Imperial College London

Department of Mathematics

Dynamic Convex Duality and Backward Stochastic Differential Equations in Utility Maximization

Author: Haoyin Lin (CID: 01788182)

A thesis submitted for the degree of

MSc in Mathematics and Finance, 2019-2020

Abstract

In this paper, we aim to solve utility maximization problems with different constraints and utility function. For each utility maximization problem, the primal and dual problems would be constructed and formulated, and then we construct their adjoint processes so as to obtain forward and backward stochastic differential equations (FBSDEs). We solve utility maximization problems by using either theoretical or numerical methods, and prove that the solutions from primal problem, dual problem, and FBSDE problem should be the same.

Acknowledgements

I would like to express my very great appreciation to Professor Zheng for his valuable and constructive suggestions during the planning and development of this research work. His willingness to give his time so generously has been very much appreciated.

I also really appreciate the support from my family and my friends.

Contents

List of Figures

List of Tables

Introduction

The utility maximization problems have been one of the focus points in the area of mathematical economics for these years. Solving such problems aim to maximize the utility for consumers under certain constraints, such as no short selling and other trading restrictions.

The concept of stochastic process was first introduced as a specific mathematical definition in a 1934 paper by Joseph Doob[4]. After that stochastic process was widely used in finance to simulate financial products and controls in trading processes. Stochastic control has a significant meaning in dealing with dynamic portfolio optimization problems, and it has developed greatly since 1970s. During this time, Robert Merton published two landmark papers in this field, which is about Hamilton-Jacobi-Bellman equation and the requirement of an underlying Markov state process [11, 12]. Based on his work, Pliska[14], Cox and Huang[2], Karatzas et al.[8] have extensive research in optimal investment problems with non-Markov setting.

The stochastic duality theory of Bismut[1] was first applied in solving the constrained optimal investment problem in Xu and Shreve's paper [15]. After that, due to efficiency of the convex dual method, it was widely used in dealing with problems in incomplete market models in the works of Karatzas et al.[9], He and Pearson[6, 7], Cvitanic and Karatzas[3]. The purpose of using convex duality theory is to convert a primal constrained problem into an unconstrained one, and by solving the dual problem, the dual solution can be reverted so as to obtain the optimal solution for primal problem. However, in general, even the dual problems can be difficult to solve, and the explicit close form result is hard to obtain. Therefore, numerical method is often used to simulate and approximate the solution in solving optimal problems.

The theory of backward stochastic differential equations (BSDEs) was proposed and introduced by Pardoux and Peng[5]. It becomes popular and significant in the field of mathematical economics and finance later on because its connections with stochastic controls and partial differential equation. According to the theory of BSDEs, the non-linear PDEs can be written in probabilistic forms, which extends the Faynman-Kac formula for linear PDEs. The BSDEs provide a way to solve non-linear PDEs by using numerical methods. The BSDEs are then combined with the forward stochastic differential equations (FSDEs) and become forward and backward stochastic differential equations (FBSDEs), which is a powerful modelling tool in solving stochastic control problems. Li and Zheng[10] proposed the necessary and sufficient conditions for primal and dual problems in terms of forward and backward stochastic differential equations. The necessary and sufficient conditions build up the connections among primal problems, dual problems, and their FBSDEs. Under the circumstance of conditions satisfied, the optimal primal problem agrees with the adjoint process of the dual problem and vice verse, and thus it provides more flexibility to obtain an optimal primal solution.

In this paper, we study the utility maximization problems with different constraints and utility functions, and this paper is mainly divided by three parts. The background setting and theorems used in this paper are mainly referred to the paper by Li and Zheng [10]. In the first part, we introduce the background and set up the market model, wealth process, and value function. After the primal problem is formulated, we convert it to a dual problem by using supermartingle approach, and introduce the corresponding adjoint processes for both primal and dual problems based on Pham's book[13]. There are no constraints for the first part, which means the control set is the whole space and all parameters are deterministic. We will show that the solution of dual problem coincides with the dual FBSDEs' for both power utility function and non-Hara utility function.

In the second part, all the premises remain the same except for the Brownian motion term of the stock process, which becomes an OU process. The value functions can be solved from primal and dual HJB equations explicitly with Ansatz. Numerical algorithms are designed to prove that primal problem, dual problem, and their corresponding FBSDEs problems give the same solution to wealth process. In the third part, the control set becomes a positive one dimensional space, and thus short-selling is not permitted and the OU process has a time dependent Brownian motion term instead of constant. We will solve the dual FBSDE to obtain optimal controls and verify the controls by checking the value of value functions.

The rest of the paper is organized as follows. There are three Chapters in total, which corresponds to each part mentioned above. The last section concludes the paper.

Chapter 1

Unconstrained Utility Maximization

Assumption:
- 1-dimensional geometric Brownian motion asset price process
- All coefficients constant
- Control set K=$\mathbb{R}$
- Maximize utility of wealth at time T

1.1 Market Model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which is defined a 1-dimension Brownian motion $\{W(t), t \in [0,T]\}$ with $T > 0$ denoting a fixed terminal time. Let $\{\mathcal{F}_t, t \in [0,T]\}$ be the standard filtration induced by W.

Denote by $\mathcal{F}^*$ the σ-algebra of $\mathcal{F}_t$ progressively measurable sets on $\Omega \times [0,T]$. For any stochastic process $v : \Omega \times [0,T] \to \mathbb{R}^m$, $m \in \mathcal{N}^+$, we write $v \in \mathcal{F}^*$ to indicate v is $\mathcal{F}^*$ measurable. We introduce the following notation:

$$\mathcal{H}^p(0,T;\mathbb{R}^m) = \{\xi : \Omega \times [0,T] \to \mathbb{R}^m | \xi \in \mathcal{F}^*, E[\int_0^T |\xi(t)|^p dt] < \infty\}$$

Consider a market consisting of a bank account with price $S_0(t)$ and risky assets with price $S(t)$ satisfying SDE:

$$\begin{cases} dS_0(t) = rS_0(t)dt \\ dS(t) = S(t)(\mu dt + \sigma dW(t)) \end{cases} \tag{1.1.1}$$

with $S_0(0) = 1, S(0) = S > 0$, where $r > 0, \mu > 0, \sigma > 0$ are all constant, W is a standard Brownian motion.

Consider a small investor with initial wealth $x > 0$ and a self-financing strategy. Define the set of admissible portfolio strategies by:

$$\mathcal{A} := \{\pi \in \mathcal{H}^2(0,T;\mathbb{R}) : \pi(t) \in K = \mathbb{R} \text{ for } t \in [0,T] \text{ a.e.}\}$$

where $\pi(t)$ is a portfolio process defined as the fraction of the wealth invested in the stock at time t.

Given any $\pi \in \mathcal{A}$, the investor's total wealth X^π satisfies the following dynamics:

$$dX^\pi(t) = X^\pi(t)\{[r + \pi(t)\sigma\theta]dt + \pi(t)\sigma dW_t\} \tag{1.1.2}$$

$$X^\pi(0) = x \tag{1.1.3}$$

where $\theta = \frac{b-r}{\sigma}$ is the market price of risk

Let $U : [0, \infty) \to \mathbb{R}$ be a given utility function that is twice continuously differentiable, strictly increasing, strictly concave and satisfies the following conditions:

$$U(0) = \lim_{x \to 0} U(x) > -\infty, \quad \lim_{x \to 0} U'(x) = \infty, \quad \lim_{x \to \infty} U'(x) = 0$$

The utility functions that we will discuss in the following steps have the form:

$$\begin{cases} \text{power} : \frac{x^p}{p} \\ \text{non-Hara} : \frac{1}{3}H(x)^{-3} + H(x)^{-1} + xH(x), \quad H(x) = \sqrt{2}(-1 + \sqrt{1+4x})^{-\frac{1}{2}} \end{cases}$$

Define the value function as:

$$V = \sup_{\pi \in \mathcal{A}} E[U(X^{\pi}(T))] \tag{1.1.4}$$

$$V(t, x) = \sup_{\pi \in \mathcal{A}} E[U(X^{\pi}(T))|X^{\pi}(t) = x] \tag{1.1.5}$$

where $\mathcal{A} = \{\pi \in \mathcal{H}^2(0, T; \mathbb{R}^n) : \pi(t) \in K \text{ for } t \in [0, T] \text{ a.e.}\}$

To avoid trivialities, we assume that $-\infty < V < \infty$

1.2 Dual Problem and HJB equation

From (1.1.5), we can get the HJB function for this value function is:

$$\partial_t V + \sup_{\pi \in \mathcal{A}} [D_x V x(r + \pi^T \sigma \theta) + \frac{1}{2} tr((\pi^T \sigma)(\pi^T \sigma)^T x^2 D_x^2 V)] = 0 \tag{1.2.1}$$

The dual function of U is defined as:

$$\widetilde{U} = \sup_{x>0}(U(x) - xy) \tag{1.2.2}$$

It is clear that $\widetilde{U} = \infty$ if $y < 0$ and $\widetilde{U}$ is twice continuously differentiable, strictly decreasing and strcitly convex on $(0, \infty)$.

The dual process Y is a strictly positive process and has following semimartingale decomposition:

$$\begin{aligned} dY(t) &= Y(t)\{\alpha dt + \beta^T dW(t)\} \\ Y(0) &= y \end{aligned}$$

We need to find α and β such that $X^{\pi}Y$ is a supermartingale for all admissible control processes $\pi \in \mathcal{A}$.

By applying Ito's lemma, we have:

$$d(X^{\pi}(t)Y(t)) = X^{\pi}(t)Y(t)\{[r + \pi^T(t)\sigma(t)\theta(t) + \alpha(t) + \pi^T\sigma(t)\beta(t)]dt + [\pi^T(t)\sigma(t) + \beta^T(t)]dW(t)\}$$

To make $X^T Y$ a supermartingale, we must have

$$r + \pi^T(t)\sigma(t)\theta(t) + \alpha(t) + \pi^T\sigma(t)\beta(t) \leq 0$$

for all $\pi \in K$ a.s. for a.e. $t \in [0, T]$, which is equivalent to

$$r + \alpha(t) + \delta_K(-\sigma(t)(\theta(t) + \beta(t))) \leq 0$$

where $\delta_K(z) = \sup_{\pi \in K}\{-\pi^T z\}$ is a support function of the set -K.

Define $v(t) = -\sigma(t)(\theta(t) + \beta(t))$. We have

$$\alpha(t) \leq -(r + \delta_K((v(t)))), \quad \beta(t) = -(\sigma^{-1}(t)v(t) + \theta(t))$$

From the definition of the dual function, we have

$$E[U(X^{\pi}(T))] \leq E[\widetilde{U}(Y(T))] + E[X^{\pi}(T)Y(T)] \leq E[\widetilde{U}(Y(T))] + xy$$

The second inequality above is due to $X^{\pi}Y$ being a supermartingale. This leads to

$$\sup_{\pi} E[U(X^{\pi}(T))] \leq \inf_{y,\alpha,v}(E[\widetilde{U}(Y(T))] + xy)$$

For any fix y, v, we can get the solution of the SDE of Y is bounded above by the process $Y^{(y,v)}$ satisfying the SDE:

$$dY^{(y,v)}(t) = -Y^{(y,v)}(t)\{[r + \delta_K(v(t))]dt + [\theta + \sigma^{-1}v(t)]^T dW(t)\}, \quad 0 \leq t \leq T$$

$$Y^{(y,v)}(0) = y \tag{1.2.3}$$

that is, $Y(t) \leq Y^{(y,v)}(t)$ a.s. for $0 \leq t \leq T$.

Since $\widetilde{U}$ is a strictly decreasing function, we have $E[\widetilde{U}(Y(T))] \geq E[\widetilde{U}(Y^{(y,v)}(T))]$ for any fixed y, v, which implies the optimal α is determined by $\alpha(t) = -(r + \delta_K((v(t))))$. The process $Y^{(y,v)}$ is a dual process and $v \in \mathcal{D}$ is a dual control process, where the set $\mathcal{D}$ is defined by

$$\mathcal{D} = \{v = \Omega \times [0, T] \to \mathbb{R} | v \in \mathcal{F}^* \ and \ \int_0^T [\delta_K(v(t)) + |v(t)|^2]dt < \infty \quad a.s.\}$$

According to the assumption $K = \mathbb{R}$, so $\delta_K(z) = +\infty$ for arbitrary z except for $z = 0$.
To make $r + \alpha(t) + \delta_K(-\sigma(t)(\theta(t) + \beta(t))) \leq 0$, we need $v(t) = -\sigma(t)(\theta(t) + \beta(t)) = 0$ for any π.
Also, since all coefficients are constant, we have $\alpha = -r$ and $\beta = -\theta$.

Therefore, the dual process $Y^{(y,v)}(t)$ satisfies the SDE:

$$dY(t) = -Y(t)\{rdt + \theta dW(t)\}, \quad 0 \leq t \leq T$$

$$Y(0) = y$$

The optimal value of the dual minimization problem is defined by:

$$V = \inf_{y \in (0,\infty)} (xy + E[\widetilde{U}(Y(T))]) \tag{1.2.4}$$

Since r, b, σ are deterministic, the wealth process X^{π} is a Markov controlled process and the stochastic optimal control theory may be used to solve the first stage problem.

Define $V(t,x) = \inf_{y \in (0,\infty)}(xy + \widetilde{V}(t,y))$, we have $\widetilde{V}$ is $C([0,T] \times R_+)$ and $C^{1,\infty}([0,T] \times R_+)$, $y \to \widetilde{V}(t,y)$ is strictly convex, and C^{∞} for $t \in [0,T)$, but $\widetilde{V}(T,y)$ is only convex and continuous. For $0 \leq t \leq T$, since $\widetilde{V}(t,.)$ is C^{∞}, minimum point is obtained by solving

$$\frac{\partial \widetilde{V}(t,y)}{\partial y} + x = 0 \tag{1.2.5}$$

Since $\widetilde{V}(t,.)$ is strictly convex, so $\widetilde{V}_y(t,.)$ is strictly increasing.
For every $x > 0$, there exists unique y solving (1.2.4), write it $y = y(t,x)$.
We have

$$V(t,x) = \widetilde{V}(t, y(t,x)) + xy(t,x) \tag{1.2.6}$$

By (1.2.5),

$$\begin{aligned} V_t &= \widetilde{V}_t + \widetilde{V}_y \frac{\partial y}{\partial t} + x\frac{\partial y}{\partial t} \\ &= \widetilde{V}_t + (\widetilde{V}_y + x)\frac{\partial y}{\partial t} \\ &= \widetilde{V}_t \end{aligned} \tag{1.2.7}$$

$$V_x = \widetilde{V}_y \frac{\partial y}{\partial x} + y + x\frac{\partial y}{\partial x} = y \tag{1.2.8}$$

$$V_{xx} = \frac{\partial y}{\partial x} \tag{1.2.9}$$

By (1.2.4),

$$\frac{\partial(\widetilde{V}_y + x)}{\partial x} = \widetilde{V}_{yy}\frac{\partial y}{\partial x} + 1 = 0 \tag{1.2.10}$$

We can get

$$\frac{\partial y}{\partial x} = -\frac{1}{\widetilde{V}_{yy}}$$

$$V_{xx} = -\frac{1}{\widetilde{V}_{yy}} \tag{1.2.11}$$

1.3 Necessary and sufficient conditions for primal problems

We now state the necessary and sufficient optimality for the primal problem.[10]

Theorem 3.5 (Primal problem and associated FBSDE): Let $\hat{\pi} \in \mathcal{A}$. Then $\hat{\pi}$ is optimal for the primal problem if and only if the solution $(X^{\hat{\pi}}, \hat{p_1}, \hat{q_1})$ of FBSDE

$$\begin{aligned}
&dX^{\hat{\pi}(t)}(t) = X^{\hat{\pi}}(t)[(r(t) + \hat{\pi}^T(t)\sigma(t)\theta(t))dt + \hat{\pi}^T(t)\sigma(t)dW(t)]\\
&X^{\hat{\pi}}(0) = x_0\\
&d\hat{p_1}(t) = -[(r(t) + \hat{\pi}^T(t)\sigma(t)\theta(t))\hat{p_1}(t) + \hat{q_1}^T(t)\sigma^T(t)\hat{\pi}(t)]dt + \hat{q_1}^T(t)dW(t)\\
&\hat{p_1}(T) = -U^{'}(X^{\hat{\pi}}(T))
\end{aligned} \tag{1.3.1}$$

satisfies the condition

$$-X^{\hat{\pi}}(t)\sigma(t)[\theta(t)\hat{p_1}(t) + \hat{q_1}(t)] \in N_K(\hat{\pi}(t)), \quad \forall t \in [0,T], \mathbb{P} - a.s. \tag{1.3.2}$$

where $N_K(x)$ is the normal cone of the closed convex set K at $x \in K$, defined as

$$N_K(x) = \{y \in \mathbb{R}^N : \forall x^* \in K, y(x^* - x) \leq 0\}$$

1.4 Necessary and sufficient conditions for dual problems

Next we address the dual problem. To ensure the existence of an optimal solution, we impose the following condition:

Assumption 3.6: for any $(y, v) \in (0, \infty) \times \mathcal{D}$, we have

$$E[\tilde{U}(Y^{(y,v)}(T))^2] < \infty$$

Given an admissible dual control $(\hat{y}, \hat{v}) \in (0, \infty) \times \mathcal{D}$ with the dual process $Y^{(\hat{y},\hat{v})}$ that solve the SDE (1.7), the associated adjoint equation for dual problem is the following linear BSDE in the unkonwn processes $\hat{p_2} \in \mathcal{H}^2(0,T;\mathbb{R})$ and $\hat{q_2} \in \mathcal{H}^2(0,T;\mathbb{R}^N)$:

$$\begin{aligned}
&d\hat{p_2}(t) = ([r(t) + \delta_K(\hat{v}(t))]\hat{p_2}(t) + \hat{q_2}^T(t)[\theta(t) + \sigma^{-1}(t)\hat{v}(t)])dt + \hat{q_2}^T(t)dW(t)\\
&\hat{p_2}(T) = -\tilde{U}^{'}(Y^{(\hat{y},\hat{v})}(T))
\end{aligned} \tag{1.4.1}$$

Since $\hat{p_2}Y^{(\hat{y},\hat{v})}$ is a martingale, we can find $\hat{p_2}(t), 0 \leq t \leq T$ from the relation

$$\hat{p_2}(t)Y^{(\hat{y},\hat{v})}(t) = E[\hat{p_2}(T)Y^{(\hat{y},\hat{v})}(T)|\mathcal{F}_t] = -E[\tilde{U}^{'}(Y^{(\hat{y},\hat{v})}(T))Y^{(\hat{y},\hat{v})}(T)|\mathcal{F}_t] \tag{1.4.2}$$

We now state the necessary and sufficient optimality conditions for the dual problem.

Theorem 3.9 (Dual problem and associated FBSDE): Let $(\hat{y}, \hat{v}) \in (0, \infty) \times \mathcal{D}$. Then $(\hat{y}, \hat{v})$ is optimal for the dual problem if and only if the solution $(Y^{(\hat{y},\hat{v})}, \hat{p_2}, \hat{q_2})$ of FBSDE

$$\begin{aligned}
&dY^{(\hat{y},\hat{v})}(t) = -Y^{(\hat{y},\hat{v})}(t)\{[r(t) + \delta_K(\hat{v}(t))]dt + [\theta(t) + \sigma^{-1}(t)\hat{v}(t)]^T dW(t)\} \\
&Y^{(\hat{y},\hat{v})}(0) = \hat{y} \\
&d\hat{p_2}(t) = \{[r(t) + \delta_K(\hat{v}(t))]^T \hat{p_2}(t) + \hat{q_2}^T(t)[\theta(t) + \sigma^{-1}(t)\hat{v}(t)]\}dt + \hat{q_2}^T(t)dW(t) \\
&\hat{p_2}(T) = -\tilde{U}'(Y^{(\hat{y},\hat{v})}(T))
\end{aligned} \tag{1.4.3}$$

satisfies the condition

$$\begin{aligned}
&\hat{p_2}(0) = x_0 \\
&\hat{p_2}(t)^{-1}[\sigma^T(t)]^{-1}\hat{q_2}(t) \in K \\
&\hat{p_2}(t)\delta_K(\hat{v}(t)) + \hat{q_2}^T(t)\sigma^{-1}(t)\hat{v}(t) = 0, \forall t \in [0, T]\mathbb{P} - a.s.
\end{aligned} \tag{1.4.4}$$

1.5 Dynamic relations of primal and dual problems

We can now state the dynamic relations of the optimal portfolio and wealth processes of the primal problem and the adjoint processes of the dual problem and vice versa.[10]

Theorem 3.10 (From dual problem to primal problem): Suppose that $(\hat{y}, \hat{v}) \in (0, \infty) \times \mathcal{D}$ is optimal for the dual problem. Let $(Y^{(\hat{y},\hat{v})}, \hat{p_2}, \hat{q_2})$ be the associated process that solve the FBSDE (1.4.3) and satisfies condition (1.4.4). Define

$$\hat{\pi}(t) = \frac{[\sigma^T(t)]^{-1}\hat{q_2}(t)}{\hat{p_2}(t)}, \quad t \in [0, T] \tag{1.5.1}$$

Then $\hat{\pi}$ is the optimal control for the primal problem with initial wealth x_0. The optimal wealth process and associated adjoint process are given by

$$\begin{aligned}
&X^{\hat{\pi}}(t) = \hat{p_2}(t) \\
&\hat{p_1}(t) = -Y^{(\hat{y},\hat{v})}(t) \\
&\hat{q_1}(t) = Y^{(\hat{y},\hat{v})}(t)[\sigma^{-1}(t)\hat{v}(t) + \theta(t)]
\end{aligned} \tag{1.5.2}$$

Theorem 3.11 (From primal problem to dual problem): Suppose that $\pi \in \mathcal{A}$ is optimal for the primal problem with initial wealth x_0. Let $(X^{\hat{\pi}}, \hat{p_1}, \hat{q_1})$ be the associated process that satisfies the FBSDE (1.3.1) and conditions (1.3.2). Define

$$\begin{aligned}
&\hat{y} = -\hat{p_1}(0) \\
&\hat{v}(t) = -\sigma(t)[\frac{\hat{q_1}(t)}{\hat{p_1}(t)} + \theta(t)], \quad t \in [0, T]
\end{aligned} \tag{1.5.3}$$

Then $(\hat{y}, \hat{v})$ is an optimal control for the dual problem. The optimal dual process and associated adjoint process are given by

$$\begin{aligned}
&Y^{\hat{y},\hat{v}}(t) = -\hat{p_1}(t), \\
&\hat{p_2} = X^{\hat{\pi}}(t), \\
&\hat{q_2} = \sigma^T(t)\hat{\pi}(t)X^{\hat{\pi}}(t).
\end{aligned} \tag{1.5.4}$$

1.6 Power Utility Function

- Solving from dual

In this subsection, we assume U is a power utility function defined by $U(x) = (1/\beta)x^\beta,\ x \in (0, \infty)$, where $\beta \in (0, 1)$ is a constant. In this case, the dual problem can be written as:

$$\begin{aligned}
\widetilde{V} &= \inf_{y \in (0,\infty)} (xy + E[\widetilde{U}(Y(T))|Y(t) = y]) \\
&= \inf_{y \in (0,\infty)} (xy + \hat{V})
\end{aligned}$$

where $\hat{V} = E[\widetilde{U}(Y(T))|Y(t) = y]$
Since $\widetilde{U}(y) = \sup_{x>0}(U(x) - xy)$, taking derivative w.r.t x, we have:

$$x^{\beta-1} = y$$

So $\widetilde{U}(y) = \frac{1-\beta}{\beta} y^{\frac{\beta}{\beta-1}}, \quad y \in (0, \infty)$

Recall that Y satisfies the SDE $dY(t) = -Y(t)\{rdt + \theta dW(t)\}$
Then for the dual HJB equation has the form:

$$\frac{\partial}{\partial t} v(t, y) - ryv_y(t, y) + \frac{1}{2}\theta^2 y^2 v_{yy}(t, y) = 0$$

$$v(T, y) = \widetilde{U}(y)$$

It is trivial to solve Y, and we have $Y(T) = y \exp(-(r + \frac{\theta^2}{2})(T - t) - \theta W_{T-t})$.
And thus, $\hat{V}(t, y) = \widetilde{U}(y) e^{-\frac{\beta}{\beta-1}((r+\frac{\theta^2}{2})(T-t))} e^{\frac{\beta}{\beta-1}^2 \theta^2 (T-t)/2}$.
To solve

$$\inf_{y \in (0,\infty)} (xy + \hat{V})$$

We take the derivative w.r.t y and we have

$$y(t, x) = x^{\beta-1} \exp(\beta(r + \frac{\theta^2}{2})(T - t)) \exp(-\frac{\beta^2}{\beta - 1}\theta^2(T - t)/2)$$

Since the process starts from time 0, by setting $t = 0$, we have:

$$\hat{y} = x^{\beta-1} \exp(\beta(r + \frac{\theta^2}{2})T) \exp(-\frac{\beta^2}{\beta - 1}\theta^2 T/2)$$

Since

$$\begin{aligned}
V(t, X) &= \hat{V}(t, y(t, x)) + xy(t, x) \\
&= \frac{1 - \beta}{\beta} y^{\frac{\beta}{\beta-1}} \exp(-\frac{\beta}{\beta - 1}(r + \frac{\theta^2}{2})(T - t)) \exp((\frac{\beta}{\beta - 1})^2 \theta^2 (T - t)/2) \\
&+ x^\beta \exp(\beta(r + \frac{\theta^2}{2})(T - t)) \exp(-\frac{\beta^2}{\beta - 1}\theta^2(T - t)/2) \\
&= \frac{1}{\beta} x^\beta \exp(\beta(r + \frac{\theta^2}{2})(T - t)) \exp(-\frac{\beta^2}{\beta - 1}\theta^2(T - t)/2)
\end{aligned}$$

Recall that $\pi^* = -\frac{\partial_x V \theta}{x\sigma \partial_{xx} V}$, and $\partial_x V = x^{\beta-1} \exp(\beta(r + \frac{\theta^2}{2})(T - t)) \exp(-\frac{\beta^2}{\beta-1}\theta^2(T - t)/2)$, $\partial_{xx} V = (\beta - 1)x^{\beta-2} \exp(\beta(r + \frac{\theta^2}{2})(T - t)) \exp(-\frac{\beta^2}{\beta-1}\theta^2(T - t)/2)$. As a result, we have:

$$\pi^* = \frac{\theta}{(1 - \beta)\sigma}$$

Given that X follows a geometric Brownian motion, we have:

$$\begin{aligned}
X^{\pi^*}(t) &= x \exp((r + \pi\sigma\theta - \frac{\pi^2\sigma^2}{2})t + \pi\sigma W_t) \\
&= x \exp((r + \frac{(1 - 2\beta)\theta^2}{2(1 - \beta)^2})t + \frac{\theta}{1 - \beta} W_t)
\end{aligned}$$

- Solving from primal and dual FBSDE

Given an admissible dual control $(\hat{y}, \hat{v}) \in (0, \infty) \times \mathcal{D}$ with the dual process $Y^{(\hat{y},\hat{v})}$ that solves the SDE of Y and condition $E[\widetilde{U}(Y^{(\hat{y},\hat{v})}(T))^2] < \infty$ holds with $(y, v) = (\hat{y}, \hat{v})$, the associated adjoint equation for the dial problem is the following linear BSDE in the unknown processes

$\hat{p_2} \in \mathcal{H}^2(0,T;\mathbb{R})$ and $\hat{q_2} \in \mathcal{H}(0,T;\mathbb{R}^N)$:

$$d\hat{p_2}(t) = \{r\hat{p_2}(t) + \hat{q_2}(t)\theta\}dt + \hat{q_2}(t)dW(t)$$
$$\hat{p_2}(T) = -\widetilde{U}'(Y^{(\hat{y},\hat{v})}(T))$$

Since $\hat{p_2}Y^{(\hat{y},\hat{v})}(t)$ is a martingale, we can find $\hat{p_2}$, $0 \le t \le T$, from the relation

$$\hat{p_2}(t)Y^{(\hat{y},\hat{v})}(t) = E[\hat{p_2}(T)Y^{(\hat{y},\hat{v})}(T)|\mathcal{F}_t] = -E[\widetilde{U}'(Y^{(\hat{y},\hat{v})}(T))Y^{(\hat{y},\hat{v})}(T)|\mathcal{F}_t]$$

Recall that $\widetilde{U}(y) = \frac{1-\beta}{\beta}y^{\frac{\beta}{\beta-1}}$, and we can get the derivative of $\widetilde{U} = -y^{\frac{1}{\beta-1}}$. Since in the last section, we solved that $Y^{\hat{y}}(T) = \hat{y}\exp(-(r+\frac{\theta^2}{2})T - \theta W_T)$
By applying Theorem 3.9, we see that

$$\begin{aligned}\hat{p_2}(T) &= -\tilde{U}'(Y^{(\hat{y})}(T)) \\ &= \hat{y}^{\frac{1}{\beta-1}}\exp(-\frac{1}{\beta-1}(r+\frac{\theta^2}{2})T - \frac{1}{\beta-1}\theta W_T)\end{aligned}$$

Since $\hat{p_2}Y^{(\hat{y})}$ is a martingale, then we have:

$$\begin{aligned}\hat{p_2}(t)Y^{(\hat{y},\hat{v})}(t) &= \hat{p_2}(t)Y^{\hat{y}}(t) \\ &= -E[\widetilde{U}'(Y^{\hat{y}}(T))Y^{\hat{y}}(T)|\mathcal{F}_t] \\ &= -E[-Y^{\hat{y}}(T)^{\frac{1}{\beta-1}}Y^{\hat{y}}(T)|\mathcal{F}_t] \\ &= E[Y^{\hat{y}}(T)^{\frac{\beta}{\beta-1}}|\mathcal{F}_t] \\ &= \hat{y}^{\frac{\beta}{\beta-1}}\exp(-\frac{\beta}{\beta-1}(r+\frac{\theta^2}{2})T - \frac{\beta}{\beta-1}\theta W_t)\exp(\frac{1}{2}\theta^2(\frac{\beta}{\beta-1})^2(T-t))\end{aligned}$$

Therefore, for $\hat{p_2}(t)$, we have:

$$\hat{p_2}(t) = \hat{y}^{\frac{1}{\beta-1}}\exp(-\frac{\beta}{\beta-1}rT + \frac{1}{2}\theta^2\frac{\beta}{(\beta-1)^2}T)\exp(-\frac{1}{\beta-1}\theta W_t)\exp(rt + \frac{1}{2}\theta^2(\frac{1-2\beta}{(\beta-1)^2})t)$$

According to the Theorem 3.9, to get to optimal value, we need to satisfies the following conditions

$$\begin{aligned}&\hat{p_2}(0) = x \\ &\hat{p_2}(t)^{-1}[\sigma^T(t)]^{-1}\hat{q_2}(t) \in K \\ &\hat{p_2}(t)\delta_K(\hat{v}(t)) + \hat{q_2}^T(t)\sigma^{-1}(t)\hat{v}(t) = 0, \forall t \in [0,T] \quad \mathbb{P}-a.s.\end{aligned} \tag{1.6.1}$$

where the second and third conditions are satisfied automatically given that $K = \mathbb{R}$ and $\hat{v}(t) = 0$.
For the first condition, we have:

$$\hat{p_2}(0) = \hat{y}^{\frac{1}{\beta-1}}\exp(-\frac{\beta}{\beta-1}rT + \frac{1}{2}\theta^2\frac{\beta}{(\beta-1)^2}T) = x$$

And thus

$$\hat{y} = x^{\beta-1}\exp(\beta rT - \frac{1}{2}\theta^2\frac{\beta}{\beta-1}T)$$

$$\hat{p_2}(t) = x\exp rt + \frac{1}{2}\theta^2\frac{1-2\beta}{(\beta-1)^2}t - \frac{1}{\beta-1}W_t$$

By applying Ito's formula on $\hat{p_2}(t)$, we have:

$$d\hat{p_2}(t) = \hat{p_2}(t)(r + \frac{1}{2}\frac{1-2\beta}{(\beta-1)^2}\theta^2)dt + \hat{p_2}(t)\frac{1}{1-\beta}\theta dW_t + \frac{1}{2}\hat{p_2}(t)^2(\frac{1}{\beta-1}\theta)^2dt$$

According to the adjoint BSDE of dual problem, we observe that $\hat{q_2}(t) = \hat{p_2}(t)\frac{1}{1-\beta}\theta$
By applying the theorem 3.10, we get:

$$\hat{\pi} = \frac{\hat{q_2}(t)}{\sigma\hat{p_2}(t)} = \frac{\theta}{(1-\beta)\sigma}$$

$$\begin{aligned} X^{\hat{\pi}}(t) &= x\exp((r+\pi\sigma\theta-\frac{\pi^2\sigma^2}{2})t+\pi\sigma W_t) \\ &= x\exp((r+\frac{(1-2\beta)\theta^2}{2(1-\beta)^2})t+\frac{\theta}{1-\beta}W_t) \end{aligned}$$

We can also use Theorem 3.11 to verify if the optimal weight and wealth process are correct. Define $y=-\hat{p_1}(0)$, and $\hat{v}(t)=-\sigma(t)[\frac{\hat{q_1}(t)}{\hat{p_1}(t)}+\theta(t)]=0$ given that $\frac{\hat{q_1}(t)}{\hat{p_1}(t)}+\theta(t)=0$ by Theorem 3.5.

Since $dX^{\pi}(t)=X^{\pi}(t)\{[r+\pi(t)\sigma\theta]dt+\pi(t)\sigma dW_t\}$, we have:

$$dX^{\hat{\pi}}(t)=X^{\hat{\pi}}(t)\{[r+\frac{\theta^2}{1-\beta}]dt+\frac{\theta}{1-\beta}dW_t\}$$

$$\mathcal{H}(t,x,a,p_1,q_1)=x(r+\frac{\theta^2}{1-\beta})p_1+x\frac{\theta}{1-\beta}q_1$$

Then we can get the primal adjoint equation (BSDE):

$$\begin{aligned} -d\hat{p_1}(t) &= ((r+\frac{\theta^2}{1-\beta})\hat{p_1}(t)+\frac{\theta}{1-\beta}\hat{q_1}(t))dt-\hat{q_1}(t)dW_t \\ &= r\hat{p_1}(t)dt+\theta\hat{p_1}(t)dW_t \end{aligned}$$

Therefore,

$$\hat{p_1}(t)=\hat{p_1}(0)\exp(-(r+\frac{\theta^2}{2})t-\theta W_t)$$

Recall that $Y^{\hat{y}}=\hat{y}\exp(-(r+\frac{\theta^2}{2})t-\theta W_t)$, so we have:

$$Y^{\hat{y}}(t)=-\hat{p_1}(t)$$

By observing the equations of $X^{\pi}(t),\hat{p_2}(t),\hat{q_2}(t)$, we can easily know that the second and third conditions are satisfied.

1.7 Non-Hara Utility Function

- Solving from dual

The Non-Hara utility function has the form:

$$U(x)=\frac{1}{3}H(x)^{-3}+H(x)^{-1}+xH(x)$$

for $x>0$, where $H(x)=\sqrt{2}(-1+\sqrt{1+4x})^{-\frac{1}{2}}$

Then for the dual function, we have $\widetilde{U}(y)=\sup_{x>0}(U(x)-xy)$. Taking derivative with respect to x and let the equation equals to zero, we have $\widetilde{U}(y)=\frac{1}{3}y^{-3}+y^{-1}$.
The same as the power utility function we have solved before, the dual problem can be written as:

$$\begin{aligned} \widetilde{V} &= \inf_{y\in(0,\infty)}(xy+E[\widetilde{U}(Y(T))]) \\ &= \inf_{y\in(0,\infty)}(xy+\hat{V}) \end{aligned}$$

where $\hat{V}=E[\widetilde{U}(Y(T)]$.

Recall that Y follows the same geometric brownian motion as it is in the power utility function example, as thus we have $Y(T)=Y(t)\exp(-(r+\frac{\theta^2}{2})(T-t)-\theta W_{T-t})$.
Since $Y(0)=y$, we have $Y(t)=y\exp(-(r+\frac{\theta^2}{2})(t)-\theta W_t)$.
Therefore, for $\hat{V}$, we have:

$$\begin{aligned} \hat{V}(t,y) &= E[\frac{1}{3}y^{-3}\exp(3(r+\frac{\theta^2}{2})(T-t)+3\theta W_{T-t})+y^{-1}\exp((r+\frac{\theta^2}{2})(T-t)+\theta W_{T-t})] \\ &= \frac{1}{3}y^{-3}\exp(3(r+\frac{\theta^2}{2})(T-t))\exp(\frac{9\theta^2(T-t)}{2})+y^{-1}\exp((r+\frac{\theta^2}{2})(T-t))\exp(\frac{\theta^2(T-t)}{2}) \end{aligned}$$

So, for $\widetilde{V}$, we have:

$$\widetilde{V}(t,y) = xy + \frac{1}{3}y^{-3}\exp(3(r+\frac{\theta^2}{2})(T-t))\exp(\frac{9\theta^2(T-t)}{2}) + y^{-1}\exp((r+\frac{\theta^2}{2})(T-t))\exp(\frac{\theta^2(T-t)}{2})$$

Taking derivative with respect to y, we have:

$$x - y^{-4}\exp(3(r+\frac{\theta^2}{2})(T-t))\exp(\frac{9\theta^2(T-t)}{2}) - y^{-2}\exp((r+\frac{\theta^2}{2})(T-t))\exp(\frac{\theta^2(T-t)}{2}) = 0$$

$$xy^4 - \exp(3(r+\frac{\theta^2}{2})(T-t))\exp(\frac{9\theta^2(T-t)}{2}) - y^2\exp((r+\frac{\theta^2}{2})(T-t))\exp(\frac{\theta^2(T-t)}{2}) = 0$$

$$y^2 = \frac{\exp((r+\theta^2)(T-t)) + \sqrt{\exp(2(r+\theta^2)(T-t)) + 4x\exp(3(r+2\theta^2)(T-t))}}{2x}$$

$$y = \frac{1}{\sqrt{2x}}[\exp((r+\theta^2)(T-t)) + \sqrt{\exp(2(r+\theta^2)(T-t)) + 4x\exp(3(r+2\theta^2)(T-t))}]^{\frac{1}{2}}$$

Since the process starts from time 0, by setting $t = 0$, we have:

$$\hat{y} = \frac{1}{\sqrt{2x}}[\exp((r+\theta^2)T) + \sqrt{\exp(2(r+\theta^2)T) + 4x\exp(3(r+2\theta^2)T)}]^{\frac{1}{2}}$$

Recall that we can write the equation for optimal π as:

$$\begin{aligned}\pi^*(t) &= -\frac{\theta\partial\widetilde{V}_x}{x\sigma\partial\widetilde{V}_{xx}} \\ &= \frac{y\theta}{x\sigma}\hat{V}_{yy} \\ &= \frac{\theta}{\sigma}\frac{4y^{-4}\exp(3(r+2\theta^2)(T-t)) + 2y^{-2}\exp((r+\theta^2)(T-t))}{x}\end{aligned}$$

Also recall the equation (1.2.5), we have:

$$\begin{aligned}X^{\pi^*(t)} &= -\frac{\partial\hat{V}(t,\hat{Y}(t))}{\partial\hat{Y}(t)} \\ &= \hat{Y}(t)^{-4}\exp(3(r+2\theta^2)(T-t)) + \hat{Y}(t)^{-2}\exp((r+\theta^2)(T-t)) \\ &= \hat{y}^{-4}\exp(4(r+\frac{\theta^2}{2})t + 4\theta W_t)\exp(3(r+2\theta^2)(T-t)) \\ &+ \hat{y}^{-2}\exp(2(r+\frac{\theta^2}{2})t + 2\theta W_t)\exp((r+\theta^2)(T-t)) \\ &= \hat{y}^{-4}\exp(3(r+2\theta^2)T) + 4\theta W_t)\exp(r-4\theta^2)t + \hat{y}^{-2}\exp(r+\theta^2)T + 2\theta W_t)\exp rt\end{aligned}$$

- Solving from primal and dual FBSDE

Recall that the optimal state process for the dual problem is given by $Y^{\hat{y}}(T) = \hat{y}\exp -(r+\frac{\theta^2}{2})T - \theta W_T$
Using the martingale property of $\hat{p_2}\hat{Y}$, we have

$$\begin{aligned}\hat{p_2}(t)\hat{Y}(t) &= -E[\widetilde{U}'(Y(T))Y(T)|\mathcal{F}_t] \\ &= E[\hat{Y}(T)^{-3} + \hat{Y}(T)^{-1}|\mathcal{F}_t] \\ &= \hat{y}^{-3}\exp(3(r+\frac{\theta^2}{2})T) + 3\theta W_t)\exp\frac{9\theta^2}{2}(T-t) \\ &+ \hat{y}^{-1}\exp((r+\frac{\theta^2}{2})T) + \theta W_t)\exp\frac{\theta^2}{2}(T-t)\end{aligned}$$

And therefore, we have

$$\hat{p_2}(t) = \hat{y}^{-4}\exp(3(r+2\theta^2)T) + 4\theta W_t)\exp(r-4\theta^2)t + \hat{y}^{-2}\exp(r+\theta^2)T + 2\theta W_t)\exp rt$$

To satisfy the condition of Theorem 3.9, we have

$$\hat{p_2}(0) = x = \hat{y}^{-4}\exp 3(r+2\theta^2)T) + \hat{y}^{-2}\exp r + \theta^2)T$$

So

$$\hat{y} = \frac{1}{\sqrt{2x}}[\exp((r+\theta^2)T) + \sqrt{\exp(2(r+\theta^2)T) + 4x\exp(3(r+2\theta^2)T)}]^{\frac{1}{2}}$$

By applying Ito's formula on $\hat{p_2}(t)$, and take out the term of brownian motion, we have:

$$\hat{q_2}(t) = 4\theta\hat{y}^{-4}\exp\left(3(r+2\theta^2)T\right) + 4\theta W_t)\exp\left(r-4\theta^2\right)t + 2\theta\hat{y}^{-2}\exp\left(r+\theta^2\right)T + 2\theta W_t)\exp rt$$

By applying theorem 3.10, we have:

$$\pi^*(t) = \frac{\hat{q_2}(t)}{\sigma\hat{p_2}(t)}$$

$$X^{\pi^*(t)} = \hat{p_2}(t) = \hat{y}^{-4}\exp\left(3(r+2\theta^2)T\right) + 4\theta W_t)\exp\left(r-4\theta^2\right)t + \hat{y}^{-2}\exp\left(r+\theta^2\right)T + 2\theta W_t)\exp rt$$

We can also use Theorem 3.11 to verify if the optimal weight and wealth process are correct. Define $y = -\hat{p_1}(0)$, and $\hat{v}(t) = -\sigma(t)[\frac{\hat{q_1}(t)}{\hat{p_1}(t)} + \theta(t)] = 0$ given that $\frac{\hat{q_1}(t)}{\hat{p_1}(t)} + \theta(t) = 0$ by Theorem 3.5.

Since $dX^{\pi}(t) = X^{\pi}(t)\{[r + \pi(t)\sigma\theta]dt + \pi(t)\sigma dW_t\}$, we have

$$dX^{\hat{\pi}}(t) = X^{\hat{\pi}}(t)\{[r + \theta\frac{\hat{q_2}(t)}{\hat{p_2}(t)}]dt + \frac{\hat{q_2}(t)}{\hat{p_2}(t)}dW_t\}$$

$$\mathcal{H}(t, x, a, p_1, q_1) = x(r + \theta\frac{\hat{q_2}(t)}{\hat{p_2}(t)})p_1) + x\frac{\hat{q_2}(t)}{\hat{p_2}(t)}q_1$$

Then we can get the primal adjoint equation (BSDE):

$$\begin{aligned} -d\hat{p_1}(t) &= ((r + \theta\frac{\hat{q_2}(t)}{\hat{p_2}(t)})\hat{p_1}(t) + \frac{\hat{q_2}(t)}{\hat{p_2}(t)}\hat{q_1}(t))dt - \hat{q_1}(t)dW_t \\ &= r\hat{p_1}(t)dt + \theta\hat{p_1}(t)dW_t \end{aligned}$$

Therefore,

$$\hat{p_1}(t) = \hat{p_1}(0)\exp(-(r + \frac{\theta^2}{2})t - \theta W_t)$$

Recall that $Y^{\hat{y}} = \hat{y}\exp(-(r + \frac{\theta^2}{2})t - \theta W_t)$, so we have

$$Y^{\hat{y}}(t) = -\hat{p_1}(t)$$

By observing the equations of $X^{\pi}(t), \hat{p_2}(t), \hat{q_2}(t)$, we can easily know that the second and third conditions are satisfied.

Chapter 2

Unconstrained Utility Maximization with OU Process

Assumption:
- 1-dimensional geometric Brownian motion asset price process
- All coefficients constant
- Control set K=$\mathbb{R}$
- Maximize utility of wealth at time T
- $dS = HSdt + \sigma dW_t$
- $dH_t = k(c - H_t)dt + \sigma_1 dW_t$

2.1 Market Model

In the second stage, the drift term of the stock becomes an OU process, and thus the wealth process is written as:

$$\begin{aligned} dX_t &= (1-\pi_t)X_t r dt + \frac{\pi_t X_t}{S_t} dt \\ &= (1-\pi_t)X_t r dt + \pi_t X_t (H_t dt + \sigma dW_t) \\ &= X_t(r + \pi_t(H(t) - r))dt + \pi_t X_t \sigma dW_t \end{aligned} \tag{2.1.1}$$

Notice that $H(t)$ follows a Vasicek Model, and we have:

$$\begin{aligned} d[e^{kt}H_t] &= ke^{kt}H_t dt + e^{kt} dH_t \\ &= ke^{kt}H_t dt + e^{kt}[k(c - H_t)dt + \sigma_1 dW_t] \\ &= ke^{kt} c dt + e^{kt}\sigma_1 dW_t \end{aligned} \tag{2.1.2}$$

Integrating both sides between s and t, we have:

$$e^{kt}H_t - e^{ks}H_s = \int_s^t e^{ku} kc du + \int_s^t e^{ku}\sigma_1 dW_u$$

$$H(t) = H(s)e^{-k(t-s)} + c(1 - e^{-k(t-s)}) + \sigma_1 \int_s^t e^{-k(t-u)} dW_u$$

$$H(t) = he^{-kt} + c(1 - e^{-kt}) + \sigma_1 \int_0^t e^{-k(t-u)} dW_u$$

2.2 Primal Method

The value function has the form:

$$V(t, x, h) = \sup_{\pi \in \mathcal{A}} E[U(X^\pi(T)) | X^\pi(0) = x,\ H(0) = h] \tag{2.2.1}$$

with terminal condition: $V(T,x,h) = \frac{1}{\beta}x^\beta$
The HJB has the form:

$$\partial_t V + k(c-h)\partial_h V + \frac{1}{2}\sigma_1^2\partial_{hh}V + \sup_{\pi\in\mathcal{A}}[x(r+\pi(h-r))\partial_x V + \frac{1}{2}(\pi\sigma x)^2\partial_{xx}V + \pi x\sigma\sigma_1\partial_{xh}V] = 0 \quad (2.2.2)$$

Assume that $V(t,x,h) = U(x)f(t,h)$, and $f(t,h) = \exp A(t) + B(t)h + C(t)h^2$.

According to the terminal condition, it implies that $A(T) + B(T)h + C(T)h^2 = 0$. Since this equation has to be satisified for every h, we assume that $A(T) = B(T) = C(T) = 0$.
Then for each term in HJB, they can be written as:

$$\begin{aligned}
\partial_t V &= (A'(t) + hB'(t) + h^2C'(t))V \\
\partial_h V &= (B(t) + 2hC(t))V \\
\partial_{hh} V &= 2C(t)V + (B(t) + 2hC(t))^2 V \\
\partial_x V &= x^{\beta-1}\exp A(t) + B(t)h + C(t)h^2 = \frac{\beta}{x}V \\
\partial_{xx} V &= (\beta-1)x^{\beta-2}\exp A(t) + B(t)h + C(t)h^2 = \frac{\beta-1}{x}\partial_x V = \frac{\beta(\beta-1)}{x^2}V \\
\partial_{xh} V &= \frac{\beta}{x}(B(t) + 2hC(t))V
\end{aligned}$$

Since each term has V, we can cancel out V in the HJB.
Therefore, the term $\sup_{\pi\in\mathcal{A}}[x(r+\pi(h-r))\partial_x V + \frac{1}{2}(\pi\sigma x)^2\partial_{xx}V + \pi x\sigma\sigma_1\partial_{xh}V]$ can be written as:

$$\sup_{\pi\in\mathcal{A}}[\beta(r+\pi(h-r)) + \frac{1}{2}\pi^2\sigma^2\beta(\beta-1) + \pi\sigma\sigma_1\beta(B(t)+2hC(t))] \quad (2.2.3)$$

Taking derivative w.r.t π, we have:

$$\pi^*(t) = \frac{-(h-r) - \sigma\sigma_1(B(t)+2hC(t))}{\sigma^2(\beta-1)} \quad (2.2.4)$$

Plugging in π^*, we have:

$$\begin{aligned}
&\sup_{\pi\in\mathcal{A}}[\beta(r+\pi(h-r)) + \frac{1}{2}\pi^2\sigma^2\beta(\beta-1) + \pi\sigma\sigma_1\beta(B(t)+2hC(t))] \quad (2.2.5) \\
&= \beta r - \frac{1}{2}\frac{\beta(h-r)^2}{\sigma^2(\beta-1)} - \frac{\beta(h-r)}{\sigma(\beta-1)}\sigma_1(B(t)+2hC(t)) - \frac{\beta}{2(\beta-1)}\sigma_1^2(B(t)+2hC(t))^2
\end{aligned}$$

For the HJB equation, we have:

$$\begin{aligned}
&A'(t) + hB'(t) + h^2C'(t) + k(c-h)(B(t)+2hC(t)) + \frac{1}{2}\sigma_1^2[2C(t) + (B(t)+2hC(t))^2] \quad (2.2.6) \\
&+ \beta r - \frac{1}{2}\frac{\beta(h-r)^2}{\sigma^2(\beta-1)} - \frac{\beta(h-r)}{\sigma(\beta-1)}\sigma_1(B(t)+2hC(t)) - \frac{\beta}{2(\beta-1)}\sigma_1^2(B(t)+2hC(t))^2 = 0
\end{aligned}$$

To solve $A(t)$,$B(t)$ and $C(t)$, we classify the HJB in terms of the power of h and set each term's coefficient to zero. Thus, we have

$$A'(t) + kcB(t) + \sigma_1^2C(t) + \frac{1}{2}\sigma_1^2B(t)^2 + \beta r - \frac{r^2\beta}{2\sigma(\beta-1)} + \frac{\beta\sigma_1 rB(t)}{\sigma(\beta-1)} - \frac{\beta\sigma_1^2}{2(\beta-1)}B(t)^2 = 0 \quad (2.2.7)$$

$$B'(t) - kB(t) + 2kcC(t) + 2\sigma_1^2B(t)C(t) + \frac{r\beta}{\sigma^2(\beta-1)} - \frac{\beta\sigma_1B(t)}{\sigma(\beta-1)} + \frac{2\beta\sigma_1rC(t)}{\sigma(\beta-1)} - \frac{2\beta\sigma_1^2B(t)C(t)}{(\beta-1)} = 0 \quad (2.2.8)$$

$$C'(t) - 2kC(t) + 2\sigma_1^2C(t)^2 - \frac{\beta}{2\sigma^2(\beta-1)} - \frac{2\beta\sigma_1C(t)}{\sigma(\beta-1)} - \frac{2\beta\sigma_1^2C(t)^2}{(\beta-1)} = 0 \quad (2.2.9)$$

For the equation (2.2.9), we have:

$$C'(t) - (2k + \frac{2\beta\sigma_1}{\sigma(\beta-1)})C(t) + (2\sigma_1^2 - \frac{2\sigma_1^2\beta}{\beta-1})C(t)^2 - \frac{\beta}{2\sigma^2(\beta-1)} = 0 \quad (2.2.10)$$

Let $a_1 = (2\sigma_1^2 - \frac{2\sigma_1^2\beta}{\beta-1}), b_1 = (2k + \frac{2\beta\sigma_1}{\sigma(\beta-1)}), c_1 = \frac{\beta}{2\sigma^2(\beta-1)}$, we can simplify this equation as:

$$C'(t) + a_1 C(t)^2 - b_1 C(t) - c_1 = 0$$

$$C'(t) + a_1 (C(t) - \frac{b_1}{2a_1})^2 - (\frac{b_1^2}{4a_1} + c_1) = 0$$

Define $C(t) = \chi(t) + \frac{b_1}{2a_1}$, we have $\chi(T) = -\frac{b_1}{2a_1}$.

Let $\phi = \frac{b_1^2}{4a_1} + c_1$, $a_1 = \frac{1}{k_1}$, we have the ODE:

$$\frac{\partial_t \chi}{k_1\phi - \chi^2} = \frac{1}{k_1} \tag{2.2.11}$$

subject to $\chi(T) = -\frac{b_1}{2a_1}$.
This ODE is of Riccati type and can be integrated exactly, and thus we can easily get:

$$\chi(t) = \sqrt{k_1\phi}\frac{1 + \zeta e^{2\gamma(T-t)}}{1 - \zeta e^{2\gamma(T-t)}} \tag{2.2.12}$$

where

$$\gamma = \sqrt{\frac{\phi}{k_1}} \quad \text{and} \quad \zeta = \frac{\frac{b_1}{2a_1} + \sqrt{\phi k_1}}{\frac{b_1}{2a_1} - \sqrt{\phi k_1}} \tag{2.2.13}$$

Since we have solved $C(t)$, then for equation (2.2.8), we have

$$\partial_t B(t) + (2\sigma_1^2 C(t) - k - \frac{\beta\sigma_1}{\sigma(\beta-1)} - \frac{2\beta\sigma_1^2}{\beta-1}C(t))B(t) = -2kcC(t) - \frac{r\beta}{\sigma^2(\beta-1)} - \frac{2r\beta\sigma_1}{\sigma(\beta-1)}C(t) \tag{2.2.14}$$

Let $(2\sigma_1^2 C(t) - k - \frac{\beta\sigma_1}{\sigma(\beta-1)} - \frac{2\beta\sigma_1^2}{\beta-1}C(t)) = P(t)$, and $(-2kcC(t) - \frac{r\beta}{\sigma^2(\beta-1)} - \frac{2r\beta\sigma_1}{\sigma(\beta-1)}C(t)) = g(t)$, we have ODE

$$\partial_t B(t) + P(t)B(t) = g(t) \tag{2.2.15}$$

Then we have

$$B(t) = \frac{\int \mu(t)g(t)dt + \text{constant}}{\mu(t)} \tag{2.2.16}$$
$$\mu(t) = e^{\int P(t)dt}$$

where we can solve constant from $B(T) = 0$. Now we have get $C(t)$ and $B(t)$, so for equation (2.2.7), we have

$$A(t) = \int \{-kcB(t) - \sigma_1^2 C(t) - \frac{1}{2}\sigma_1^2 B(t)^2 - \beta r + \frac{r^2\beta}{2\sigma^2(\beta-1)} - \frac{\beta\sigma_1 r B(t)}{\sigma(\beta-1)} + \frac{\beta\sigma_1^2}{2(\beta-1)}B(t)^2\}dt + C \tag{2.2.17}$$

where C can be solved from $A(T) = 0$.

2.3 Dual Method

Define the dual function by $\tilde{U}(y) = \sup_{x>0}(U(x) + xy)$ as before. Also, the same as section one, the dual process Y is a strictly positive process and has the following semimartingale decomposition:

$$dY(t) = Y(t)\alpha(t)dt + \beta^T(t)dW(t), \quad 0 \le t \le T$$
$$Y(0) = y$$

where processes α and β are chosen such that $X^\pi Y$ is a supermartingale for all admissible control processes $\pi \in \mathcal{A}$. Using Ito's lemma, we have:

$$d(X^\pi(t)Y(t)) = X^\pi(t)Y(t)\{[r + \pi_t(H_t - r) + \alpha(t) + \pi_t\sigma\beta(t)]dt + [\pi_t\sigma + \beta^T(t)]dW(t)\}$$

To make $X^\pi Y$ a supermartingale, we must have

$$r + \pi_t(H_t - r) + \alpha(t) + \pi_t\sigma\beta(t) \le 0$$

for all $\pi \in K$ a.s. for a.e. $t \in [0, T]$, which is equivalent to

$$r + \alpha(t) + \delta_K(-(H_t - r) - \sigma\beta(t)) \le 0$$

where $\delta_K(.)$ is the support function of set $-K$, defined by $\delta_K(z) = \sup_{\pi\in K}\{-\pi z\}$.
Define $v(t) = -(H_t - r) - \sigma\beta(t)$, we have

$$\begin{aligned}\alpha(t) &\le -(r + \delta_K(v(t)))\\ \beta(t) &= -(\sigma^{-1}v(t) + \sigma^{-1}(H_t - r))\end{aligned}$$

Given that $K = \mathbb{R}$, we have $v(t) = 0$, and thus $\alpha(t) = -r$, $\beta(t) = -\sigma^{-1}(H_t - r)$.
Therefore, the dual process satisfies the SDE:

$$\begin{aligned}dY(t) &= -Y(t)\{rdt + \sigma^{-1}(H_t - r)dW(t)\} \quad (2.3.1)\\ Y(0) &= y\end{aligned}$$

The optimal value of the dual minimization problem is defined by

$$\tilde{V} = \inf_{y\in(0,\infty)}(xy + E[\tilde{U}(Y(T))]) \quad (2.3.2)$$

Define the dual value function

$$\hat{V}(t, y, h) = E[\tilde{U}(Y(T))|Y(0) = y,\ H(0) = h] \quad (2.3.3)$$

Using the dynamic programming principle (DPP), we have $\hat{V}$ satisfies the following HJB (Hamilton–Jacobi–Bellman) equation:

$$\frac{\partial\hat{V}}{\partial t} + k(c-h)\frac{\partial\hat{V}}{\partial h} + \frac{1}{2}\sigma_1^2\frac{\partial^2\hat{V}}{\partial h^2} + [-ry\frac{\partial\hat{V}}{\partial y} + \frac{1}{2}\sigma^{-2}(h-r)^2y^2\frac{\partial^2\hat{V}}{\partial y^2} - y\sigma^{-1}(h-r)\sigma_1\frac{\partial^2\hat{V}}{\partial y\partial h}] = 0 \quad (2.3.4)$$

Assume that $\hat{V} = \tilde{U}(y)f(t, h)$, where $\tilde{U}(y) = \frac{1-\beta}{\beta}y^{\frac{\beta}{\beta-1}}$ and $f(t, h) = \exp A(t) + hB(t) + h^2C(t)$ with terminal condition that $f(T, h) = 1$, which means $A(T) = B(T) = C(T) = 0$.

Then for each term in HJB, they can be written as:

$$\begin{aligned}\partial_t\hat{V} &= (A'(t) + hB'(t) + h^2C'(t))\hat{V}\\ \partial_h\hat{V} &= (B(t) + 2hC(t))\hat{V}\\ \partial_{hh}\hat{V} &= 2C(t)\hat{V} + (B(t) + 2hC(t))^2\hat{V}\\ \partial_y\hat{V} &= -\frac{\beta}{1-\beta}y^{-1}\hat{V}\\ \partial_{yy}\hat{V} &= \frac{\beta}{1-\beta}y^{-2}\hat{V} + (\frac{\beta}{1-\beta})^2y^{-2}\hat{V}\\ \partial_{yh}\hat{V} &= -\frac{\beta}{1-\beta}y^{-1}(B(t) + 2hC(t))\hat{V}\end{aligned}$$

Then, the dual HJB can be simplified as

$$\begin{aligned}&A'(t) + hB'(t) + h^2C'(t) + k(c-h)(B(t) + 2hC(t)) + \frac{1}{2}\sigma_1^2(2C(t) + (B(t) + 2hC(t))^2) + \quad (2.3.5)\\ &\frac{\beta}{1-\beta}r + \frac{1}{2}\sigma^{-2}(h-r)^2(\frac{\beta}{1-\beta} + (\frac{\beta}{1-\beta})^2) + \frac{\beta\sigma^{-1}\sigma_1}{1-\beta}(B(t)h + 2h^2C(t) - rB(t) - 2hrC(t)) = 0\end{aligned}$$

To solve $A(t)$,$B(t)$ and $C(t)$, we classify the dual HJB in terms of the power of h and set each term's coefficient to zero. Thus, we have

$$A'(t) + kcB(t) + \sigma_1^2C(t) + \frac{1}{2}\sigma_1^2B(t)^2 + \frac{\beta}{1-\beta}r + \frac{1}{2}\sigma^{-2}r^2\frac{\beta}{(1-\beta)^2} - \frac{rB(t)\beta\sigma^{-1}\sigma_1}{1-\beta} = 0 \quad (2.3.6)$$

$$B'(t) - kB(t) + 2kcC(t) + 2\sigma_1^2B(t)C(t) - \sigma^{-2}r\frac{\beta}{(1-\beta)^2} + \frac{\beta\sigma^{-1}\sigma_1}{1-\beta}(B(t) - 2rC(t)) = 0 \quad (2.3.7)$$

$$C'(t) - 2kC(t) + 2\sigma_1^2C(t)^2 + \frac{1}{2}\sigma^{-2}\frac{\beta}{(1-\beta)^2} + \frac{2\beta\sigma^{-1}\sigma_1}{1-\beta}C(t) = 0 \quad (2.3.8)$$

For the equation (2.3.8), we have:

$$C'(t) + 2\sigma_1^2 C(t)^2 + (\frac{2\beta\sigma^{-1}\sigma_1}{1-\beta} - 2k)C(t) + \frac{1}{2}\sigma^{-2}\frac{\beta}{(1-\beta)^2} = 0$$

let $a_2 = 2\sigma_1^2$, $b_2 = \frac{2\beta\sigma^{-1}\sigma_1}{1-\beta} - 2k$, $c_2 = \frac{1}{2}\sigma^{-2}\frac{\beta}{(1-\beta)^2}$, and we have:

$$C'(t) + a_2 C(t)^2 + b_2 C(t) + c_2 = 0$$

$$C'(t) + a_2(C(t) + \frac{b_2}{2a_2})^2 + c_2 - \frac{b_2^2}{4a_2} = 0$$

Let $C(t) = \chi(t) - \frac{b_2}{2a_2}$, and set $\phi = \frac{b_2^2}{4a_2} - c_2$, $a_2 = \frac{1}{k_2}$, we have the ODE:

$$\frac{\partial_t \chi}{k_2\phi - \chi^2} = \frac{1}{k_2} \tag{2.3.9}$$

subject to $\chi(T) = \frac{b_2}{2a_2}$.

This ODE is of Riccati type and can be integrated exactly, and thus we can easily get:

$$\chi(t) = \sqrt{\phi k_2}\frac{1 + \zeta e^{2\gamma(T-t)}}{1 - \zeta e^{2\gamma(T-t)}} \tag{2.3.10}$$

where

$$\gamma = \sqrt{\frac{\phi}{k_2}} \quad \text{and} \quad \zeta = \frac{\frac{-b_2}{2a_2} + \sqrt{\phi k_2}}{\frac{-b_2}{2a_2} - \sqrt{\phi k_2}} \tag{2.3.11}$$

Since we have solved $C(t)$, then for equation (2.3.7), we have

$$\partial_t B(t) + (2\sigma_1^2 C(t) - k + \frac{\beta\sigma^{-1}\sigma_1}{1-\beta})B(t) = -2kcC(t) + \sigma^{-2}r\frac{\beta}{(1-\beta)^2} + \frac{2r\beta\sigma^{-1}\sigma_1 C(t)}{1-\beta} \tag{2.3.12}$$

Let $(2\sigma_1^2 C(t) - k + \frac{\beta\sigma^{-1}\sigma_1}{1-\beta}) = P(t)$, and $(-2kcC(t) + \sigma^{-2}r\frac{\beta}{(1-\beta)^2} + \frac{2r\beta\sigma^{-1}\sigma_1 C(t)}{1-\beta}) = g(t)$, we have ODE

$$\partial_t B(t) + P(t)B(t) = g(t) \tag{2.3.13}$$

Then we have

$$B(t) = \frac{\int \mu(t)g(t)dt + \text{constant}}{\mu(t)} \tag{2.3.14}$$
$$\mu(t) = e^{\int P(t)dt}$$

where we can solve constant from $B(T) = 0$.
Now we have get $C(t)$ and $B(t)$, so for equation (2.3.6), we have

$$A(t) = \int \{-kcB(t) - \sigma_1^2 C(t) - \frac{1}{2}\sigma_1^2 B(t)^2 - \frac{\beta}{1-\beta}r - \frac{1}{2}\sigma^{-2}r^2\frac{\beta}{(1-\beta)^2} + \frac{\beta\sigma^{-1}\sigma_1}{1-\beta}rB(t)\}dt + C \tag{2.3.15}$$

where C can be solved from $A(T) = 0$.

2.4 FBSDE

For the primal FBSDE, we will compute the wealth process by using numerical method. Recalling the theorem 3.5, we have:

Theorem 3.5 (Primal problem and associated FBSDE): Let $\hat{\pi} \in \mathcal{A}$. Then $\hat{\pi}$ is optimal for the primal problem if and only if the solution $(X^{\hat{\pi}}, \hat{p_1}, \hat{q_1})$ of FBSDE

$$\begin{aligned}
&dX^{\hat{\pi}(t)}(t) = X^{\hat{\pi}}(t)[(r(t) + \hat{\pi}^T(t)\sigma(t)\theta(t))dt + \hat{\pi}^T(t)\sigma(t)dW(t)] \\
&X^{\hat{\pi}}(0) = x_0 \\
&d\hat{p_1}(t) = -[(r(t) + \hat{\pi}^T(t)\sigma(t)\theta(t))\hat{p_1}(t) + \hat{q_1}^T(t)\sigma^T(t)\hat{\pi}(t)]dt + \hat{q_1}^T(t)dW(t) \\
&\hat{p_1}(T) = -U^{'}(X^{\hat{\pi}}(T))
\end{aligned} \tag{2.4.1}$$

satisfies the condition

$$-X^{\hat{\pi}}(t)\sigma(t)[\theta(t)\hat{p_1}(t) + \hat{q_1}(t)] \in N_K(\hat{\pi}(t)), \quad \forall t \in [0,T], \mathbb{P} - a.s. \tag{2.4.2}$$

where $N_K(x)$ is the normal cone of the closed convex set K at $x \in K$, defined as

$$N_K(x) = \{y \in \mathbb{R}^N : \forall x^* \in K, y(x^* - x) \leq 0\}$$

Since $K = \mathbb{R}$, we have $\hat{q_1}(t) = -\sigma^{-1}(H_t - r)\hat{p_1}(t)$, for the primal FBSDE, we have:

$$d\hat{p_1}(t) = -r\hat{p_1}(t)dt - \sigma^{-1}(H_t - r)\hat{p_1}(t)dW_t$$
$$\hat{p_1}(T) = -X(T)^{\beta-1}$$

We want to find p_0 such that $\hat{p_1}(T) = -X(T)^{\beta-1}$. In other words, we need to optimize the equation $\min_{p_0,\pi(t)} E[\hat{p_1}(T) + X(T)^{\beta-1}]$, which is equivalent to $\min_{p_0,\pi(t)} E[(\hat{p_1}(T) + X(T)^{\beta-1})^2]$.
Using Euler scheme, we have:

$$H_{t_{i+1}} = H_{t_i} + k(c - H_{t_i})h + \sigma_1\sqrt{h}Z_{t_i}$$

$$X_{t_{i+1}} = X_{t_i} + X_{t_i}(r + \pi_{t_i}(H_{t_i} - r))h + \pi_{t_i}X_{t_i}\sigma\sqrt{h}Z_{t_i}$$
$$\hat{p_1}(t_{i+1}) = \hat{p_1}(t_i) - r\hat{p_1}(t_i)h - \sigma^{-1}(H_{t_i} - r)\hat{p_1}(t_i)\sqrt{h}Z_{t_i}$$

To solve this optimal control problem numerically, we divided interval $[0,T]$ by n intervals with step size $h = T/n$ and grid points $t_0 = 0$, $t_i = h_i, i = 1,...,n$. Assume on subinterval $[t_i;t_{i+1})$, control $\hat{\pi}_i = \alpha(i) + \beta_(i)H_{t_i}$, where $\alpha(t)$ and $\beta(t)$ are piecewise constant within each subinterval.

For the dual FBSDE, recalling the theorem 3.9, we have:

Theorem 3.9 (Dual problem and associated FBSDE): Let $(\hat{y},\hat{v}) \in (0,\infty) \times \mathcal{D}$. Then $(\hat{y},\hat{v})$ is optimal for the dual problem if and only if the solution $(Y^{(\hat{y},\hat{v})}, \hat{p_2}, \hat{q_2})$ of FBSDE

$$\begin{aligned}
&dY^{(\hat{y},\hat{v})}(t) = -Y^{(\hat{y},\hat{v})}(t)\{[r(t) + \delta_K(\hat{v}(t))]dt + [\theta(t) + \sigma^{-1}(t)\hat{v}(t)]^T dW(t)\} \\
&Y^{(\hat{y},\hat{v})}(0) = \hat{y} \\
&d\hat{p_2}(t) = \{[r(t) + \delta_K(\hat{v}(t))]^T\hat{p_2}(t) + \hat{q_2}^T(t)[\theta(t) + \sigma^{-1}(t)\hat{v}(t)]\}dt + \hat{q_2}^T(t)dW(t) \\
&\hat{p_2}(T) = -\tilde{U}^{'}(Y^{(\hat{y},\hat{v})}(T))
\end{aligned} \tag{2.4.3}$$

satisfies the condition

$$\begin{aligned}
&\hat{p_2}(0) = x_0 \\
&\hat{p_2}(t)^{-1}[\sigma^T(t)]^{-1}\hat{q_2}(t) \in K \\
&\hat{p_2}(t)\delta_K(\hat{v}(t)) + \hat{q_2}^T(t)\sigma^{-1}(t)\hat{v}(t) = 0, \forall t \in [0,T]\mathbb{P} - a.s.
\end{aligned} \tag{2.4.4}$$

And thus in this case, we can write the dual FBSDE as following:

$$\begin{aligned}
&dY(t) = -Y(t)\{rdt + \sigma^{-1}(H_t - r)dW(t)\} \qquad (2.4.5) \\
&Y(0) = y \\
&d\hat{p_2}(t) = \{r\hat{p_2}(t) + \hat{q_2}(t)\sigma^{-1}(H_t - r)\}dt + \hat{q_2}(t)dW_t \\
&\hat{p_2}(T) = -\tilde{U}'(Y^{\hat{y}}(T))
\end{aligned}$$

satisfies the condition

$$\begin{aligned}
&\hat{p_2}(0) = x_0 \\
&\hat{p_2}(t)^{-1}[\sigma^T(t)]^{-1}\hat{q_2}(t) \in K \\
&\hat{p_2}(t)\delta_K(\hat{v}(t)) + \hat{q_2}^T(t)\sigma^{-1}(t)\hat{v}(t) = 0, \forall t \in [0,T]\mathbb{P} - a.s. \qquad (2.4.6)
\end{aligned}$$

We treat $\hat{q_2}(t)$ as piecewise constant within each time interval and use Euler scheme for the SDE.

$$H_{t_{i+1}} = H_{t_i} + k(c - H_{t_i})h + \sigma_1\sqrt{h}Z_{t_i}$$

$$Y_{t_{i+1}} = Y_{t_i} - Y_{t_i}rh - Y_{t_i}\sigma^{-1}(H_{t_i} - r)\sqrt{h}Z_{t_i}$$

$$\hat{p_2}(t_{i+1}) = \hat{p_2}(t_i) + \{r\hat{p_2}(t_i) + \hat{q_2}(t_i)\sigma^{-1}(H_{t_i} - r)\}h + \hat{q_2}(t_i)\sqrt{h}Z_{t_i}$$

with the initial condition

$$\hat{p_2}(0) = x_0$$

and terminal condition

$$\hat{p_2}(T) = Y(T)^{\frac{1}{\beta-1}}$$

We want to find optimal y_0 and $q_2(t)$ such that $\hat{p_2}(T) = Y(T)^{\frac{1}{\beta-1}}$. In other words, we want to optimize the problem $\inf_{y_0,q_2(t)} E[(\hat{p_2}(T) - Y(T)^{\frac{1}{\beta-1}})^2]$.

To solve this optimal control problem numerically, we divided interval $[0,T]$ by n intervals with step size $h = T/n$ and grid points $t_i = h_i; i = 0, 1, ..., n$. Assume on subinterval $[t_i; t_{i+1})$, control $\hat{q_2}(i)$ is taken to be constant, say that $\hat{q_2}(i)$, for $i = 0, 1, ..., n-1$.

By setting the parameters as $k = 1, c = 1, r = 0.05, \sigma = 0.8, \sigma_1 = 0.3, T = 1, \beta = 0.5, t_0 = 0, h_0 = 0.5, x_0 = 10, dt = 0.01$(time step), we generate the wealth process from time zero to one with time step size 0.01, and thus there are 101 points in total.Noting that the same set of Brownian motion will be used to generate path in all methods (including primal, dual HJB and primal, dual FBSDE).

Using the primal and dual FBSDE methods to simulate wealth processes, we have:

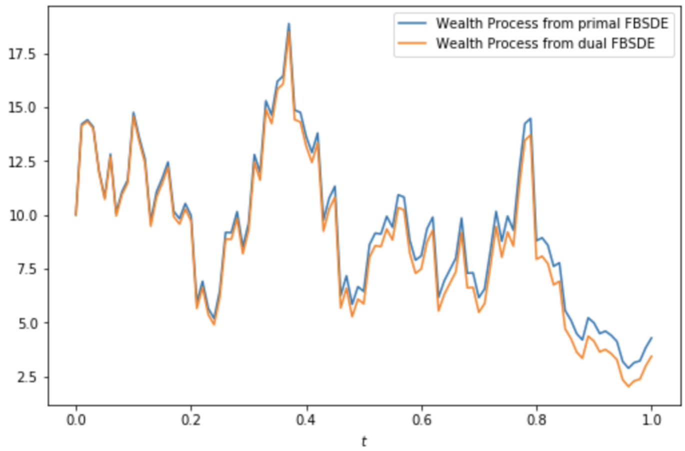

Figure 2.1: Wealth Processes from Primal and Dual FBSDE

2.5 Numerical Verification

In this section, we are going to show that the value functions and wealth processes obtained from the primal, dual and FBSDE method are the same.

Value Functions

Since we have solved the closed form solution of value function for primal method, we can plot $V(t,h) = U(x)f(t,h)$ explicitly, and we use Trapezoidal rule for estimating the integration parts in $A(t), B(t),$ and $C(t)$.

The trapezoidal rule may be viewed as the result obtained by averaging the left and right Riemann sums, and is sometimes defined this way. The integral can be even better approximated by partitioning the integration interval, applying the trapezoidal rule to each subinterval, and summing the results. Let x_k be a partition of $[a,b]$, such that $a = x_0 < x_1 < ... < x_{N_1} < x_N = b$, and Δx_k be the length of the k-th subinterval, then

$$\int_a^b f(x)dx \approx \sum_{k=1}^{N} \frac{f(x_{k+1}) + f(x_k)}{2}\Delta x_k = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{N-1}) + f(x_N))$$

For the dual method, we use the same method to solve the dual value function by assuming that $\tilde{V} = \inf_y[xy + \hat{V}] = \inf_y[xy + \tilde{U}(y)f(t,h)] = \inf_y[xy + \frac{1-\beta}{\beta}y^{\frac{\beta}{\beta-1}}f(t,h)]$.
By taking derivative w.r.t y, we have:

$$\begin{aligned}
\tilde{V} &= \inf_y[xy + \hat{V}] \\
&=> x - y^{\frac{1}{\beta-1}} \exp A(t) + hB(t) + h^2C(t) = 0 \\
&=> y = x^{\beta-1} \exp A(t) + hB(t) + h^2C(t)^{1-\beta} \\
&=> \tilde{V} = x^\beta f(t,h)^{1-\beta} + \frac{1-\beta}{\beta}x^\beta f(t,h)^{1-\beta} = \frac{1}{\beta}x^\beta f(t,h)^{1-\beta}
\end{aligned}$$

By setting the parameters as $k = 1, c = 1, r = 0.05, \sigma = 0.8, \sigma_1 = 0.3, T = 1, \beta = 0.5, t_0 = 0, h_0 = 0.5, x_0 = 10, dt = 0.01$(time step), we have:

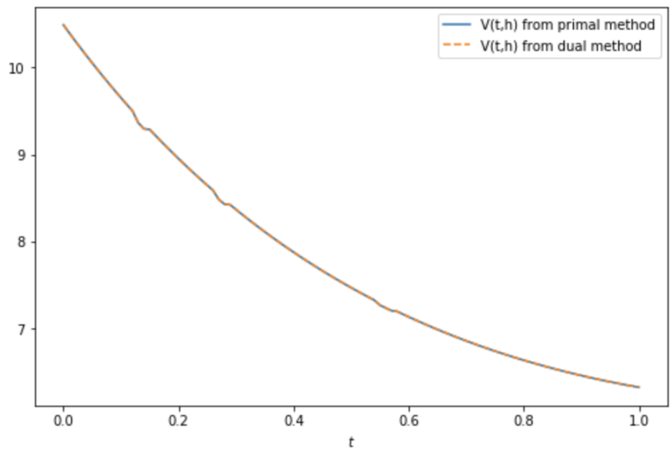

Figure 2.2: Primal and Dual Value Functions from Time Zero to One, dt = 0.01

We can see that in this graph, the two value functions overlap completely, and thus it illustrates that the value functions obtained from primal HJB and dual HJB should be the same.

Wealth Processes

For the wealth process, we use the same parameters as above, and Euler scheme is used to generate paths of X, H, and Y for primal and dual methods.
For the primal method, we have:

$$dH_t = k(c - H_t)dt + \sigma_1 dW_t$$

$$dX_t = X_t(r + \pi_t(H(t) - r))dt + \pi_t X_t \sigma dW_t$$

where $\pi(t) = \frac{-(H_t-r)-\sigma\sigma_1(B(t)+2hC(t))}{\sigma^2(\beta-1)}$. By using Euler scheme, we have:

$$H_{t_{i+1}} = H_{t_i} + k(c - H_{t_i})dt + \sigma_1\sqrt{dt}Z_{t_i}$$

$$X_{t_{i+1}} = X_{t_i} + X_{t_i}(r + \pi(H_{t_i} - r))h + \pi X_{t_i}\sigma\sqrt{h}Z_{t_i}$$

For the dual problem, recall that

$$X^{\pi^*(t)} = -\frac{\partial \hat{V}(t, \hat{Y}(t))}{\partial \hat{Y}(t)}$$

$\tilde{V} = \inf_y[xy + \hat{V}] = \inf_y[xy + \tilde{U}(y)f(t,h)]$ and $\hat{V} = \tilde{U}(y)f(t,h)$.
Since we have solved that $\hat{V} = \frac{1-\beta}{\beta}y^{\frac{\beta}{\beta-1}}\exp\left(A(t) + hB(t) + h^2C(t)\right)$, by taking derivative of $\tilde{V}$ w.r.t y, we have $\hat{y}(t) = x^{\beta-1}\exp(A(t) + B(t)h + C(t)h^2)^{1-\beta}$,
and thus $\hat{y}(0) = x^{\beta-1}\exp(A(0) + B(0)h + C(0)h^2)^{1-\beta}$.

$$\begin{aligned}\partial_y \hat{V} &= -y^{\frac{1}{\beta-1}}\exp\left(A(t) + hB(t) + h^2C(t)\right) \\ &= -x\end{aligned} \tag{2.5.1}$$

So we have:

$$X(t) = Y(t)^{\frac{1}{\beta-1}}\exp\left(A(t) + H_tB(t) + H_t^2C(t)\right)$$
$$dY(t) = -Y(t)\{rdt + \sigma^{-1}(H_t - r)dW(t)\}$$

By using Euler scheme, we have:

$$H_{t_{i+1}} = H_{t_i} + k(c - H_{t_i})dt + \sigma_1\sqrt{dt}Z_{t_i}$$

$$Y_{t_{i+1}} = Y_{t_i} - Y_{t_i}rdt - Y_{t_i}\sigma^{-1}(H_{t_i} - r)\sqrt{dt}Z_{t_i}$$

Also, we are going to use the same set of Brownian motion to generate every paths in primal, dual HJB and primal, dual FBSDE for consistent. The wealth processes simulated from the primal and dual HJB are shown in the following graph:

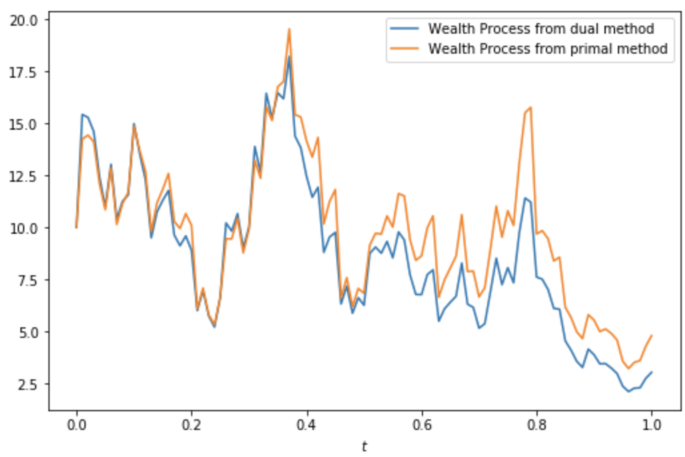

Figure 2.3: Wealth Processes from Primal and Dual HJB, dt = 0.01

By combining figure 2.3 and 2.1 into one plot we have:

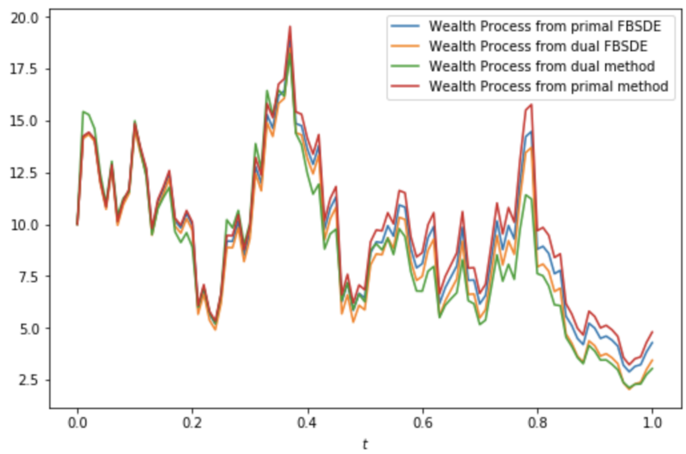

Figure 2.4: Wealth Processes from Primal, Dual HJB and Primal, Dual FBSDE, dt = 0.01

We see that at the beginning of the figure 2.4, all four methods show consistent trend and fit closely, but as time increase (after $t = 0.5$), we see that wealth processes start diverging though they still have the same trend patterns. To see whether the divergence is significant, we will perform error analysis.

Error Analysis

We use the wealth process from primal HJB as standard and calculate the mean square error of wealth processes from the other three methods in comparison to the primal HJB.
In other words, we have:

$$MSE = \frac{1}{n}\sum_{i=0}^{n}(X_i - \hat{X}_i)^2 \tag{2.5.2}$$

where X_i could be wealth processes from dual HJB, primal and dual FBSDE, and $\hat{X}_i$ is wealth process from primal HJB.
Then we have:

Method	MSE
Primal HJB	0
Dual HJB	2.375
Primal FBSDE	0.278
Dual FBSDE	1.185

Table 2.1: Mean Square Error

In comparison to the maximum (19.538) and the minimum (3.222) values of wealth process from primal HJB, the MSEs from the other three methods are acceptable, and thus the wealth processes from these four methods are considered to be the same.

To see the effect of time step size to the simulation result, we also use different time step size for simulating. To generate the paths for wealth processes, we use the same parameters and methods as above, and a same set of Brownian Motion is used within each time step size.

We have the wealth processes for $dt = 0.02$:

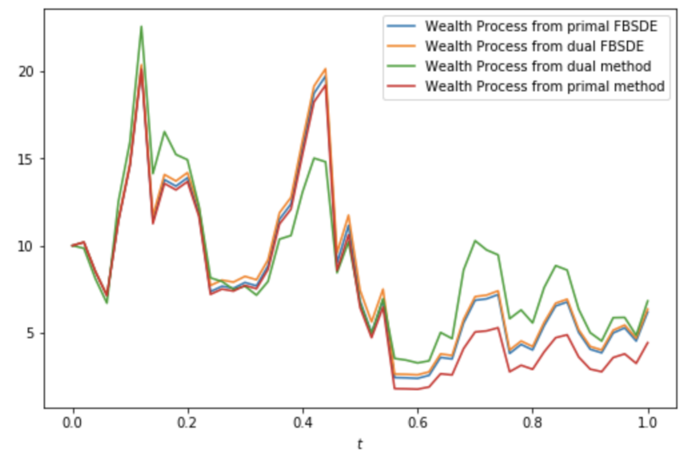

Figure 2.5: Wealth Processes from Primal, Dual HJB and Primal, Dual FBSDE, dt = 0.02

And the wealth processes for $dt = 0.05$:

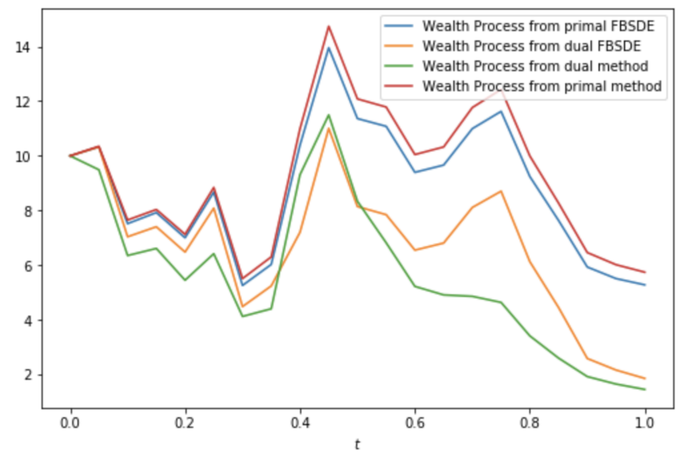

Figure 2.6: Wealth Processes from Primal, Dual HJB and Primal, Dual FBSDE, dt = 0.05

In figure 2.4, 2.5, and 2.6, we see that the wealth processes from all four methods have the same trend pattern, though as time step size increases the differences among each method become larger as t moving from time 0 to time 1. This is because, as time step size becomes larger, the simulation result becomes less accurate and error could be accumulated as t increases, and thus large time step size gives larger difference on the simulation result of wealth processes.

Performing error analysis as before, we have:

Method	MSE dt = 0.01	MSE dt = 0.02	MSE dt = 0.05
Primal HJB	0	0	0
Dual HJB	2.375	5.547	17.389
Primal FBSDE	0.278	0.865	0.287
Dual FBSDE	1.185	1.274	9.027

Table 2.2: Mean Square Error

From table 2.2, we see that as time step size increases, simulated wealth processes have greater error.

Simulation from Dual Value Function

Also, we can use simulation to get dual value directly for fixed y and h.
Recall that

$$\tilde{V} = \inf_{y\in(0,\infty)}(xy + E[\tilde{U}(Y(T))])$$

$$\tilde{U}(Y) = \frac{1-\beta}{\beta}Y^{\frac{\beta}{\beta-1}}$$

And the initial value of Y has to satisfy the condition

$$x + E[\tilde{U}'(Y(T))]) = 0$$

Therefore, we can write a function which takes y_0 and h as input values and return $x+E[\tilde{U}'(Y(T))])$, where $Y(T)$ can be generated by using Euler Scheme as following:
Recall that $dY_t = -Y(t)\{rdt + \sigma^{-1}(H_t - r)dW_t\}$, so we can have:

$$\begin{aligned} d(\log(Y(t))) &= \frac{1}{Y(t)}dY(t) - \frac{1}{2}\frac{1}{Y(t)^2}(d(Y(t)))^2 \\ &= (-rdt - \sigma^{-1}(H_t - r)dW_t) - \frac{1}{2}\sigma^{-2}(H_t - r)^2 dt \end{aligned}$$

Thus, we have:

$$\begin{aligned} \log(\frac{Y_t}{Y_0}) &= \int(-r - \frac{1}{2}\sigma^{-2}(H_t - r)^2)dt + \int -\sigma^{-1}(H_t - r)dW_t \\ Y(T) &= y\exp\int_0^T(-r - \frac{1}{2}\sigma^{-2}(H_t - r)^2)dt + \int_0^T -\sigma^{-1}(H_t - r)dW_t \end{aligned}$$

$$\begin{aligned} \tilde{U}(Y(T)) &= \frac{1-\beta}{\beta}Y(T)^{\frac{\beta}{\beta-1}} \qquad (2.5.3) \\ &= \frac{1-\beta}{\beta}(y\exp\int_0^T(-r - \frac{1}{2}\sigma^{-2}(H_t - r)^2)dt + \int_0^T -\sigma^{-1}(H_t - r)dW_t)^{\frac{\beta}{\beta-1}} \end{aligned}$$

By taking derivative with respect to y in equation 2.43, we have:

$$\tilde{U}'(Y(T)) = -y^{\frac{1}{\beta-1}}\exp\int_0^T(-r - \frac{1}{2}\sigma^{-2}(H_t - r)^2)dt + \int_0^T -\sigma^{-1}(H_t - r)dW_t)^{\frac{\beta}{\beta-1}} \qquad (2.5.4)$$

Then we can apply Euler Scheme and Trapezoidal Rule for the integration part, and we have:

$$H_{t_{i+1}} = H_{t_i} + k(c - H_{t_i})dt + \sigma_1\sqrt{dt}Z_{t_i}$$

$$\int_0^T(-r - \frac{1}{2}\sigma^{-2}(H_t - r)^2)dt \approx \Sigma_{i=0}^n\frac{(-r - \frac{1}{2}\sigma^{-2}(H_{t_{i+1}} - r)^2) + (-r - \frac{1}{2}\sigma^{-2}(H_{t_i} - r)^2)}{2}dt$$

$$\int_0^T -\sigma^{-1}(H_t - r)dW_t \approx \Sigma_{i=0}^n\frac{(-\sigma^{-1}(H_{t_{i+1}} - r)) + (-\sigma^{-1}(H_{t_i} - r))}{2}dW_t$$

where $dW_t = W_{t_{i+1}} - W_{t_i} \sim N(0, dt)$.

Then we simulate the value of $\tilde{U}'(Y(T))$ for 1000 times and approximate the expectation by taking average over the sum. Once we have the value of $x + E[\tilde{U}'(Y(T))])$, we can apply bisection method to find the root of the function $x + E[\tilde{U}'(Y(T))])$, which makes the condition $x + E[\tilde{U}'(Y(T))] = 0$ satisfy.

The optimal y from the bisection method is 0.5500, and recalling in the previous section, we have $\hat{y}(0) = x^{\beta-1}\exp(A(0) + B(0)h + C(0)h^2)^{1-\beta} = 0.5246$. After we get y_0 from bisection method, we can use it to simulate wealth process by using dual HJB, and compare with the other four results.

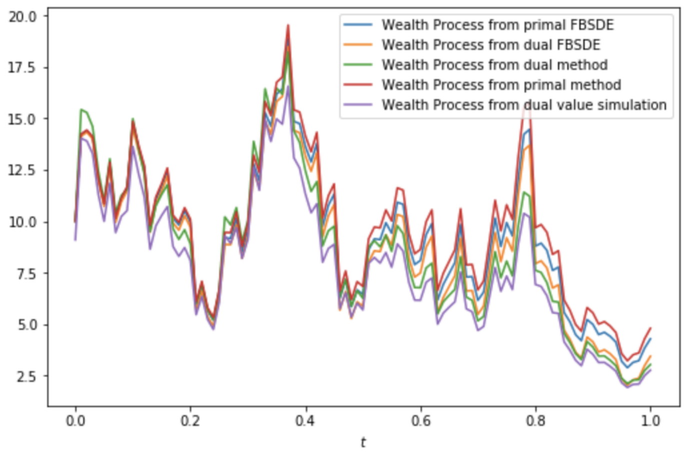

Figure 2.7: Wealth Processes from Primal, Dual HJB and Primal, Dual FBSDE, and Direct Dual Method, dt = 0.01

Given that the optimal y from bisection is close to the optimal y obtained from ansatz before, the generated wealth process performs similar pattern as the other four.

Chapter 3

Constrained Utility Maximization with OU Process

Assumption:
- 1-dimensional geometric Brownian motion asset price process
- All coefficients constant
- Control set K=$\mathbb{R}^+$
- Maximize utility of wealth at time T
- $dS = HSdt + \sigma dW_t$
- $dH_t = k(c - H_t)dt + \sigma_1(t)dW_t$

where
- $\sigma_1(t) = \frac{1}{\sigma}\Sigma(t) + \rho\sigma_H$
- $\Sigma'(t) = (1-\rho^2)\sigma_H^2 - 2(k + \frac{\rho\sigma_H}{\sigma})\Sigma(t) - \frac{1}{\sigma^2}\Sigma(t)^2$
- σ, σ_H, k, c are positive constants, and $\rho \in (-1, 1)$

3.1 Market Model

In stage three, the wealth process is:

$$dX_t = X_t(r + \pi_t(H_t - r))dt + \pi_t X_t dW_t \tag{3.1.1}$$

Define the dual function by $\tilde{U}(y) = \sup_{x>0}(U(x) - xy)$. The dual process Y is a strictly positive process and has the following semimartingale decomposition:

$$dY(t) = Y(t)\alpha(t)dt + \beta^T(t)dW(t), \quad 0 \le t \le T$$
$$Y(0) = y$$

where processes α and β are chosen such that $X^\pi Y$ is a supermartingale for all admissible control processes $\pi \in \mathcal{A}$. Using Ito's lemma, we have:

$$d(X^\pi(t)Y(t)) = X^\pi(t)Y(t)\{[r + \pi_t(H_t - r) + \alpha(t) + \pi_t\sigma\beta(t)]dt + [\pi_t\sigma + \beta^T(t)]dW(t)\}$$

To make $X^\pi Y$ a supermartingale, we must have

$$r + \pi_t(H_t - r) + \alpha(t) + \pi_t\sigma\beta(t) \le 0$$

for all $\pi \in K$ a.s. for a.e. $t \in [0, T]$, which is equivalent to

$$r + \alpha(t) + \delta_K(-(H_t - r) - \sigma\beta(t)) \le 0$$

where $\delta_K(.)$ is the support function of set $-K$, defined by $\delta_K(z) = \sup_{\pi\in K}\{-\pi z\}$.
Define $v(t) = -(H_t - r) - \sigma\beta(t)$, we have

$$\alpha(t) \le -(r + \delta_K(v(t)))$$
$$\beta(t) = -(\sigma^{-1}v(t) + \sigma^{-1}(H_t - r))$$

Since K is a closed convex cone, then $\delta_K(z) = 0$ if $z \in \tilde{K}$ and ∞ otherwise, where $\tilde{K} = \{z : z^T\pi \geq 0, \forall \pi \in K\}$ is the positive polar cone of K. Therefore, the dual process becomes:

$$dY(t) = -Y(t)\{rdt + (\sigma^{-1}v(t) + \sigma^{-1}(H_t - r))dW_t\} \tag{3.1.2}$$

with $Y(0) = y$.

3.2 Solving with dual FBSDE

Since $K = R^+$, the ansatz we used in stage two is no longer satisfied given that $\pi(t) = \frac{-(H_t-r)-\sigma\sigma_1(B(t)+2hC(t))}{\sigma^2(\beta-1)}$ could be negative, and thus HJB may not be solved in this case. Therefore, we use dual FBSDE in this section.

Recalling Theorem 3.9, the dual FBSDE has the form:

$$d\hat{p_2}(t) = \{r\hat{p_2}(t) + \hat{q_2}(t)[\sigma^{-1}v(t) + \sigma^{-1}(H_t - r)]\}dt + \hat{p_2}(t)dW_t \tag{3.2.1}$$

with $\hat{p_2}(T) = -\tilde{U}'(Y(T))$ satisfies the condition

$$\begin{aligned}
&\hat{p_2}(0) = x_0 \\
&\hat{p_2}^{-1}(t)\frac{1}{\sigma}\hat{q_2}(t) \in K \\
&\hat{q_2}(t)\frac{1}{\sigma}v(t) = 0
\end{aligned}$$

with $v(t) \in \tilde{K}$.

From theorem 3.10, we have:

$$\hat{\pi}(t) = \frac{\hat{q_2}(t)}{\hat{p_2}(t)\sigma} \tag{3.2.2}$$

Denote $\hat{q_2}(t) = \hat{\pi}(t)\hat{p_2}(t)\sigma$ and $\hat{p_2} = p$, the dual FBSDE has the form:

$$\begin{aligned}
&dY(t) = -Y(t)\{rdt + (\sigma^{-1}v(t) + \sigma^{-1}(H_t - r))dW_t\} \qquad (3.2.3)\\
&dp(t) = rp(t)dt + \pi(t)p(t)(H_t - r)dt + \sigma\pi(t)p(t)dW_t \\
&Y(0) = y, p(T) = -\tilde{U}'(Y(T)) \\
&p(0) = x_0, \pi(t) \in K, v(t) \in \tilde{K}, \pi(t)v(t) = 0
\end{aligned}$$

The solution is equivalent to find y, π, v such that

$$\begin{aligned}
&dY(t) = -Y(t)\{rdt + (\sigma^{-1}v(t) + \sigma^{-1}(H_t - r))dW_t\} \qquad (3.2.4)\\
&dp(t) = rp(t)dt + \pi(t)p(t)(H_t - r)dt + \sigma\pi(t)p(t)dW_t \\
&Y(0) = y, p(0) = x_0, \pi(t) \in K, v(t) \in \tilde{K}
\end{aligned}$$

and

$$E[(p(T) + \tilde{U}'(Y(T)))^2 + \int_0^T \pi(t)v(t)dt] = 0 \tag{3.2.5}$$

Hence, we can solve

$$\hat{V} = \text{minimize}_{y,\pi,v} E[(p(T) + \tilde{U}'(Y(T)))^2 + \int_0^T \pi(t)v(t)dt] \tag{3.2.6}$$

subject to equations (3.2.4).

To solve this problem numerically, at first we need to solve $\Sigma(t)$.
Recall that:

$$\sigma_1(t) = \frac{1}{\sigma}\Sigma(t) + \rho\sigma_H$$

$$\Sigma'(t) = (1-\rho^2)\sigma_H^2 - 2(k + \frac{\rho\sigma_H}{\sigma})\Sigma(t) - \frac{1}{\sigma^2}\Sigma(t)^2$$

where $\Sigma(0) = \sigma_0$.
Since $\Sigma(t)$ has Riccati form, we can solve it explicitly.

$$\Sigma'(t) + 2(k + \frac{\rho\sigma_H}{\sigma})\Sigma(t) + \frac{1}{\sigma^2}\Sigma(t)^2 = (1-\rho^2)\sigma_H^2 \tag{3.2.7}$$

Let $a_3 = \frac{1}{\sigma^2}, b_3 = 2(k + \frac{\rho\sigma_H}{\sigma}), c_3 = (1-\rho^2)\sigma_H^2$, we have:

$$\Sigma'(t) + b_3\Sigma(t) + a_3\Sigma(t)^2 = c_3 \tag{3.2.8}$$

$$=> \Sigma'(t) + a_3(\Sigma(t) + \frac{b_3}{2a_3})^2 = c_3 + \frac{b_3^2}{4a_3} \tag{3.2.9}$$

Let $\Sigma(t) = \chi(t) - \frac{b_3}{2a_3}$, and set $\phi = \frac{b_3^2}{4a_3} + c_3$, $a_3 = \frac{1}{k_3}$, we have the ODE:

$$\frac{\partial_t \chi}{k_3\phi - \chi^2} = \frac{1}{k_3} \tag{3.2.10}$$

subject to $\chi(0) = \sigma_0 + \frac{b_3}{2a_3}$.

This ODE is of Riccati type and can be integrated exactly. Integrating both sides of above over $[0, t]$:

$$log\frac{\sqrt{k_3\phi} + \chi(t)}{\sqrt{k_3\phi} - \chi(t)} - log\frac{\sqrt{k_3\phi} + \chi(0)}{\sqrt{k_3\phi} - \chi(0)} = 2\gamma t$$

$$\frac{\sqrt{k_3\phi} + \chi(t)}{\sqrt{k_3\phi} - \chi(t)} = \frac{\sqrt{k_3\phi} + \chi(0)}{\sqrt{k_3\phi} - \chi(0)}e^{2\gamma t}$$

Let $\zeta = \frac{\sqrt{k_3\phi} + \chi(0)}{\sqrt{k_3\phi} - \chi(0)}$, we have

$$\chi(t) = \sqrt{k_3\phi}\frac{\zeta e^{2\gamma t} - 1}{\zeta e^{2\gamma t} + 1} \tag{3.2.11}$$

3.3 Implementation of Numerical Algorithm

After we get $\Sigma(t)$, we can use it to generate path for H_t, and use H_t for generating $p(t), Y(t)$. Setting $T = 1, N = 10$, and $dt = T/N = 0.1$. The parameters that we are going to optimize are $y_0, \pi(t_0), ..., \pi(t_9), v(0), ..., v(9)$.

First Model

Assuming that π and v have the form $\pi = a(t) + bH(t)$ and $v(t) = c(t) + dH(t)$, where $a(t), c(t)$ are piecewise constants within each subinterval and b, d are constants.
Let the parameters be $k = 1, c = 1, r = 0.05, \sigma = 0.8, \sigma_H = 0.5, \Sigma_0 = 0.1, \rho = 0, T = 1, \beta = 0.5, t_0 = 0, h_0 = 0.5, x_0 = 10, dt = 0.1$ (time step).

Since we have solved $\Sigma(t)$, by applying Euler Scheme, we have:

$$H_{t_{i+1}} = H_{t_i} + k(c - H_{t_i})dt + \sigma_1(t)\sqrt{dt}Z$$
$$Y_{t_{i+1}} = Y_{t_i} - Y_{t_i}rdt - Y_{t_i}(\sigma^{-1}v(t) + \sigma^{-1}(H_{t_i} - r))\sqrt{dt}Z$$
$$p_{i+1} = p_i + rp_i dt + (\pi(t))p_i(H_i - r)dt + \sigma\pi(t)p_i\sqrt{dt}Z$$

By substitute $\pi(t) = a(t) + bH(t)$, $v(t) = c(t) + dH(t)$ and $\sigma_1(t) = \frac{1}{\sigma}\Sigma(t) + \rho\sigma_H$, we have:

$$H_{t_{i+1}} = H_{t_i} + k(c - H_{t_i})dt + (\frac{1}{\sigma}\Sigma(t_i) + \rho\sigma_H)\sqrt{dt}Z_{t_i}$$
$$Y_{t_{i+1}} = Y_{t_i} - Y_{t_i}rdt - Y_{t_i}(\sigma^{-1}(c(t_i) + dH_{t_i})^+ + \sigma^{-1}(H_{t_i} - r))\sqrt{dt}Z_{t_i}$$
$$p_{i+1} = p_i + rp_i dt + (a(t_i) + bH_{t_i})^+ p_i(H_i - r)dt + \sigma(a(t_i) + bH_{t_i})^+ p_i\sqrt{dt}Z_{t_i}$$

Using numerical method, we can find the optimal parameters $(y, b, d, a(t_i), c(t_i), i = 0, ..., 9)$ such that minizing the function $E[(p(T) + \tilde{U}'(Y(T)))^2 + \int_0^T \pi(t)v(t)dt]$, where $p(T)$ and $\tilde{U}'(Y(T))$ are generated by using Euler Scheme, and the integration is estimated by using Trapezoidal Rule. To ensure that $\pi \in K$ and $v \in \tilde{K}$, we make $\pi(t_i) = \max(a(t_i)+bH_{t_i}, 0)$ and $v(t_i) = \max(c(t_i)+dH_{t_i}, 0)$ during optimization.

After we have the optimal parameters, we have:

t	0	1	2	3	4	5	6	7	8	9
$a(t)$	-0.0496	0.0051	-0.0132	0.0484	-0.0189	-0.0091	0.0102	-0.2172	-0.0782	-0.0803
$c(t)$	-0.0507	-0.1529	0.0025	-0.0068	0.1428	-0.0786	0.1237	0.0190	-0.0285	-0.0280

Table 3.1: Optimal parameters $a(t)$ and $c(t)$, N = 10

And optimal y, b, d:

$$y = 0.3369, b = 0.0397, d = 0.0175$$

Plotting $a(t)$ and $c(t)$ on graph, we have:

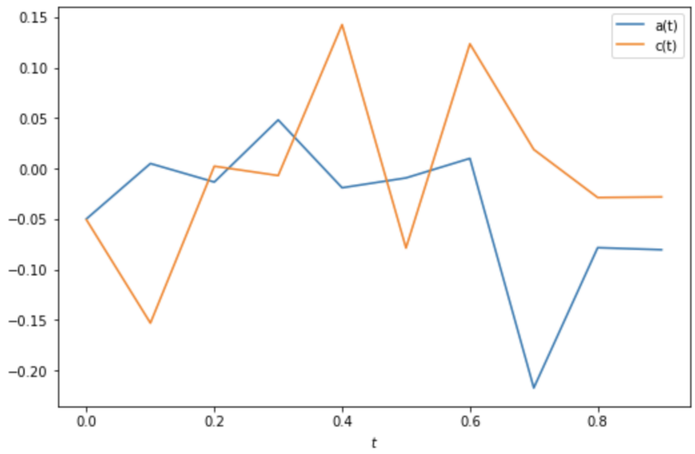

Figure 3.1: Optimal $a(t)$ and $c(t)$, N = 10

Using the optimal parameters to generate paths for H_t, we can get $\pi(t_i)$ and $\check{(}t_i)$ for $i = 0, 1, ..., 9$:

t	0	1	2	3	4	5	6	7	8	9
π	0	0.0208	0.0022	0.0613	0	0.0010	0.0222	0	0	0
v	0	0	0.0093	0	0.1477	0	0.1290	0.0244	0	0

Table 3.2: $\pi(t)$ and v generated by optimal parameters, N = 10

Recalling that from Theorem 3.10, $X^{\hat{\pi}}(t) = p(t)$. Thus, $p(t)$ is the wealth process, and $Y(t)$ is the dual process. Since $\sup_\pi E[U(X(T))] = \sup_\pi E[U(p(T))] = \min_{y,v} E[\tilde{U}(Y(T))]$, and

$$E[U(p(T))] \leq \sup_\pi E[U(p(T))] = \min_{y,v}(E[\tilde{U}(Y(T))] + xy) \leq E[\tilde{U}(Y(T))] + xy$$

Using the optimal parameters, we can generate paths for $p(T)$ and $Y(T)$, and simulate the values of $E[U(p(T))]$ and $E[\tilde{U}(Y(T))]$. Given that $E[U(p(T))]$ is the lower bound, and $E[\tilde{U}(Y(T))] + xy$ is the upper bound, if these two values are close to each other, we can approximate the true value of $\sup_\pi E[U(p(T))]$.

By simulating, we have:

$E[U(p(T))]$	6.9946
$E[\tilde{U}(Y(T))] + xy$	7.3011

Table 3.3: Values of upper bound and lower bound, N = 10

The value of $\sup_\pi E[U(p(T))]$ should fall within the interval $[6.9946, 7.3011]$.

Second Model

To enhance the model, we assume that $\pi(t) = a(t) + b(t)H_t$ and $v(t) = c(t) + d(t)H_t$, where $a(t), b(t), c(t)$, and $d(t)$ are piecewise constants within each subinterval. Using the same algorithm as before to compute the optimal value of y, π, and v, we have:

t	0	1	2	3	4	5	6	7	8	9
$a(t)$	-0.0102	-0.0281	0.0959	-0.0396	0.1259	-0.0326	0.0292	-0.0220	-0.0468	-0.0349
$b(t)$	-0.2129	-0.0219	-0.0382	-0.0438	-0.0195	0.1008	-0.0413	0.0165	-0.1293	-0.0281
$c(t)$	0.1024	0.0747	-0.0101	-0.0103	0.0313	-0.0021	0.0559	-0.0479	0.0431	-0.0073
$d(t)$	-0.0199	-0.0100	-0.0104	0.0284	-0.0319	0.0556	-0.0134	0.0662	-0.2088	-0.0255

Table 3.4: Optimal parameters $a(t), b(t), c(t)$, and $d(t)$, N = 10

And optimal y:

$$y = 0.3215$$

Plotting $a(t), b(t), c(t)$, and $d(t)$ on graph, we have:

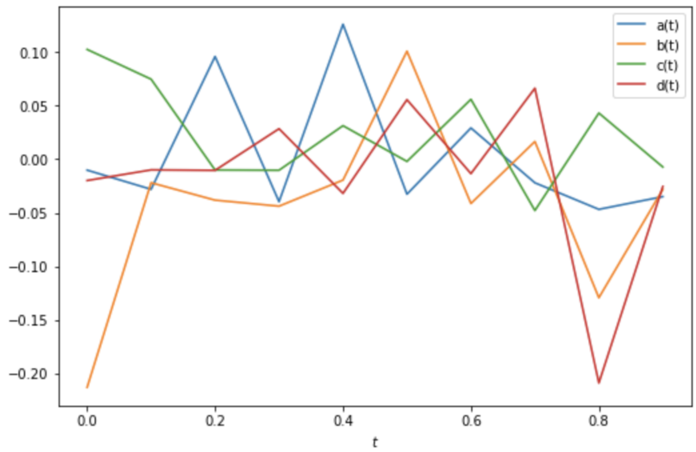

Figure 3.2: Optimal $a(t), b(t), c(t)$, and $d(t)$, N = 10

Using the optimal parameters to generate paths for H_t, we can get $\pi(t_i)$ and $v(t_i)$ for $i = 0, 1, ..., 9$:

	0	1	2	3	4	5	6	7	8	9
π	0	0	0.0796	0	0.1185	0	0.0149	0	0	0
v	0.0925	0.0700	0	0.0006	0.0191	0.0152	0.0513	0	0	0

Table 3.5: Optimal π and v, N = 10

The values of $E[U(p(T))]$ and $E[\tilde{U}(Y(T))] + xy$ are:

$E[U(p(T))]$	6.9979
$E[\tilde{U}(Y(T))] + xy$	7.3129

Table 3.6: Values of upper bound and lower bound, N = 10

By comparing the figures 3.1 and 3.2, we see that the optimal parameters are fluctuate around -0.2 and 0.15, and no special trend pattern is shown. Aiming to minimize the equation $E[(p(T) + \tilde{U}'(Y(T)))^2 + \int_0^T \pi(t)v(t)dt]$, it seems reasonable that the optimal parameters are small.

Theoretically saying, the optimal parameters should compute y, π, and v such that $E[(p(T) + \tilde{U}'(Y(T)))^2 + \int_0^T \pi(t)v(t)dt] = 0$. Therefore, the closer the value of $E[(p(T) + \tilde{U}'(Y(T)))^2 + \int_0^T \pi(t)v(t)dt]$ to 0, the better the model performs.

To verify whether the optimal parameters are reliable, we use the optimal parameters to generate paths for Y, H and p, and compute the value of $\hat{V} = \text{minimize}_{y,\pi,v} E[(p(T) + \tilde{U}'(Y(T)))^2 + \int_0^T \pi(t)v(t)dt]$. The same set of Brownian motion would be used for generating paths for Y, H, and p based on two different sets of optimal parameters, which makes the result of comparison more accurate. Thus, we have:

	$\hat{V}$
First Model	0.0050
Second Model	0.0048

Table 3.7: Values of value function, N = 10

From table 3.7, we see that the values of $\hat{V}$ from both models are quite close to zero, which means both models' optimal parameters are reasonable. Also, the second models give a better result given that its value is closer to zero in comparison to the first one.

First Model, N=20

To see the effect of time step size to models, we let $N = 20$, which means $dt = 0.05$. Then for the first model, where $\pi(t_i) = a(t_i) + bH_{t_i}$ and $v(t_i) = c(t_i) + dH_{t_i}$, $i = 0, 1, ..., 19$. Thus we have:

$$y = 0.3235, b = -0.1315, d = -0.0323$$

Considering that showing tables for $a(t)$ and $c(t)$ will be cumbersome, the figure for optimal parameter is shown as following:

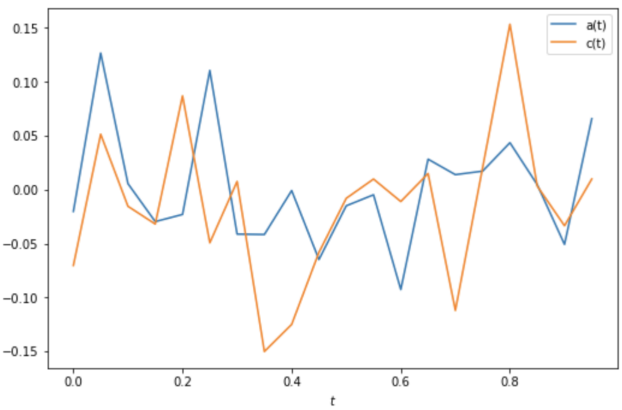

Figure 3.3: Optimal $a(t)$ and $c(t)$, N = 20

The values of $E[U(p(T))]$ and $E[\tilde{U}(Y(T))] + xy$ are:

$E[U(p(T))]$	6.9921
$E[\tilde{U}(Y(T))] + xy$	7.2600

Table 3.8: Values of upper bound and lower bound, N = 20

By comparing the first model of N = 10 and N = 20, we see that for N = 20, $a(t)$ and $c(t)$ fluctuate around -0.15 and 0.15, which have smaller volatility. Also, the interval becomes smaller, where $[6.9921, 7.2600]$ is more precise than $[6.9946, 7.3011]$.

Second Model, N =20

For the second model, where $\pi(t) = a(t) + b(t)H_t$ and $v(t) = c(t) + d(t)H_t$, we have:

$$y = 0.3161$$

And optimal $a(t), b(t), c(t)$ and $d(t)$ is shown as following:

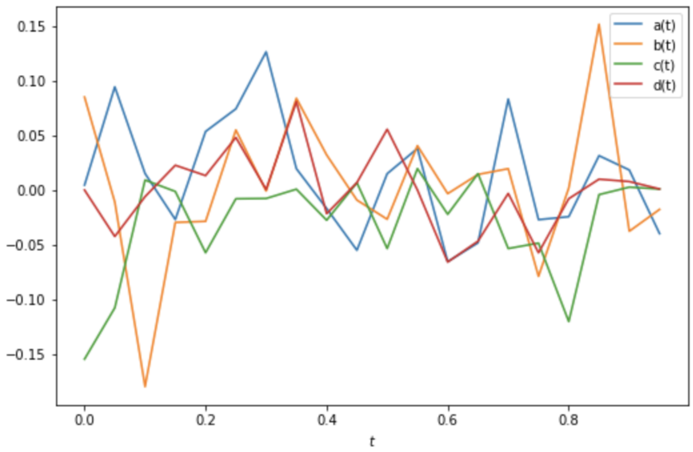

Figure 3.4: Optimal $a(t), b(t), c(t)$ and $d(t)$, N = 20

The values of $E[U(p(T))]$ and $E[\tilde{U}(Y(T))] + xy$ are:

$E[U(p(T))]$	7.0084
$E[\tilde{U}(Y(T))] + xy$	7.2419

Table 3.9: Values of upper bound and lower bound, N = 20

And the values of value function $\hat{V}$ are:

	$\hat{V}$
First Model	0.0040
Second Model	0.0035

Table 3.10: Values of value function, N = 20

By comparing the models from two different time step size, the model with smaller time step size gives a slightly better result overall, where the lower bound $E[U(p(T))]$ and upper bound $E[\tilde{U}(Y(T))] + xy$ are getting closer to each other. For tables 3.10 and 3.6, the accuracy increases from $\frac{7.3129-6.9979}{6.9979} \times 100\% = 4.5\%$ to $\frac{7.2419-7.0084}{7.0084} \times 100\% = 3.3\%$. Thus, the true value of $\sup_\pi E[U(p(T))]$ can be estimated in a more accurate way. The values of value function are still very close to 0, which ensure that the optimal parameters are reasonable.

Conclusion

In this paper, we study the utility maximization problems with and without the constraints. For each utility maximization problem, after the primal problem is formulated, we convert the primal problem to dual problem by using supermartingale approach. After that, we write down the corresponding adjoint processes for both primal and dual problems according to the book of Pham[13]. In the next step, the FBSDEs and conditions are formulated based on Li and Zheng's work[10]. The value functinos and wealth processes are solved either theoretically or by using numerical algorithms. In all three chapters, given the same model setting, the wealth processes and value functions obtained from primal, dual, and FBSDEs problems show consistence or similar pattern. Thus, we believe that we have successfully verified the connections among primal, dual, and FBSDEs in the specific utility maximization problems provided in this paper.

Bibliography

[1] J. Bismut. Conjugate convex functions in optimal stochastic control. *Journal of Mathematical Analysis and Applications - J MATH ANAL APPL*, 44:384–404, 1973.

[2] J.C. Cox and C.F. Huang. Optimal consumption and portfolio policies when asset prices follow a diffusion process. *J. Econom. Theory*, 49:33–83, 1989.

[3] J. Cvitanic and I. Karatzas. Convex duality in convex portfolio optimization. *The Annals of Applied Probability*, 2:767–818, 1992.

[4] J. Doob. Stochastic processes and statistics. *Proceedings of the National Academy of Sciences of the United States of America*, 20:376–379, 1934.

[5] E.Pardoux and S.G.Peng. Adapted solution of a backward stochastic differential equation. *Systems Control Letters*, 14:55–61, 1990.

[6] H. He and N.D. Pearson. Consumption and portfolio policies with incomplete markets and short-sale constraints: the finite-dimensional case. *Mathematical Finance*, 1:1–10, 1991.

[7] H. He and Ne. Pearson. Consumption and portfolio policies with incomplete markets and short-sale constraints: The infinite dimensional case. *Journal of Economic Theory*, 54:259–304, 1991.

[8] I. Karatzas, J. Lehoczky, and S. Shreve. Martingale and duality methods for utility maximization in incomplete markets. *Siam Journal on Control and Optimization - SIAM*, 29, 1991.

[9] I. Karatzas and S. Shreve. *Methods of Mathematical Finance*. Springer, New York, 2001.

[10] Y. Li and H. Zheng. Dynamic convex duality in constrained utility maximization. *Stochastics An International Journal of Probability and Stochastic Processes*, 90:1145–1169, 2018.

[11] R. Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics*, 51:247–57, 1969.

[12] R. Merton. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory - J ECON THEOR*, 3:373–413, 1971.

[13] H. Pham. *Continuous-Time Stochastic Control and Optimization with Financial Applications*, volume 61. Springer, New York, 2009.

[14] S.R. Pliska. A stochastic calculus model of continuous-time trading: Optimal protfolios. *Math. Oper. Res.*, 11:371–382, 1986.

[15] G.L. Xu and S.E. Shreve. A duality method for optimal consumption and investment under short- selling prohibition. i. general market coefficients. *Ann. Appl. Prob.*, 2:87–112, 1992.