [32] for more detail.

After short introductory part above, a more mathematical section is presented in the following paragraphs. First, the definition of Homogeneous Poisson process, which taken from [64, Definition 5.1, page 313] will be explained below.

Definition 1.1 (Homogeneous Poisson Process)**.** A counting process $\{N(t), t \geq 0\}$ is called a Poisson process if its initial value is 0, the process has independent increments, and the number of defaults in any period of length ΔT has Poisson distribution with parameter $\lambda \Delta T$.

The term λ is defined as Poisson process rate, or in this context known as *hazard rate* whose value is always positive. If we look at the definition carefully, it also infers that the process has not only independent increments but also stationary increments. A small modification of the former process leads us to the Non-Homogeneous Poisson process. The latter is more popular since it is more likely to occur in real situations. Indeed, we can convert the Homogeneous Poison process into Non-Homogeneous one by making hazard rate as a function of time. Next, we also define *cumulative hazard rate* (or *hazard function*) as is defined in [20, page 66], reads

$$t \to \int_0^t \lambda_u du =: \Lambda_t.$$

Until now, we have not clearly described the link between the Poisson process and default time. In fact, this is absolutely clear. Again, we follow [20, page 66] and [13, page 662] by defining default time as the inverse of hazard function on an independent exponential random variable $\xi \sim$Exp(1).

Mathematically, we write the previous definition as

$$\tau := \Lambda^{-1}(\xi).$$

From basic probability theory, exponentially distributed random variable has cumulative distribution function: $\mathbb{P}\{\xi \leq u\} = 1 - e^{-u}$, or write it in different way as $\mathbb{P}\{\xi > u\} = e^{-u}$. Therefore, by applying simple transformation we can easily see that

$$\mathbb{P}\{\tau > t\} = \mathbb{P}\{\Lambda(\tau) > \Lambda(t)\} = \mathbb{P}\{\xi > \Lambda(t)\} = e^{-\Lambda(t)} = e^{-\int_0^t \lambda(s)ds}.$$

It might be recognised without going too far from our main point that default probability under Intensity models has a relatively similar form with the continuously compounded discount factor in Interest Rate theory. We refer interested reader to [19] for more detail on this topic.

1.2.2 Multivariate Lack of Memory (MLOM)

Reliability theory is a part of statistics focused on modelling the probability of a system to work properly under a predetermined time frame. The Multivariate Lack of Memory (MLOM) property, which is used as our foundation to build default probability model in this thesis, is the core concept of reliability theory. Most materials in this part are taken from [55, page 30-44], [54], and [16]. Random variable τ has an univariate lack of memory property if

$$\mathbb{P}(\tau \geq a + b | \tau \geq b) = \mathbb{P}(\tau \geq a), \quad \text{for any } a, b \in \mathcal{T}. \tag{1.2}$$

In particular, Equation (1.2) could be generalised to the multivariate case (MLOM) easily. We illustrate 2-dimensional case for the sake of simplicity in the following equation:

$$\mathbb{P}(\tau_1 \geq a + b, \tau_2 \geq c + b | \tau_1 \geq b, \tau_2 \geq b) = \mathbb{P}(\tau_1 \geq a, \tau_2 \geq c) \tag{1.3}$$

for all a, b, $c \in \mathcal{T}$. If we simplify Equation (1.3) by setting $a = c$, the bivariate vector (τ_1, τ_2) is then said to fulfill a Multivariate Homogeneous Lack of Memory (MHLOM) condition. For the rest of this study, we utilise MHLOM instead of MLOM property. We define MHLOM concept formally, in that

$$\mathbb{P}(\tau_1 \geq a + b, \tau_2 \geq a + b, | \tau_1 \geq b, \tau_2 \geq b) = \mathbb{P}(\tau_1 \geq a, \tau_2 \geq a)$$

for any $a, b \in T$. Readers may be questioning what kind of multivariate distribution has MHLOM property. The answer is obvious, Marshall-Olkin distribution.

Marshall-Olkin Distribution

The exponential distribution is a remarkable distribution function as it has the lack of memory property which is suitable for reliability (or survival) modelling. In 1967, Professor Marshall and Olkin [55] made an advance breakthrough in this area by proposing a generalisation of the

exponential distribution with particular exponential margins. This is the reason why sometimes Marshall-Olkin distribution is also known as multivariate exponential distribution. Marshall-Olkin bivariate survival function evaluated at non-negative point (t_1, t_2) is defined as

$$S(t_1, t_2) = e^{(-\lambda_1 t_1 - \lambda_2 t_2 - \lambda_{12} \max(t_1, t_2))},$$

where λ_1, λ_2, and λ_{12} are non-negative parameters. It also has exponential margins with survival function

$$S(t_1) = e^{-(\lambda_1+\lambda_{12})t_1} \quad \text{and} \quad S(t_2) = e^{-(\lambda_2+\lambda_{12})t_2}.$$

As Marshall-Olkin distribution is derived from the exponential distribution, it still has the lack of memory feature. This fact is summarised by Marshall and Olkin in [55] as a simple yet brilliant theorem as follows.

Theorem 1.2. *Consider a default time vector of m companies* $(\tau_1, ..., \tau_m)$.

$(\tau_1, ..., \tau_m)$ *has Marshall Olkin multivariate distribution* $\leftrightarrow$ *MLOM condition holds.*

Proof. See [55, page 37]. □

The study of MLOM concept including its relation with Marshall-Olkin distribution is a bottom line of this paper. Under particular MLOM condition, we can obtain similar results from those 2 simulation procedures described earlier in this chapter (see [16]). This will be explained precisely in chapter 2.

Remark 1.3. In some literature, parameters of bivariate Marshall-Olkin distribution are described in terms of α_1 and α_2, where

$$\alpha_1 = \frac{\lambda_{12}}{\lambda_{12} + \lambda_1} \text{ and } \alpha_2 = \frac{\lambda_{12}}{\lambda_{12} + \lambda_2}.$$

1.2.3 Copula

Copula is a joint distribution function of several random variables with uniform margins. This concept has been widely used since the 19^{th} century due to its powerful feature, especially in financial modelling. The concept of copula itself has been introduced by Fréchet [37] in 1951 to overcome some problems with linear correlation theory. In the next 8 years, Sklar [68] came out with the basic definition of copula which is still used until now. As copula will be our primary tool here, it is more convenient for readers to gain more understanding about this topic. So, we shall point out several important basic concepts which mostly quoted from [58, page 184-234].

For the rest of this thesis, we would denote n-dimensional copula with $C(u_1, u_2, ...u_n)$. Referring to [58] and [63], a mapping $C : [0, 1]^n \to [0, 1]$ can be identified as an n-dimensional copula if it satisfies all of the following:

- $C(u_1, \ldots, u_n)$ is an increasing function for each u_i with $i \in \{1, \ldots, n\}$.
- Let $u_j = 1$ for all $j \in \{1, \ldots, n\}$, $j \neq i$. Then the following equation holds

$$C(u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_n) = u_i.$$

- For any $(a_1, ...a_n), (b_1, ..., b_n) \in [0,1]^n$ with $a_i <= b_i$,

$$\Sigma_{i_1=1}^{2} ... \Sigma_{i_1=1}^{2} (-1)^{\Sigma_{k=1}^{n} i_k} C(u_{1i_1}, ..., u_{ni_n}) >= 0 \tag{1.4}$$

  where $u_{j1} = a_j$ and $u_{j2} = b_j$ for any $j = 1, ..., n$.

Remark 1.4. Equation (1.4) can be found in [58, page 185] or [63, page 5]. In general, condition (1.4) is related to Schwartz distribution in the framework of measure theory, see [16, page 7].

The next theorem, which is known as Sklar's theorem, links multivariate distribution function F with a copula function C (proof of this theorem could be seen in [58, page 187]).

Theorem 1.5 (Sklar's Theorem (1959))**.** *Let F be an n-dimensional distribution function with marginals $F_1, ..., F_n$ and also let $\bar{\mathbb{R}} = [-\infty, \infty]$. There is a copula $C : [0,1]^n \to [0,1]$ such that :*

$$F(x_1, x_2, ...x_n) = C(F_1(x_1), ..., F_n(x_n)), \quad x_1, ...x_n \in \bar{\mathbb{R}}. \tag{1.5}$$

Equation (1.5) is then defined as a joint distribution function with marginals $F_1, ..., F_n$. It is also worth mentioning that a discrete copula function does exist. The interested reader can refer to Mesiar (2005) for more detailed explanation in discrete copula case. Furthermore, we would like to introduce copula's boundaries which are usually recognised as Fréchet-Hoeffding bounds.

Proposition 1.6. *(Fréchet-Hoeffding bounds) The following condition holds for every copula $C(u_1, ..., u_n)$:*

$$min\ \{u_1, ..., u_n\} \geq C(u_1, ..., u_n) \geq max\ \{\Sigma_{i=1}^{n} u_i + 1 - d, 0\}.$$

The upper and lower copula boundaries are usually known as a comonotonicity and counter-monotonicity copula respectively. The comonotonicity copula describes a condition where all elements are perfectly positive dependent on each other. On the other hand, the counter-monotonicity copula is associated with perfect negative dependence structure.

Copula theory is one of the famous mathematical tools which are still developed by many scholars, especially in the financial industry, see for instance [25] and [48]. A plenty of copula types are arising and sometimes they resemble each other. As for clarity reason, we properly restrict this discussion to some particular copula classes. There are three categories of copulas based on their construction procedure: fundamental copulas, implicit copulas, and explicit copulas. The main differences between these three types are described below.

- Implicit copulas

  According to the name, copulas in this category do not have simple explicit expressions. Therefore, simulation method is needed to assess such copula. Gaussian copula,

  $$C_{\rho}^{Ga}(u_1, u_2) = \Phi_{\rho}(\Phi^{-1}(u_1), \Phi^{-1}(u_2)),$$

  is a famous example of implicit copulas, where Φ_{ρ} is a bivariate normal distribution function with a correlation matrix ρ. Another familiar example is t copula with ν degrees of freedom,

  $$C_{\nu,\rho}^{t}(u_1, u_2) = \mathbf{t}_{\nu,\rho}(\mathbf{t}^{-1}(u_1), \mathbf{t}^{-1}(u_2)) \tag{1.6}$$

  where $\mathbf{t}_{\nu,\rho}$ is a bivariate t distribution function with a correlation matrix ρ. Both Gaussian and t copula will be discussed further in the following chapter.

- Explicit copulas

  In contrast with implicit copulas, each explicit copula has a closed-form representation describing the joint distribution function. Mostly in this study, we are dealing with three well-known explicit copulas: Gumbel, Frank, and Clayton. Each of them has a single parameter called α which describes the dependence structure. See Table 1 in section 2 for more detail. Later on, we would frequently cope with the archimedean copula. It is very often the term archimedean copula and explicit copula are used interchangeably. This is because almost all of the archimedeans have explicit forms. To illustrate this, consider 2-dimensional Gumbel copula

  $$C_{\alpha}^{Gu}(u_1, u_2) = e^{-[\{-\ln(u_1)\}^{\alpha} + \{-\ln(u_2)\}^{\alpha}]^{1/\alpha}} \tag{1.7}$$

  with $\alpha \in [1, \infty)$. Compared to t copula in Equation (1.6), we could directly compute the probability on (u_1, u_2) using Gumbel copula's formula in Equation (1.7) without the necessity to do some simulations.

- Fundamental copulas

  There are three Fundamental copulas in total. Indeed, we have already mentioned two examples of fundamental copulas: comonotonicity and counter-monotonicity copula. The other member of such copulas is independence copula. In particular, bivariate independence copula is written as

  $$\hat{C}(u_1, u_2) = u_1 \cdot u_2.$$

  Fundamental copulas can be constructed by modifying dependency parameter(s) of implicit and explicit copulas. Consider 2-dimensional Gaussian copula. If the parameter ρ is fixed to be -1, we can get the counter-monotonicity copula. Similarly, if we set $\rho = 0$ and $\rho = 1$,

the independence and comonotonicity copula are obtained respectively. One can observe fundamental copulas' surface plots in Figure 1 below.

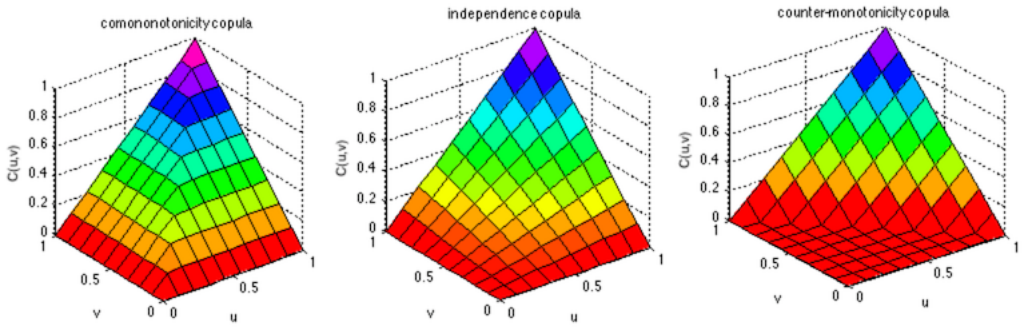

Figure 1: Surface plots of the comonotonicity copula, independence copula, and counter-monotonicity copula.

The next important concept is survival copula. Consider survival function in Equation (1.1). Using the fact that $S(t)$ is a decreasing function and applying quantile transformation theory, we then generalise the function becomes n-dimensional (joint) survival function:

$$\begin{aligned} S(t_1,...,t_n) &= \mathbb{P}(\tau_1 \geq t_1, ...\tau_n \geq t_n) = \mathbb{P}(S_1(\tau_1) < S_1(t_1), ..., S_n(\tau_n) < S_n(t_n)) \\ &= \mathbb{P}(U_1 < S_1(t_1), ..., U_n < S_n(t_n)) =: C(S_1(t_1), ..., S_n(t_n)), \end{aligned} \tag{1.8}$$

where C is a survival copula and U_i is uniform random variable. For convenience reason we set $u_i = S_i(t_i)$, so (1.8) can be written as

$$S(t_1,...,t_n) =: C(u_1,...,u_n).$$

Indeed, there is a complicated relation between the original copula and survival copula. Keen reader is encouraged to refer to [58, page 196] or [19, page 716].

Besides those aforementioned copulas, there are several other copula classes such as archimedean copulas, extreme value copulas, and self-chaining copulas. Since these three copula types are involved in our study, we briefly explain each of them in the following paragraph.

1. Archimedean copulas

   Many experts in both mathematical and finance areas have done a lot of research for this copula class. Archimedean copulas are really convenient since they allow us to do high-dimensional mathematical modelling with a single parameter. A copula is classified as archimedean group if it takes form (for 2-dimensional case)

   $$C(u,v) = \psi^{-1}(\psi(u) + \psi(v))$$

where ψ is a generator function, see [58, page 221-222] or [16]. As explained before, most of explicit copulas are also categorised as archimedean copulas, for example, Frank, Gumbel, and Clayton copula. Figure 2 illustrates surface plots of probability distribution function (pdf) of the three aforesaid archimedean copula. One can see that Frank copula's pdf is radially proportional while Gumbel and Clayton copula's show the opposite. Gumbel and Clayton have more probability concentrated in their tails than Frank copula does. It is also evident that Clayton copula exhibits lower tail dependence. In contrast, Gumbel copula indicates upper tail dependence. As Clayton and Gumbel have lower and upper tail dependence respectively, they are frequently used in risk modelling area.

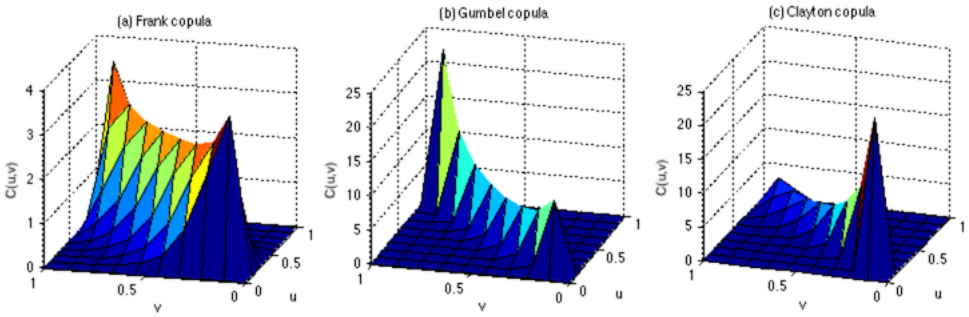

Figure 2: Surface plots showing the pdf of 3 archimedean copulas. (a) Frank copula with $\alpha = 10$. (b) Gumbel copula with $\alpha = 7$. (c) Clayton copula with $\alpha = 7$.

2. Extreme value copulas

   For this part, we use the definition from [41, page 2] and rewrite it here for reader's convenient. Supposed we have a vector $Y_i = (Y_{i1}, ..., Y_{im})$ with $i \in \{1, ..., n\}$, drawn from iid random variables with joint distribution function F and copula C_Y. Also let

$$M_n = (M_{n1}, ..., M_{nm}), \text{ with } M_{nj} = \max(Y_{1j}, Y_{2j}, ..., Y_{nj}).$$

   A copula of M_n, C_M, is defined as

$$C_M(u_1, \ldots, u_m) = C_Y(u_1^{1/n}, \ldots, u_m^{1/n})^n.$$

   Next, if we can find a copula C_Y which satisfies

$$\lim_{n\to\infty} C_Y(u_1^{1/n}, ..., u_m^{1/n})^n = C(u_1, ..., u_m)$$

   for all $(u_1, ..., u_m) \in [0, 1]^m$, the copula C is then said to be extreme value copula. Moreover, as is described in classic extreme value theory, a copula C is max-stable if and only if

$$C(u_1, ..., u_m) = C(u_1^{1/k}, ..., u_m^{1/k})^k$$

holds for all positive integer k larger than 1 and $(u_1, ..., u_m) \in [0,1]^m$. The connection between extreme value and max-stable copula is also explained in [41, Theorem 2.1, page 3], namely if a copula is max-stable, it must be an extreme value copula. Similarly, every extreme value copula is max-stable. Although extreme value copula is not our main concern here, one shall realise that the characteristic of max-stable copula (or extreme value copula) coincides with self-chaining copula, see Definition 1.7 or [16, page 18]. In addition, Genest and Rives [39] as well as McNeil and Neslehova [57] state that Gumbel-Hougaard (or Gumbel) copula is the only copula which can be classified as both archimedean and extreme value copula.

3. Self-chaining copula

   The term self-chaining copula was first introduced by Brigo and Chourdakis in [16]. Recall our purpose of this study, we intend to analyse errors between the single and iterated default probability simulations. This problem is related to MHLOM concept described earlier in this chapter. To apply MHLOM theory in default simulation procedure, one shall have MHLOM representation for copula. We outline Brigo and Chourdakis' [16, page 9] brilliant concept which is addressed to link MHLOM and copula theory. Consider a single default simulation of d different companies whose dependency structures are described by copula C over the period of T years, write

$$C(u_1..., u_d), \quad u_1 = S_1(T), \ldots, u_d = S_d(T). \tag{1.9}$$

   The time interval is then split into N equally sized sub-intervals so that we have

$$C(u_1^{1/N}, ..., u_d^{1/N})^N, \quad u_1 = S_1(T), \ldots, u_d = S_d(T). \tag{1.10}$$

   In order to obtain consistency in both methods, Equation (1.9) and (1.10) should come up with similar results. This brings us to the definition of self-chaining copula which is formally defined in the following.

   Definition 1.7. A copula is said to be self-chaining if it satisfies

$$C(u_1..., u_d) = C(u_1^{1/N}, ..., u_d^{1/N})^N. \tag{1.11}$$

   Not only defining a self-chaining term, Brigo and Chourdakis in [16, page 11] were also presenting their main invention in the form of proposition below.

   Proposition 1.8. *A survival copula satisfies MHLOM condition if and only if it is self-chaining.*

   Furthermore, one shall recognise that extreme-value copula indeed satisfies MHLOM criteria. Hence, Gumbel-Hougaard copula belongs to self-chaining copula class. The other example of self-chaining copula is Marshall-Olkin copula. Figure 3 illustrates Marshall Olkin copula surface and contour plot.

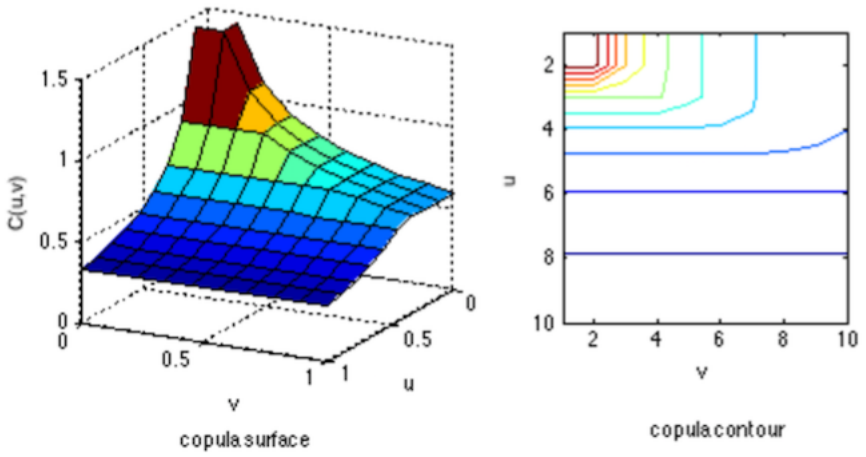

Figure 3: Surface plot and contour plot of Marshall Olkin copula with $\lambda_1 = 0.01$, $\lambda_2 = 0.03$, and $\lambda_3 = 0.02$.

Copula Simulation

People who work in risk and financial areas should be familiar with copula simulation. Here, we properly explain what exactly copula simulation is. Then, we also demonstrate how to do such kind of simulation as we need this technique for computing default probabilities, especially for implicit copulas. The most basic yet important concept for copula simulation is probability and quantile transformation, see [27, page 203-212], [58, page 186], [62], and [63, page 6]. We illustrate a case where this concept is required. Suppose we would like to simulate iid observations of Exponential distribution. However, there only exists Standard Normal iid random variables generator. By applying probability transformation followed by quantile transformation, we finally have solved the problem.

Next, assume we want to simulate two random variables X and Y whose marginal distributions are given by F_X and F_Y respectively, with copula C. Broadly speaking, there are 2 main steps involved in copula simulation process.

1. Generate a pair of Uniform random variables U and V from copula C.

2. Do the quantile transformations

$$X = F_X^{\leftarrow}(U) \quad \text{and} \quad Y = F_Y^{\leftarrow}(V).$$

At first glance, this procedure looks really straightforward. However, generating Uniform random variables from specified copula is not as easy as it seems. For instance, we need to put more effort when working with archimedean copula and Marshall-Olkin copula. Let we demonstrate the first simulation step (i.e. generate a pair of random variables U and V) for Gaussian, Clayton, and Marshall-Olkin copula.

1. Gaussian copula

   Gaussian copula is the most known copula due to its simplicity. The procedure below is taken from [6, page 1].

   - Firstly, perform Choleski's decomposition to get a pair of correlated random variables X and Y, which are normally distributed.
   - Secondly, by applying probability transformation, we have had a couple of uniform random variables $U = \Phi(X)$ and $V = \Phi(Y)$ drawn from Gaussian copula. The symbol Φ represents cumulative distribution function of Normal distribution.

2. Clayton copula

   As for this copula, the procedure is also borrowed from [6, page 2] using conditional distribution approach.

   - Generate two iid samples U and W from Uniform distribution.
   - With α being the archimedean's parameter, let

   $$V = \left[U^{-\alpha}\left(W^{-\frac{\alpha}{1+\alpha}} - 1\right) + 1\right]^{-\frac{1}{\alpha}}.$$

   For another archimedean copula, Gumbel copula say, the procedure is more complicated. See [62] and [39] for clear explanation.

3. Marshall-Olkin copula

   Devroye's [29] algorithm is applied to generate a pair of Uniform random variables of Marshall-Olkin copula. Let we first denote λ_1, λ_2, and λ_{12} as Marshall-Olkin's parameters. After that, draw three iid random variables a, b, c from Uniform$(0, 1)$ distribution. Lastly, set

   $$U = \min\left(-\ln(a)/\lambda_1, -\ln(c)/\lambda_{12}\right) \text{ and } V = \min\left(-\ln(b)/\lambda_1, -\ln(c)/\lambda_{12}\right).$$

As we can see from Figure 4, all copulas depict different dependency patterns. Gumbel copula has upper tail dependence while Clayton copula has lower tail dependence. This means, for Gumbel copula, U and V are more closely related to each other when their values grow larger. Moreover, one may identify that t copula has both upper and lower tail dependence; on the contrary, Gaussian and Frank copula do not show any sign of tail dependency. Interestingly, Marshall-Olkin copula shows unique dependency pattern which resembles a smooth curve, see [62, page 54]. Eventually, we transformed those points in Figure 4 using quantile transformations of Exponential distribution function to get Figure 5.

Remark 1.9. In order to produce Figure 4 and 5, we initially fixed Kendall's tau to be 0.5. Copulas' parameters can then be obtained according to the predetermined Kendall's coefficient. As a consequence, various patterns in both Figure 4 and Figure 5 are purely triggered by dependency structure of each copula.

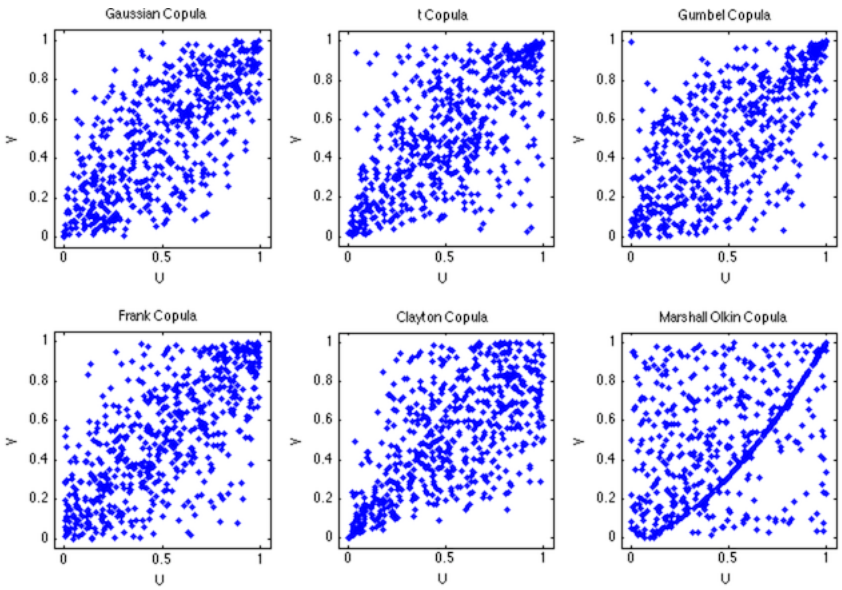

Figure 4: Six hundred simulated points from six bivariate copulas with $\lambda_1 = 0.01$, $\lambda_2 = 0.03$, and $\lambda_{12} = 0.02$ in 10 years time frame. (a) Gaussian copula with $\rho = 0.7071$. (b) t copula with $\rho = 0.7071$ and $\nu = 4$. (c) Gumbel copula with $\alpha = 2$. (d) Frank copula with $\alpha = 5.7363$. (e) Clayton copula with $\alpha = 2$. (f) Marshall-Olkin copula with $\alpha_1 = 0.8$, $\alpha_2 = 0.5714$.

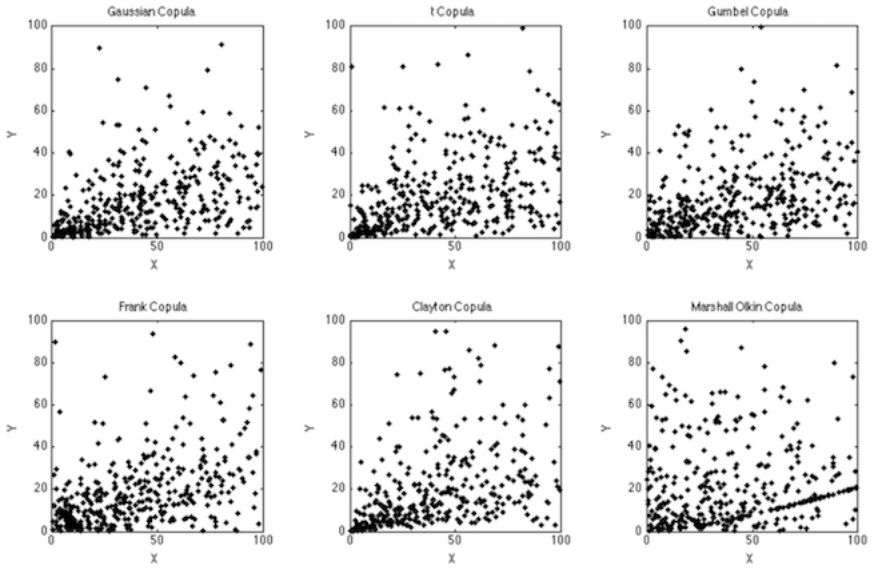

Figure 5: Six hundred simulated points from six bivariate copulas with Exponential marginals for $\lambda_1 = 0.01$, $\lambda_2 = 0.03$, and $\lambda_{12} = 0.02$ in 10 years time frame, transformed from points in Figure 4. (a) Gaussian copula with $\rho = 0.7071$. (b) t copula with $\rho = 0.7071$ and $\nu = 4$. (c) Gumbel copula with $\alpha = 2$. (d) Frank copula with $\alpha = 5.7363$. (e) Clayton copula with $\alpha = 2$. (f) Marshall-Olkin copula with $\alpha_1 = 0.8$, $\alpha_2 = 0.5714$.

1.2.4 Kendall's Tau

When it comes to financial modelling, we always deal with dependency problems. In the previous section, we show that copula is the ultimate solution for such problems as long as people use it correctly. Besides linear correlation (i.e. Pearson coefficient) and copula function, there also exists a simple dependence measure called rank correlation. It allows us to measure dependency level according to the order of the data. Similar with linear correlation, the value of rank correlation can vary between -1 and 1. For particular copulas, Gumbel copula for example, Kendall's tau values might only lie within $[0, 1]$ range.

It is well known that Kendall's tau and Spearman's rho are two of the most famous examples of rank correlation. Before going too far, it is better to know the importance of this concept to our study. Having mentioned the purpose of this paper earlier, the reader might notice that we would compare some copula functions later. By picking particular Kendall's tau quantities for the copulas, their dependence parameters can be determined directly. Off course there exist a lot more advanced methods in parameter selection process rather than the one we used here (i.e. methods-of-moment), see Genest and Rives paper in [39]. Despite copula parameterisation is not our primary concern here, it is really an important area, especially in Financial industry. See [1], [50], and [54] for instance.

We would briefly discuss Kendall's tau quantity since we utilise it for 2-dimensional copula case later in this study. The measure which was introduced by Sir Maurice Kendall in 1938 evaluates concordance level for random vectors. We shall then see the formal definition of Kendall's tau of a pair of random variables which is borrowed from [57, page 206].

Definition 1.10 (Kendall's Tau)**.** Let (X_1, X_2) be a random vector and (Y_1, Y_2) be an independent copy of the former pair. Kendall's rank correlation is defined as

$$\mathcal{K}_\tau(X_1, X_2) = \mathbb{P}\{(X_1 - Y_1)(X_2 - Y_2) > 0\} - \mathbb{P}\{(X_1 - Y_1)(X_2 - Y_2) < 0\}.$$

Moreover, if X_1, X_2 are continuous random variables and has a unique copula C, then

$$\mathcal{K}_\tau(X_1, X_2) = 4 \int \int_{[0,1]^2} C(u, v)\ dC(u, v) - 1. \tag{1.12}$$

From the definition above, a calculation of Kendall's tau value for each copula seems too abstract to be applied. Fortunately, literature and MATLAB help us solving this problem. Kendall's tau explicit formulas of the 6 copulas involved in this study are shown later in chapter 3.

1.3 Layout of Thesis

The rest of this study is organised as follows: section 2 introduces the problems in a more technical way (i.e. mathematical expressions), the cases are classified into two main groups: all survival and mixed survival. Within each group, we try to solve the 2-dimensional (bivariate) case and 3-dimensional (trivariate) case. Moreover, there are six copulas examined in this analysis: Gaussian, t, Clayton, Frank, Gumbel, and Marshall-Olkin. Problem-solving procedures are also explained carefully.

The results of our simulations are shown and discussed intensively in the following section 3. As expected, there are numerous graphs, tables, and figures to visualise simulation errors. Additionally, this section presents some formulas used to obtain the results. Compared to the previous section, this part contains more complex materials including computation of copula parameter(s), copula functions, and so on. Lastly, section 4 concludes the paper and gives some recommendations to the further researches.

2 Problem Description

In this section, all problems are explained and illustrated step by step. In addition, technical procedures and some mathematical tools which are applied to analyse the errors can also be found. To start with, we describe two simulation procedures used in this thesis as follows: let C be a copula of m-dimensional random default time vector, $\tau = (\tau_1, \tau_2, ..., \tau_m)$, with exponential margins and default parameters $\lambda_1, ..., \lambda_m$. Also, let T as terminal simulation time and define $\Delta T = T/N$.

1. In the first procedure (or single simulation), we compute the probability $\mathbb{P}(\tau_1 \geq T, \ldots, \tau_m \geq T)$ via

$$C(S_1(T), ..., S_m(T)) = C(e^{-\lambda_1 T}, ..., e^{-\lambda_m T}).$$

2. While for the second procedure (or iterated simulation), the probability is approximated using $\mathbb{P}\left(\tau_1 \geq \Delta T, \ldots, \tau_m \geq \Delta T\right)^N$, in terms of copula function we have

$$C(S_1(\Delta T), \ldots, S_m(\Delta T))^N = C(e^{-\lambda_1 \Delta T}, ..., e^{-\lambda_m \Delta T})^N.$$

Figure 6 illustrates how the first and second simulation are held. As described above, Figure 6(a) depicts a direct simulation over the period of T years while Figure 6(b) illustrates N times sequential local simulations. Next, our main purposes are simplified into 2 points:

1. Studying errors between the single and iterated simulation for all survival cases. As for this objective, we only calculate the probability that all entities survive until final time T. For example, in bivariate case, we calculate the probability $\mathbb{P}(\tau_1 \geq T, \tau_2 \geq T)$.

2. Evaluating differences between single and iterated simulation for mixed cases (some entities default and others survive at terminal time T). For instance, in trivariate case, we compute the probability that company 1, 2 survive but company 3 default, namely

$$\mathbb{P}(\tau_1 \geq T, \tau_2 \geq T, \tau_3 < T)$$

   or company 1 survives and company 2, 3 default, reads

$$\mathbb{P}(\tau_1 \geq T, \tau_2 < T, \tau_3 < T).$$

   Brigo, Mai, and Scherer [18] show that Markovian survival indicator process is a core concept for evaluating survival probability in mixed default problems. On the other hand, we apply a basic probability rule to transform mixed default case into all survival case. Hence, we can implement the similar method to solve both the first and second cases.

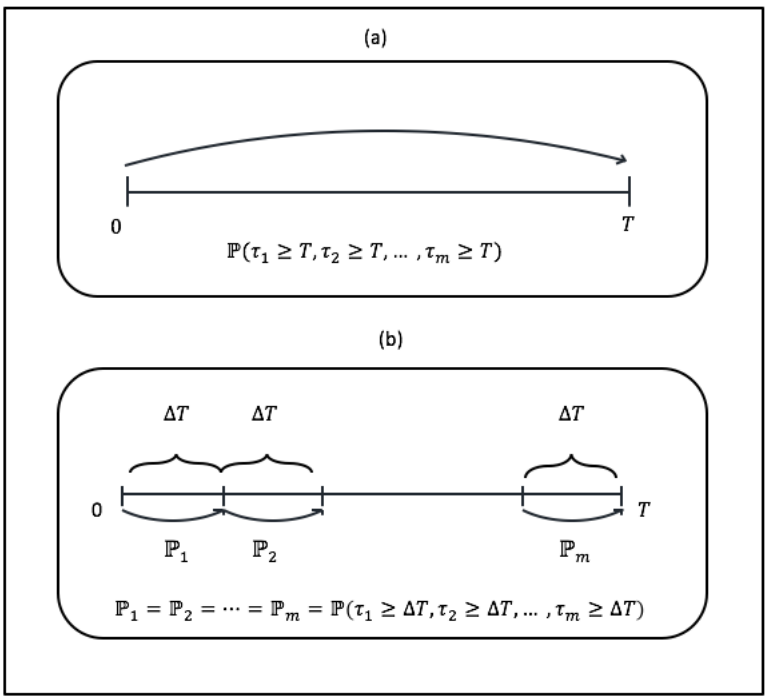

Figure 6: Illustration of: (a) simulation procedure 1 and (b) simulation procedure 2 for T years period and m entities involved.

According to Brigo and Chourdakis [16], those two simulation methods described above coincide if the multivariate distribution function of default times has MHLOM property. When the presence of MHLOM characteristic is neglected, the following inequality holds.

$$\mathbb{P}(\tau_1 \geq T, ..., \tau_m \geq T) \geq \mathbb{P}(\tau_1 \geq \Delta T, ..., \tau_m \geq \Delta T)^N.$$

Our next task is to transform MHLOM property into copula (or survival function) representation. For simplicity reason, we illustrate 1-dimensional case here. From basic probability theory, it is clear that

$$\mathbb{P}(\tau \geq A + B | \tau > A) = \frac{\mathbb{P}(\tau \geq A + B, \tau \geq A)}{\mathbb{P}(\tau \geq A)} = \frac{\mathbb{P}(\tau \geq A + B)}{\mathbb{P}(\tau \geq A)} \tag{2.1}$$

where $A, B \in \mathcal{T}$. By applying MHLOM property described in Equation (1.2) to Equation (2.1), we could then write

$$\mathbb{P}(\tau \geq A + B) = \mathbb{P}(\tau \geq B) \cdot \mathbb{P}(\tau \geq A)$$

which means, in terms of survival function,

$$S(A + B) = S(A) \cdot S(B).$$

Another way to link MHLOM concept and copula is by applying Proposition (1.8). Consequently, we know that self-chaining (or extreme-value) copula possesses MHLOM property. From previous studies ([16] and [18]), it has been known that Marshall-Olkin copula and Gumbel copula are the examples of self-chaining copulas. We then want to justify this statement by checking the consistency levels of procedure 1 and 2 with various T, N, and parameter values. The same analyses are conducted for non-self-chaining copulas such as Gaussian, t, Clayton, and Frank.

The following Table 1, which is summarised from [19, page 716], describes six copulas parameters range and their value regarding dependence structures. It is evident that parameter values vary significantly across the copulas. According to the table, Frank copula has the widest range of parameter while Gaussian and t copula have the narrowest ones. Fundamental copulas such as comonotonicity, independence, and counter-monotonicity copula could be constructed by varying other copulas' parameters. This is why copula parameter is considered as a degree of dependence. But sometimes, there exist some copulas which could not be transformed into one of the fundamental copulas, for example, Gumbel copula.

Table 1: Parameter values of copulas

Copula	**Parameter Range**	**Parameter Value**		
		comonotonicity	**counter-monotonicity**	**independence**
Gaussian	$\rho \in (-1, 1)$	1	-1	0
t	$\rho \in (-1, 1)$	1	-1	0
Gumbel	$\alpha \in [1, \infty)$	∞	-	1
Clayton	$\alpha \in [-1, \infty), \alpha \neq 0$	∞	-1	0
Frank	$\alpha \in (-\infty, \infty), \alpha \neq 0$	∞	$-\infty$	0

It has been mentioned that almost all of the problems in this study are solved quantitatively using MATLAB software. We refer the keen reader to have a look at [16] as well as [18] for analytical explanations.

Remark 2.1. In the one-dimensional case, it is obvious that procedure 1 and procedure 2 would match each other under exponential distribution. It comes as no surprise since exponential distribution has lack of memory property, in that

$$S(T) = e^{-\lambda T} = (e^{-\lambda \Delta T})^N = S(\Delta T)^N.$$

2.1 All Survival Case

As mentioned before, the first task in this thesis is studying errors of survival probability of some names at terminal time T. We first examine the bivariate case before moving to the more challenging task, trivariate case. MHLOM concept and self-chaining copula are indeed our basic foundation in interpreting simulation results. For both bivariate and trivariate case, the effects of modifying copula parameters on iterated errors are analysed. Next, we try to find a connection between self-chaining copulas and fundamental copulas. Regarding the bivariate case, particularly, the changes in dependency structure of iterated copulas for large N are also our main concern.

2.1.1 Bivariate Problem

Consider bivariate copula function of the first simulation procedure with T as maturity time and λ_1, λ_2 as non-negative parameters, namely

$$C(S_1(T), S_2(T)) = C(e^{-\lambda_1 T}, e^{-\lambda_2 T}).$$

Before going too far, it is worth mentioning that survival function has this particular form since default times are exponentially distributed. As for the second procedure, the time path is divided into N identical sub-intervals, in that

$$[0, \Delta T], [\Delta T, 2\Delta T], \ldots, [(N-1)\Delta T, N\Delta T].$$

Thus, copula function for the latter simulation procedure can be written as

$$C(S_1(\Delta T), S_2(\Delta T))^N = C(e^{-\lambda_1 \Delta T}, e^{-\lambda_2 \Delta T})^N.$$

Another essential term is relative error (ε_r) between the two simulation procedures which is described in the following:

$$\varepsilon_r := \frac{\mathbb{P}(\tau_1 \geq T, \tau_2 \geq T) - \mathbb{P}(\tau_1 \geq \Delta T, \tau_2 \geq \Delta T)^N}{\mathbb{P}(\tau_1 \geq T, \tau_2 \geq T)} = \frac{C(e^{-\lambda_1 T}, e^{-\lambda_2 T}) - C(e^{-\lambda_1 \Delta T}, e^{-\lambda_2 \Delta T})^N}{C(e^{-\lambda_1 T}, e^{-\lambda_2 T})}.$$

It has been proved by Brigo and Chourdakis [16] that using self-chaining copulas in both simulation procedures would generate the same results. However, self-chaining copulas are not as popular as Gaussian or t copula. Hence, in practice, people often ignore self-chaining copula and use Gaussian or t copula instead. We present a simple example of how large the iteration errors can be for Gaussian copula and t copula with 4 degrees of freedom in Table 2. As we can see from the table, relative errors between the first and second method for those 2 copulas are relatively significant. It also can be observed that for all three parameter values, Gaussian copula always has larger errors than t copula, which indicates that t copula is more likely to resemble self-chaining copula than

Gaussian copula does. Lastly, iteration errors tend to be smaller as parameter values go to either 0 or 1. Indeed, selecting copula for default modelling is a critical issue even though there still a lot of workers who underestimate this problem.

Table 2: Bivariate error tabel for Gaussian and t ($\nu = 4$) copula with $T = 10$y and $N = 500$.

Copula	λ_1	λ_2	**Parameter**	(%) **Relative Error**
Gaussian	0.05	0.02	$\rho = 0.1$	1.99
t				1.55
Gaussian			$\rho = 0.5$	8.61
t				6.56
Gaussian			$\rho = 0.9$	6.472
t				4.45

The next principal concept in the bivariate case is Kendall's tau rank correlation. One standard measure needs to be specified since we intend to compare iteration errors of six different copulas. To this end, one of the most well-known rank correlation measures, Kendall's tau, is chosen. By fixing a particular value of Kendall's tau, one can calculate the copula parameter(s) using a specific formula. For instance, the following is Kendall's formula for Clayton copula:

$$\mathcal{K}_\tau = \frac{\alpha}{\alpha + 2}$$

where $\mathcal{K}_\tau$ denotes Kendall's value and $\alpha \in [-1, \infty)$ is Clayton's parameter. Undoubtedly, this formula is obtained by deriving Equation (1.12). Kendall's tau formulas of the six copulas used in this study are summed up in Table 4. For further illustration, Figure 7 represents the relations between several copula parameters and concordance measure, Kendall's tau. It is evident that rho (ρ) and alpha (α) have different relations with Kendall's tau. On the left-hand side, we can see a convex curve of t copula's parameter while on the right-hand side there are two concave curves of Clayton and Gumbel copula's parameters. Concerning the latter, it can be seen that Clayton and Gumbel's alphas almost resemble each other for any Kendall's tau quantities. This is in line with Table 1 which shows the range of Gumbel and Clayton's alphas are relatively close.

Eventually, we move to dependency problems. In real world matters, we could not foresee any unpredictable events that confront us. Hence, academicians always anticipate such things by finding more and more problems to be explored including this part of study. Limiting behaviour copula studies are received an enormous amount of consideration presently. For example, see [24], [41], [66]. Particularly in this section, we eager to know the impact of iterated simulation on dependency issue. If the number of sub-intervals (N) increase to infinity or ΔT becomes smaller, naturally the difference between iterated copula and the independence copula would decrease. This

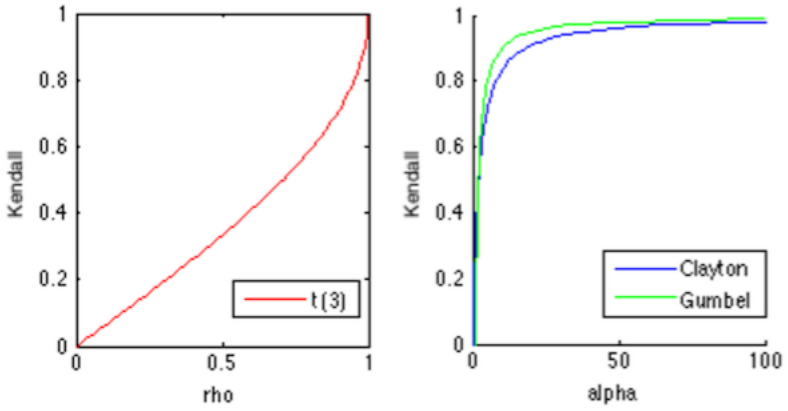

Figure 7: t, Clayton, and Gumbel copula parameters with respect to Kendall's tau.

is true for several copula classes but self-chaining copulas. Mathematically, we write this problem as

$$\varepsilon_l := |\lim_{N\to\infty} C(S_1(\Delta T), S_2(\Delta T))^N - \hat{C}(S_1(T), S_2(T))| \tag{2.2}$$

where $\hat{C}$ is the independence copula and $\varepsilon_l \in \mathbb{R}_+ \cup \{0\}$. Knowing the value of ε_l would help us to justify whether dependence structure is destroyed by iteration or not. Indeed, ε_l must be large enough to assure that iterated simulation does not eliminate dependence structure. For convenient, we would then name ε_l as *iterated limit error* for the rest of this study. Also for this part, we present both analytical and numerical approach to conclude the analyses.

2.1.2 Trivariate Problem

Generally speaking, in this part, we would examine simulation error which is defined similarly as above, reads

$$\varepsilon_r := \frac{\mathbb{P}(\tau_1 \geq T, \tau_2 \geq T, \tau_3 \geq T) - \mathbb{P}(\tau_1 \geq \Delta T, \tau_2 \geq \Delta T, \tau_3 \geq \Delta T)^N}{\mathbb{P}(\tau_1 \geq T, \tau_2 \geq T, \tau_3 \geq T)}.$$

In contrast to the previous case, Kendall's tau measure is not utilised while evaluating trivariate errors since multivariate Kendall's tau (for $n \geq 3$) calculation is not as simple as in the bivariate cases, see [33]. Hence, comparison among copula types are not conducted here and we focus on comparing errors between the bivariate and trivariate case instead. For instance, relative errors of trivariate Gaussian and t copula can be observed in Table 3. As is expected, the errors between those two simulation approaches with 3 entities included are higher than the former case. This indicates that greater number of entities involved in simulation leads us to the more significant

errors. In chapter 3, we would illustrate more examples regarding such problems, including the errors with various dependency levels.

Table 3: Trivariate error table for Gaussian and t ($\nu = 4$) copula with $T = 10$y and $N = 500$.

Copula	λ_1	λ_2	λ_3	**Parameter**	**(%) Relative Error**
Gaussian	0.05	0.02	0.03	$\rho = 0.1$	5.68
t					1.19
Gaussian				$\rho = 0.5$	20.31
t					3.51
Gaussian				$\rho = 0.9$	14.52
t					2.77

2.2 Mixed Survival Case

Mixed case is indeed a more challenging task compared to the previous one. This topic was first raised up by Brigo, Mai, and Scherer [18]. In their paper, Markovian survival indicator process is being suggested to be used as a tool to resolve mixed simulation problems. One thing needs to be mentioned, the term "mixed" in [18] is defined slightly differently compared to ours as the elimination of defaulted or liquidated entities cases is included in their models. The use of markovian survival indicator process is incorporated with looping default model which is first introduced by Jarrow and Yu [45]. Surprisingly, looping default model for bivariate case coincides with Freund distribution, see [38]. As for the general n-dimensional case with $n \geq 3$, it might be more complicated since only a few multivariate distribution functions that are stable under marginalisation (i.e. nested margining). This is an important issue considering stability is required to conserve MHLOM characteristic. To this end, Brigo, Mai, and Scherer proved that Marshall-Olkin distribution characteristics are preserved under marginalisation [18, page 11-13].

On the other hand, we try to alter mixed cases into all survival cases by utilising Venn diagram (for low-dimensional cases) and the rule of inclusion for probability, see [61, page 53]. Similar to all survival cases, mixed cases would be split into bivariate and trivariate cases. As for the trivariate case, we would emphasise the effect of adding up numbers of entities involved on errors' magnitude. Intensities and terminal time, however, also accounts for the rise of iteration errors.

2.2.1 Bivariate Problem

In this last part of section 2, we directly focus on error's definition instead of discussing Kendall's tau measure as in the previous section. Firstly, by looking at Figure 8 we could write mixed

survival function in terms of all survival function, in that

$$\begin{aligned}\mathbb{P}(\tau_1 \geq T, \tau_2 < T) &= \mathbb{P}\left\{(\tau_1 \geq T) \cap (\Omega \backslash (\tau_2 \geq T))\right\} \\ &= \mathbb{P}(\tau_1 \geq T) - \mathbb{P}(\tau_1 \geq T, \tau_2 \geq T).\end{aligned}$$

Next, we describe the error as

$$\varepsilon_r := \frac{[\mathbb{P}(\tau_1 \geq T) - \mathbb{P}(\tau_1 \geq T, \tau_2 \geq T)] - [\mathbb{P}(\tau_1 \geq \Delta T)^N - \mathbb{P}(\tau_1 \geq \Delta T, \tau_2 \geq \Delta T)^N]}{\mathbb{P}(\tau_1 \geq T) - \mathbb{P}(\tau_1 \geq T, \tau_2 \geq T)}.$$

One shall notice that if the copula is self-chaining, the error becomes extremely small (or zero). With regard to non-self-chaining copulas, iteration errors vary depending on several factors such as the final time T, Kendall's tau values, and so on.

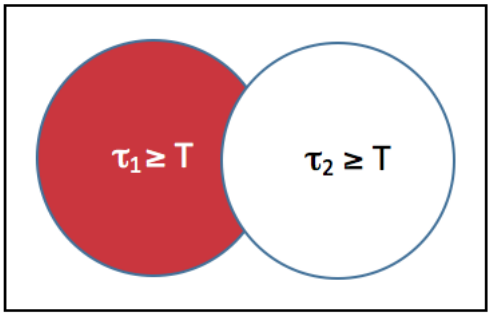

Figure 8: Venn diagram of the bivariate mixed case.

2.2.2 Trivariate Problem

In trivariate case, there are 2 problems that would be looked into. Not only the descriptions but also the procedures taken to solve these two problems are almost similar to the bivariate case. To start with, consider the case where one company survives and 2 others default. With the help of Figure 9(a) and 3-dimensional rule of inclusion, we can write

$$\begin{aligned}\mathbb{P}(\tau_1 \geq T, \tau_2 < T, \tau_3 < T) =& \mathbb{P}\left\{(\tau_1 \geq T) \cap (\Omega \backslash (\tau_2 \geq T)) \cap (\Omega \backslash (\tau_3 \geq T)\right\} \\ =& \mathbb{P}(\tau_1 \geq T) - \mathbb{P}(\tau_1 \geq T, \tau_2 \geq T) - \mathbb{P}(\tau_1 \geq T, \tau_3 \geq T) \\ &+ \mathbb{P}(\tau_1 \geq T, \tau_2 \geq T, \tau_3 \geq T).\end{aligned}$$

Hence, for error term we have

$$\varepsilon_r := \frac{\mathbb{P}_{single} - \mathbb{P}_{multi}}{\mathbb{P}_{single}}, \tag{2.3}$$

where

$$\mathbb{P}_{single} = \mathbb{P}(\tau_1 \geq T) - \mathbb{P}(\tau_1 \geq T, \tau_2 \geq T) - \mathbb{P}(\tau_1 \geq T, \tau_3 \geq T) + \mathbb{P}(\tau_1 \geq T, \tau_2 \geq T, \tau_3 \geq T) \text{ and}$$
$$\mathbb{P}_{multi} = \mathbb{P}(\tau_1 \geq \Delta T)^N - \mathbb{P}(\tau_1 \geq \Delta T, \tau_2 \geq \Delta T)^N - \mathbb{P}(\tau_1 \geq \Delta T, \tau_3 \geq \Delta T)^N$$
$$+\mathbb{P}(\tau_1 \geq \Delta T, \tau_2 \geq \Delta T, \tau_3 \geq \Delta T)^N.$$

Regarding the second problem (i.e. 2 companies survive and 1 company default), again, by observing Figure 9(b) and applying the rule of inclusion one could describe the following:

$$\begin{aligned}\mathbb{P}(\tau_1 \geq T, \tau_2 \geq T, \tau_3 < T) =& \mathbb{P}\{(\tau_1 \geq T) \cap (\tau_2 \geq T) \cap (\Omega \backslash (\tau_3 \geq T)\} \\ =& \mathbb{P}(\tau_1 \geq T, \tau_2 \geq T) - \mathbb{P}(\tau_1 \geq T, \tau_2 \geq T, \tau_3 \geq T).\end{aligned}$$

Error term is defined similarly as in Equation 2.3 but with

$$\mathbb{P}_{single} = \mathbb{P}(\tau_1 \geq T, \tau_2 \geq T) - \mathbb{P}(\tau_1 \geq T, \tau_2 \geq T, \tau_3 \geq T) \text{ and}$$
$$\mathbb{P}_{multi} = \mathbb{P}(\tau_1 \geq \Delta T, \tau_2 \geq \Delta T)^N - \mathbb{P}(\tau_1 \geq \Delta T, \tau_2 \geq \Delta T, \tau_3 \geq \Delta T)^N.$$

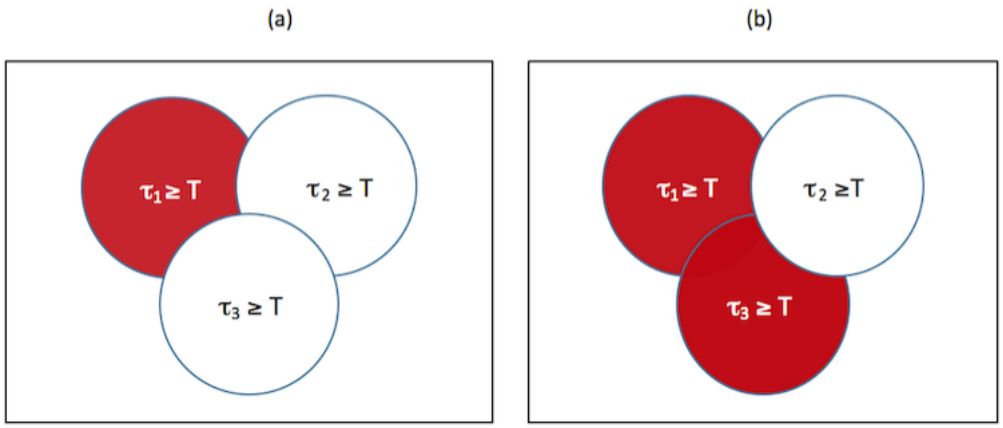

Figure 9: Venn diagram of the trivariate mixed case. (a) 1 survival and 2 defaults. (b) 2 survivals and 1 default.

3 Results and Discussion

In this section, we show default probability calculation results of both cases including all survival and mixed survival in the form of tables and graphs. Some basic technical stuff such as Kendall's tau coefficient calculation and explicit copula distribution function are also clearly presented. Additionally, comments and discussions are provided for each sub-case.

3.1 All Survival Case

To simulate survival copulas, we first need to determine their parameters. There exist some copula parameter estimation methods and particularly, in this study, Kendall's tau measure is utilised to specify dependence parameter values. Table 4, which is summarised from [50, page 20], shows the formulas of Kendall's tau rank correlation of some bivariate copulas.

Table 4: Kendall's tau of Gaussian, t, Gumbel, Frank, Clayton, and Marshall-Olkin copula.

Copula	**Parameter(s)**	$\mathcal{K}_\tau$
Gaussian / t	$\rho \in (-1, 1)$	$\frac{2}{\pi}\arcsin(\rho)$
Gumbel	$\alpha \in [1, \infty)$	$1 - \frac{1}{\alpha}$
Frank	$\alpha \in (-\infty, \infty), \alpha \neq 0$	$1 + \frac{4(D(\alpha)-1)}{\alpha}$, with $D(\alpha) = \frac{1}{\alpha}\int_0^\alpha \frac{u}{e^u-1}$
Clayton	$\alpha \in [-1, \infty), \alpha \neq 0$	$\frac{\alpha}{\alpha+2}$
Marshall-Olkin	$\alpha_1, \alpha_2 \in [0, 1]$	$\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2-\alpha_1\alpha_2}$

After fixing Kendall's Tau and obtaining dependency parameter, survival probability can be calculated easily. As for explicit copulas such as Gumbel, Clayton, Frank, and Marshall-Olkin, we could calculate the numbers by directly substitute the parameter (α) into the following formulas:

1. Gumbel copula
$$C_\alpha^{Gu}(u_1, u_2) = e^{-[\{-\ln(u_1)\}^\alpha + \{-\ln(u_2)\}^\alpha]^{1/\alpha}}.$$

2. Clayton copula
$$C_\alpha^{Cl}(u_1, u_2) = \max(\, u_1^{-\alpha} + u_2^{-\alpha} - 1, 0)^{-1/\alpha}.$$

3. Frank copula
$$C_\alpha^{Fr}(u_1, u_2) = -\frac{1}{\alpha}\ln\left\{1 + \frac{(e^{-\alpha u_1} - 1)(e^{-\alpha u_2} - 1)}{e^{-\alpha} - 1}\right\}.$$

4. Marshall-Olkin copula
$$C(u_1, u_2) = \min\,(u_1^{1-\alpha_1}u_2, u_1u_2^{1-\alpha_2})$$

However, implicit copulas such as Gaussian and t copula require a little more effort to obtain the results, see copula simulation in Chapter 1. As an illustration, Table 5 presents relative errors

of six copulas with a specific Kendall's tau value ($\mathcal{K}_\tau = 0.5$). Dependence parameter values are calculated using the formulas provided in Table 4. Gaussian copula, as expected, has the highest relative error at 2.38%. It is then followed by Frank copula at 2.20% and Clayton copula at 1.24%. Moreover, the error of t copula with 4 degrees of freedom is slightly higher than the one produced by t copula with 3 degrees of freedom. Unsurprisingly, both Gumbel and Marshall-Olkin errors are the lowest among others as they own self-chaining property. For a deeper understanding, we compare all 6 copula errors as a function of $\mathcal{K}_\tau$ in the following paragraph.

Table 5: Relative error table of 6 different copulas with $T = 5$y, $N = 1000$, $\lambda_1 = 0.01$, $\lambda_2 = 0.03$, and $\mathcal{K}_\tau = 0.5$.

Copula	**Parameter(s)**	**(%) Relative Error**
Gaussian	$\rho = 0.7071$	2.38
t	$\nu = 3$, $\rho = 0.7071$	0.23
	$\nu = 4$, $\rho = 0.7071$	0.48
Gumbel	$\alpha = 2$	2.99E-12
Frank	$\alpha = 5.7362827$	2.20
Clayton	$\alpha = 2$	1.24
MO	$\alpha_1 = 0.8$, $\alpha_2 = 0.5714$	2.59E-12

The line chart in Figure 10 shows relative errors between simulation procedure 1 and procedure 2 for Gaussian, t, Gumbel, Frank, Clayton, and Marshall-Olkin copula between $\mathcal{K}_\tau = 0$ and $\mathcal{K}_\tau = 1$. From this graph, we can observe whether a copula holds self-chaining property or not. Indeed, we expect errors of self-chaining copulas as small as possible. Overall, the copulas show relatively similar patterns for whole Kendall's tau values. They start at around 0 for $\mathcal{K}_\tau = 0$, keep increasing until particular levels, and begin decreasing before finally reaching their endpoints.

Gaussian copula has the highest errors for $\mathcal{K}_\tau$ smaller than 0.5 while for $\mathcal{K}_\tau$ exceed 0.55, both Frank and Clayton copula overtake Gaussian's errors. From its relative errors, one might indicate that Gaussian copula is not a proper model to be used for iterated simulation. Moreover, both t copula with 3 and 4 degrees of freedom exhibit better performances than the Gaussian one since their errors are smaller. Regarding t copula itself, the graph depicts that the lower degrees of freedom it has, the more self-chaining it becomes.

As for Gumbel and Marshall-Olkin copula, they remain unchanged at extremely low error levels for whole Kendall's tau ranges. These results are not surprising since both Gumbel and Marshall-Olkin copula are self-chaining. Next, we are moving to Clayton and Frank copula which show moderately different patterns than other copulas. From $\mathcal{K}_\tau = 0$ to around $\mathcal{K}_\tau = 0.55$, we can infer

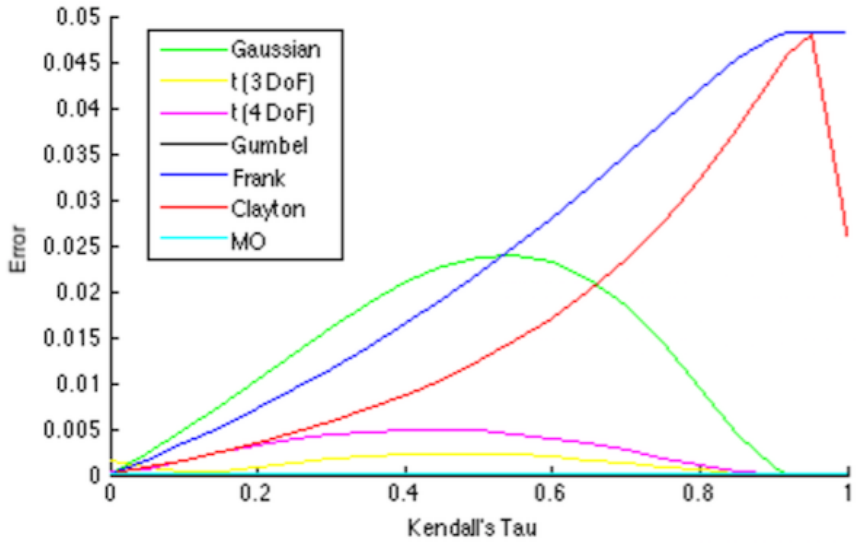

Figure 10: Relative error line charts of several different copulas: Gaussian, t($\nu = 3$), t($\nu = 4$), Gumbel, Frank, Clayton, and Marshall-Olkin with respect to Kendall's tau ($\mathcal{K}_\tau$) values for $T = 5$y, $N = 1000$, $\lambda_1 = 0.01$, and $\lambda_2 = 0.03$.

that both Clayton and Frank are more self-chaining than the Gaussian does. However, there are significant increments in Clayton and Frank's errors from $\mathcal{K}_\tau = 0.55$ until $\mathcal{K}_\tau = 0.99$ before they start falling to particular points at the end of the graph.

Broadly speaking, it is clear that all copulas resemble self-chaining copula for both $\mathcal{K}_\tau = 0$ and $\mathcal{K}_\tau = 1$. One may relate this fact to the fundamental copula terms by converting Kendall's tau values using the formulas provided in Table 4. Once parameters are obtained, we shall see Table 1 to conclude that the independence and comonotonicity copula are indeed self-chaining.

Furthermore, we would like to discuss Clayton and Frank copula cases. According to Table 1, the two copulas shall tend to be closer to self-chaining copula (i.e. errors become extremely smaller) as Kendall's tau approaches 1, although we could not see this fact clearly from Figure 10. This is due to numerical error problems for higher Kendall's tau values, especially for Clayton and Frank case. Therefore we illustrate relative error graphs for these two copulas with respect to their parameter values, instead of Kendall's tau quantities.

Figure 11 illustrates Clayton and Frank's errors as a function of α. Generally, these two plots indicate that as alpha goes larger, error terms are also lessening. This is exactly what we expect from Clayton and Frank copula according to Table 1. In Figure 11(a), there is a significant rise in the error's magnitude of Clayton copula from around 0 to just under 0.05. This increasing part

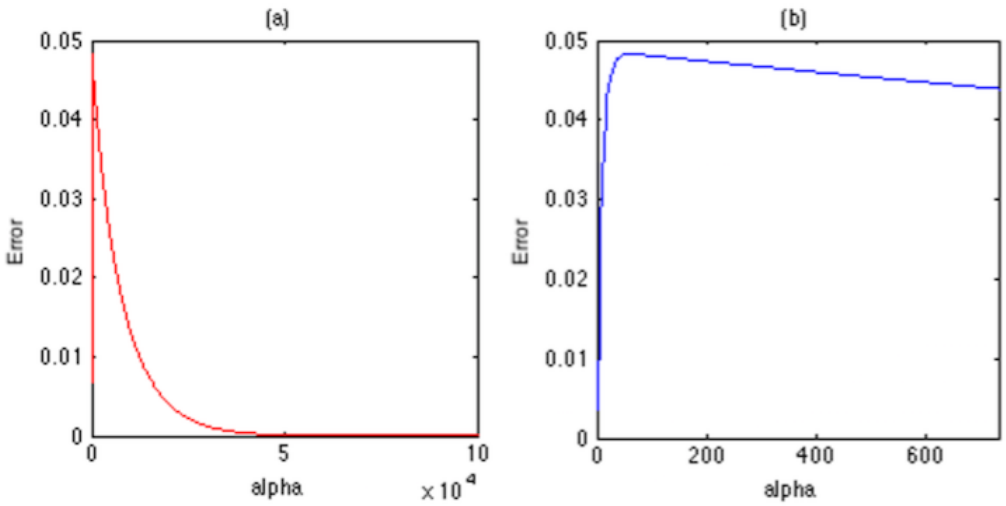

Figure 11: Relative error line charts of: (a) Clayton copula and (b) Frank copula with respect to α for $T = 5$y, $N = 1000$, $\lambda_1 = 0.01$, and $\lambda_2 = 0.03$.

corresponds to the previous line chart in Figure 10. Moreover, Clayton's deviation starts dropping to its lowest point when its parameter reaches the value of 50000 and remains stable for the rest of alpha values. From Table 1 we know that for α relatively high, Clayton copula becomes the comonotonicity copula, which is actually self-chaining. When it comes to Frank's copula, decreasing pattern could still be observed although it is not as substantial as in Clayton's graph. Needless to say, this occurs as a consequent of numerical problems since we could only observe the plot until $\alpha = 740$. Therefore, an analytical approach is more favourable to solve Frank's copula issue. At last, we shall mention that similar errors analysis using Spearman's rho instead of Kendall's Tau were conducted in [36].

We are now moving to iterated limit problems, where error term ε_l is described in Equation (2.2). Later on, it can be seen that dependence structures of some copulas are destroyed whenever N grows larger. Therefore, the choice of copulas for our default modelling could be determined by analysing iterated limit. To begin with, Gaussian and t copula's limit errors are illustrated in Figure 12 below.

Figure 12(a) illustrates the errors of t copula with 3 degrees of freedom while Figure 12(b) depicts 4 degrees of freedom t copula's errors. Both graphs show relatively identical pattern. Starting at their highest points, then plunge to particular points and stay at almost the same level until the end of their x-axis. However, if we look at these two figures cautiously, we shall notice a slight difference between these two graphs. The second graph shows a minor decreasing trend which

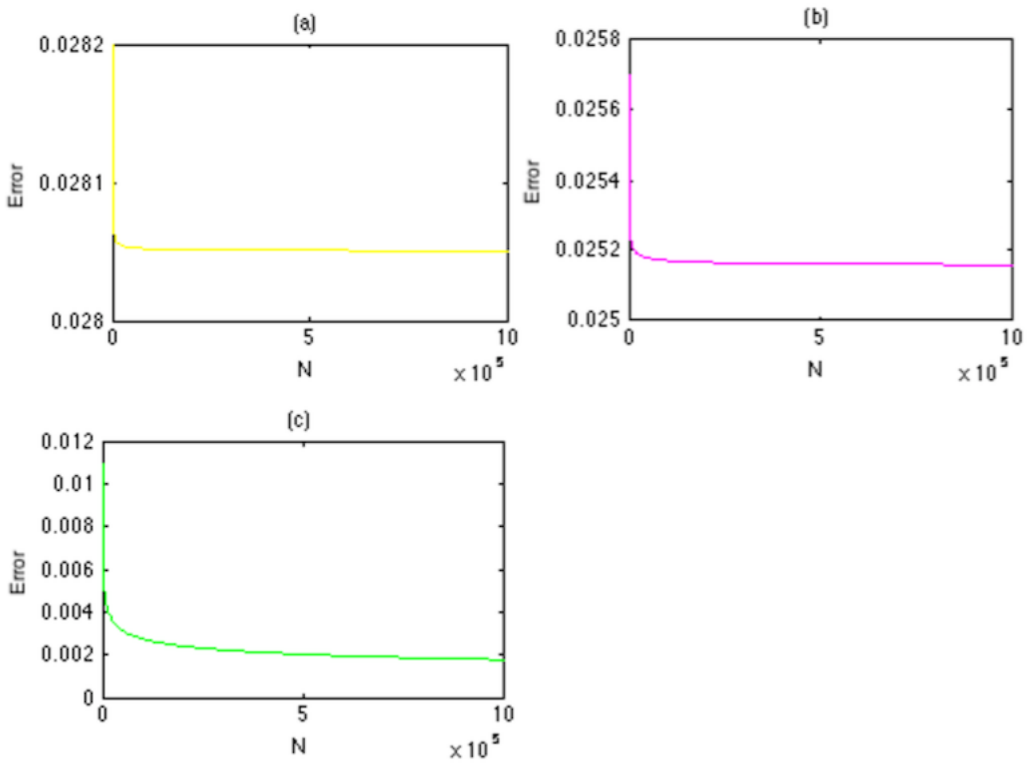

Figure 12: Iterated limit error line charts of: (a) t copula ($\nu = 3$), (b) t copula ($\nu = 4$), and (c) Gaussian copula with $\mathcal{K}_\tau = 0.5$, $\lambda_1 = 0.01$, and $\lambda_2 = 0.03$.

indicates its limit error goes smaller as N increases. Moreover, it also can be seen that t copula with 4 degrees of freedom always has fewer errors than the ones with 3 degrees of freedom. Thus, iterating t copula with higher degrees of freedom terminates dependence structure.

Regarding Gaussian copula in Figure 12(c), one might notice that its errors are evidently lower than both t copula with 3 and 4 degrees of freedom. Furthermore, Gaussian graph experiences downward trend for all N values which, again, indicates its dependence structure is destroyed by iteration. This is in line with Brigo and Chourdakis' [16, page 9-10] simulation results on Gaussian copula. Hence, we might urge that this undeniable fact regarding Gaussian copula is really important and shall be recognised by the industry in order to prevent unwanted mistakes as in the recent financial crisis.

Secondly, Clayton and Frank's iterated limit errors can be observed in Figure 13. From the very first glance, it is evident that both Clayton and Frank copula limit seem rather close to the independence limit. At the beginning, Clayton and Frank's errors start at relatively low points, $1.4 \cdot 10^{-4}$ and $3 \cdot 10^{-4}$ respectively. Eventually, there is a sudden drop in Clayton's graph to near

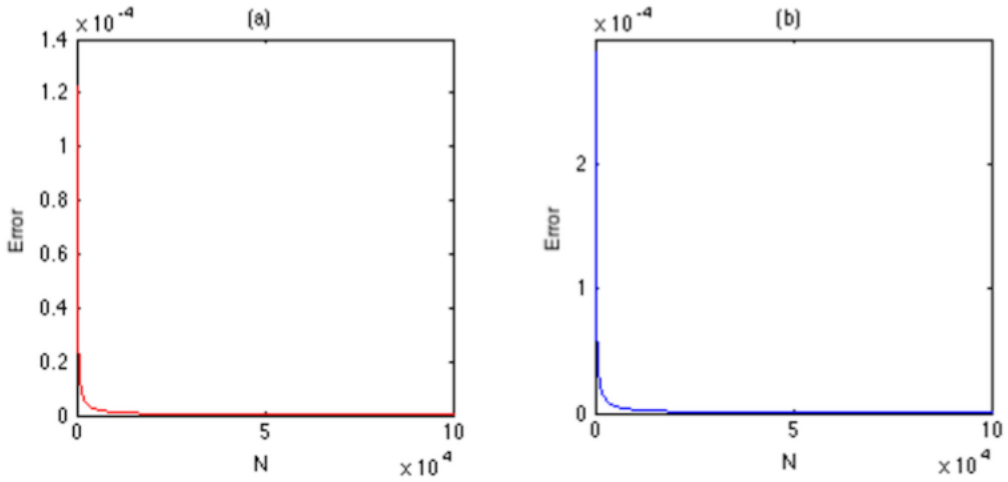

Figure 13: Iterated limit error line charts of: (a) Clayton copula and (b) Frank copula with $\mathcal{K}_\tau = 0.5$, $\lambda_1 = 0.01$, and $\lambda_2 = 0.03$.

0 and remain stable for the rest of the graph. The same thing is also happened for Frank's limit, before reaching its lowest point, it falls significantly from $3 \cdot 10^{-4}$ to around 0. To recapitulate, we might say that both Clayton and Frank copula are not proper models to be considered in iterated simulation procedure.

Besides numerical approach explained above, analytical way to resolve the questions are also provided. Generally, we would like to prove that iterated Clayton and Frank copula tend to the independence copula as N approaches infinity. Now, for iterated limit of Clayton copula, write the equation as

$$\lim_{N\to\infty} C_\alpha^{Cl}(S_1(\Delta T), S_2(\Delta T))^N = \hat{C}(S_1(T), S_2(T))$$

where $\hat{C}(S_1(T)_2(T))$ is the independence copula and also let $\lambda_1 = \lambda_2 = \lambda$ for simplicity reason.

Proof. Consider a Clayton copula $C_\alpha^{Cl}(S_1(T), S_2(T))$ and write the iterated version as

$$\begin{aligned}\lim_{N\to\infty} C_\alpha^{Cl}(S_1(\Delta T), S_2(\Delta T))^N &= \lim_{N\to\infty} \left(e^{\lambda \frac{T}{N}\alpha} + e^{\lambda \frac{T}{N}\alpha} - 1\right)^{-N/\alpha} \\ &= \lim_{N\to\infty} \left(2e^{\lambda \frac{T}{N}\alpha} - 1\right)^{-N/\alpha}.\end{aligned}$$

Next, let $y := (2e^{\lambda \frac{T}{N}\alpha} - 1)^{-N/\alpha}$ and take the natural logarithm of both sides to get

$$\ln y = -\frac{N}{\alpha} \ln \left(2e^{\lambda \frac{T}{N}\alpha} - 1\right).$$

Apply limit on each side of the equation

$$\begin{aligned}\lim_{N\to\infty} \ln y &= \lim_{N\to\infty} \frac{-N}{\alpha} \ln \left(2e^{\lambda \frac{T}{N}\alpha} - 1\right) \\ &= \lim_{N\to\infty} \frac{\ln \left(2e^{\lambda \frac{T}{N}\alpha} - 1\right)}{-\alpha/N} \\ &\overset{\text{L}}{=} \lim_{N\to\infty} \frac{1}{\left(2e^{\lambda\alpha \frac{T}{N}} - 1\right)} \cdot \frac{\left(2e^{\lambda\alpha \frac{T}{N}}\right) \cdot \left(-\lambda \frac{T}{N^2}\alpha\right)}{\alpha/N^2} \\ &= -2\lambda T.\end{aligned}$$

Finally, we obtain

$$\lim_{N\to\infty} \left(2e^{\lambda \frac{T}{N}\alpha} - 1\right)^{-N/\alpha} = e^{-2\lambda T} = e^{-\lambda T} \cdot e^{-\lambda T} = S_1(T) \cdot S_2(T) = \hat{C}(S_1(T)_2(T)).$$

□

After calculating Clayton copula's limit, we found out the confirmation of this result later in a textbook edited by Dey and Yan's [47, Remark 6.2.3]. Not only Clayton copula, but they also prove that Gaussian copula is belong to the domain of attraction of independence copula in [47, Example 6.2.4]. As for Frank copula, the same technique is applied to solve the limit. Similarly, write the equation as

$$\lim_{N\to\infty} C^{Fr}(S_1(\Delta T), S_2(\Delta T))^N = \hat{C}(S_1(T), S_2(T))$$

where $\hat{C}(S_1(T)_2(T))$ is the independence copula and set $\lambda_1 = \lambda_2 = \lambda$ for the sake of simplicity.

Proof. Consider a Frank copula $C_\alpha^{Fr}(S_1(T), S_2(T))$ and write its iterated limit as

$$\lim_{N\to\infty} C_\alpha^{Fr}(S_1(\Delta T), S_2(\Delta T))^N = \lim_{N\to\infty} \left\{ -\frac{1}{\alpha} \ln \left(1 + \frac{(e^{-\alpha e^{-\lambda \frac{T}{N}}} - 1)(e^{-\alpha e^{-\lambda \frac{T}{N}}} - 1)}{e^{-\alpha} - 1}\right)\right\}^N.$$

Now, by letting

$$y := \left\{ -\frac{1}{\alpha} \ln \left(1 + \frac{(e^{-\alpha e^{-\lambda \frac{T}{N}}} - 1)(e^{-\alpha e^{-\lambda \frac{T}{N}}} - 1)}{e^{-\alpha} - 1}\right)\right\}^N$$

and applying the natural logarithm of each side, we would obtain

$$\ln y = N \left[\ln \left(-\frac{1}{\alpha}\right) + \ln \left\{ \ln \left(1 + \frac{(e^{-\alpha e^{-\lambda \frac{T}{N}}} - 1)(e^{-\alpha e^{-\lambda \frac{T}{N}}} - 1)}{e^{-\alpha} - 1}\right)\right\}\right].$$

The next step is to take the limit of both sides, in that

$$\begin{aligned}
\lim_{N\to\infty} \ln y &= \lim_{N\to\infty} \frac{\ln\left(-\frac{1}{\alpha}\right) + \ln\left\{\ln\left(1+\frac{(e^{-\alpha e^{-\lambda\frac{T}{N}}}-1)(e^{-\alpha e^{-\lambda\frac{T}{N}}}-1)}{e^{-\alpha}-1}\right)\right\}}{1/N} \\
&\overset{\text{L}}{=} \lim_{N\to\infty} \frac{1}{\ln\left\{1+\frac{(e^{-\alpha e^{-\lambda\frac{T}{N}}}-1)^2}{e^{-\alpha}-1}\right\}} \cdot \frac{1}{1+\frac{(e^{-\alpha e^{-\lambda\frac{T}{N}}}-1)^2}{e^{-\alpha}-1}} \cdot \frac{2(e^{-\alpha e^{-\lambda\frac{T}{N}}}-1)}{e^{-\alpha}-1} \\
&\quad \cdot (e^{-\alpha e^{-\lambda\frac{T}{N}}}) \cdot (-\alpha e^{-\lambda\frac{T}{N}}) \cdot \lambda\frac{T}{N^2} \cdot (-N^2) \\
&= -\frac{1}{\alpha} \cdot \frac{1}{e^{-\alpha}} \cdot 2 \cdot e^{-\alpha} \cdot (-\alpha) \cdot (-\lambda T) \cdot \\
&= -2\lambda T.
\end{aligned}$$

Thus, at last we get

$$\begin{aligned}
\lim_{N\to\infty} \left\{-\frac{1}{\alpha} \ln\left(1+\frac{(e^{-\alpha e^{-\lambda\frac{T}{N}}}-1)(e^{-\alpha e^{-\lambda\frac{T}{N}}}-1)}{e^{-\alpha}-1}\right)\right\}^N &= e^{-2\lambda T} \\
&= S_1(T) \cdot S_2(T) \\
&= \hat{C}(S_1(T) \cdot S_2(T)).
\end{aligned}$$

□

Thirdly, Marshall-Olkin and Gumbel copula limit error are presented in Figure 14. As illustrated in Figure 14(a), Marshall-Olkin's error fluctuates near 0.0334 for the whole N values. It is also evident that the oscillation becomes larger when the value of N is raising. Thus, limit of Marshall-Olkin copula does not resemble the independence copula. In terms of Gumbel copula, its limit error goes up and down widely, especially for large N. As Gumbel's general pattern is rather similar with the one shown by Marshall-Olkin, the same conclusion might be drawn directly from Marshall-Olkin's case, in that Gumbel copula does not tend to the independence copula as the number of sub-intervals goes to infinity.

One might notice a relation between iterated limit problem and self-chaining copula concept. Figure 12 illustrates that Gaussian copula has smaller limit errors than t copula. As for t copula, the lower degrees of freedom it has, the higher its errors can be. This behaviour shall seem familiar to the reader. Moreover, Figure 13 shows that limit errors of Clayton and Frank copula are extremely small compared to others. On the other hand, Marshall-Olkin and Gumbel copula have the most critical limit errors. All of these are pieces of evidence that self-chaining copulas tend to have the more significant limit errors than other copulas without self-chaining property. Actually, this fact is completely obvious as limit of iterated self-chaining copula is self-chaining copula itself, namely

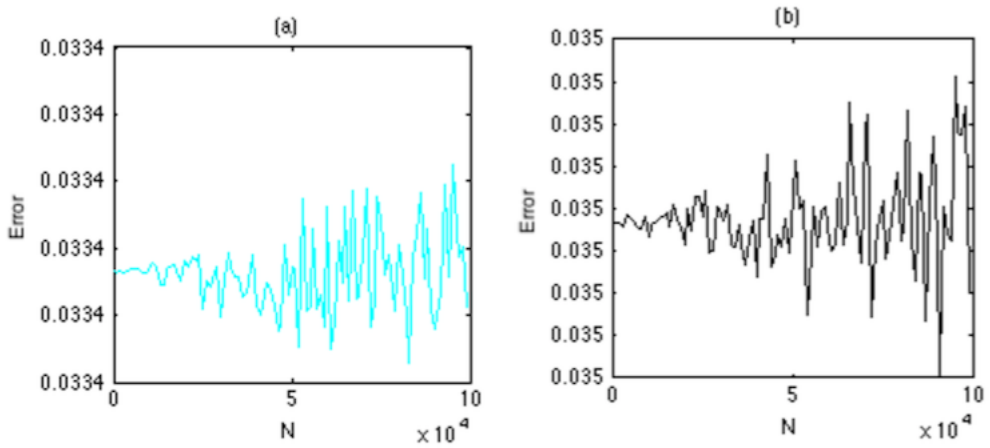

Figure 14: Iterated limit error line charts of: (a) Marshall-Olkin copula and (b) Gumbel copula with $\mathcal{K}_\tau = 0.5$, $\lambda_1 = 0.01$, $\lambda_2 = 0.03$, and $\lambda_{12} = 0.04$.

$$\lim_{N\to\infty} C^{SC}(S_1(\Delta T), S_2(\Delta T))^N = C^{SC}(S_1(T), S_2(T))$$

where $C^{SC}(S_1(T), S_2(T))$ denotes self-chaining copula. For instance, we will prove bivariate Gumbel copula case with intensities λ_1 and λ_2.

Proof. Consider a Gumbel copula $C_\alpha^{Gu}(S_1(T), S_2(T))$ and write its iterated limit as

$$\lim_{N\to\infty} C_\alpha^{Gu}(S_1(\Delta T), S_2(\Delta T))^N = \lim_{N\to\infty} e^{-N\left[\left(\lambda_1 \frac{T}{N}\right)^\alpha + \left(\lambda_2 \frac{T}{N}\right)^\alpha\right]^{1/\alpha}}.$$

The next step is letting $y := e^{-N\left[\left(\lambda_1 \frac{T}{N}\right)^\alpha + \left(\lambda_2 \frac{T}{N}\right)^\alpha\right]^{1/\alpha}}$ and taking the natural logarithm of all sides, hence we get

$$\ln y = -N\left[\left(\lambda_1 \frac{T}{N}\right)^\alpha + \left(\lambda_2 \frac{T}{N}\right)^\alpha\right]^{1/\alpha}.$$

Take the limit on the left and right side of the equation, in particular

$$\begin{aligned}
\lim_{N\to\infty} \ln y &= \lim_{N\to\infty} \frac{[(\lambda_1 \frac{T}{N})^\alpha + (\lambda_2 \frac{T}{N})^\alpha]^{1/\alpha}}{-1/N} \\
&\overset{\text{L}}{=} \lim_{N\to\infty} \frac{1/\alpha \cdot [(\lambda_1 \frac{T}{N})^\alpha + (\lambda_2 \frac{T}{N})^\alpha]^{\frac{1-\alpha}{\alpha}} \cdot (-\alpha T/N^2) \cdot [(\lambda_1 \frac{T}{N})^{\alpha-1} \cdot \lambda_1 + (\lambda_2 \frac{T}{N})^{\alpha-1} \cdot \lambda_2]}{1/N^2} \\
&= -(\lambda_1^\alpha + \lambda_2^\alpha)^{\frac{1-\alpha}{\alpha}} \cdot (\lambda_1^\alpha + \lambda_2^\alpha) \cdot T \\
&= -[(\lambda_1 T)^\alpha + (\lambda_2 T)^\alpha]^{1/\alpha}.
\end{aligned}$$

At last, we obtain the following

$$\lim_{N\to\infty} e^{-N\left[\left(\lambda_1 \frac{T}{N}\right)^\alpha + \left(\lambda_2 \frac{T}{N}\right)^\alpha\right]^{1/\alpha}} = e^{-[(\lambda_1 T)^\alpha + (\lambda_2 T)^\alpha]^{1/\alpha}} = C_\alpha^{Gu}(S_1(T), S_2(T)).$$

□

Additionally, we present an example of self-chaining copula iterated limit from the numerical point of view. Now, define the difference between iterated self-chaining copula limit and self-chaining copula itself as

$$\varepsilon_{SC} = |\lim_{N\to\infty} C^{SC}(S_1(\Delta T), S_2(\Delta T))^N - C^{SC}(S_1(T), S_2(T))|.$$

The following table shows percentage ε_{SC} of Gumbel copula and Marshall-Olkin copula with 3 different N quantities. As we can observe from the table, the percentage errors of iterated self-chaining copulas relative to themselves are extremely small and remain steady. Hence, the result follows.

Table 6: Percentage limit error (ε_{SC}) of self-chaining copulas: Gumbel and Marshall-Olkin, with $\lambda_1 = 0.01$, $\lambda_2 = 0.03$, $\lambda_{12} = 0.04$, and $\mathcal{K}_\tau = 0.5$.

Copula	**N=1000**	**N=10000**	**N=1000000**
Gumbel	1.9839E-12	1.59E-11	3.72E-11
Marshall-Olkin	4.73E-12	3.56E-11	4.41E-10

The next problem is the trivariate case of all survival entities. As in the bivariate case, we examine errors for several copulas: Gaussian, t, Gumbel, Frank, Clayton, and Marshall-Olkin. The following are explicit formulas for some trivariate copulas:

1. Gumbel copula

$$C_\alpha^{Gu}(u_1, u_2, u_3) = e^{-[(-\ln(u_1))^\alpha + (-\ln(u_2))^\alpha + (-\ln(u_3))^\alpha]^{1/\alpha}}.$$

2. Clayton copula

$$C_\alpha^{Cl}(u_1, u_2, u_3) = \max\left(u_1^{-\alpha} + u_2^{-\alpha} + u_3^{\alpha} - 2, 0\right)^{-1/\alpha}.$$

3. Frank copula

$$C_\alpha^{Fr}(u_1, u_2, u_3) = -\frac{1}{\alpha}\ln\left\{1 + \frac{(e^{-\alpha u_1} - 1)(e^{-\alpha u_2} - 1)(e^{-\alpha u_3} - 1)}{(e^{-\alpha} - 1)^2}\right\}.$$

4. Marshall-Olkin copula

   Trivariate Marshall-Olkin formula in the following is taken from Li (2006: 5-7), reads

$$\begin{aligned} C(u_1, u_2, u_3) =& u_1 u_2 u_3 \cdot \min\left(u_1^{-\alpha_1^{12}}, u_2^{-\alpha_2^{12}}\right) \cdot \min\left(u_1^{-\alpha_1^{13}}, u_3^{-\alpha_3^{13}}\right) \\ & \cdot \min\left(u_2^{-\alpha_2^{23}}, u_3^{-\alpha_3^{23}}\right) \cdot \max\left(u_1^{\alpha_1^{123}}, u_2^{\alpha_2^{123}}, u_3^{\alpha_3^{123}}\right) \end{aligned}$$

   where $\alpha_i^J \geq 0$, $i \in J$, and $J \subseteq \{1, 2, 3\}$.

(a)

Copula	Parameter(s)	Relative Error (%)	
		N=1000	N=10000
Gaussian	ρ=0.1	16.687	16.824
t	ν=3, ρ= 0.1	0.131	0.183
	ν=4, ρ= 0.1	0.683	0.668
Gumbel	α=2	7.17E-12	1.17E-11
Frank	α=5	10.039	10.06
Clayton	α=2	4.533	4.538
Marshall-Olkin	$\alpha_1^{12}=0.35\ \alpha_2^{12}=0.15\ \alpha_1^{13}=0.12$ $\alpha_3^{13}=0.035\ \alpha_2^{23}=0.15\ \alpha_3^{23}=0.11$ $\alpha_1^{123}=0.059\ \alpha_2^{123}=0.024\ \alpha_3^{123}=0.02$	8.71E-12	1.65E-11

(b)

Copula	Parameter(s)	Relative Error (%)	
		N=1000	N=10000
Gaussian	ρ=0.5	77.877	82.892
t	ν=3, ρ= 0.5	0.971	0.995
	ν=4, ρ= 0.5	2.124	2.173
Gumbel	α=10	4.68E-12	1.08E-11
Frank	α=20	17.057	17.142
Clayton	α=20	17.494	17.579
Marshall-Olkin	$\alpha_1^{12}=0.5\ \alpha_2^{12}=0.33\ \alpha_1^{13}=0.5$ $\alpha_3^{13}=0.27\ \alpha_2^{23}=0.44\ \alpha_3^{23}=0.36$ $\alpha_1^{123}=0.17\ \alpha_2^{123}=0.11\ \alpha_3^{123}=0.09$	1.49E-11	2.29E-11

(c)

Copula	Parameter(s)	Relative Error (%)	
		N=1000	N=10000
Gaussian	ρ=0.9	6.41	7.896
t	ν=3, ρ= 0.9	0.825	0.836
	ν=4, ρ= 0.9	1.413	1.457
Gumbel	α=80	3.55E-12	2.38E-11
Frank	α=1000	14.032	15.256
Clayton	α=500	15.997	16.894
Marshall-Olkin	$\alpha_1^{12}=0.60\ \alpha_2^{12}=0.75\ \alpha_1^{13}=0.80$ $\alpha_3^{13}=0.80\ \alpha_2^{23}=0.75\ \alpha_3^{23}=0.60$ $\alpha_1^{123}=0.40\ \alpha_2^{123}=0.50\ \alpha_3^{123}=0.40$	9.13E-12	3.20E-11

Figure 15: Relative error ε_r tables Gaussian, t, Gumbel, Frank, Clayton, and Marshall-Olkin copula with $\lambda_1 = 0.01$, $\lambda_2 = 0.03$, $\lambda_3 = 0.05$, and $T = 5$y. (a) Copulas with low-level dependency. (b) Copulas with moderate-level dependency. (c) Copulas with high-level dependency.

Figure 15 shows the relative errors of six different copulas with different dependency levels. Generally, all copulas' errors grow larger when the number of time steps increases. The increments of errors are also be affected by the choice of dependency parameter(s). As can be seen in the figure above, the raises of errors for copulas with moderate-level dependency are more substantial than others. However, Gumbel and Marshall-Olkin's errors are considerably unchanged for whole dependence parameters' values. It has been shown that Gaussian, Frank, and Clayton copula tend

to come closer to the independence copula as N approaches infinity. Therefore, their errors go larger when the value of N is switched, from 1000 to 10000. The important point here is errors become smaller if we set dependence parameters to particular values so that the copulas tend to resemble either independence or comonotonicity copula. This fact is unsurprising as we have already discussed this issue in terms of Kendall's tau framework in bivariate case. One might also notice that trivariate errors are always more significant than bivariate errors. Thus, the more entities involved in this simulation, the higher error we expect.

3.2 Mixed Survival Case

After discussing all survival cases, we start examining the more ambitious tasks in mixed survival case. It has been explained that in solving these problems, same technique as in all survival case is utilised, by initially converting mixed survival case into all survival case. Regarding the bivariate case, we would compare simulation errors with various dependence structures and times. Whereas in the trivariate case, the effect of changing time period as well as intensities on relative errors are thoroughly examined.

Table 7: Relative error table of bivariate copulas with $\lambda_1 = 0.01$, $\lambda_2 = 0.03$, $N = 1000$, and $T = 5y$.

Copula	**(%) Relative Error**		
	$\mathcal{K}_\tau = 0.1$	$\mathcal{K}_\tau = 0.5$	$\mathcal{K}_\tau = 0.9$
Gaussian	0.3958	2.0129	0.0730
t($\nu = 3$)	0.0217	0.1962	0.0050
Gumbel	3.6970E-12	2.5979E-12	5.9952E-13
Clayton	0.1335	1.0275	3.7404
Frank	0.2742	2.2440	4.1068
MO	6.7945E-12	8.2045E-12	9.90319E-12

Percentage relative errors of six aforementioned copulas with various attributes are shown in Table 7 and Table 8. Firstly, from Table 7, it can be clearly seen that in 5 years period, iteration errors across different copula types and tau quantities diverge considerably.

According to Table 7, as tau enlarges to particular points, relative errors of all copulas rise. Gaussian and Frank copula have the most significant error increments from around 0% to 2% when their tau values are ascended to 0.5, whereas Gumbel and Marshall-Olkin copula errors do not exhibit any notable changes. Next, to see whether there is a changing pattern in simulation error, the value of tau is raised again by 0.4, from $\mathcal{K}_\tau = 0.5$ to $\mathcal{K}_\tau = 0.9$. Interestingly, there are three different effects on copula errors. Gaussian and t copula experience downward trend while Clayton

and Frank copula's errors level up to around 4%. As for Gumbel and MO copula, their errors remain stable at very low level. One might be familiar with such kind of patterns. Indeed, it is roughly similar to Figure 10 in the previous case. Numerical problems are also encountered as in all survival case, especially for Clayton and Frank copula with high Kendall's tau values. This issue is addressed later in this chapter.

Table 8: Relative error table of bivariate copulas with $\lambda_1 = 0.01$, $\lambda_2 = 0.03$, $N = 1000$, and $T = 50y$.

Copula	**(%) Relative Error**		
	$\mathcal{K}_\tau = 0.1$	$\mathcal{K}_\tau = 0.5$	$\mathcal{K}_\tau = 0.9$
Gaussian	1.7299	5.5881	0.0746
t(3)	0.6004	1.4136	0.0286
Gumbel	1.998E-13	1.3988E-12	5.9952E-13
Clayton	1.722	7.8620	8.5988
Frank	1.6176	7.5209	8.4101
MO	1.58E-13	9.9920E-13	8.99281E-13

Table 8 depicts relative errors of copulas with different degrees of dependency for $T = 50$ years. It is obvious that if the time period is added up, the errors elevate as well. In particular, error patterns are reasonably identical with the ones in Table 7. Gaussian copula still has the greatest amount of discrepancies for $\mathcal{K}_\tau = 0.1$. It is then followed by Clayton and Frank copula with quite substantial errors. t copula has relatively small errors, although it is not as insignificant as Gumbel and Marshall-Olkin's. When Kendall's value reach 0.5, copula errors become more considerable, except for self-chaining copulas whose errors remain stable near 0. However, as $\mathcal{K}_\tau$ is set up to be 0.9, the errors of all copulas but Clayton and Frank finally go down. Again, a numerical issue comes up for Clayton and Frank copula. In order to overcome this problem, we calculate their relative errors with respect to parameter values instead.

Table 9: Relative error table of bivariate Clayton and Frank copula relative to parameter (α) value with $\lambda_1 = 0.01$, $\lambda_2 = 0.03$, $N = 1000$, and $T = 5y$.

Clayton		**Frank**	
α	**(%) Relative Error**	α	**(%) Relative Error**
10000	1.3427599	340	4.6428781
30000	0.1264161	540	4.5115475
50000	0.0123309	740	4.4789916

Table 9 shows errors of Clayton and Frank copula with high parameter values. It is evident that Clayton's error is prominently decreasing as its parameter value grows. Similar results hold for Frank copula. Hence, the same conclusion is obtained as in all survival case, in that copula errors reach their lowest whenever $\mathcal{K}_\tau$ closest to 0 and 1. Furthermore, we can see that errors for the bivariate mixed case are moderately higher than all survival case. Thus, involving default events in copula simulation leads us to the less persistent results.

Regarding the trivariate case, there are several comparisons that need to be discussed. As in all survival case, Kendall's tau measure is not utilised due to numerical complexity. It has been mentioned that the case is divided into 2 problems: simulation with 1 default (2 survivals) and simulation with 2 defaults (1 survival).

Table 10: Relative error of trivariate copulas (1 default case) with $\lambda_1 = 0.01$, $\lambda_2 = 0.03$, $\lambda_3 = 0.05$, and $N = 1000$.

Copula	**Parameter(s)**	**(%) Relative Error**	
		$T = 5$y	$T = 10$y
Gaussian	$\rho_{12} = 0.3$, $\rho_{23} = 0.5$, $\rho_{13} = 0.9$	3.823	5.505
t($\nu = 3$)		0.551	1.187
t($\nu = 4$)		1.011	1.861
Gumbel	$\alpha = 10$	4.207E-12	4.297E-12
Frank		7.977	11.599
Clayton		8.803	12.688
MO	$\alpha_1^{12} = 0.58$ $\alpha_2^{12} = 0.48$ $\alpha_1^{13} = 0.58$ $\alpha_3^{13} = 0.50$ $\alpha_2^{23} = 0.63$ $\alpha_3^{23} = 0.66$ $\alpha_1^{123} = 0.19$ $\alpha_2^{123} = 0.16$ $\alpha_3^{123} = 0.16$	6.095E-12	1.550E-12
Comonotonicity	perfectly dependent	1.350E-12	9.470E-12
Independence	independent	3.708E-12	3.997E-12

Table 10 illustrates copula errors in 1 default simulation case for $T = 5$ years and $T = 10$ years. Instead of doing comparison across copulas, studying errors within similar copulas with different time frame is more desirable. Generally, errors for longer time period simulations are greater than the ones with shorter time period, which is in line with the previous bivariate mixed survival case results. Frank copula has the most notable error rising by around 45.5 %. The second biggest error increments come from Clayton and Gaussian copula, with elevations as high as 44.8 % and 44.1 % respectively. A common thing we can observe is t copula with 4 degrees of freedom has relatively bigger errors than t copula with 3 degrees of freedom. Both t copula errors have relatively similar increases of around 1%. Needless to say, Gumbel and Marshall-Olkin errors are not affected

by the change of time period. Similarly, the comonotonicity and independence copula which are categorised as self-chaining copula also have insignificant errors for both $T = 5$y and $T = 10$y.

Table 11: Relative error of trivariate copulas (2 defaults case) with $T = 5$y and $N = 1000$.

Copula	λ_1	λ_2	λ_3	**Parameter(s)**	(%) **Relative Error**
Gaussian	0.01	0.01	0.01	$\rho_{12} = 0.3$ $\rho_{23} = 0.5$ $\rho_{13} = 0.9$	0.335
	0.01	0.01	0.03		0.979
	0.01	0.03	0.05		2.661
t($\nu = 3$)	0.01	0.01	0.01		0.0193989
	0.01	0.01	0.03		0.0878247
	0.01	0.03	0.05		0.4848366
Gumbel	0.01	0.01	0.01	$\alpha = 10$	1.4877E-12
	0.01	0.01	0.03		4.50751E-12
	0.01	0.03	0.05		4.30767E-12
Frank	0.01	0.01	0.01	$\alpha = 10$	0.5792797
	0.01	0.01	0.03		1.3466057
	0.01	0.03	0.05		4.8530275
Clayton	0.01	0.01	0.01	$\alpha = 10$	0.6158312
	0.01	0.01	0.03		1.4478716
	0.01	0.03	0.05		5.4179188
MO	0.01	0.01	0.01	$\alpha_1^{12} = 0.58$ $\alpha_2^{12} = 0.48$ $\alpha_1^{13} = 0.58$	8.99281E-12
	0.01	0.01	0.03	$\alpha_3^{13} = 0.50$ $\alpha_2^{23} = 0.63$ $\alpha_3^{23} = 0.66$	2.46025E-12
	0.01	0.03	0.05	$\alpha_1^{123} = 0.19$ $\alpha_2^{123} = 0.16$ $\alpha_3^{123} = 0.16$	7.69385E-12

Relative errors for the last case in this study, trivariate copulas with 2 companies default, can be found in Table 11. In this table, we present each copula with various intensities (λ) in order to show that a slight change in intensity values could result in significant difference. However, one need to keep in mind that the difference is not only affected by intensities but also the copula parameters. We can see that Gaussian copula with all intensities equal to 0.01 has relative error 0.335%. If one of the intensities is modified, in this case $\lambda_3 = 0.03$, its error grows three times larger. The same results hold for t, Frank, and Clayton copula although the changes vary depending on their parameters. As is expected, the alteration of intensities does not have a big impact on self-chaining copulas. It can be observed that Gumbel and Marshall-Olkin copula errors are consistently unchanged around 0.

Table 12: Relative error of trivariate Gaussian copula (2 defaults case) with $T = 5$y and $N = 1000$.

λ_1	λ_2	λ_3	ρ	(%) **Relative Error**
0.0001	0.0001	0.0001	0	2.998E-12
			0.99999	8.538E-07
0.1	0.1	0.1	0	1.998E-12
			0.99999	5.862E-07

More examples regarding self-chaining copulas can be seen in Table 12. To begin with, the trivariate Gaussian copula parameters are set to be zeros, or, in other words, we obtain independence copula. As one might predict, relative errors for both intensity variations are incredibly insignificant. The same conclusion holds for Gaussian copula whose parameter tends to 1 (i.e. the comonotonicity copula), although numerical constraint causes the errors are slightly bigger than the ones in the independence case. Lastly, it is evident that relative errors of trivariate copulas in mixed survival case are moderately higher than the errors in all survival case. This finding is in compliance with the previous bivariate case result.

4 Summary, Conclusions, and Further Work

4.1 Summary and Conclusions

The aim of this study is to examine errors of dependence iteration for particular bivariate and trivariate copulas. Most of the problems are evaluated using numerical approach while for few bivariate copulas we also compute the probabilities analytically. There are two main procedures used in this thesis, the first one is to simulate the copula function through the whole period, from the beginning to terminal time T, directly. As for the second procedure, the time path is divided into N uniform sub-intervals, in that $[0, \Delta T], [\Delta T, 2\Delta T], \ldots, [(N-1)\Delta T, N\Delta T]$.

In chapter 1, all basic concepts such as copula, intensity model, and Kendall's tau are explained in detail. We utilise intensity model as our default probability model, then combine this with copula function to describe dependence structures among entities. Furthermore, multivariate homogeneous lack of memory (MHLOM) property is also introduced as the most basic requirement needed to obtain similar simulation results from the aforementioned procedures. This theory is a generalisation of univariate lack of memory concept which is usually recognised as exponential distribution's main characteristic. Eventually, Marshall-Olkin distribution is known as multivariate distribution function embedded with MHLOM property. This property is then transformed into a copula function which brings us to the definition of self-chaining copula as described in [16]. Since Gumbel and Marshall-Olkin copula are examples of self-chaining copulas, we expect insignificant errors from simulations of these two aforesaid copulas.

In the following chapter 2, the problems we encounter are described in the form of mathematical expressions. Firstly, the problems are broken down into all survival and mixed survival case. For each case, we study both bivariate and trivariate copula errors. In particular, Kendall's tau measure is also applied to compare all bivariate copulas. Generally, copula errors are defined as

$$\varepsilon_r := \frac{\mathbb{P}_{single} - \mathbb{P}_{multi}}{\mathbb{P}_{single}},$$

where $\mathbb{P}_{single}$ denotes the first procedure and $\mathbb{P}_{multi}$ stands for the second procedure. In addition, we also investigate the effect of bivariate copula iteration on dependency structure.

Generally, all simulation results are shown in Chapter 3. As for the bivariate copula in all survival case, it is evident that as $\mathcal{K}_\tau$ reach 0 (independent) or 1 (perfectly dependent), all copulas tends to resemble self-chaining copula (i.e. errors become completely negligible). Broadly speaking, Gaussian copula has the highest error among others for $\mathcal{K}_\tau$ less than or equal 0.55 while Frank copula's error dominates others for $\mathcal{K}_\tau$ more than 0.55. It has also been shown that iterating non-self-chaining copulas such as Gaussian, Frank, Clayton, and higher degrees of t copula might

terminate dependence structure since they tend to the independence copula as N rises. Similarly for the trivariate case, if dependence parameters are set such that the copulas become either independence or positive dependence, iteration errors become more inconsiderable.

The impact of dependence parameters on iteration error is also true for the mixed survival problems, both in bivariate and trivariate case. Apart from self-chaining copulas, it is obvious that the longer terminal time we set, the higher incidents of errors occur. Additionally, intensities are known to have quite significant effects on consistency levels of the second procedure. Final conclusion we can point out from this study is that adding up the number of companies involved in the simulation and engaging default events in default probability calculation would increase iteration errors of all copulas but the self-chaining ones.

4.2 Further Work

There are still a lot of aspects in this study that can be explored. The next research shall extend all cases to higher dimensional larger than 3. Furthermore, involving more extreme value copulas like Galambos or t-extreme-value is more ideal, see [41]. In terms of estimating copula parameter (s), it would be better if maximum likelihood method is implemented. It is also possible to utilise other rank correlation statistics such as Goodman and Kruskal's gamma or Somer's D to compare iteration error across copulas. Another research idea is trying to find the limit of iterated non-self-chaining copulas as N tends to infinity.

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