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Uncertain Volatility Model for Option Pricing

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Abstract

The Black-Scholes Model has long been a concrete foundation for option pricing. However, the classic Black-Scholes model operates on a static volatility assumption, leading to an inaccuracy when dealing the real market. Therefore, we want to analyze an extension of the classic Black-Scholes model, the Uncertain Volatility Model (UVM). The UVM assumes that the volatility is no longer a known constant, but still lies within a certain range.
In this study, we will present the Uncertain Volatility Model in detail, and introduce the core idea, which is the worst-case scenario approach, proposed by the model. Additionally, we will also compare the Uncertain Volatility Model with standard Black-Scholes Model both though hypothetical cases and implementation on market data.

Keywords: Uncertain Volatility Model, Black-Scholes-Barenblatt equation, Worst-case scenario approach, option pricing

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Chapter 1

Introduction

In finance, an European option is a derivative, which grants the owner the right but not the obligation to buy (call option) or sell (put option) the underlying at a determined strike price on a specific date, which is usually called as the maturity. The origin of options can be traced back to the concept "opsies", which is introduced in 1688 on the Amsterdam stock exchange. After more than three-hundred-years development, several kinds of more complicated option have been derived from the European option. One example is the barrier option, which will be activated or expired when the underlying hits a certain threshold. In modern finance, options are mostly used as an insurance method, to avoid some unforeseeable consequences with risky underlying.

The theoretical study of pricing options has been existed for a long time. The very first theory on option pricing is the elementary option pricing theory, which is published by Charles Castelli in 1877. This theory is then further developed by S.A. Nelson in 1904 and Leonard Higgins in 1906. Certainly enough, the elementary option pricing theory is too primitive to model the option market. In order to model the option market better, Louis Bachelier became the first scholar who introduced and analyzed the Brownian motion in financial area. Bachelier developed the theory of option pricing in 1900, in which he introduced the concept of martingale into option pricing and used it to model the stochastic time evolution of price change of the underlying. After Bachelier, some more theories were published, but they all failed to make new breakthrough. The awkward situation lasted for almost 70 years until 1970s. Early of that decade, a rather complete valuation method for option came out eventually. This method, the Black-Scholes-Merton model, is based on equilibrium theoretical hypothesis, and it is innovative to assume that options can be replicated by a leveraged position in the underlying asset. Not only has the Black-Scholes-Merton model built a concrete foundation for the option pricing studies, and also had it left space for the future scholar to develop and extend it. Additionally, the market had proven that the Black-Scholes-Merton model and its extension are really useful and work fine with the reality. Then in 1976, Stephen Ross developed the Arbitrage-Pricing Theory (APT), which was another milestone in finance area. The Arbitrage-Pricing Theory also provided more theoretical supports for Black-Scholes-Merton theory and its extended model.

Although Black-Scholes Method is widely accepted by scholars and financial practitioners, the method itself and most of its early extension are based on a strong assumption that the volatility of the market is a constant. However, in reality, this is not always the case. The volatility of the market is a reflection of people‘s expectation of the future market, and it will always react to new information. Therefore it is uncertain and unpredictable, which will cause trouble when constructing parameters. In order to solve this problem, some new stochastic volatility models had been invented since 1980s, such as Hull and White model (1987), Stein and Stein model (1991), and Heston model (1993). These mentioned models all have innovative ideas on modelling the movement of the volatility, while at the same time a new Black-Scholes extension had been proposed in mathematical finance. The method proposed by M. Avellaneda, A. Levy and A. Paras is known as the Uncertain Volatility Model (UVM). The model assumes a less strict condition that the volatility of the market is restricted within a bounded set, but is otherwise not determined.

Under this assumption, the arbitrage-free price acquired by this method will not be unique. Those outcomes can be viewed as two worst-case scenario prices for buy-side and sell-side separately. The core idea of the Uncertain Volatility Model is to solve a nonlinear partial differential equation, which is a generalization of the classic Black-Sholes equation, known as the Black-Scholes-Barenblatt equations.

Because of the non-linearity, the Black-Scholes-Barenblatt equation does not have an analytical solution. However, in 1995, Avellaneda proposed an approaching numerical method which can calculate the price for an European option. In 1996, the application of the UVM on hedging a European options‘ portfolio was carried out by Avellaneda and Paras. Afte that the numerical solution of UVM for barrier options were solved by Avellaneda and Buff in 1998. Then the extension for many other derivatives of the UVM were published. Later on, in 2007, Liu founded a new uncertainty theory, which is culminant of normality, duality, subadditivity and product axioms. After the proposing of the uncertainty theory, Liu also raised the notion of uncertain process, which soon developed into the canonical Liu process. Based on the theoretical work done by Liu, a new kind of uncertain volatility model was proposed. This uncertain volatility follows a canonical Liu process and the whole model is considered as application of both the UVM and the Heston model.

The rest of the thesis is organized as follows: In Section. 2, some reviews on famous option pricing model will be presented. In Section. 3, we will do a thorough introduction to Black-Scholes model and its extension, which are the core part of solving a Uncertain Volatility Model. Then in Section. 4, the Uncertain Volatility Model and its application will be presented. Section. 5 will be the exhibition of numerical implementation, whereas the last section will be the conclusion part.

Chapter 2

Literature Review

In this section, we are going to review some famous models for option pricing.

2.1 Expected-value Theory of Bachelier

The core idea of Bachelier's method for option pricing is based on equilibrium. Bachelier proposed that the mathematical expectation of a speculator should be zero, which is a prototype of the modern notion, the efficient market.
The Expected-value Theory from Bachelier claims that the price for an European option with a determined maturity and strike price should be the expected value of that option at expiry date:

$$E\left(C_T\right) = \int_K^{\infty} (s-K)\, f(s)\, ds$$

Where C_T is the value of the option at maturity, S_T is the underlying price at expiry date, K is the strike price, and $f(S_T)$ is the future density function of the underlying stock, which assumed by Bachelier is a normal distribution
The first refinement of the Bachelier model introduces the geometric Brownian motion into the model. It assumes that the underlying stock follows a geometric Brownian diffusion process, which is characterised by

$$dS_t/S_t = \alpha dt + \sigma dW_t$$

Where W_t is a Wiener process, and σ is a parameter characterising the amplitude of the Wiener process.
With the above refinement, the future price density $f(S_T)$ follows a log-normal distribution at time T:

$$f(s) = \frac{1}{\sigma s\sqrt{2\pi T}} \exp\left(\frac{-\left\{\ln\left[s/S_0\right] - \left(\alpha - \sigma^2/2\right)T\right\}^2}{2\sigma^2 T}\right)$$

Once we plug-in the log-normal density into the Bachelier pricing model, we will get the pricing formula as follows:

$$E\left(C_T\right) = S_0 \exp(\alpha T)\Phi\left(\delta_1\right) - K\Phi\left(\delta_{-1}\right)$$
$$\delta_n = \left\{\ln\left(S_0/K\right) + \left(\alpha + n\sigma^2/2\right)T\right\}/(\sigma\sqrt{T})$$
$$n \in \{1, -1\}$$

Where $\Phi(x)$ is the cumulative density function of a standard normal distribution.
After the introduction of Brownian motion, the further refinement involves the discounting, which defines the present value of an European option should be given by:

$$V_{\text{EV}} = \exp(-rT)E\left(V_T\right)$$

where r is the risk-free interest rate.
The Bachelier model is the first mathematical finance model that analyse Brownian motion, which contributes a good start for the study of option pricing. However, the Bachelier model is quite primitive and according to [1] it loses its accuracy when modelling longer maturity options.

2.2 Black-Scholes option pricing

In 1973, Black, Scholes and Merton proposed this famous option pricing method with stochastic differential equations. In this part, we will give a short introduction to it. More detailed derivation and critics will be presented in the next chapter.
The Black-Scholes model assumes that the complete market is consist of only two asset, one risk-free asset and one risky asset. In addition, the market is assumed to have no transaction cost, which is called frictionless. The movement of the price of the risky asset follows the following stochastic diffusion:

$$dS_t = S_t \left[\mu dt + \sigma dW_t\right], \quad 0 \le t \le T$$

Where μ and σ are two constants.
The value of an European option is given by the solution of the following partial differential equation:

$$\frac{\partial V}{\partial t}(t, S_t) + \frac{\partial V}{\partial S}(t, S_t)\, rS_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(t, S_t)\, \sigma^2 S_t^2 = rV(t, S_t)$$

The solution, which is famous for its name as the Black-Scholes formula, was derived by Fischer Black and Myron Scholes. The establishment of Black-Scholes model is indeed a milestone for option pricing. Many of the following proposed models are an extension of Black-Scholes model, including our Uncertain Volatility Model.

2.3 Arbitrage Pricing Model

In 1976, Stephen Ross proposed a justification that, under arbitrage-free situation in a hypothetical market with infinite many assets, the expected return on a given asset should be a linear function of its covariance with the market portfolio. The justification from Ross set a firm foundation for the Arbitrage Pricing Model, which later became a necessary content for every finance textbook.

Before discussing the model, we want to introduce the concept of arbitrage first. Consider a sequence of portfolios $\phi(\,k)$ in the kth market segment. If the return of the portfolio $V(\,\phi(\,k)\,)$ has the following properties as $k \to \infty$:

$$EV(\psi(k)) \to \infty, \quad \mathrm{var}(V(\psi(k))) \to 0$$

we say that there exist an asymptotic arbitrage. If no sequence of portfolio satisfies those two properties, we can conclude that the market is arbitrage-free. Once we conclude that no arbitrage exists in a given market, we can in the same time confirm the existence of risk-neutral measures. Risk-neutral evaluation is an important part of the Arbitrage Pricing Model. It can set up a fair pricing rule for derivatives in an arbitrage-free market, and also can be an essential method for utility maximization process, which is an important application of the arbitrage pricing model.
According to [2], the general form of the model can be described as follows:
Consider a hypothetical market with a countably infinite number of assets. Then the Arbitrage Pricing Model gives the return R of any set of assets is

$$R = r_f + \Delta f + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \Psi)$$

where $R \in \mathbf{R}^m$, Δ is a $m \times n$ parameter matrix for the factors, ϵ follows a multi-normal distribution, and is called the idiosyncratic risk. f, as the vector of factors, is also assumed to follow a multi-normal distribution

$$f \sim \mathcal{N}(\mu, \Omega)$$

Note μ is the expected risk premium vector and Ω is the covariance matrix.
Now we further assume that the idiosyncratic risk terms are independent, then the mean and variance for R is

$$\mathbb{E}(R) = r_f + \Lambda\mu$$
$$\operatorname{Cov}(R) = \Delta\Omega\Delta^T + \Psi$$

In most of the application of the Arbitrage-Pricing-Model, we will assume that the factors can be known or at least can be observed from the data. However, the size of the factor matrix and the nature of it can change over time or over different market. Therefore, the model does have inadequacy when information is not sufficient or not up-to-date.

2.4 Heston Model

In the last decade of the 20^{th} century, many stochastic volatility models have been proposed, and also have proven to be flexible and suitable for pricing and hedging. Among many stochastic volatility models, the Heston model is the most outstanding. To describe the model, consider a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P}\right)$. Let S_t be the log of the forward underlying price, and it follows the following process:

$$\mathrm{d}S_t = \sqrt{v_t}dW_t - \frac{v_t}{2}\,\mathrm{d}t, \quad S_0 = x_0$$
$$\mathrm{d}v_t = \kappa\left(\theta_t - v_t\right)\mathrm{d}t + \xi_t\sqrt{v_t}\ \mathrm{d}B_t,$$

where v_t is the square of volatility with zero initial value and it follows a Cox–Ingersoll–Ross process. κ is a positive integer, which represents the mean reversion. θ_t is the positive long-term level. ξ is the volatility of volatility. (B_t, W_t) is a two-dimensional correlated Brownian motion on our filtered space and their correlation is

$$\mathrm{d}\langle W, B\rangle_t = \rho_t\ \mathrm{d}t$$

For further introduction, we want to discuss here more about the v_t, the square of the volatility. As we have mentioned above, v_t follows a Cox-Ingersoll-Ross process (CIR process). The CIR process is an extension of an interest rate model, which is named as Vasicek Model. Under the CIR assumption, v_t satisfies the above stochastic differential equation. $\kappa\left(\theta_t - v_t\right)$ is actually a drift factor, and it ensures that in the long run the mean reversion of the volatility will go towards θ_t with the speed κ. That is also why κ should be a strictly positive value. The second part of the differential equation $\xi_t\sqrt{v_t}$ is the standard deviation factor part. It prevents the existence of negative volatility value. In fact, if the parameters satisfy the condition that

$$2\kappa\theta_t > \xi^2$$

We will have a strictly positive volatility.
Similar to the Black-Scholes Model, Heston Model has few closed form formula when pricing European puts and calls. Among all the closed form formulas, we are going to introduce the Heston formula and the Lewis formula, which are the most famous and accepted solution in option pricing problem under the Heston Model.
Consider the underlying asset S_t:

$$S_t = e^{\int_0^t (r_s - q_s)ds} e^{X_t}$$

where the above representation is a solution to the Heston Model‘s differential equations and X_t is the characteristic function of the log-asset price.
According to Heston, the solution to the option pricing problem will be similar to Black-Scholes form:

$$\text{Call}_{\text{ Heston}}\ (t, S_t, v_t; T, K) = S_t e^{-\int_t^T q_s ds} P_1 - K e^{-\int_t^T r_s ds} P_2$$

where P_1 and P_2 are two probabilities, and they are equal to the one-dimensional integration of characteristic function.

Similar to Heston, Lewis also worked with the characteristic function, but he also applied the generalized Fourier transform. Lewis' work apply the complex integration along a line, which lies in the complex plane and is parallel to the real axis. In order to guarantee a safe integration, Lewis introduce the usage of complex number z, whose imaginary part $Im(z) = \frac{1}{2}$.
The Lewis formula for European call is:

$$\text{Call}_{\text{Heston}}\ (t, S_t, v_t; T, K) = S_t e^{-\int_t^T q_s ds} - \frac{K e^{-\int_t^T r_s ds}}{2\pi} \int_{\frac{i}{2}-\infty}^{\frac{i}{2}+\infty} e^{-izX} \phi_T(-z) \frac{dz}{z^2 - iz}$$

$$X = \log\left(\frac{S_t e^{-\int_t^T q_s ds}}{K e^{-\int_t^T r_s ds}}\right)$$

$$\phi_T(z) = \mathbb{E}\left(e^{z(X_T - X_t)} \mid \mathcal{F}_t\right)$$

Where $\phi_T(z)$ can have an explicit formula when all parameters for the Heston Model is constant. The integration part of the formula can be computed by numerical method. And the whole formula is more preferable when implementing numerical result.

Chapter 3

Black-Scholes Model and its generalization

The Black-Scholes option pricing method is regarded as the first complete option valuation under certain reasonable assumptions. The Black-Scholes option pricing method is nowadays still well-known and provides a concrete basic for the further development, including the uncertain volatility model.

This section will present the core idea of the Black-Scholes equation and its model, as well as an important generalization of it, which is an essential component of the uncertain volatility model.

3.1 Black-Scholes Model

The famous Black-Scholes Model is established on several market and asset assumptions. The assumptions of the market are:

- The market is free from arbitrage.
- Free to borrow and lend any amount of cash with the same risk-free rate.
- Free to long and short any amount of Stock.
- There is no transaction cost for any trading.

And the assumptions for the assets are:

- The risk-free rate and the volatility is constant and can be observed.
- No dividend distributed for the underlying stocks.
- The return of the asset follows a log-normal distribution.

Under the framework, consisting of the above assumptions, the idea of the Black-Scholes model can be interpreted as two core concepts. The first one is that the underlying price is a random walk evolving with time. The second one is that the market is efficient. This assumption is quite strong and might cause trouble in the application of the model in real market, and we will discuss it further when comparing with UVM (Uncertain Volatility Model).

Now, we want to express the model in the mathematical way. Since we do not want to dive too deep into the Black-Scholes model, which is not the main part of this thesis, the Black-Scholes model we will introduce below is built in an economy only with two tradable securities, the bond and the underlying stock.
Consider there exists a probability space with a filtration

$$(\Omega, \mathcal{F}, (\mathcal{F}_t : 0 \leq t \leq T), P)$$

And according to our assumption, we have a constant risk-free interest rate r. The first tradable security is the riskless bond B and its price evolution is

$$dB_t = B_t r dt, \quad B_0 = 1 \tag{3.1.1}$$

equivalently, we can write it as

$$B_t = e^{rt} \tag{3.1.2}$$

For the second security, given W_t as an $(\mathcal{F}_t, P)$-Wiener process, the value evolution of the security follows the below stochastic differential equation

$$dS_t = S_t[\mu dt + \sigma dW_t], \quad 0 \leq t \leq T \tag{3.1.3}$$

where μ and σ are positive constants, and the initial value for S is greater than zero.
The stochastic differential equation has a unique solution, which can be derive simply by using Ito's formula, which is

$$S_t \quad = \quad S_0 exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\} \quad 0 \leq t \leq T \tag{3.1.4}$$

Now we consider a candidate claim Y, whose value depends on the price of the underlying stock S. With the setting of Black-Scholes model, we can derive the famous pricing method for Y, the Black-Scholes equation and the Black-Scholes formula, which will be discussed in detail in the following subsection.

3.2 Black-Scholes Equation

Let‘s set $Y = f(S_T)$ and $V_t = V(t, S_t)$ be the value of the candidate claim at time t.
($f(x)$ is the determine function for the candidate claim. For example, if the candidate claim is an European call option, then $f(x) = (x - K)^+$, where K is the strike price for the European option.)
There are multiple ways to derive the Black-Scholes equation. Two famous derivation process are done via Ito stochastic differential equations and via Stratonovich stochastic differential equation separately. The former one is widely accepted in the mathematical finance area, while the latter one is more popular among physicists. These two derivation process are actually equivalent with only some differences in some detailed interpretation. Therefore, we will only introduce Ito‘s stochastic-differential- equation way.
To derive the famous formula, let‘s first consider a self-financing portfolio that consist of longing α unit of the candidate claim, shorting Θ unit of the stock S and Φ unit of the risk-less bond B, which is similar to the one proposed by Merton.
In the above case, the value P of the mentioned portfolio at time t is:

$$P_t = \alpha V_t - \Theta_t S_t - \Phi_t B_t \tag{3.2.1}$$

where S_t satisfies condition A and B_t satisfies condition B.

Now we make further assumption that the portfolio needs zero net investment, hence we can complete this portfolio by short-selling, borrowing and all of these incur zero transaction fee. Under this assumption, we have $P = 0$ for any $t \in [0, T]$. Then we can transform the equation (3.2.1) into

$$0 = V_t - \frac{\Theta_t}{\alpha} S_t - \frac{\Phi_t}{\alpha} B_t \tag{3.2.2}$$

$$V_t = \theta_t S_t + \phi_t B_t \tag{3.2.3}$$

where $\theta_t = \frac{\Theta_t}{\alpha}$ and $\phi_t = \frac{\Phi_t}{\alpha}$.
The equation (3.2.3) is also called as the value process.
Furthermore, by our assumption, the whole portfolio is self-financing. To express it in differential terms, we will have:

$$dV_t = d(\theta_t S_t + \phi_t B_t) \tag{3.2.4}$$

$$dV_t = \theta_t dS_t + \phi_t dB_t \tag{3.2.5}$$

Meanwhile, we should recall that V is the value of a candidate claim, which is a function depends on the underlying price S and the time. That is to say, V can be written as $V(S, t)$, and S follows the Ito SDE. Therefore we can apply Ito lemma to dV and get:
$dV(t, S_t) =$

$$(\frac{\partial V}{\partial t}(t, S_t) + \frac{\partial V}{\partial S}(t, S_t)\,\mu S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(t, S_t)\,\sigma^2 S_t^2)\,dt + \frac{\partial V}{\partial S}(t, S_t)\,\sigma S_t dW_t \tag{3.2.6}$$

Also, by combining equation (3.2.1) and equation (3.2.3), we can get that:
$\theta_t = \frac{\partial V}{\partial S}(t, S_t)$, $\phi_t = (V_t - \theta_t S_t)/B_t$
Now by applying substitution to the self-financing outcome, we will have:
$dV(t, S_t) =$

$$[V(t, S_t) - \frac{\partial V}{\partial S}(t, S_t)\,S_t]\,rdt + \frac{\partial t}{\partial S_t}(t, S_t)\,S_t(\mu dt + \sigma dW_t) \tag{3.2.7}$$

Then by combining equation (3.2.6) and equation (3.2.7), we arrive at our destination, the famous Black-Scholes partial differential equation:

$$\frac{\partial V}{\partial t}(t, S_t) + \frac{\partial V}{\partial S}(t, S_t)\,rS_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(t, S_t)\,\sigma^2 S_t^2 = rV(t, S_t) \tag{3.2.8}$$

3.3 Black-Scholes formula

We have presented the derivation of the famous Black-Scholes partial differential equation, and now we can deduce the price formula for European option, which is known as the Black-Scholes formula.
The celebrated Black-Scholes formula calculates the price for European call and put options. The derivation processes are similar for both put and call options, therefore we will take call option as example and present the deduction process for call option in details. In this case,

$$V(t, S) = C(t, S) = (S - K)^+ = max(S - K, 0) \tag{3.3.1}$$

where the K is the strike price for the option.
Before the beginning of deduction, we need to note down some essential condition for the formula. We note that the Black-Scholes PDE has the domain $[0, \infty)$, and it is essential to set up the boundary condition in order to solve the PDE. Since the strike price is also confined within positive range, it is quite obvious to see that if the price of stock drops to zero, then the value of the option will also become zero. In another word, the boundary value for S = 0 is

$$V(t, 0) = C(t, 0) = 0 \tag{3.3.2}$$

For the other boundary condition, we have slightly different situation that the price of the underlying can rise to infinity. In this case, we need to figure out the boundary condition for $S \to \infty$. For this boundary condition, we notice that K is always a constant for a given option, therefore as the value of the stock S goes to infinity, S will satisfy the condition $K \ll S$. Then the value of V, which is $(S-K)^{+}$, will gradually converge to S. Now we have the second boundary condition:

$$\lim_{x \to \infty} \frac{V(x,t)}{x} = 1 \tag{3.3.3}$$

In addition to those two boundary conditions, we still have two more parameters to clarify. The first one is the strike price K, which is a constant in the pricing function that we have mentioned before. The second one is T, the maturity, at which the owner have the right but not the obligation to buy the underlying stock.

Now we are going to present the solving process for equation (3.2.8) under the constraint (3.3.1)-(3.3.3), which is introduced by[3].

We firstly apply the change of variable on equation (3.2.8), in order to turn it into a forward parabolic equation. We let:

$$z = \ln(S/K), \quad t' = T - t \tag{3.3.4}$$

Note that the range of z and t' is

$$(-\infty < z < \infty) \quad (0 < t' < T) \tag{3.3.5}$$

After we change the variables of the equation, we get:

$$\frac{\partial V}{\partial t'} = -rV(z,t') + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial V}{\partial z} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial z^2} \tag{3.3.6}$$

Then we create a new dependent variable u:

$$u(z,t') = \exp\left[-\frac{1}{2}\left(1 - \frac{2r}{\sigma^2}\right)z + \frac{1}{8}\sigma^2\left(1 + \frac{2r}{\sigma^2}\right)(T - t')\right] V(z,t') \tag{3.3.7}$$

The newly created variable will turn the equation into the ordinary diffusion equation.

$$\frac{\partial u}{\partial t'} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial z^2} \tag{3.3.8}$$

Under the assumptions of the Black-Scholes model, the above diffusion equation has a constant diffusion coefficient $\frac{\sigma^2}{2}$. Also its initial condition is

$$u(z,0) = K\exp\left[-\frac{1}{2}\left(1 - \frac{2r}{\sigma^2}\right)z + \frac{1}{8}\sigma^2\left(1 + \frac{2r}{\sigma^2}\right)T\right]\max\left(\mathrm{e}^{z} - 1, 0\right) \tag{3.3.9}$$

And by [3], the solution to this kind of diffusion equation is standard:

$$u(z,t') = \frac{1}{\sqrt{2\pi\sigma^2 t'}} \int_{-\infty}^{\infty} u(y,0)\mathrm{e}^{-(z-y)^2/2\sigma^2 t'}\,\mathrm{d}y \tag{3.3.10}$$

Now we can reverse back the change of variable we have done in the beginning, and we get the celebrated Black-Scholes formula for European call options:

$$V_{Call}(S,t) = SN(d_1) - K\mathrm{e}^{-r(T-t)}N(d_2)$$

$$(0 \le t \le T)$$

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \mathrm{e}^{-u^2/2}\,\mathrm{d}u$$

$$d_1 = \frac{\ln(x/K) + \left(r + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

The Black-Scholes formula for European put option has a similar derivation, so we do not present it in details here. The analytical outcome for put option is

$$V_{Put}(S,t) = Ke^{-r(T-t)}N(-d_2) - S_t N(-d_1)$$

$$(0 \le t \le T)$$

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \mathrm{e}^{-u^2/2}\, \mathrm{d}u$$

$$d_1 = \frac{\ln(x/K) + \left(r + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

3.4 Critics on Black-Scholes

We have presented the financial basics and the mathematical derivation of Black-Scholes model. As the whole model has been introduced thoroughly, we are going to make some rational judgement on this famous model.

The Black-Scholes Model is indeed a milestone in the history of option pricing. It satisfies most of the basic rules in the finance world. Also it is also a good application of arbitrage-free-pricing method, martingale measure, and risk neutral probability measure. However, most of its achievements are built on the concrete assumptions, while some of the assumptions do not suit the reality perfectly. For example, there are two parameters in the Black-Scholes equation, the volatility σ and the risk-free interest rate r. According to the assumptions we mentioned in the first part of the section, both of them are presumed to be constant and observable. However, this is not always the case.

We take a look at the volatility first. The volatility is to some extent similar as the option price, because both of them reflect the expectation of the future of the market. Due to this similarity, the volatility will act like the price of the option. When new information appears in the market, the volatility reacts to it and change happens inevitably.

The other parameter, the interest rate, is facing the same situation. Although the risk-free interest rate is firmly connected to the spot rate, which makes it observable. However, when it comes to option with longer maturity, it turns to be unclear how the turn structure will be like. There are indeed many assumptions for long term interest rate model, but it is clear that we can not use one single value to represent it.

To conclude, though the Black-Scholes model performs well under all of those strong assumptions, it does have drawbacks on the construction of the parameters. Those two important parameters, the volatility and the risk-free interest rate, can not be pinned down by a fixed value, or even a deterministic function. Therefore, we are going to introduce a generalization of the Black-Scholes model in the next part.

3.5 Black-Scholes-Barenblatt Equation

In this section, we are going to introduce a developed Black-Scholes equation, which is also the fundamental part of the uncertain volatility model. As we have mentioned in the last section, the classic Black-Scholes model has problems when dealing with the pinned-down volatility. Therefore, we have new assumptions for the generalization of the classic Black-Scholes model.

To make it straightforward, still we consider our option is based on only one liquidly traded stock S, and S follows the Ito process,

$$dS_t = S_t\left(rdt + \sigma_t dB_t\right) \tag{3.5.1}$$

where B is a one-dimensional Brownian motion, and the time to maturity is T. Although we have mentioned in last section that the interest rate r can not be decided without trouble when T is long enough, still we assume in this generalization that our option is a short term option and r can be viewed as a fixed value. However, we now consider the volatility σ no long as a pinned-down value, instead a non-anticipative function such that

$$0 < \sigma_{min} \leq \sigma_t \leq \sigma_{max} \tag{3.5.2}$$

Now we denote by **P** the class of all probability measures on the set of paths $\{S_t\}_{0 \leq t \leq T}$, such that (S ito process) holds for some $\sigma \in [\sigma_{min}, \sigma_{max}]$. Also, we denote the pay-off function for the derivative at maturity by $\phi(S_T)$. Note that normally $\phi(S_T)$ should be function that calculate the discounted value for cash flows, for example

$$\phi(S_T) = \sum_{j=1}^{N} \mathrm{e}^{-r(t_j - t)} F_j\left(S_{t_j}\right) \tag{3.5.3}$$

where F_j is known as the functions of the price of the underlying stock, like $(S_{t_j} - K)^+$ for call options. Then according to the arbitrage-free pricing theorem, the value of our derivative under the uncertain volatility assumption should lie in the following range

$$W^+(S_t, t) = \sup_{P \in \mathbf{P}} \mathrm{E}_t^P\left[\phi(S_T)\right] \tag{3.5.4}$$

and

$$W^-(S_t, t) = \inf_{P \in \mathbf{P}} \mathrm{E}_t^P\left[\phi(S_T)\right] \tag{3.5.5}$$

These two values are considered to be optimal risk-averse price of the derivative at time t for ask and bid separately. And as it is pointed out by [4] that

$$W^+(\phi(S_T)) = -W^-(-\phi(S_T)) \tag{3.5.6}$$

Then we can focus on only one of them, and avoid the problem of losing the generosity. Thus, we will focus on W^+ for the rest of the section. According to [5], the solution of W^+ can be obtained by solving a dynamical programming partial differential equations with σ as the control variable. And this equation are known as the Black-Scholes-Barenblatt equation,

$$\frac{\partial W(S,t)}{\partial t} + r\left(S\frac{\partial W(S,t)}{\partial S} - W(S,t)\right) + \frac{1}{2}\sigma^2\left[\frac{\partial^2 W(S,t)}{\partial S^2}\right] S^2 \frac{\partial^2 W(S,t)}{\partial S^2} = 0 \tag{3.5.7}$$

$$W(S,T) = \phi(S)$$

$$\sigma\left[\frac{\partial^2 W}{\partial S^2}\right] = \begin{cases} \sigma_{\max} & \text{if } \frac{\partial^2 W}{\partial S^2} \geq 0 \\ \sigma_{\min} & \text{if } \frac{\partial^2 W}{\partial S^2} < 0 \end{cases}$$

The above equation can be written in the form, which is similar to the classic Black-Scholes equation

$$\begin{cases} \frac{\partial W(S,t)}{\partial t} + r\left(S\frac{\partial W(S,t)}{\partial S} - W(S,t)\right) + G\left(S^2\frac{\partial^2 W(S,t)}{\partial S^2}\right) = 0, & (S,t) \in \mathbb{R}_+ \times [0,T) \\ W^\phi(S,T) = \phi(S), & x \in \mathbb{R}_+ \end{cases}$$

where

$$G(\alpha) := \frac{1}{2}\left(\sigma_{\max}^2 \alpha^+ - \sigma_{\min}^2 \alpha^-\right) = \frac{1}{2}\left(\sigma_{\max}^2 - \sigma_{\min}^2\right)|\alpha| + \frac{1}{2}\left(\sigma_{\max}^2 + \sigma_{\min}^2\right)\alpha, \quad \alpha \in \mathbb{R}$$

and
$\alpha^+ = \max(\alpha, 0)$, $\alpha^- = \min(\alpha, 0)$
Note that, when $\sigma_{min} = \sigma_{max} = \sigma$, the above formula turns into the classic Black-Scholes formula we have discussed in the above sections. By observation, we figure out that the Black-Scholes-Barenblatt partial differential equation is fully non-linear of second order. Therefore, there exists

no close form solution, and [5] raised that it can be solved by using method of finite difference, which we will dive deep into it later on.
However, though that [5] claimed that the Black-Scholes-Barenblatt equation can be solved by method of finite difference, still we have to discover whether the Black-Scholes-Barenblatt will have solutions or even unique solution or not.
To solve this issue, we want to introduce the notion of viscosity solution, following [4]. The viscosity solution lets merely continuous functions be the solutions of the (PDE), and provides theorems for existence and uniqueness problem.

Definition 3.1
A viscosity subsolution (resp. supersolution) of the Black-Scholes-Barenblatt equation (3.5.7) on $(0, T) \times \mathbb{R}$ is an upper (resp. lower) semicontinuous function u such that, for all $(t, x) \in (0, T) \times \mathbb{R}$, $\phi \in \mathbb{C}^2((0, T) \times \mathbb{R})$ such that $u(t, x) = \phi(t, x)$ and $u < \phi$ (resp. $u > \phi$) on $(0, T) \times \mathbb{R} \setminus (t, x)$, we have

$$\partial_t u^{\varphi} + G\left(x^2 \partial_{xx} u^{\varphi}\right) \geq 0 \quad (\text{ resp. } \leq 0)$$

Definition 3.2
A continuous function is a viscosity solution of the Black-Scholes-Barenblatt equation on $(0, T) \times \mathbb{R}$, if it is both a supersolution and a subsolution.

If we set the interest rate to be zero, which is the simplest case. We can observe that the Black-Scholes-Barenblatt partial differential equation satisfies the equation condition above. Therefore, in this case, the viscosity solutions of the Black-Scholes-Barenblatt PDE on $(0, T) \times \mathbb{R}$ is automatically a viscosity subsolution and a viscosity supsolution on $(0, T) \times \mathbb{R}$. The general case can be introduced by this base case, see [4].
Then we are going to introduce the theorem for the solvability of the Black-Scholes-Barenblatt partial differential equation.
We first let $\mathbf{C}_{l,Lip}(\mathbb{R}^+)$ represent the collection of all locally Lipschitz functions ϕ on $\mathbb{R}^+$, which satisfy

$$|\varphi(x) - \varphi(y)| \leq C\left(1 + |x|^m + |y|^m\right)|x - y|, \quad \forall x, y \in \mathbb{R}_+$$

where C is a positive constant and $m \in \mathbb{N}$ is a parameter depending on ϕ. Then by the theorem introduced by [4], we have the following result.

Theorem 3.3
For each $\phi \in \mathbf{C}_{l,Lip}(\mathbb{R}^+)$ as the boundary condition, the Black-Scholes-Barenblatt equation has a unique viscosity solution u^{ϕ}

With the theorem introduced above, we can see that for each uncertain volatility model equation, we can indeed find a unique viscosity solution by numerical method. Now, the only thing we left for Black-Scholes-Barenblatt equation is the method to solve it, and we are going to introduce it in the coming section of the uncertain volatility model.

Chapter 4

Uncertain Volatility Model

The invention and development of the uncertain volatility model are the attempts to solve the deficiency of the classic pricing model. We have taken a glance at the uncertain volatility model in the last section, when we are introducing the Black-Scholes-Barenblatt equation, and now we are going to present this model and its applications in detail.

4.1 Model Description

First of all, the model of uncertain volatility have almost the same assumptions on market and assets as the Black-Scholes model, except for the volatility part. Following the last section, we believe that the price of the underlying asset is a diffusion process with dynamic coefficients.

$$dS_t = S_t \left(\mu dt + \sigma dB_t\right)$$

The first coefficient, μ, represents the spot drift and can be set as different value under different assumptions. In the setting of [6], it is a spot drift function $\mu(S,t)$, whose range lies in a certain area. By [6], the spot drift function can be set as $\mu(S,t) = r_t - d_t$, where r_t and d_t are the risk-free interest rate and the dividend rate of the asset respectively. However, for simplicity purpose, we will assume $\mu_t = r_t$, which is the risk-free interest rate and is a constant.
The second coefficient, σ, represents the volatility risk process. The assumption of σ is the most important part of the model, and the main concern for the rest of this section. It is assumed to fluctuate within a band, $[\sigma_{min}, \sigma_{max}]$, where both σ_{min} and σ_{max} are constants.
Then for each pair of μ and σ that make S satisfy the diffusion process, we will have a unique corresponding probability measure $P(\mu, \sigma)$. Let $\mathbf{P}$ denote as the set of all defined P.
Let us assume that there exists a portfolio, whose pay-off is $F_1(S_{t_1})$, $F_2(S_{t_2})$, $F_3(S_{t_3})$, ...,$F_N(S_{t_N})$ at time t_1, t_2, t_3, ..., t_N respectively. (Note that we can reduce the case to one single European option by setting $N = 1$)
Then the uncertain volatility model gives that the worst-case-scenario value for the sell-side is

$$V_{sell-side}\left(S_t, t\right) = \sup_{P \in \mathbf{P}} \mathrm{E}_t^P \left[\sum_{j=1}^{N} \mathrm{e}^{-r(t_j - t)} F_j\left(S_{t_j}\right)\right]$$

and the worst-case-scenario value for the buy-side is

$$V_{buy-side}\left(S_t, t\right) = \inf_{P \in \mathbf{P}} \mathrm{E}_t^P \left[\sum_{j=1}^{N} \mathrm{e}^{-r(t_j - t)} F_j\left(S_{t_j}\right)\right]$$

As we have introduced in the last section, both $V_{sell-side}$ and $V_{buy-side}$ are the solution to the Black-Scholes-Barenblatt partial differential equation, but with different conditional parameter σ.

$$\frac{\partial V}{\partial t} + \frac{1}{2}S^2\sigma^2\left\{\frac{\partial^2 V}{\partial S^2}\right\}\frac{\partial^2 V}{\partial S^2} + r_t S\frac{\partial V}{\partial S} - r_t V = 0$$

where $\sigma\{x\}$ is the control variable

$$\sigma^2_{sell-side}\{x\} = \begin{cases} \sigma^2_{\max} & \text{if } x \geq 0 \\ \sigma^2_{\min} & \text{if } x < 0 \end{cases}$$

and

$$\sigma^2_{buy-side}\{x\} = \begin{cases} \sigma^2_{\max} & \text{if } x \leq 0 \\ \sigma^2_{\min} & \text{if } x > 0 \end{cases}$$

Now we are going to interpret the model financially. The core idea of the financial interpretation is the concept of the worst case scenario, which actually has already appeared between lines when we are describing the model. As it is introduced by [5] and explained further by (Topper Extra), we can view the worst case scenario from two different sides.

For the worst case scenario of the sell-side, the uncertain volatility model will first maximize the value of the portfolio, and then set the case-dependent price for it. Under the construction of the model, if we have correctly captured the volatility in our band, which is $[\sigma_{min}, \sigma_{max}]$, then the sellers will be covered against adverse market behavior for sure.

As for the worst case scenario of the buy-side, the uncertain volatility model computes the minimal portfolio value, which is the opposite of the sell-side case. When the payment from the buyer is less than the pay-off valued of the portfolio, they will be covered against the adverse market behavior.

To explain this core concept more in detail, we are going to perform the delta-hedge method, in order to present this concept mathematically.

Let us study the sell-side case first and assume that the derivative can be hedged by Δ_t units of underlying stocks, which satisfies

$$dS_t = S_t\left(\mu dt + \sigma dB_t\right)$$

$$\mu = r$$

and B_t units of bonds, which satisfies

$$dB_t = B_t r dt$$

Then we have

$$B_t + \Delta S_t = V$$

Consider the portfolio in the following form, which we just move the underlying term to the right side for calculation convenience

$$B_t = V - \Delta S_t$$

Apply the Ito's Lemma, we can get

$$\mathrm{d}B_t = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)\mathrm{d}t + \left(\frac{\partial V}{\partial S} - \Delta\right)\mathrm{d}S$$

By choosing the required $\Delta = \frac{\partial V_{sell-side}(S_t,t)}{\partial S}$, we have

$$\mathrm{d}B_t = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)\mathrm{d}t$$

$$B_t = V_{sell-side}(S_t,t) - S_t \times \frac{\partial V_{sell-side}(S_t,t)}{\partial S}$$

Then by the explanation given by Avellaneda [5], if σ satisfies our initial condition, then the portfolio, which has an initial value $V = V_{sell-side}(S_t,t)$ and stick to our constraints, will have a non-negative final value after all the cash-flows get distributed. Also our portfolio is assumed to be self-financed as usual, then the portfolio can create a risk-free hedge for short position. It is also the optimal ways of hedging, since it costs the least among all the other strategies. Under this construction, the corresponding $\sigma_{sell-side}$ formulates the worst-case-scenario volatility path. Similarly, for the buy-side, we just modify the value assigned to Δ. Replacing it with

$$\frac{\partial V_{buy-side}(S_t, t)}{\partial S}$$

Then we can get a similar result that the uncertain volatility model helps us to create a long position hedge and provides a maximal initial bid price. In this case, the corresponding $\sigma_{buy-side}$ presents the worst-case-scenario path.

4.2 Strict sub-additivity and risk-diversification

In this section, we are going to introduce one of the important characteristics of the model, the strict sub-additivity of the solution. Then we plan to present the financial interpretation of this characteristic.
First of all, let us consider two derivative products Φ and Ω, which are not identical to each other.

$$\Phi = \sum_{j=1}^{N} \mathrm{e}^{-r(t_j - t)} F_j\left(S_{t_j}\right)$$

$$\Omega = \sum_{i=1}^{N} \mathrm{e}^{-r(t_i - t)} G_i\left(S_{t_i}\right)$$

Then we construct supremum of the expectation for both of them and get

$$\sup_{P\in\mathbf{P}} \mathrm{E}_t^P[\Phi] = \sup_{P\in\mathbf{P}} \mathrm{E}_t^P\left[\sum_{j=1}^{N} \mathrm{e}^{-r(t_j - t)} F_j\left(S_{t_j}\right)\right]$$

$$\sup_{P\in\mathbf{P}} \mathrm{E}_t^P[\Omega] = \sup_{P\in\mathbf{P}} \mathrm{E}_t^P\left[\sum_{i=1}^{N} \mathrm{e}^{-r(t_i - t)} G_i\left(S_{t_i}\right)\right]$$

Note that the supremum of the expectation is actually the solution of the Black-Scholes-Barenblatt partial differential equation, so they share the properties of solution of Black-Scholes-Barenblatt equation. Needless to say, $\sup_{P\in\mathbf{P}} \mathrm{E}_t^P[\Phi]$ and $\sup_{P\in\mathbf{P}} \mathrm{E}_t^P[\Omega]$ have their own Black-Scholes-Barenblatt equation respectively. As we have introduced in the last section, Φ and Ω can be considered as the boundary condition, and they are actually locally Lipschitz functions, so that there exists unique solutions for their own corresponding Black-Scholes-Barenblatt equations.
Now we are going to introduce the strict sub-additivity theorem, carried out by (BSB).
Theorem
Fix $T > 0$. For $\Phi, \Omega \in C_{l,Lip}(\mathbf{R}^+)$, we have

$$\sup_{P\in\mathbf{P}} \mathrm{E}_t^P[\Phi + \Omega] < \sup_{P\in\mathbf{P}} \mathrm{E}_t^P[\Phi] + \sup_{P\in\mathbf{P}} \mathrm{E}_t^P[\Omega]$$

if and only if there exists $(x_0, t_0) \in \mathbf{R}^+ \times (0, T)$ such that

$$u^{\Phi+\Omega}(x_0, t_0) < u^{\Phi}(x_0, t_0) + u^{\Omega}(x_0, t_0)$$

where $u^{\Phi+\Omega} = \sup_{P\in\mathbf{P}} \mathrm{E}_t^P[\Phi + \Omega]$ and u^{Φ}, u^{Ω} similarly.
The detailed proof for this theorem is presented nicely in (reference BSB).

This theorem provides us a straightforward way to check the strict-additivity property for the solution of the uncertain volatility model. In some cases, we can simply check point $(x_0, t_0) = (1, 0)$ or any other easy and special points, and we work the result out.

However, in most of the cases, the relationship for special points fails to meet the requirement and the condition turns out to be difficult to check, so, to get a more general method, we are going to introduce another theorem, which provides, instead of if-and-only-if condition, only the sufficient conditions. This simpler checking theorem is also carried out by [4].

Theorem

Fix $T > 0$. For $\Phi, \Omega \in C_{l,Lip}(\mathbf{R}^+)$, if there exists $x_0 \in \mathbf{R}^+$, such that

$$\Phi''(x_0)\,\Omega''(x_0) < 0$$

Then we have

$$\sup_{P \in \mathbf{P}} \mathrm{E}_t^P[\Phi + \Omega] < \sup_{P \in \mathbf{P}} \mathrm{E}_t^P[\Phi] + \sup_{P \in \mathbf{P}} \mathrm{E}_t^P[\Omega]$$

The detailed proof for this theorem and the counter example of the inverse statement can also be found in [4].

This theorem connects the convexity of the derivative and the strict sub-additivity, and convexity has always been an important indicator in the uncertain volatility model. Therefore, we can spend less time on checking the strict sub-additive condition, while we are processing the model.
Since, by presenting the theorems, we have shown that the strict sub-additive relationship between the solutions of the Black-Scholes-Barenblatt equations, now we want to get back to the inequality itself, to discuss the financial interpretation of it.
Similar to the situation that Black-Scholes-Barenblatt equation has two solution, one for the sell-side and one for the buy-side, there are two interpretations for inequalities for each side respectively. As we have already built a solid mathematical foundation for the sell-side inequality, we are going to discuss it first.
The sell-side inequality is

$$\sup_{P \in \mathbf{P}} \mathrm{E}_t^P[\Phi + \Omega] < \sup_{P \in \mathbf{P}} \mathrm{E}_t^P[\Phi] + \sup_{P \in \mathbf{P}} \mathrm{E}_t^P[\Omega]$$

The inequality can be understood in the aspect of diversification of volatility risk in a portfolio. More precisely speaking, it shows that if we consider two derivatives as a whole instead of individuals, the ask price will be lower. In another word, ask price of a portfolio is lower than the sum of ask price of each components in the portfolio.
Similarly, we can interpret the buy-side inequality, which follows a strict super-additivity.

$$\inf_{P \in \mathbf{P}} \mathrm{E}_t^P[\Phi + \Omega] > \inf_{P \in \mathbf{P}} \mathrm{E}_t^P[\Phi] + \inf_{P \in \mathbf{P}} \mathrm{E}_t^P[\Omega]$$

Also from the aspect of diversification, it can be viewed as which the bid price for the portfolio is higher than the sum of its components.
To get a further insight of this result in the aspect of efficiency, we can get back to the portfolio and the uncertain volatility model. When we define the equation for the uncertain volatility model, we actually set the volatility σ as a control variable, and it depends on the convexity of the value function, and takes from only two values, σ_{min} and σ_{max}. In another word, once the convexity is determined, the calculation process takes a single constant value of volatility. Hence, the supremum and the infimum in the model are calculated under a fix volatility situation respectively. If the two bounds are for standard European options, they will be carried out by standard Black-Scholes formula with two extreme volatility value. This is because that the separate calculations only involve a fix volatility, and as we have mentioned in last section, the standard Black-Scholes formula is just a special case of Black-Scholes-Barenblatt equation, where $\sigma_{max} = \sigma_{min}$. Financially speaking, the risk-averse agent will always sell the option at the highest volatility time and buy at the lowest volatility situation. However, a portfolio with more complex derivatives will have different interpretation. When the portfolio constructs of more than only European options, it can not be done separately by the classic Black-Scholes formula. Instead, we have to apply the Black-Scholes-Barenblatt on the whole set of the portfolio, and then get result for both ask and bid price simultaneously. Under this circumstance, uncertain volatility model shows its efficiency in pricing portfolios.

4.3 Volatility Risk Hedging

In the last section, we have discussed the property of uncertain volatility model and its application the diversifying risk. Now we are going to present more on the topics of managing risk.
First, let us consider a simple case of volatility risk hedging. In this case, we assume that the hedging portfolio can be built from not only the European options but any other liquid derivatives. Let Φ be the derivative that is offered to the risk-averse agent, and we want to hedge the risk. Now, suppose another derivative Ω with market price P is an asset we can use to hedge, and the option is also dependent on S. Since we need to offer Φ to the agent, we buy λ unit of the mentioned derivative Ω to hedge the short position we have, and use the delta-hedge method for the possible exposure. When we consider this above hedging portfolio, our cost rises from the price of buying λ units of Ω and the coverage of any mismatch between our hedge and the offer we need to make. To present it mathematically, it should be

$$\lambda P + \mathrm{Sup}_P \, \mathrm{E}_t^P[\Phi - \lambda\Omega]$$

Note that we definitely want to minimize our cost, so it turns out to be a minimization problem

$$\mathrm{Inf}_\lambda\{\lambda P + \mathrm{Sup}_P \, \mathrm{E}_t^P[\Phi - \lambda\Omega]\}$$

By [5] and [7], we can apply the Black-scholes-Barenblatt equation on the second term for the minimization problem. However, as we have discussed before, the Black-Scholes-Barenblatt equation has the advantage of efficiency, the another solution of the BSB system is the solution for another scenario.
Now suppose that we are not offering Φ, but holding an derivative portfolio with the combination of Φ and λ unit of Ω. In this case, what we want to do is to maximize our pay-off. Therefore, the corresponding optimization problem is

$$\mathrm{Sup}_\lambda\{\lambda P + \mathrm{Inf}_P \, \mathrm{E}_t^P[\Phi - \lambda\Omega]\}$$

By solving the optimization system, we can obtain the optimal price range. Note that, as we increase our input dimension, which in another word is that we have more derivative to use for hedging and input vector for the optimization problems, we can have narrower price range.

Since we have finished presenting the simple case, now let us take at look at more complex case with multiple options for hedging.
Suppose now we are offering another derivative Ψ, which needs to pay cash-flows $F_j(S_{t_j})$, where F is a value function of S and $j \in [1, 2, 3, ..., N]$. Since we are constructing a more complex case, we now have M European option for hedging. Their expiration dates are denoted by τ_i, where $i \in [1, 2, 3, ..., M]$, and their pay-off functions are given by $G(S_{\tau_i})$. Further, we assume that each European option can be traded in market freely at the price P_i, and we neglect the difference between bid-offer for simplicity.
Consider the same question as the simple case, we are going to buy all of those M European options, each with λ_i unit, in order to hedge our risk. Then the price for these heding option is

$$\sum_{i=1}^{M} \lambda_i C_i$$

(which can also be written in vector form.)
After the purchase for hedging, we can view the pay-off of the options as the discount for the cash-flows we need to deliver for Ψ. So if we view the liability and the portfolio as a whole, the present value can be represented by

$$\sum_{j=1}^{N} e^{-r(t_j - t)} F_j\left(S_{t_j}\right) - \sum_{i=1}^{M} \lambda_i e^{-r(\tau_i - t)} G_i\left(S_{\tau_i}\right)$$

Then we turn to the total cost of the hedging. Similar to the simple case, we consider the cost we spent on buying these options and the mismatch between our liability and the pay-off we received, which is the above formula. Let $\mathcal{C}(S_t, t, \vec{\lambda})$ denote the total cost.

$$\mathcal{C}(S_t, t, \vec{\lambda}) =$$

$$\sup_{P \in \mathcal{P}} \mathbf{E}^P \left\{ \sum_{j=1}^{N} e^{-r(t_j - t)} F_j\left(S_{t_j}\right) - \sum_{i=1}^{M} \lambda_i e^{-r(\tau_i - t)} G_i\left(S_{\tau_i}\right) \right\}$$
$$+ \sum_{i=1}^{M} \lambda_i C_i$$

As the agent, we do want to minimize the cost, so once again we have the minimization problem:

$$\mathrm{Inf}_{\vec{\lambda}} \mathcal{C}(S_t, t, \vec{\lambda}) =$$

$$\mathrm{Inf}_{\vec{\lambda}} \{ \sup_{P \in \mathcal{P}} \mathbf{E}^P \left\{ \sum_{j=1}^{N} e^{-r(t_j - t)} F_j\left(S_{t_j}\right) - \sum_{i=1}^{M} \lambda_i e^{-r(\tau_i - t)} G_i\left(S_{\tau_i}\right) \right\} + \sum_{i=1}^{M} \lambda_i C_i \}$$

The above optimization is what we called the Lagrangian Uncertain Volatility Model, or in short λ-UVM.
As $G_i(S_{\tau_i})$ and C_i are defined to be non-negative, and the total cost function $\mathcal{C}(S_t, t, \vec{\lambda})$ is defined as an upper bound, we can conclude that the total cost function is convex in $\vec{\lambda}$. Now consider the special situation that $|\vec{\lambda}| = 0$, where we do not buy any options, and recall that we are using delta-hedge method to deal with the mismatch. Therefore, in this situation, the optimization problem is simplified to the standard Black-Scholes-Barenblatt problem.

$$\mathcal{C}(S_t, t, \vec{\lambda} | \vec{\lambda} = 0) = \sup_{P \in \mathcal{P}} \mathbf{E}^P \left\{ \sum_{j=1}^{N} e^{-r(t_j - t)} F_j\left(S_{t_j}\right) \right\}$$

Note that the above optimization problem is the standard solution for sell-side of the Black-Scholes-Barenblatt equation.
Although the problem can be simplified when $|\vec{\lambda}| = 0$, the minimum of the cost function is actually obtained when $|\vec{\lambda}| \neq 0$. Recall the result we got previously that the extremum is always obtained at σ_{min} or σ_{max}, and the volatility of the options portfolio lies inside the band, which makes those options become cheaper when buying and more expensive when selling. Therefore, hedging with options can result in a cheaper cost than delta-heding only.

So far, our optimization problems are always built on an ideally Black-Scholes market, where there is no transaction cost, no limitation for buying and no bid-and-ask spread. However the risk-hedging problem is more of a real world application than a ideal market model, we want to modify the situation such that our optimization system can suit the real market better.
The first thing we want to add is the constrain on buying and selling. In the Black-Scholes assumptions, we can buy and short-sell the underlying or the derivatives without limitations, while in the real market, we will face the situation where there is not enough offer on the market. And also it is not realistic to buy or short sell infinite amount of derivatives, which could be a solution to our optimization problems. Thus it is reasonable for us to limit the variation of λ_i, and one of the rational settings for it could be

$$\Lambda_i^- \leq \lambda_i \leq \Lambda_i^+$$
$$\Lambda_i^- \leq 0$$
$$\Lambda_i^+ \geq 0$$

We let $\Lambda_i^- \leq 0$ is because we want to include short position when constructing the hedging portfolio. Also the specific range for each option can be decided by historical data, like observing the maximum daily trading volume of the option in a selected time range. Besides that, we keep the

assumption that we can trade even fractional options on the market, because it can be approached to integer level after applying a suitable scale.
The second aspect we want to talk about is the incomplete market. Transaction cost is one of the side-effects caused by the incompletenss of the market. Normally, transaction cost is proportion to the amount of derivatives we trade, and this cause the cost of hedging to rise. Another side-effect is the difference between bid and offer price. In Black-Scholes assumptions, agent can buy and sell the same underlying or derivative always at the same price. However, this is not the case in reality, the existence of bid/offer spreads also gives a rise to the cost of hedging, since you have to spend more when doing short sell. Therefore, we want to include these two factors into our hedging model.
To deal with these two factors, we consider the case for constant volatility case first. In the constant volatility case, if we want to account for the transaction cost and the bid/offer spread, we have to modify the volatility model. When the underlying asset has a constant volatility, the cost of Delta-hedging is obtained by solving a Black-Scholes-Barenblatt equation with control variable σ to be

$$\sigma\left[\frac{\partial^2\mathcal{C}}{\partial S^2}\right]=\begin{cases}\sigma\sqrt{1+\sqrt{\frac{2}{\pi}}\frac{k}{\sigma\sqrt{dt}}}, & \text{if } \frac{\partial^2\mathcal{C}}{\partial S^2}\geq 0.\\ \sigma\sqrt{1-\sqrt{\frac{2}{\pi}}\frac{k}{\sigma\sqrt{dt}}}, & \text{if } \frac{\partial^2\mathcal{C}}{\partial S^2}< 0.\end{cases}$$

Where k is the expected roundtrip transaction cost and dt is the time-lag between adjustments.
The above constant case result has been applied to the uncertain volatility case by (reference Avellaneda and Paras). Their work claims that with given the volatility band $[\sigma_{min}, \sigma_{max}]$, the adjusted volatility will have a wider range. Accordingly, the partial different equation for solving the delta hedging cost is

$$\frac{\partial\mathcal{C}}{\partial t}+\frac{1}{2}\hat{\sigma}^2\left\{\frac{\partial^2\mathcal{C}}{\partial S^2}\right\}\cdot\frac{\partial^2\mathcal{C}}{\partial S^2}+\mu S\frac{\partial\mathcal{C}}{\partial S}-r\mathcal{C}=0$$

$$\hat{\sigma}\left[\frac{\partial^2\mathcal{C}}{\partial S^2}\right]=\begin{cases}\max_{\sigma_{min}\leq\sigma\leq\sigma_{max}}\left\{\sigma\sqrt{1+\sqrt{\frac{2}{\pi}}\frac{k}{\sigma\sqrt{dt}}}\right\}, & \text{if } \frac{\partial^2\mathcal{C}}{\partial S^2}\geq 0.\\ \min_{\sigma_{min}\leq\sigma\leq\sigma_{max}}\left\{\sigma\sqrt{Max\left[1-\sqrt{\frac{2}{\pi}}\frac{k}{\sigma\sqrt{dt}}\cdot 0\right]}\right\}, & \text{if } \frac{\partial^2\mathcal{C}}{\partial S^2}< 0.\end{cases}$$

So far we have finished the modification for delta-hedging part, and now we turn to the modification for the derivative hedging part.
To include the bid/offer spread into the cost function of λ-UVM, we first assume that for the i^{th} option the bid price for it is represented by $C_i^{(b)}$, similarly the offer price is $C_i^{(o)}$.
Now in our model, the options are traded at different price at the market. To be specific, the agent buy the option with the offer price and sell the option with the bid price. Additionally, we have constraints on the amount of trading for each option.

$$\Lambda_i^- \leq \lambda_i \leq \Lambda_i^+$$

Then, to modify the cost term , we use offer price for $\lambda_i > 0$ and bid price for $\lambda_i < 0$. Thus for each option, the cost of holding the corresponding position of it is

$$\lambda_i \times C_i^o \quad if \quad \lambda_i > 0$$

$$|\lambda_i| \times C_i^b \quad if \quad \lambda_i < 0$$

To combine them, we have

$$\lambda_i\left(\frac{C_i^{(o)}+C_i^{(b)}}{2}\right)+|\lambda_i|\left(\frac{C_i^{(o)}-C_i^{(b)}}{2}\right)$$

And our total cost function is

$$\mathcal{C}^{(b/o)}\left(S_t, t; \vec{\lambda}\right) =$$

$$\sup_{P\in\mathcal{P}} \mathbf{E}^P \left\{ \sum_{j=1}^{N} e^{-r(t_j-t)} F_j\left(S_{t_j}\right) - \sum_{i=1}^{M} \lambda_i e^{-r(\tau_i - t)} G_i\left(S_{\tau_i}\right) \right\}$$
$$+\sum_{i=1}^{M}\left[\lambda_i\left(\frac{C_i^{(o)}+C_i^{(b)}}{2}\right)+|\lambda_i|\left(\frac{C_i^{(o)}-C_i^{(b)}}{2}\right)\right]$$

$$\Lambda_i^- \le \lambda_i \le \Lambda_i^+$$

So far, we have made two modifications for our Lagrangian uncertain volatility model, the constrained in λ and the adjustment for incomplete market. The model seems to be improved a lot in the aspect of fitting the real market, but it is still not enough. There are still spaces for improvement, for example replacing the static hedging strategy with a dynamic one. However, the remaining issues are not so easy to solve, so we will not discuss them here.

4.4 Calibration of the volatility band

We have discussed an important application of the uncertain volatility model in the last section, and now we want to go back to the model itself. There is an essential part we need to focus on before we go further with the model, which is the calibration of the volatility band. Although we are using a range instead of a single number to represent the volatility, over-and-under estimation can still happen when we fail to calibrate a suitable range. We might also discover some fake chance of arbitrage if we use a way too narrow band. Therefore, a just fine calibration of the volatility is a fundamental requirement for the uncertain volatility model.
In the following paragraphs, we are going to introduce four methods of calibrating the volatility. The first three are introduced by [7], and the last one is introduced by [8]
The first method is the simplest calibration, which we use two constant as the upper bound and lower bound for the volatility band. This method is closely related to the classic Black-Scholes formula. If we know certain information of an option, namely maturity, price, risk-free interest rate and the underlying price, we can calculate the implied volatility by using the classic Black-Scholes formula. By using this method, we are actually making another assumption that all the implied volatilities that come from the options, which lie in our interest range, will always stay in a certain range which we are going to figure out. Then we can use the maximum and minimum value we get from the set of implied volatility as the boundaries of our band. Mathematically speaking, it is

$$\sigma_{min} \le \sigma_{implied}(t, T_i) \le \sigma_{max}$$

where t is the current time and T_i is the maturity of this option.
In this case, we have to be cautious about the options we take into account. Since our band represents the upper and lower bound of the implied volatility set, the set we get should be as complete as we could reach, which means that we should calculate implied volatilities of all the options that could be considered as target input. An insufficient set may result in a narrower band.
The second method defines the boundaries as the functions of time. This method involves the mathematical relation between the spot volatility and the implied volatility. In this case, the boundaries can be represented as

$$\sigma_{\min} = \sigma_{\min}(t) \quad , \quad \sigma_{\max} = \sigma_{\max}(t)$$

Then we apply the relationship between forward spot volatility and the implied volatility to get the band.

$$\frac{1}{T-t}\int_t^T \sigma_{min}^2(s)ds \le \sigma_{impl.}^2(t,T) \le \frac{1}{T-t}\int_t^T \sigma_{\max}^2(s)ds$$

Note that, in this case, we can actually apply modification of transaction cost to the model, which we have just discussed in the last section. By applying the modification, we can get a strict wider volatility band.
The third method is introduced to deal with markets where the variation of the implied volatility are significantly large, when comparing options with different maturity. The method use the notion of *implied-forward-forward volatility.* Given an example, if at time t, we have two at-the-money options with different maturity, namely T_1 and T_2, we can then calculate the *implied* T_1*-to-*T_2 *volatility*, $\sigma_{implied}(T_1, T_2)$ via the following equation

$$(T_2 - t) \cdot \sigma^2_{impl.}(t, T_2) = (T_1 - t) \cdot \sigma^2_{impl.}(t, T_1) + (T_2 - T_1) \cdot \sigma^2_{impl.}(T_1, T_2)$$

Then the band can be observed by picking the supremum and the infimum of all the *implied-forward-forward volatilities.*
The last method we are going to introduce does not create an interval for the uncertain volatility, instead it assumes that the uncertain volatility follows an uncertain process. This method works for a modified uncertain volatility model, which can be consider as an uncertain Heston model. Since we have not introduced this model yet, we will only give a short insight for it here.
The model assumes that the volatility follows canonical Liu process and the stock model with uncertain volatility is

$$\begin{cases} dB_t = rB_t dt \\ dS_t = S_t \left(\mu dt + \sqrt{\sigma_t} dC_{1t}\right) \\ d\sigma_t = \kappa\left(\theta - \sigma_t\right) dt + \sigma\sqrt{\sigma_t} dC_{2t} \end{cases}$$

where C_{1t} and C_{2t} are two independent canonical Liu-process. σ_t is the volatility of the stock price, σ is the volatility of the volatility, μ is the log-drift of the stock price, and κ is the rate of reversion to the long-term price variance.
Like our model, this model does not have closed solution either. Details about the model will be introduced later on.

4.5 Application on Barrier Options

In this section, we are going to introduce the application of the uncertain volatility model to barrier options. The barrier options are path-dependent derivatives, and can be traded in call or put position normally as European options. The only different is that they become activated or invalid when the underlying hits a predetermined barrier. Four typical kinds of barrier options are introduced below.
Up-and-Out-Option: The current price of underlying is below the threshold. Once it hits the threshold, the option becomes invalid.
Up-and-in-Option: The current price of the underlying is less than the threshold. The holder can exercise the option only if the price reach above the threshold.
Down-and-Out-Option: The current price of the underlying starts above the barrier. The option is invalid when the price drops below the barrier.
Down-and-in-Option: The current price of the underlying starts above the barrier. One the price goes below the barrier, the option is activated.

Due to the nature of barrier option, its value function will have discontinuity, which cause trouble for normal delta-hedge method, especially when time is approaching expiry. Therefore, researchers usually choose portfolio of European options to hedge with barrier option. Though hedging with European option eliminate some problems, like large delta and inversed gamma exposure, we are still facing a problem of hitting the barrier. The issue happens when the barrier options in our hedging portfolio is activated or knocked-out. Because when the states of one or several barrier options change, then the portfolio is also changed. In this case, the volatility risk of the portfolio might change a lot, because it contains more or less elements.
To illustrate this problem, let us consider a simple case. Suppose our portfolio consist of one Up-and-Out option and M European options for hedging. The Up-and-Out option has constant

barrier B, and a value function without the barrier $F(S_{t_i})$, which depends on underlying stock S. The value function for European options is $G_i(S_{t_i})$, where $i \in \{1, 2, ..., M\}$, and the hedge ratio is λ_i for each option. Normally the λ-UVM value function should be

$$V^{bar}(S, t, \vec{\lambda}) =$$

$$\sup_{P \in \mathcal{P}} \mathbf{E}^P \left\{ (\sum_{j=1}^{N} e^{-r(t_j - t)} F_j(S_{t_j})) \mathbf{I}_{\{S_t < B\}} - \sum_{i=1}^{M} \lambda_i e^{-r(\tau_i - t)} G_i(S_{\tau_i}) \right\}$$

When the price of the underlying rises above the barrier, the barrier option gets knockout and the corresponding value function is

$$V^{knockout}(S, t, \vec{\lambda}) =$$

$$\sup_{P \in \mathcal{P}} \mathbf{E}^P \left\{ - \sum_{i=1}^{M} \lambda_i e^{-r(\tau_i - t)} G_i(S_{\tau_i}) \right\}$$

Since the value function has changed, we actually have two different Black-Scholes-Barenblatt equations to solve. The solution to the partial differential equation with the second function is called the residual liability and can be used as the boundary condition for V^{bar}. Then the pricing process for the whole portfolio has turned into a boundary-value problem.
However, when we have more than one barrier options in our portfolio, the above procedure must be done in a certain order. By [6], the number of boundary condition, which need to be found separately, grows quadratically in the number of barrier options. Detailed proof for it can be found in [6].

4.6 UVM with Uncertain Process

In this section, we are going to present a new model, introduced by [8]. This model is an uncertain volatility version of the Heston model. It provides us a new way of constructing the uncertain volatility. Instead of using a static or time dependent interval , it defines the volatility as an uncertain process. By doing so, the uncertain volatility is a dynamical process and can capture the unforeseeable shock that happens in the market.

4.6.1 Uncertain Process

Before the introduction of the model, we need to clarify some notion that we are going to use.
Definition
An uncertain process C_t is said to be a canonical Liu process if
(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous.
(ii) C_t has stationary and independent increments.
(iii) Every increment $C_{s+\delta t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2. The uncertainty distribution of C_t is

$$\Phi_t(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}t}\right)\right)^{-1}, x \in \mathbb{R}$$

and the following is the cumulative distribution function

$$\Phi_t^{-1}(\alpha) = \frac{t\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}$$

Definition
Given that C_t is a canonical Liu Process, and μ, $\sigma > 0$ are two real numbers.

The arithmetic Liu process is defined as

$$A_t = \mu t + \sigma C_t$$

where μ represents the drift and σ is the diffusion.
The geometric Liu process is defined as

$$G_t = exp(\mu t + \sigma C_t)$$

where the μ and σ here stands for the log-drift and log-diffusion respectively. **Definition** Given that C_t is a canonical Liu process, and $\mathbf{f}$ and $\mathbf{g}$ are two functions. Then the uncertain differential equation is defined as

$$dX_t = \mathbf{f}(t, X_t)\, dt + \mathbf{g}(t, X_t)\, dC_t$$

where the solution of it should also be a Liu process.

Some useful theorems, like existence of solution, existence and uniqueness theorem, are well-presented in [8]

4.6.2 European Option Pricing

We have briefly introduced the uncertain volatility model with uncertain process when we were discussing the calibration of volatility. In this section, we are going to present how this model works with European option pricing.
Recall that the model assumes that the volatility follows canonical Liu process and the stock model with uncertain volatility is

$$\begin{cases} dB_t = rB_t dt \\ dS_t = S_t \left(\mu dt + \sqrt{\sigma_t} dC_{1t}\right) \\ d\sigma_t = \kappa\left(\theta - \sigma_t\right) dt + \sigma\sqrt{\sigma_t} dC_{2t} \end{cases}$$

where C_{1t} and C_{2t} are two independent canonical Liu-process. σ_t is the volatility of the stock price, σ is the volatility of the volatility, μ is the log-drift of the stock price,θ is the variation of the long-term price, and κ is the rate of reversion to the long-term price variance.
Note that the system of equation above can be written in matrix form.

$$\begin{pmatrix} dB_t \\ dS_t \\ d\sigma_t \end{pmatrix} = \begin{pmatrix} rB_t \\ \mu S_t \\ \kappa\left(\theta - \sigma_t\right) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sqrt{\sigma_t} S_t \\ \sigma\sqrt{\sigma_t} \end{pmatrix} \begin{pmatrix} 0 \\ dC_{1t} \\ dC_{2t} \end{pmatrix}$$

We define

$$dX_t = (dB_t, dS_t, d\sigma_t)^T$$

$$F(t, X_t) = (rB_t, \mu S_t, \kappa(\theta - \sigma_t))$$

$$G(t, X_t) = (0, \sqrt{\sigma_t} S_t, \sigma\sqrt{\sigma_t})$$

$$C_t = (0, C_{1t}, C_{2t})$$

Then the matrix form can be written as an uncertain differential equation

$$dX_t = F(t, X_t)\, dt + G(t, X_t)\, dC_t$$

The analytical solution theorem, raised by Wang in 2012, says that the uncertain differential equation has an analytical solution only if $G(t, X_t)$ can be written as a product of an integrable process and a canonical Liu process. Apparently, the uncertain differential equation above does not have an analytical solution. Therefore, we will present the numerical solution from α-path method, which is introduced by [8]. According to α-path method,

$$S_T^\alpha = S_0 \exp\left(\mu T + \int_0^T \frac{\sqrt{\sigma_t^\alpha}\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} dt\right)$$

and σ_t^α solves the following ordinary differential equation

$$d\sigma_t^\alpha = \kappa\left(\theta - \sigma_t^\alpha\right) dt + \frac{\sigma\sqrt{\sigma_t^\alpha}\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} dt$$

Detailed proof of the α-path method can be found in [8].
Since we now have a explicit formula for the underlying S_T, we can price the European option by simply plug-in the formula.
For an European call option with strike price K and maturity T, the pricing formula is

$$V_{call} = e^{-rT} E\left[(S_T - K)^+\right]$$

By plug-in S_T^α we have

$$V_{call} = e^{-rT} \int_0^1 \left(S_T^\alpha - K\right)^+ d\alpha$$

Similarly for European put option

$$V_{put} = e^{-rT} E\left[(K - S_T)^+\right]$$

By plug-in S_T^α we have

$$V_{put} = e^{-rT} \int_0^1 \left(K - S_T^\alpha\right)^+ d\alpha$$

Chapter 5

Numerical Implementation

In this section, we are going to introduce the numerical implementation of the uncertain volatility model. We will first introduce the algorithm and present some simple cases. Then we will apply it to the real market data.

5.1 European Option Pricing

As we have discussed before, there is no analytical solution for the Black-Scholes-Barenblatt equation. Hence we can only use numerical approximation for the European option pricing. In this section, we are going to introduce an approach, introduced in [5]. This approach can be divided into two steps.
Step.1 We are going to build a *trinomial tree* to simulate the stochastic differential process of the underlying stock S_t, which satisfies

$$\mathrm{d}S_t = S_t \left(\sigma_t \ \mathrm{d}Z_t + r \ \mathrm{d}t\right)$$

where $\sigma_t \in [\sigma_{min}, \sigma_{max}]$
Step.2 Then we use the *finite-difference method* to calculate the price of the European option. We will discuss these two steps in the following part.

5.1.1 Trinomial Trees

Building trinomial trees is an efficient approach to simulate the motions of underlying. Recall that in binomial tree model, we set the parameters u and d for up-situation and down-situation, with respect to the probability p_u and p_d. The basic rule of building a binomial tree model is that we want those parameters match the assumption of geometric Brownian motion. Similarly, we also want the trinomial tree match our assumptions for uncertain volatility model.
To present the idea of trinomial tree, let us consider a model with discrete time. Under this model assumption, we have in total N trading period, and the price S of the underlying stock can jump to another state after each trading period. Mathematically speaking, S can turn into uS, mS or dS, where $d < m < u$, $m^2 = ud$. The constraints on u, m, d reduce the complexity of the trinomial tree and allow us to have a recombining tree. Now we further assume that the initial price for S is S_0, the time period is $[0, T]$, hence each trading period is $\Delta t = T/N$. Figure 5.1 shows us a simple trinomial tree with two trading periods.

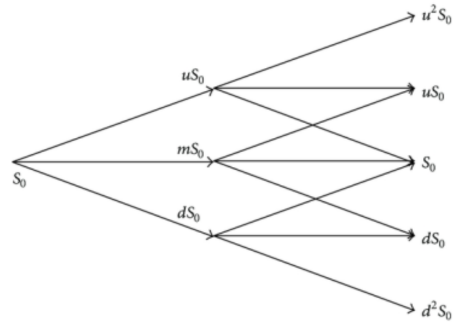

Figure 5.1: A simple trinomial tree with two trading periods

Furthermore, we let

$$
\begin{aligned}
u &= e^{\sigma_{\max}\sqrt{\Delta t}+r\Delta t}\\
m &= e^{r\Delta t}\\
d &= e^{-\sigma_{\max}\sqrt{\Delta t}+r\Delta t}
\end{aligned}
$$

The corresponding one-parameter family of pricing probability is given by

$$
\begin{aligned}
P_u(p) &= p\cdot\left(1-\frac{\sigma_{\max}\sqrt{\Delta t}}{2}\right)\\
P_m(p) &= 1-2p\\
P_d(p) &= p\cdot\left(\frac{1+\sigma_{\max}\sqrt{\Delta t}}{2}\right)
\end{aligned}
$$

where $p \in [\frac{\sigma_{min}^2}{2\sigma_{max}^2}]$
Now we are going to present a fairly simple model. Consider the underlying stock, with initial price S_0, will go though 3 trading period. According to our trinomial model, the corresponding tree is shown in graph 5.2

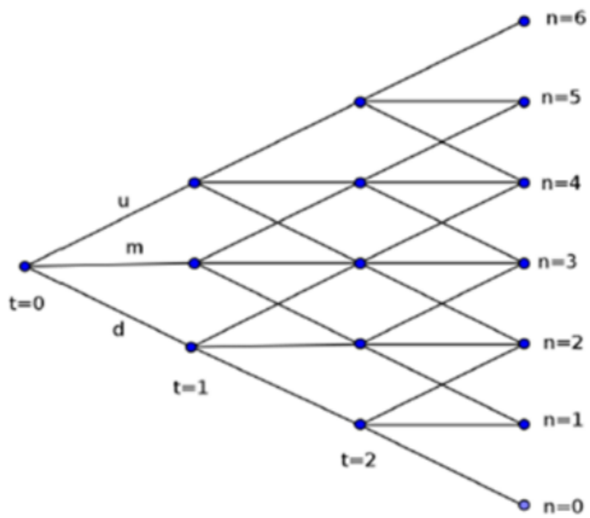

Figure 5.2: A simple trinomial tree example

To represent it in matrix form, it should be like

$$\begin{bmatrix} S_0 & S_1^{-1} & S_2^{-2} & S_3^{-3} \\ 0 & S_1^0 & S_2^{-1} & S_3^{-2} \\ 0 & S_1^1 & S_2^0 & S_3^{-1} \\ 0 & 0 & S_2^1 & S_3^0 \\ 0 & 0 & S_2^2 & S_3^1 \\ 0 & 0 & 0 & S_3^2 \\ 0 & 0 & 0 & S_3^3 \end{bmatrix}$$

Where S_n^m represents the underlying price. m is the state and n shows that it is after the n^{th} trading period. The formula for S_n^m is

$$S_n^m = S_0 \mathrm{e}^{m \cdot \sigma_{\max} \sqrt{\Delta t} + n \cdot r \Delta t}$$

To make it easy for matrix calculation, we define $S(j, n)$ to represent its location in the matrix, so our corresponding calculation formula turns into

$$S(j, n) = S_0 \mathrm{e}^{(j-n) \cdot \sigma_{\max} \sqrt{\Delta t} + (n-1) \cdot r \Delta t}$$

Consider a simple numerical example $S_0 = 100$, $T = 30/365 = 6/73$, $N = 3$ (Three trading period), $\sigma_{max} = 0.3$, and $r = 0.05$. The corresponding matrix is

$$\begin{bmatrix} 100 & 95.29 & 90.79 & 86.51 \\ 0 & 100.13 & 95.42 & 90.92 \\ 0 & 105.24 & 100.27 & 95.55 \\ 0 & 0 & 105.38 & 100.41 \\ 0 & 0 & 110.74 & 105.52 \\ 0 & 0 & 0 & 110.90 \\ 0 & 0 & 0 & 116.54 \end{bmatrix}$$

5.1.2 Finite Difference

In the last section, we have conclude that the price of the stock at each node is

$$S(j,n) = S_0 e^{(j-n)\cdot\sigma_{\max}\sqrt{\Delta t}+(n-1)\cdot r\Delta t}$$

Now consider the pricing process for an European Option with value function $F(x)$. For each node (j,n), the value of the option should be

$$F_n^j = F_n(S(j,n))$$

By the uncertain volatility model, the corresponding worst-case scenario price is

$$W_n^{\pm,j} = \text{ Sup or Inf E}\left[\sum_{k=j+1}^{N} e^{-r(t_k-t_n)}F_k(S_k)\right]$$

According to the structure of trinomial tree, the equation for $W_n^{\pm,j}$ should be like

$$W_n^{\pm,j} = F_n^j + e^{-r\Delta t}\times \text{Sup}_p \text{ or } \text{Inf}_p\left[P_u(p)W_{n+1}^{\pm,j+1} + P_m(p)W_{n+1}^{\pm,j} + P_d(p)W_{n+1}^{\pm,j+1}\right]$$

Recall that $W_n^{\pm,j}$ are the solutions of Black-Barenblatt equation, and the parameters p have maximum value $\frac{1}{2}$ and minimum value $\frac{\sigma_{min}^2}{2\sigma_{max}^2}$. By applying the finite difference method, we have:

$$W_n^{+,j} = F_n^j + e^{-r\Delta t}\begin{cases} W_{n+1}^{+,j} + \frac{1}{2}L_{n+1}^{+,j} & \text{if } L_{n+1}^{+,j} \ge 0 \\ W_{n+1}^{+,j} + \frac{\sigma_{\min}^2}{2\sigma_{\max}^2}L_{n+1}^{+,j} & \text{if } L_{n+1}^{+,j} < 0 \end{cases}$$

$$W_n^{-,j} = F_n^j + e^{-r\Delta t}\begin{cases} W_{n+1}^{-,j} + \frac{1}{2}L_{n+1}^{-,j} & \text{if } L_{n+1}^{-,j} < 0 \\ W_{n+1}^{-,j} + \frac{\sigma_{\min}^2}{2\sigma_{\max}^2}L_{n+1}^{-,j} & \text{if } L_{n+1}^{-,j} \ge 0 \end{cases}$$

where

$$L_{n+1}^{\pm,j} = \left(1 - \frac{\sigma_{\max}\sqrt{\Delta t}}{2}\right)W_{n+1}^{\pm,j+1} + \left(1 + \frac{\sigma_{\max}\sqrt{\Delta t}}{2}\right)W_{n+1}^{\pm,j-1} - 2W_{n+1}^{\pm,j}$$

Note that the above numerical solutions are actually general solutions for Black-Scholes-Barenblatt equations. When we apply the solution to European option cases, we need to note that European options can only be exercised at its maturity. Therefore, for any $W_n^{\pm,j}$, the F_n^j part is zero, except for $W_{N+1}^{\pm}$, which is the stock price at maturity. Then the adjusted formula for European option should be

$$W_{n,Euro}^{+,j} = e^{-r\Delta t}\begin{cases} W_{n+1}^{+,j} + \frac{1}{2}L_{n+1}^{+,j} & \text{if } L_{n+1}^{+,j} \ge 0 \\ W_{n+1}^{+,j} + \frac{\sigma_{\min}^2}{2\sigma_{\max}^2}L_{n+1}^{+,j} & \text{if } L_{n+1}^{+,j} < 0 \end{cases}$$

$$W_{n,Euro}^{-,j} = e^{-r\Delta t}\begin{cases} W_{n+1}^{-,j} + \frac{1}{2}L_{n+1}^{-,j} & \text{if } L_{n+1}^{-,j} < 0 \\ W_{n+1}^{-,j} + \frac{\sigma_{\min}^2}{2\sigma_{\max}^2}L_{n+1}^{-,j} & \text{if } L_{n+1}^{-,j} \ge 0 \end{cases}$$

To illustrate the process explicitly, we can represent it in matrix form.

$$\begin{bmatrix} W_3^{-3} & W_2^{-2} & W_1^{-1} & W \\ W_3^{-2} & W_2^{-1} & W_1^{0} & 0 \\ W_3^{-1} & W_2^{0} & W_1' & 0 \\ W_3^{0} & W_2^{1} & 0 & 0 \\ W_3' & W_2^{2} & 0 & 0 \\ W_3^{2} & 0 & 0 & 0 \\ W_3^{3} & 0 & 0 & 0 \end{bmatrix}$$

The first column of the matrix contains the value at maturity of the estimating derivative. In our case, we just simply apply the value function on the last column of the trinomial-tree matrix. Then we apply the backward induction process to get the value for each column until we reach the last one.
To present this method, we are going to complete the simple numerical example from last section.

S_0	100
T	$30/365 = 6/73$
K	95
σ_{max}	0.3
σ_{min}	0.15
r	0.05
N (Trading period)	3

Table 5.1: Numerical example 5.1

Then the corresponding W-matrix of the sell-side worst- case-scenario is

$$\begin{bmatrix} 0 & 0.86 & 2.99 & 5.83 \\ 0 & 2.56 & 5.61 & 0 \\ 1.75 & 5.20 & 8.77 & 0 \\ 5.21 & 8.78 & 0 & 0 \\ 8.79 & 12.49 & 0 & 0 \\ 12.50 & 0 & 0 & 0 \\ 16.34 & 0 & 0 & 0 \end{bmatrix}$$

According to the matrix, the sell-side worst-case price is 5.83
The corresponding W-matrix of the buy-side worst-case-scenario is

$$\begin{bmatrix} 0 & 0.21 & 2.13 & 5.26 \\ 0 & 1.95 & 5.22 & 0 \\ 1.75 & 5.20 & 8.77 & 0 \\ 5.21 & 8.78 & 0 & 0 \\ 8.79 & 12.49 & 0 & 0 \\ 12.50 & 0 & 0 & 0 \\ 16.34 & 0 & 0 & 0 \end{bmatrix}$$

Accordingly, the buy-side worst-case price is 5.26
Now we want to examine the stability of the method. According to our assumption, the outputs for two worst case price should become stable while we increase the size of the trinomial-tree. We will try N ranging from 2 to 100, to see whether the method converges.

Figure 5.3: Test of stability of the algorithm

Clear enough, we can observe a pattern of stability in Figure 5.3 when the N is increasing. Additionally, we can also discover that the gap between two prices become narrower during the period when they are stabilizing.

5.2 Risk diversification

As we have discussed in Chapter 4, the Black-Scholes-Barenblatt algorithm will generate the most efficient volatility path to calculate the arbitrage-free price range. In this section, we will prove this statement numerically and additionally we are also going to show that hedging with the Uncertain Volatility Model will give a better protection for future volatility movements.
To illustrate these statement, we will use a Bull-Shape call option bundle, which is the simplest case of portfolio consisting of only one long position and only one short position. Consider that we hold an European call with strike price 80, and short sell another European call with strike price 105. Both of them have the same maturity and the same underlying. The detailed market setting is listed below in Table 5.2, and the value of the bundle at maturity is illustrated clearly in Figure 5.4

Maturity T	1 (year)
long position K	80
short position K	105
σ_{max}	0.6
σ_{min}	0.1
risk-free interest rate	0.05
Trading period	50

Table 5.2: Scenario details

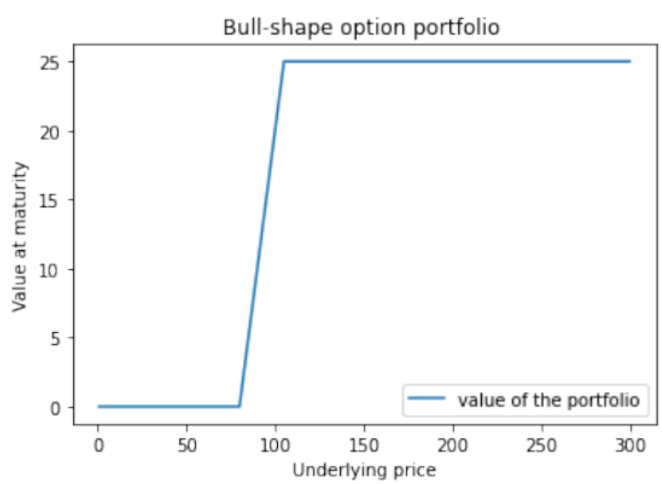

Figure 5.4: Value at maturity for the Bull-shape bundle

To demonstrate the efficiency of Black-Scholes-Barenblatt algorithm, we calculate the extreme ask- and bid-price for the bundle via BSB-algorithm and Black-Scholes formula separately. With BSB-algorithm, we are able to calculate the value of the bundle as a whole. However, when using the classic Black-Scholes formula, we need to calculate the price of those two options separately and then apply them to the extreme ask- or bid-case. When calculating the extreme case for ask, we need to use the maximum volatility for the long position and the minimum volatility for the short position, while the extreme case for bid is the opposite. Therefore, we have to apply the Black-Scholes formula four times in each iteration, which is less efficient than the BSB-algorithm. Also, as it is demonstrated by Figure 5.5, the BSB-algorithm produces more accurate range of arbitrage-free prices.

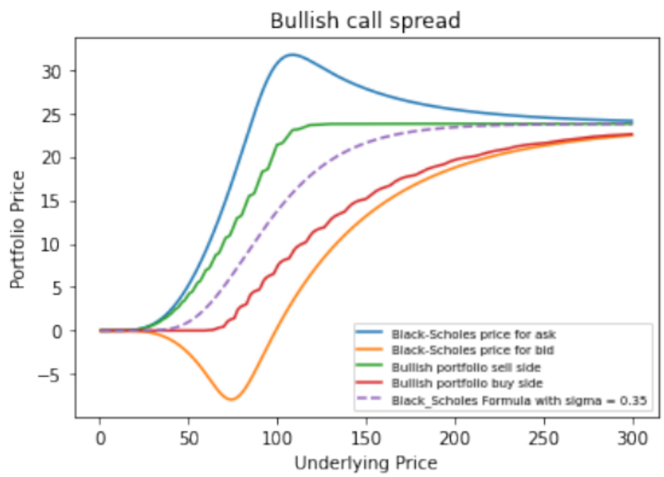

Figure 5.5: Bullish call-spread for ask and bid

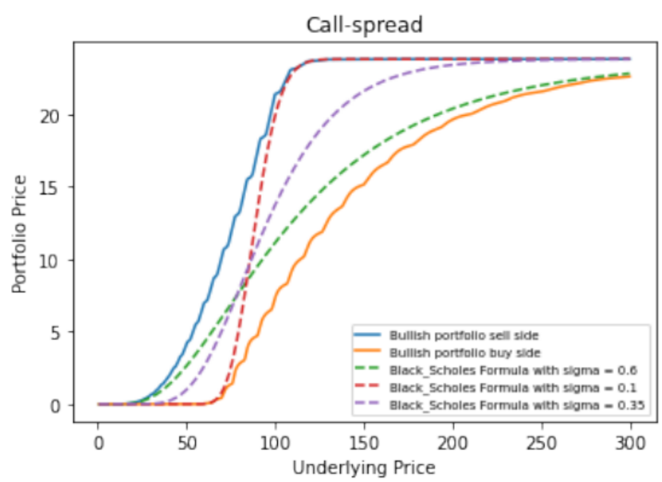

Figure 5.6: Risk-Diversification

Since we have shown that the BSB-algorithm is more efficient and accurate, now we want to prove that the BSB-algorithm also provides more help for avoiding the risk. In Figure 5.6, instead of pricing the ask- and bid- base, we use constant volatility. We price the bull-shape bundle with only the maximum volatility, the minimum volatility or the average of them. As it is shown by Figure 5.6, the maximum and minimum volatility case for Black-Scholes formula is contained by an envelope created by the BSB-algorithm. Since we assumed that the volatility can be any value within the predetermined band, the outcome from Black-Scholes formula clearly can not cover all the risk, which the portfolio might encounter. Therefore we can conclude that if the band is correctly calibrated, hedging strategy using the Uncertain Volatility Model will provide a better protection against the future volatility movement.

5.3 Calibration of the volatility band

Starting from this section, we will begin to apply the model to market data. In this section, we will calibrate the volatility band from the European options that depend on the S&P 500 index and traded at Chicago Board Options Exchange in 2019. Since the "Volatility Smile" of the implied volatility among the in-, at- and out-of-money options has been thoroughly discussed in different papers, we are going to calibrate the volatility band from another aspect. Instead of establishing a volatility band for the same maturity among different strike price, our volatility band will take boundary from the same strike price but with different time to maturity.
The call options we selected are all in-the-money options. Also, since the original data‘s time to maturity ranges from less than a week to over two years, and the implied volatility can change dramatically when the option is approaching the maturity, we decide to construct the band only for options whose time to maturity is longer than a month. The data we finally selected are the call option on S&P 500 index with strike price $K = 1000, \quad 2000, \quad 3000$.
The method of calibrating the volatility band has been introduced in the previous section. In our application, we adapt the constant boundary method and assume that all the implied volatility should be in a certain bounded set. In mathematical interpretation

$$\sigma_{min} \leq \sigma_{impl.}(t) \leq \sigma_{max}$$

Therefore, we decide to use the Black-Scholes formula to calculate the implied volatility for each option, and then observe the maximum and the minimum. To calculate the implied volatility via Black-Scholes formula, we have to first set the market price equal to the Black-Scholes option price, and then find the root of the equation, which is the implied volatility. There are multiple methods to complete the second step. In our study, we will apply the Newton‘s method. We first choose an initial value σ_0 of the implied volatility, which is equal to

$$\sigma_0 = \sqrt{\frac{2\pi}{T}} \frac{C(S,t)}{S}$$

Then we will process iteration on σ with the iteration function until we get an accurate enough outcome.

$$\sigma_{n+1} = \sigma_n - \frac{C_{\sigma_n}(S,t) - C^{mkt}}{Vega}$$

where C^{mkt} is the market price of the option and $Vega = \frac{\partial C}{\partial \sigma} = SN(d_1)\sqrt{T-t}$
In addition, since the trading date spreads over the whole year, we use a time dependent risk-free interest rate, which has a corresponding value for each month. The data for the risk-free interest rate is the Overnight Index Swap, which is provided in the appendix.
The detailed outcomes of the volatility band are presented below from Table 5.3 to Table 5.5.
For strike price = 1000:

Range of T	31/365 - 717/365
K	1000
σ_{max}	0.6669
σ_{min}	0.0996
data volume	565

Table 5.3: volatility band for strike price = 1000

For strike price = 2000:

Range of T	31/365 - 717/365
K	2000
σ_{max}	0.6202
σ_{min}	0.1010
data volume	565

Table 5.4: volatility band for strike price = 2000

For strike price = 3000:

Range of T	31/365 - 717/365
K	3000
σ_{max}	0.5884
σ_{min}	0.0988
data volume	564

Table 5.5: volatility band for strike price = 3000

5.4 Comparison with Black-Scholes Model

5.4.1 Upper and Lower difference

In this section, we want to make a simple comparison between the Uncertain Volatility Model and the Black-Scholes Model.
We will first calculate two worst-case scenario prices for different options. The size of the trinomial tree is set to be 50. Then we will use the maximum and minimum of the volatility band to calculate the Black-Scholes price separately, which correspond to the upper Black-Scholes price and the lower Black-Scholes price. Since the prices of the option change dramatically along the axis of time to maturity. It is hard to observe the an envelope directly from the price plot. Therefore, we define two terms, namely upper difference Δ_{upper} and lower difference Δ_{lower}.

$$\Delta_{upper} = P_{sup,UVM} - P_{upper,BS}$$

$$\Delta_{lower} = P_{lower,BS} - P_{sup,UVM}$$

If we have calibrated a proper volatility band, then interval between two Black-Scholes prices should fall into the interval between two worst-case scenario prices. In another word, the Black-Scholes price is contained in the envelope created by two worst-case scenarios. Hence, we should get non-nagative values for both upper difference and lower difference.

For the strike price = 1000 case:

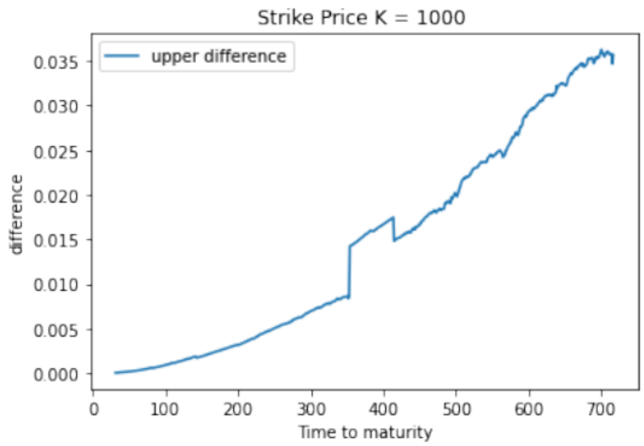

Figure 5.7: Upper-difference for strike price K = 1000

We can observe from Figure 5.7 that the upper difference Δ_{upper} increases against the time to maturity. Also the difference is always above zero, which indicates that the upper Black-Scholes price is always less than the Sell-side worst-case Scenario price.

Figure 5.8: Upper- and Lower- difference for strike price K = 1000

When comparing two differences in one graph, we can observe from Figure 5.8 that both differences are greater than zero. Hence the BS interval is inside the UVM interval, which proves our theoretical guess. Also, we can observe that the lower difference is larger than the upper difference, which indicates that the Uncertain Volatility Model cover more risk when considering the lower bound. Although both difference are not so significant when comparing the absolute value of difference to the level of the option price, which is usually above 2000 in the case $K = 1000$, the Uncertain Volatility Model can provide outstanding protection against risk when the trading is done in large amount.

For the strike price = 2000 case:

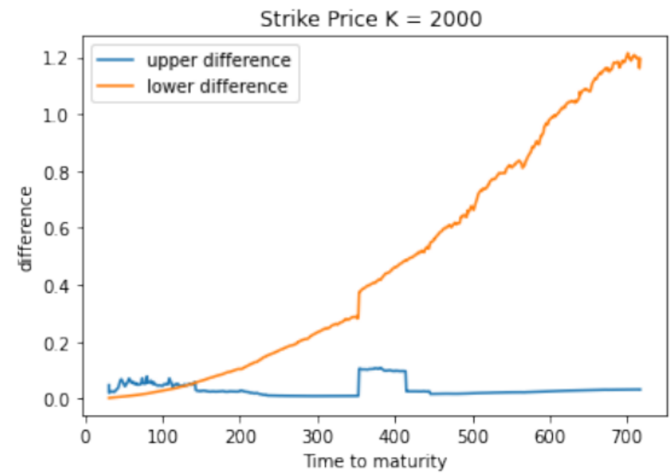

Figure 5.9: Upper- and Lower- difference for strike price K = 2000

We can observe a similar outcome in the K = 2000 case. Both upper difference and lower difference are above zero, which indicates that the outcome from Black-Scholes formula is contained by the envelope created by the Uncertain Volatility Model. Also the lower difference is significantly larger than the upper difference and the difference is small when considering from the option price scale. Additionally, we observe that the gap in $K = 2000$ case is generally smaller than those in $K = 1000$ case. This is due to a narrower volatility band for $K = 2000case$.

For the strike price = 3000 case:

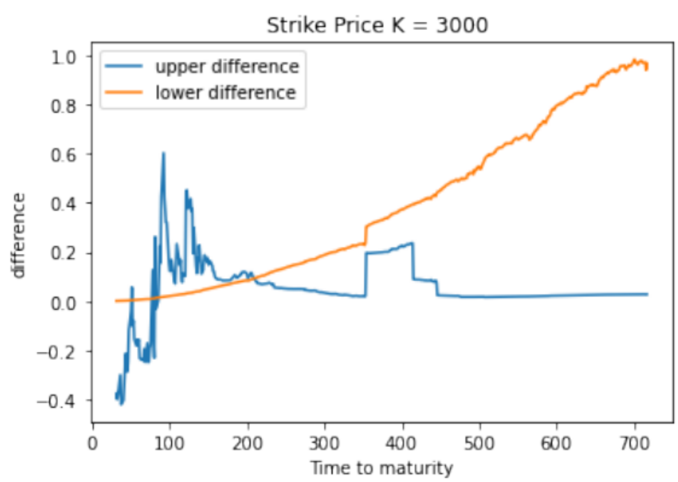

Figure 5.10: Upper- and Lower- difference for strike price K = 3000

We find a slightly different case when K increases to 3000. Especially when the time to maturity T is less than 100, the upper difference shows a negative value in most of the cases. The explanation for this anomaly might be that the strike price in this case is closer to the underlying index value, comparing with the previous two, and they are close to maturity, which indicates a more volatile environment. Therefore inaccuracy might occur when we are calculating the volatility and price. Despite the difference, we also observe a larger lower difference similar to previous two cases. And the absolute value of the difference is smaller than the previous two cases, since the volatility band for $K = 3000$ case is the narrowest.

5.4.2 Pricing Option Strategy with Market Data

We have shown via using an hypothetical example that Uncertain Volatility Model will provide an efficient and robust help when pricing and hedging portfolio. Now we are going to examine this with real market data.

In this section, we will use three S&P500 options to construct a butterfly option strategy. We will hold one call with strike price 1000 and another call with strike price 3000, and short sell two unit of the third call with strike price 2000. All of them have the same maturity date. The detailed information is listed below in Table 5.6, and the value of the bundle at maturity is demonstrated in Figure 5.11

Time to maturity T	359/365
Strike Price1 K_1	1000
Strike Price2 K_1	2000
Strike Price3 K_1	3000
σ_{max}	0.6669
σ_{min}	0.0996
Risk-free rate	1.534%

Table 5.6: Butterfly Strategy

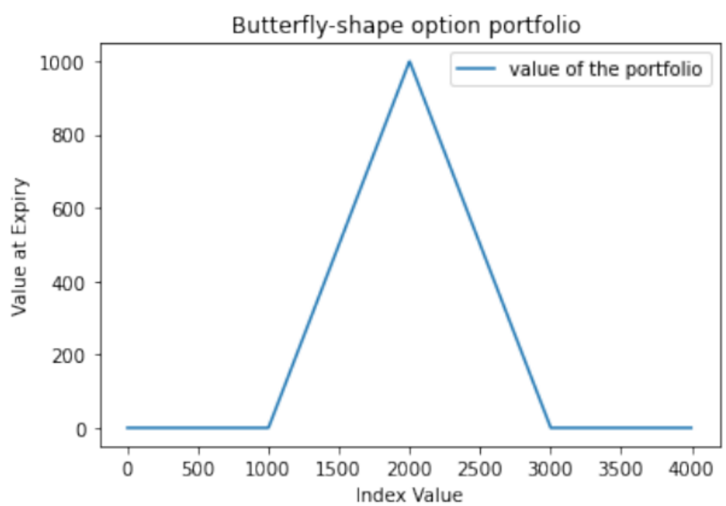

Figure 5.11: Butterfly Strategy Option Value at Maturity

Similarly, to illustrate the efficiency of the Uncertain Volatility Model and the Black-Scholes-Barenblatt algorithm, we calculate the extreme ask- and bid- case for the portfolio by using classic Black-Scholes formula. Note that we have to use different volatility for different position to get the extreme ask- and bid-spread for Black-Scholes formula. In another word, the number of applications of the Black-Scholes formula is proportional to the number of the options insider the portfolio. However, we only need to apply the BSB-algorithm once for each extreme case. Similar to our example in Section 5.2, the BSB-algorithm provides more accurate range of arbitrage-free prices, which is illustrated by Figure 5.12.

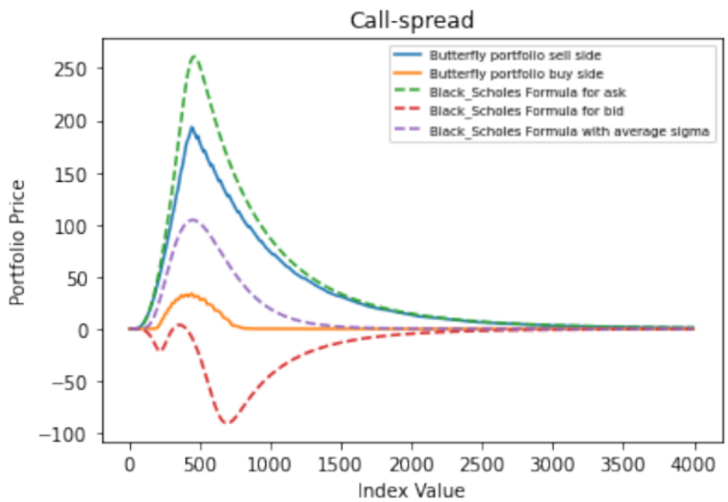

Figure 5.12: Butterfly Ask-Bid Spread

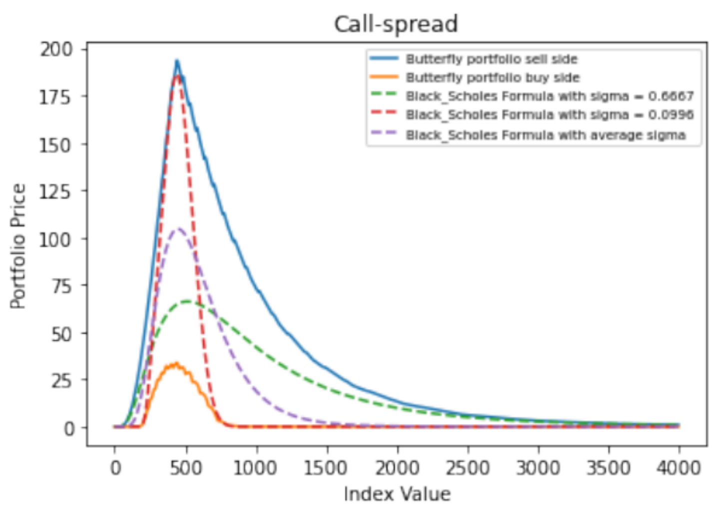

Figure 5.13: Butterfly Strategy Spread

The second step is to check whether the BSB-algorithm is more helpful when dealing with risk. Once again we use constant volatility for the whole portfolio under Black-Scholes Model. As it is demonstrated by Figure 5.13, all of those three constant volatility cases are contained by an envelop created by the BSB-algorithm. Recall that we assume that the volatility can volatile with a certain range, the Black-Scholes Model failed to provide a complete protection against risk. Once again, we have proven that Uncertain Volatility Model is a robust and efficient method to price and hedge option portfolios.

Chapter 6

Conclusion

In this study, we have introduced the Uncertain Volatility Model and have presented the numerical algorithm to price options and portfolios under the Model's construction. The worst-case scenario approach is introduced when constructing the model. By analyzing the sell-side and the buy-side worst-case scenarios, we can create an arbitrage-free price range. In order to calculate the price range, the Black-Scholes-Barenblatt equation, which is a generalization of Black-Scholes equation, is applied and the numerical method for sovling it is also presented in this study.
We also compare the Black-Scholes Model and the Uncertain Volatility Model both theoretically and numerically. In our hypothetical cases, we get perfectly expected result that the Black-Scholes-Barenblatt algorithm can provide a more accurate arbitrage free price range and a more powerful protection against risk. Although the results from market implementation encounter some exceptions when exploring relationship between price and maturity, we can still conclude that the Uncertain Volatility Model is a robust method to price and hedge options.

Appendix A

Code Script and Data

A.1 Python Script

Link for Python Script:
https://drive.google.com/file/d/1YgMJVeyePYnUXO993Ql8FXonw0lpHXJB/view?usp=sharing

A.2 Data Source

Download link for modified data:
https://docs.google.com/spreadsheets/d/1wM2XxHehwmQ7mFHeqjSY4SjO9ampPSWm/edit?usp=sharing&ouid=114412632178760823402&rtpof=true&sd=true
Download link for primitive data of S&P500 option:
https://drive.google.com/file/d/154nL3OGTGVg7zPj0SenI4kqfxjdey5LB/view?usp=sharing
Downloard link for data of risk-free interest rate:
https://docs.google.com/spreadsheets/d/1zsxwvHNdzvqIbYF8ln70u6-HUifWYSZ4/edit?usp=sharing&ouid=114412632178760823402&rtpof=true&sd=true

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GRADEMARK REPORT

FINAL GRADE

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