Imperial College London

Imperial College London

Department of Mathematics

Utility Maximization in Regime-Switching Markets with Full and Partial Information

Author:
Jiarou Li (CID: 01891595)

Supervisor:
Harry Zheng

A thesis submitted for the degree of

MSc in Mathematics and Finance, 2019-2020

Declaration

I certify that the work contained in this thesis is my own work. Any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

Acknowledgements

I would like to express my sincere gratitude to my supervisor Professor Zheng for his instructive advice and valuable suggestions on my thesis. Without his patient instruction, expert guidance, and support, the completion of this thesis would not be possible.

I also really appreciate the support from my family and my friends. It is their encouragement that helps me to finish my thesis.

Abstract

This thesis studies the utility maximization problem in a finite state regime switching market with full and partial information. In the regime switching market, the market state follows a finite state Markov chain process. And the assets in the market are subject to dynamic processes with different parameter sets under each market state. Given the incomplete market, the only information available to investors is the asset prices, while the return processes are unknown in advance. Then the asset returns depend on some hidden processes, which complex the optimal control problem that it cannot be solved by the Hamilton-Jacobi-Bellman equation or the dual control method. We first transform the original partial information problem into a full information problem by using Wonham filter. Instead of giving some analytical solutions for the optimal control problem, we suggest a dual control based Monte Carlo method to compute the lower and upper bounds of the optimal expected utility income (i.e. the optimal value function) subject to a system of forward backward stochastic differential equations (FBSDEs). The method is efficient shown with various utilty functions such as power, and non-HARA utilties.

Contents

List of Tables

Chapter 1

Introduction

1.1 Research Background

As an important field of financial engineering research, the optimal investment problem has been the focus of academic and industry research. Compared with the asset pricing problem and the risk management problem, the optimal investment problem describes the most general decision-making problem that every investor faces, and covers more aspects that need to be quantified. For example, what indicators are chosen to measure the return and risk of investments, what operational strategies are developed to allocate limited assets, and what warning lines are set to determine the timing of entry and exit, all of these are very real and urgent problems that have plagued generations of financiers and investors.

In the early days of the market, when there was no relevant theoretical analysis and quantitative tools, the traders of the major exchanges and securities firms used their accumulated experience to determine the appropriate allocation ratio and manage their portfolio assets in an extensive way. The cycle law of the real economy, the sudden events of the political economy and the development of the Society of Science and technology are all important references for investment decision-making in that era. However, with the development of theory and technology, people are increasingly not satisfied with passively accepting the arrangement of the market, expecting more reasonable quantitative indicators and investment strategies, and winning more abundant profits, the Modern Portfolio Theory (MPT) came into the market.

Modern portfolio theory, also known as mean-variance analysis, was proposed by economist Harry Markowitz (1968) [1]. Markowitz used the variance of portfolio return to characterize risk, and established a mathematical framework to maximize the average return of portfolio under a given risk. The central idea of this theory is that the risk and return of an asset should not be determined by itself, but should be taken into account its impact on the overall risk and return of the portfolio.

Efficient frontier theory is an important achievement of modern portfolio theory. By Merton (1972) [2], he takes the standard deviation of the return on the asset as the horizontal axis, and the expectation of the return as the vertical axis, and he creates a risk expectation return space, so that every possible portfolio of risky assets can be represented under that space, such that all the possible portfolios in the market define a feasible region in this space. The left boundary of this region is a hyperbola. The upper part of the hyperbola is called the efficient frontier. The portfolio on this boundary is the portfolio with the least risk given the expected return.

Capital Asset Pricing Model (CAPM) is an important derivative theory of modern portfolio theory, which was put forward independently by Sharpe (1964) [3], Lintner (1975) [4] and Mossin (1966) [5] based on Harry Markowitz's theory. For a single security, the model uses the capital market line and its relationship to the expected return and systematic risk (beta) of a single security to show how market's price affect individual securities based on the risk of a basket of securities, which establishes the link between individual securities and market portfolios.

Since the modern portfolio theory was put into use, it has been improved continuously by several generations of scholars, although it is becoming more mature, it has also been criticized. First of all, the empirical results show that the theory cannot fit the real market completely, and the assumption of efficient market and the assumption that the risk of individual securities obeys Gaussian distribution are far from the reality, which cannot effectively interpret market data and solve the actual investment problems. Secondly, the theory is based on the static analysis of multiple securities. The assumption of income distribution is not helpful for people to understand the dynamic process of investment. Moreover, due to the lack of adaptation to the new market environment, people can only use historical data to estimate parameters. In addition, use standard deviation to describe the risk are too general, as it includes both the rising and losing in returns. In reality, investors are more concerned about the risk of falling losses than the volatility of earnings.

In the dynamic management of assets, the introduction of stochastic control theory has opened a new era of financial analysis. Stochastic control theory is a branch of cybernetics, which mainly deals with the optimization problems of dynamical systems driven by observable uncertainties or noises. In the application of stochastic control theory, it is generally assumed that the dynamical system satisfies the Bayesian probability, that the development and observation of state variables are affected by the random noise with a definite probability distribution. The main goal is to design the path of the control variable with respect to time, so that the optimization problem can be solved with minimum cost without the influence of noise. From the application form, generally divided into discrete time and continuous time two categories, which can fit the frequency of data well.

When applying stochastic control theory to continuous time financial markets, the state variable in the Stochastic differential equation is usually wealth or net market value, and the control variable is usually the distribution of different assets at each moment. At any given moment, the wealth process usually consists of two parts, one is the stochastic return on the risky assets, the other is the interest return on the riskless assets. Since the 1970s, the application of stochastic control theory in finance has made a breakthrough development. Merton (1975) [6] first used stochastic control theory to study the optimal portfolio of risky assets and riskless assets, his work had a profound impact on the financial field. These theories and methods were then applied by J.L. Stein to the analysis of the financial crisis of 2007-2008.

Stochastic control theory has been developed and improved in the application of financial markets, but there are still many research gaps, such as how to describe the optimal investment problem in incomplete markets; whether there is a simple analytic solution to the optimal strategy of utility function and the return on investment; how to calculate the non-linear partial differential equation effectively when there is no analytic solution; how to extend the low dimensional solution to high dimension, these problems have inspired subsequent scholars to continue to challenge.

1.2 Current research

Stochastic control strategy is generally used to take care of the powerful enhancement issue of monetary resources, and has great execution in other application fields. Its fundamental thought is to apply the standard of dynamic programming to the ideal worth capacity and infer the halfway differential condition it fulfills, this condition is known as Hamilton Jacobi Bellman (HJB) condition. If we can find the classical solution of the HJB equation, we can prove that it is a value function of martingale principle optimization, and find an optimal control with feedback. In addition, we can prove that the value function is a unique solution for the HJB equation and approach it by numerical method.

HJB equation is a complete nonlinear partial differential equation, its solvability depends greatly on the terminal condition. For the power or logarithmic utility function under the complete market assumption of Black Scholes, we know that there is a classical closed solution for the HJB equation. Bian and Zheng (2015) [7] use the dual control method to show there is a classical solution to the HJB equation for a broad class of utility functions and give a representation of the solution to the HJB equation in terms of that of the dual HJB equation.

The regime-switching model is popular in the modeling and analysis of financial data because it allows the parameters of the asset price process to satisfy a finite state Markov chain. It has been shown in the literature that Markov chain process (MCP) is effective in providing information of the market environment. For example, Hamilton (1989) [8] introduces a regime-switching model for non-stationary time series and business cycles. Moreover, the regime-switching model not only has a strong adaptability in describing the macro market uncertainty, but also has a strong operability in the analysis of the underlying asset dynamic process. For example, Zhang et al. (2005) [9] and Yin et al. (2006) [10] study the trading rules in the regime-switching market. Honda (2003) [11], Sass and Haussmann (2004) [12] solve portfolio optimization problems with partial information and regime-switching drift processes.

In Ma et al. (2017) [13], this paper focus on extend the dual control method to the regime-switching model. And it introduces a new numerical method to find the value of the optimal problem. The dual value function in the dual space is a pricing problem that can be simply computed by Monte Carlo method by derive the tight upper and lower bounds for the HJB equaiton with state transition. Moreover, in Zhu and Zheng (2021) [14], they extend the general market model into a partial informed market, and instead of giving analytical solutions for general settings, they propose efficient methods for giving tight lower and upper bounds of the primal value functions and estimated controls based on the primal-dual frameworks.

Inspired by these work, we will extend the problem into the regime-switching model with partial information and try to derive the tight bounds for the optimal value functions.

1.3 Purpose and significance of the study

Financial markets are full of cyclical changes. The growth and harvest of agricultural products in different seasons affects the prices of options and futures; the boom and bust of the real economy causes the alternation of bull and bear markets in the stock market; the election of a new government and the change of policies indirectly cause the rise or decline of the industry, all this reveals a cyclical trend in security returns and risk, which investors have to take into account when investing.

In describing market periodicity, the regime-switching model is a mature model. Taking the continuous-time and finite-state regime-switching model as an example, it simulates the market state through the continuous-time finite-state Markov chain. The transition between different market states satisfies Poisson jump, and the subsequent state is determined by the generating matrix in different market conditions; while the return of risk assets and risk, risk-free asset returns will be different. For example, when the number of states in the security is two, it can simulate the different performances of the bear market and the bull market. When the number of states in the Markov chain is four, it can simulate the economic cycles of boom, recession, depression and recovery, the performance of security.

The aim of this paper is to study the optimal investment problem in the regime-switching markets. For certain utility function, an efficient algorithm for calculating the optimal investment return and the optimal investment strategy is designed by means of dual control theory and Monte Carlo method. In the hope to reveal the trade-off and investment behavior of different investors through the numerical calculation of different utility functions, so as to provide reasonable and feasible strategies for their investment.

From a theoretical point of view, the method studied in this paper broadens the application scope of stochastic control theory. For utility function, which does not have an analytic solution, provides a fast and efficient numerical method for solving a class of problems that can not be optimized before. From the point of view of application, the methods in this paper provide effective investment strategies and suggestions for medium-and long-term investors. For short-term investors, we can discretize the range of the volatility of risk asset's short-term rate of return and set it as a finite state, and use this paper's method to achieve the optimal investment strategy and returns.

1.4 Main research

In this paper, we mainly study the optimal investment problem in the regime-switching market both with full and partial information, and introduce the regime-switching model into the stochastic control problem.

In theory, the Dual control theory is used to transform a set of nonlinear fully coupled partial differential equation into a linear partial differential equation through a dual transformation, thus simplifying the corresponding calculation. Followed by Ma et al. (2017) [13], when dealing with full informed regime-switching markets, we use the same Monte Carlo algorithm to determine the expected payoff at the termination time and the optimal strategy in the path. The upper and lower bounds generated by this method can be well approximated for utility functions where there are analytic solutions for expected returns and investment strategies. The tight upper and lower bounds generated by this method are stable and convergent. This paper presents a feasible theory and a stable convergence algorithm for the optimal investment problem under the certain utility function in the regime-switching market.

As most papers focus on frictionless markets, while in practice due to information asymmetry and trading strategy constraints, such as short selling, borrowing and endowments, the optimal controls under perfect market assumption may not be realized. Thus, further in this paper, we will discuss the same regime-switching markets with partial information. To find an explicit form solution for utility maximization problem with partial information is quite hard. We first convert the partial information problem into a full information problem by using an innovation process and use standard filtering techniques of Wonham filter to capture the hidden processes. Under transformed full information setting, we intend to give optimal controls using stochastic control method and dual control method. However, both method failed to find an explicit solution to the problem. Instead of finding the primal control directly, we turn to use an optional method to give some tight bounds for the optimal value. This new optional method is based on the Stochastic maximum principle and Forward Backward Stochastic Differential Equations (FBSDEs). By applying this method under full space setting, we will be able to provide a tight upper and lower bounds for the optimal value function. This upper and lower bound method can be applied to general utility functions, non-HARA utilities. This upper and lower bound is very tight, and can be used as an estimate of the value function.

The paper is organized as follows. In Chapter 2, we state the general utility maximization problem under both general markets and fullly informed regime-switching markets. The stochastic control method is introduced with some examples. Chapter 3 extends into using Dual Control method to solve the optimal problems along with some analytical examples. Chapter 4 introduces in detail the Monte Carlo algorithm defined by Ma et al. (2017) [13] and use this as a benchmark method for our further research. Chapter 5 extends the market into a partially informed regime-switching markets, and provide the filtering results under Wonham filter. Two different methods for the transformed optimization problem, including stochastic control and dual control method are provided in this chapter. In Chapter 6, stochastic maximum principle and the approximated methods are provided in detail. Chapter 7 illustrates some numerical examples under both fully and partially informed regime-switching markets, with the comparison between them. Chapter 8 concludes the paper.

Chapter 2

Basic Utility Maximization Problem

In this section, we formulate the stochastic control problem in the markets, and will introduce the basic theory of utility maximization and how to solve the maximization problem.

2.1 Primal Optimization under General Market model

In this section, we first introduce the general market and the basic theory of utility primal maximization problem.

2.1.1 Market Model Set Up

Assume market has two assets, a risk-free asset (savings account) S_0, a risky asset S, satisfying the following SDE:

$$\begin{aligned} dS_0(t) &= rS_0(t)dt \\ dS(t) &= S(t)(\mu dt + \sigma dW(t)) \end{aligned} \tag{2.1.1}$$

where $S_0(0) = 1$, $S(0) = S > 0$, $r, \mu, \sigma > 0$, all constants, W is a standard Brownian motion. Assume π is an $\mathcal{F}_t$-adapted process such that $\int_0^T \pi_t^2 dt < \infty$ a.s. and π is proportional portfolio process. We denote X_t as the investor's wealth at time t. Then $\pi_t X_t$ is the amount of money invested in S and $(1 - \pi_t)X_t$ is the amount of money in savings account S_0. The wealth process X_t satisfies the following SDE:

$$\begin{aligned} dX_t &= (1 - \pi_t)X_t r dt + \pi_t X_t(\mu dt + \sigma dW_t) \\ &= rX_t dt + \pi_t X_t((\mu - r)dt + \sigma dW_t) \end{aligned} \tag{2.1.2}$$

with initial wealth $X_0 = x$.

Utility Function: $U(\cdot)$ is a utility function that $U : R_+ \triangle [0, \infty) \mapsto R$ satisfies $U \in C^1$, strictly concave, strictly increasing with Inada's condition

$$U'(0) = \lim_{x \to 0} U'(x) = \infty, \quad U'(\infty) = \lim_{x \to \infty} U'(x) = 0$$

If a utility function has constant Arrow Pratt coefficient of relative risk aversion $-\frac{xU''(x)}{U'(x)}$ for all x, then the utility function is called a **Constant relative risk aversion (CRRA)** Utility function. There are two well known CRRA utilities:

Power utility: $U(x) = \frac{1}{p}x^p$ for $x > 0$, where $p < 1$ and $p \neq 0$

Logarithmic utility: $U(x) = \ln x$ for $x > 0$.

Utility maximization problem: Define the value of the expected utility maximization problem as

$$V = sup_{\pi} E[U(X_T^{\pi})] \tag{2.1.3}$$

If there is a π^* such that $E[U(X_T^{\pi^*})] = V$, then π^* is called the optimal control strategy. One of the most commonly studied problems in mathematical finance is to find π^*.

2.1.2 Stochastic Control Method

To solve this portfolio optimization problem, we need to apply the dynamic programming principle (DPP) to the optimal value function V and show that it satisfies a partial differential equation (PDE), called the Hamilton–Jacobi–Bellman (HJB) equation, detail can be found in Bellman & Kalaba (1965)[15] and the notes from Zheng (2019)[16].

Dynamic programming principle (DPP):
For $t \geq 0, x \in R$, define the value vuction $V : [0, T] \times R \mapsto R$:

$$u(t, x) = sup_{\pi} E[U(X^{\pi}(T))|X(t) = x] \tag{2.1.4}$$

Assume on $[t, t+h]$, we choose any control process π and X evolves from (t, x) to $(t+h, X_{t+h}^{\pi})$ such that

$$\begin{aligned} E[U(X^{\pi}(T))|X(t) = x] &= E[E[U(X^{\pi}(T))|\mathcal{F}_{t+h}]|X(t) = x] \text{ (tower property)} \\ &= E[E[U(X^{\pi}(T))|X_{t+h}^{\pi}]|X(t) = x] \text{ Markov property} \\ &\leq E[u(t+h, X_{t+h}^{\pi})|X(t) = x] \end{aligned} \tag{2.1.5}$$

which implies that

$$u(t, x) \leq sup_{\pi} E[u(t+h, X_{t+h}^{\pi})|X(t) = x] \tag{2.1.6}$$

Note that now π is between $[t, t+h]$, for all $\xi > 0$, there exists $\pi^{\xi} \in [t+h, T]$ such that

$$u(t+h, X_{t+h}) - \xi < E[U(X_T^{\pi^{\xi}})|X_{t+h}] \tag{2.1.7}$$

and define

$$\hat{\pi}_s = \begin{cases} \pi_s, & t \leq s \leq t+h, \\ \pi_s^{\xi}, & t+h \leq s \leq T \end{cases} \tag{2.1.8}$$

Then apply Markov property,

$$\begin{aligned} u(t, x) &\geq E[U(X_T^{\hat{\pi}})|X_t = x] \\ &= E[E[U(X_T^{\hat{\pi}})|X_{t+h}]|X_t = x] \\ &= E[E[U(X_T^{\pi^{\xi}})|X_{t+h}]|X_t = x] \\ &\geq E[u(t+h, X_{t+h}) - \xi|X_t = x] \\ &= E[u(t+h, X_{t+h})|X_t = x] - \xi \end{aligned} \tag{2.1.9}$$

which implies

$$u(t, x) \geq sup_{\pi} E[u(t+h, X_{t+h})|X_t = x] - \xi \tag{2.1.10}$$

Let $\xi \downarrow 0$, we have

$$u(t, x) \geq sup_{\pi} E[u(t+h, X_{t+h})|X_t = x] \tag{2.1.11}$$

Then by equations (2.1.6) and (2.1.11), the DPP is defined as:

$$u(t,x) = sup_{\pi} E[u(t+h, X_{t+h})|X_t = x] \tag{2.1.12}$$

Hamilton-Jacobi-Bellman (HJB) equation:
By Bellman & Kaluba (1965)[15] and Zheng (2019)[16], using the Ito principle and the dynamic programming principle, the HJB equation is derived as the following.

Assume $u \in C^{1,2}$, meaning $u(\cdot, x) \in C^1, \quad u(t, \cdot) \in C^2$. By Ito's formula, we have

$$\begin{aligned} &u(t+h, X_{t+h}) \\ &= u(t,x) + \int_t^{t+h} (\frac{\partial u(s, X_s)}{\partial s} ds + \frac{\partial u(s, X_S)}{\partial x} dX_s + \frac{1}{2}\frac{\partial^2 u(s, X_s)}{\partial x^2} d[X,X]_s) \\ &= u(t,x) + \int_t^{t+h} (\frac{\partial u}{\partial s} + \frac{\partial u}{\partial x}(rX_s + \pi_s X_s(\mu - r)) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\pi_s^2 X_s^2 \sigma^2) ds + \int_t^{t+h} \frac{\partial u}{\partial x}\pi_s X_s \sigma dW_s \end{aligned} \tag{2.1.13}$$

Substituting (2.1.13) into DPP (2.1.12), and cancelling $u(t,x)$, we get

$$0 = sup_{\pi} E[\int_t^{t+h} (\frac{\partial u}{\partial s} + \frac{\partial u}{\partial x}(rX_s + \pi_s X_s(\mu - r)) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\pi_s^2 X_s^2 \sigma^2) ds | X_t = x] \tag{2.1.14}$$

Note that π in (2.1.14) is a process on $[t, t+h]$, thus as $h \to 0$, $\pi_s \to \pi_t \doteq \pi$ (a constant).
Divide by $h > 0$ and let $h \to 0$, we derive the HJB equation as

$$0 = sup_{\pi \in \mathcal{R}} (\frac{\partial u(t,x)}{\partial t}) + \frac{\partial u}{x}(rx + \pi x(\mu - r)) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\pi^2 x^2 \sigma^2) \tag{2.1.15}$$

with the terminal condition given by $u(T,x) = U(x)$.
Our goal is to solve PDE (2.1.15), if u is strictly concave in x, then $u_{xx} < 0$. The maximum point π in (2.1.15) satisfies

$$x(\mu - r)u_x + \pi x^2 \sigma^2 u_x x = 0$$

which gives the optimal portfolio strategy as

$$\pi^*(t,x) = -\frac{\theta}{\sigma}\frac{u_x(t,x)}{x u_{xx}(t,x)} \tag{2.1.16}$$

Substituting π^* into (2.1.15), the HJB equation is derived as:

$$\frac{\partial u(t,x)}{\partial t} + rxu_x - \frac{1}{2}\theta^2 \frac{u_x^2}{u_{xx}} = 0 \tag{2.1.17}$$

where $\theta = \frac{\mu - r}{\sigma}$, with terminal condition $u(T,x) = U(x)$.

There are several ways to solve the utility maximization problem defined in (2.1.3). Stochastic control is the standard solution to the problem; another method is the convex dual martingale method, details can be found in Karatzas et al. (1998) [17]; and dual control method, our main focus will be in this method which will be discussed in the later section.

We now first introduce a simple trial and error method that only works when utility function $U(\cdot)$ a CRRA utility function. We can assume that equation (2.1.3) $V = U(x)f(t)$ or $V = U(x) + f(t)$, such that we can reduce the HJB equation (2.1.17) into a linear differential equation system with unknown function f.

Example 2.1.1 Power Utility

Assume power utility $U(x) = \frac{1}{p}x^p$ for $x > 0$, where $p < 1$ and $p \neq 0$. The solution of the HJB equation (2.1.17) would have the following form:

$$u(t, x) = f(t)U(x)$$

Since the HJB equation would have the terminal condition $u(T, x) = U(x)$, such that we can say that the function $f(\cdot)$ also has the terminal condition as $f(T) = 1$.
First we find the partial derivatives of $u(t, x)$ as the following:

$$\frac{\partial u}{\partial t} = f'(t)U(x),$$
$$u_x = f(t)x^{p-1},$$
$$u_{xx} = f(t)(p-1)x^{p-2}$$

Substituting the derivatives into HJB (2.1.17), then we have the following linear ODE:

$$f'(t) + (rp - \frac{1}{2}\theta^2 \frac{p}{p-1})f(t) = 0$$

With the given terminal condition of $f(\cdot)$, we can solve the above equation and get

$$f(t) = \exp\left[(rp - \frac{1}{2}\theta^2 \frac{p}{p-1})(T-t)\right]$$

Example 2.1.2 Logarithmic utility
Assume logarithmic utility $U(x) = lnx$ for $x > 0$. The solution of the HJB (2.1.17) has the following form:

$$u(t, x) = U(x) + f(t)$$

Since the HJB equation would have the terminal condition $u(T, x) = U(x)$, such that we can say that the function $f(\cdot)$ also has the terminal condition as $f(T) = 0$.
The partial derivatives of $u(t, x)$ is derived as:

$$\frac{\partial u}{\partial t} = f'(t),$$
$$u_x = \frac{1}{x},$$
$$u_{xx} = -\frac{1}{x^2}$$

Substituting the derivatives into HJB (2.1.17), then f satisfies a linear ODE:

$$f'(t) + (r + \frac{1}{2}\theta^2) = 0$$

With the given terminal condition of $f(\cdot)$, we can solve the above equation and get

$$f(t) = (r + \frac{1}{2}\theta^2)(T-t)$$

2.2 Primal Optimization under Regime-Switching Market model

The regime-switching model is popular in the modeling and analysis of financial data because it allows the parameters of the asset price process to satisfy a finite state Markov chain. In this section, we describe the stochastic control problem under the regime-switching model as an extension to the genral market model.

2.2.1 Regime-Switching Market Model Set Up

Consider a fixed time period $[0, T]$, and let $(\Omega, \mathcal{F}, P)$ be a complete probability space, where P denotes physical probability, W a standard Brownian motion. We model the market by a continuous-time stationary Markov chain process (MCP) by $\{\alpha(t)\}$.
We identify the state space of $\{\alpha(t)\}$ as a finite set of unit vectors $\mathbb{E}\{e_1, e_2, \cdots, e_d\}$ where $e_i \in \mathbb{R}^d$ is a column vectors with unity in the ith position and zeros elsewhere, $i = 1, \cdots, d$. Denote by $\mathbf{Q} = (q_{ij})_{dxd}$ the generator of the Marov chain $\{\alpha(t)\}$ with $q_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^{d} q_{ij} = 0$ for each $i \in \mathbb{D}\{1, \cdots, d\}$. Then the MCP $\{\alpha(t)\}$ has a semi-martingale representation

$$\alpha_t = \alpha_0 + \int_0^t \mathbf{Q}'\alpha_v dv + M_t, \quad 0 \leq t \leq T \tag{2.2.1}$$

where $\mathbf{Q}'$ is the transpose of $\mathbf{Q}$, M_t is an $\mathbb{R}^d$-valued martingale with respect to the filtration generated by $\{\alpha(t)\}$.

The setting is the same as the previous section, that is, a financial market consists of a risk-free asset (savings account) S_0, a risky asset S, satisfying the following SDEs:

$$\begin{aligned} dS_0(t) &= r_t S_0(t)dt \\ dS(t) &= S(t)(\mu(t)dt + \sigma(t)dW(t)) \end{aligned} \tag{2.2.2}$$

where $r(t) = r\alpha(t), \mu(t) = \mu\alpha(t), \sigma(t) = \sigma\alpha(t)$, and $r = (r_1, \cdots, r_d), \mu = (\mu_1, \cdots, \mu_d), \sigma_{dxd} = (\sigma_1, \cdots, \sigma_d)$ are vectors of risk-free interest rates, return rates and volatility rates respectively, with r_i, μ_i, σ_i being the rates in regime i. Denote by $\theta = (\theta_1, \cdots, \theta_d)$ the market prices of risk with $\theta_i = \frac{\mu_i - r_i}{\sigma_i}$ for $i \in \mathbb{D}\{1, \cdots, d\}$.
Assume π is an $\mathcal{F}_t$-adapted process such that $\int_0^T \pi_t^2 dt < \infty$ a.s. and π is proportional portfolio process. We denote X_t as the investor's wealth at time t. Then $\pi_t X_t$ is the amount of money invested in S and $(1 - \pi_t)X_t$ is the amount of money in savings account S_0. The wealth process X_t satisfies the following SDE:

$$dX(t) = X(t)(r(t)dt + \pi(t)\sigma(t)(\theta(t)dt + dW(t))), \quad 0 \leq t \leq T \tag{2.2.3}$$

The utility maximization problem is defined by

$$sup_\pi E[U(X^\pi(T))] \tag{2.2.4}$$

where U is a utility function that mentioned in previous section.
To solve the optimization problem, we define a primal value function

$$V(t, x, i) = sup_{\pi \in \Pi_t} E_{t,x,i}[U(X(T))], \quad i \in \mathbb{D} \tag{2.2.5}$$

2.2.2 Stochastic Control Method

We now introduce a functional operator denoted by $\mathcal{M}$, which was first introduced by [18]:

$$\begin{aligned} \mathcal{M}V(t, x, i) &= sup_{\pi \in \Pi_t} E_{t,x,i}[V(\xi_{t,1} \wedge T, X_\pi(\xi_{t,1} \wedge T), \alpha(\xi_{t,1} \wedge T))] \\ &= sup_{\pi \in \Pi_t} E_{t,x,i}[1_{\xi_{t,1} < T} V(\xi_{t,1}, X_\pi(\xi_{t,1}), \alpha(\xi_{t,1})) + 1_{\xi_{t,1} \geq T} U(X_\pi(T))] \\ &= V \end{aligned} \tag{2.2.6}$$

where $\xi_{t,n}$ denotes the nth jump time of Markov chain $\{\alpha_t\}$ for $n \in \mathbb{N}$ with $\xi_{t,0} = t$.
This functional operator $V = \mathcal{M}V$ is the **dynamic programming equation (DPP)**, and the optimal value function V is the smallest non-negative solution to this equation.

Based on the DPP (2.2.6), the optimal value function has the HJB equations as following. An additional term needs to be added to the formula to ensure that the jumps of the process are correctly given by applying Ito's Lemma for semimartingales by Jacod & Shiryaev (1987) [19],

$$\begin{aligned}\mathcal{M}V(T,x,i) = \mathcal{M}V(t,x,i) + \int_t^T [\mathcal{M}V_t(s,x,i) + \mathcal{M}V_x(s,x,i)x(r(s) + \pi_s(\mu(s) - r(s)))\\ + \frac{1}{2}\mathcal{M}V_{xx}(s,x,i)(x\pi_s\sigma(s))^2]ds\\ + \int_t^T \mathcal{M}V_x(s,x,i)x\pi_s\sigma(s)dW_s\\ + \sum_{t\le s\le T}[\mathcal{M}V(s,x,i+) - \mathcal{M}V(s,x,i)]\end{aligned} \tag{2.2.7}$$

Denote by q_0 the jump measure of the Markov chain process $\alpha(t)$, and by T_n the successive jump time points. Then

$$q_0([0,t]\times\{j\}) = \sum_{n\in\mathbb{N}} \mathbf{I}_{[\alpha(T_n)=j, T_n\le t]}$$

The compensator of q_0 is given by

$$v([0,t]\times\{j\}) = \int_0^t \sum_{i\neq j} q_{ij}\mathbf{I}_{[\alpha(s)=i]}ds$$

Hence, we have

$$\begin{aligned}&\sum_{t\le s\le T}[\mathcal{M}V(s,x,i+) - \mathcal{M}V(s,x,i)] =\\ &= \int_t^T \sum_{j\in E}[\mathcal{M}V(s,x,j) - \mathcal{M}V(s,x,i)](q_0 - v)(ds,j)\\ &+ \int_t^T \sum_{j\in E}[\mathcal{M}V(s,x,j) - \mathcal{M}V(s,x,i)]q_{ij}ds\end{aligned} \tag{2.2.8}$$

Then by substituting (2.2.7) into DPP (2.2.6), we would have the following **HJB equation** defined as:

$$\frac{\partial V(\cdot,\cdot,i)}{\partial t} + r_i x\frac{\partial V(\cdot,\cdot,i)}{\partial x} - \frac{1}{2}\theta_i^2(\frac{\partial V(\cdot,\cdot,i)}{x})^2/\frac{\partial^2 V(\cdot,\cdot,i)}{\partial x^2} + \sum_{j=1}^d q_{ij}V(\cdot,\cdot,j) = 0, \quad i\in\mathbb{D} \tag{2.2.9}$$

with terminal condition $V(T,x,i) = U(x)$ for $i\in\mathbb{D}$

Similarly as before, since we are dealing with CRRA utility functions (i.e. power and logarithmic), there exists a closed form function to solve the HJB (2.2.9). We will try to use simple trial and error method to derive the solution of those utilies.

First, an important proposition was introduced by Fu et al. (2014) (detailed proof in [18]) to help us find the closed form solution:

Theorem 2.2.1. *The optimal value function V is the smallest non-negative solution of the dynamic programming equation $V = \mathcal{M}V$, i.e., for any $(t,x,i)\in[0,T]\times[0,\infty[\times\mathbb{D}]$, V satisfies*

$$V(t,x,i) = sup_{\pi\in\Pi_t}E_{t,x,i}[\mathbf{1}_{\xi_{t,1}<T}V(\xi_{t,1}, X^\pi(\xi_{t,1}), \alpha(\xi_{t,1})) + \mathbf{1}_{\xi_{t,1}\ge T}U(X^\pi(T))] \tag{2.2.10}$$

Based on Theorem (2.2.1), it shows that V is the solution to the dynamic programming equation (2.2.6) and the HJB equation (2.2.9). Then we define for each $i\in\mathbb{D}$ on $[0,T[\times]0,+\infty[$ the function of $M(t,x,i)$ satisfies

$$M(t,x,i) = \sigma_i^2[M_t(t,x,i) + r_i x M_x(t,x,i) - q_i M(t,x,i) + \sum_{j\neq i} q_{ij} v(t,x,j)]$$
$$M_{xx}(t,x,i) = \frac{1}{2}(\mu_i - r_i)^2 M_x^2(t,x,i) \tag{2.2.11}$$

and has the boundary condition $M(T,x,i) = U(x)$
Similarly, from HJB equation (2.2.9), for each $i \in \mathbb{D}$ on $[0,T[\times]0,+\infty[$ the function of $V(t,x,i)$ satisfies

$$V(t,x,i) = \sigma_i^2[V_t(t,x,i) + r_i x V_x(t,x,i) + \sum_{j=1}^{d} q_{ij} V(t,x,i)]$$
$$V_{xx}(t,x,i) = \frac{1}{2}(\mu_i - r_i)^2 V_x^2(t,x,i) \tag{2.2.12}$$

Exact Solution For CRRA Utility
When $U(\cdot)$ is a CRRA utility, the optimal value function $V(t,x,i)$ can be solved by using equation (2.2.11) and equation (2.2.12), which was first introduced by [18].

Example 2.2.1 Power utility: Recall that power utility is defined as

$$U(x) = \frac{1}{p}x^p$$

for $x > 0$, where $p < 1$ and $p \neq 0$, then the value function $V(t,x,i)$ has a solution given by

$$V(t,x,i) = a(t,i)\frac{x^p}{p}, \quad \forall (t,x,i) \in [0,T]x[0,\infty[x\mathbb{D} \tag{2.2.13}$$

where $a(t,i)$ is the ith component of an $\mathbb{R}d$-valued column of

$$a(t) = (a(t,1), a(t,2), \cdots, a(t,d))^T$$

Since $a(\cdot,i)$ is a continuous function on $[0,T]$, with $a(T,i) = 1$ for each $i \in \mathbb{D}$, then equation (2.2.13) follows that

$$\begin{aligned} V_t(t,x,i) &= \frac{1}{p}x^p a_t(t,i) \\ V_x(t,x,i) &= x^{p-1}a(t,i) \\ V_{xx}(t,x,i) &= (p-1)x^{p-2}a(t,i) \end{aligned} \tag{2.2.14}$$

If $a(\cdot) \geq 0$, the mapping $x \mapsto V(t,x,i)$ is concave and the maximizer π^* of the HJB equation is given by

$$\pi^*(t,x,i) = -\frac{\theta(t)}{\sigma(t)}\frac{V_x(t,x,i)}{xV_{xx}(t,x,i)} \tag{2.2.15}$$

Inserting the derivatives and π^* into the HJB equation (2.2.12)gives

$$\frac{1}{p}a_t(t,i) + a(t,i)(r(t) + \frac{1}{2(1-p)}\theta^2) + \frac{1}{p}\sum_{j=1}^{d} q_{ij}[a(t,j) - a(t,i)] = 0$$

and boundary condition $a(T,i) = 1$, rearranging we get:

$$a_t(t,i) = [\frac{1}{2}(\frac{\mu_i - r_i}{\sigma_i})^2\frac{p}{p-1} - r_i p]a(t,i) - \sum_{j=1}^{d} q_{ij}a(t,j) \tag{2.2.16}$$

Equation (2.2.16) can be rewrite as

$$\begin{aligned} a_t(t) &= (\Lambda - Q)a(t) \\ a_t(t) &= (a_t(t,1), a_t(t,2), \cdots, a_t(t,d))^T \end{aligned} \tag{2.2.17}$$

where

$$\begin{aligned} \Lambda &= diag(\lambda(1), \lambda(2), \cdots, \lambda(d)) \in \mathbb{R}^{dxd}, \\ \lambda(i) &= \frac{1}{2}(\frac{\mu_i - r_i}{\sigma_i})^2 \frac{p}{p-1} - r_i p \end{aligned} \tag{2.2.18}$$

Then we can conclude that the optimal value function $V(t, x, i)$ has a solution given by

$$V(t,x,i) = a(t,i)\frac{x^p}{p}, \quad \forall (t,x,i) \in [0,T]x[0,\infty[x\mathbb{D}$$

where $a(t) = \exp\left[-(\Lambda - Q)(T-t)\right] \cdot 1$

Example 2.2.2 Logarithmic utility: Recall that log utility is defined as $U(x) = lnx$ for $x > 0$, then the value function $V(t, x, i)$ has a solution given by

$$V(t,x,i) = \tilde{a}(t,i)lnx + g(t,i), \ \forall (t,x,i) \in [0,T]x[0,\infty[x\mathbb{D} \tag{2.2.19}$$

where $\tilde{a}(t, i)$ is the ith component of an $\mathbb{R}d$-valued column of

$$\tilde{a}(t) = (\tilde{a}(t,1), \tilde{a}(t,2), \cdots, \tilde{a}(t,d))^T$$

and $g(t, i)$ is the ith component of an $\mathbb{R}^d$-valued column of

$$g(t) = (g(t,1), g(t,2), \cdots, g(t,d))^T$$

Similarly as before, since $\tilde{a}(\cdot, i)$ and $g(\cdot, i)$ are both continuous functions on $[0, T]$, with $\tilde{a}(T, i) = 1, \ g(T, i) = 0$ for each $i \in \mathbb{D}$, then equation (2.2.19) follows that

$$\begin{aligned} V_t(t,x,i) &= \tilde{a}_t(t,i)lnx + g_t(t,i) \\ V_x(t,x,i) &= \tilde{a}(t,i)\frac{1}{x} \\ V_{xx}(t,x,i) &= -\tilde{a}(t,i)\frac{1}{x^2} \end{aligned} \tag{2.2.20}$$

Substituting equation (2.2.20) into equation (2.2.12), we would get

$$\begin{aligned} \tilde{a}_t(t,i) &= -\sum_{j=1}^{d} q_{ij}\tilde{a}(t,j) \\ g_t(t,i) &= -sum_{j=1}^{d} q_{ij} g(t,j) - \frac{1}{2}(\frac{\mu_i - r_i}{\sigma_i})^2 \tilde{a}(t,i) - r_i\tilde{a}(t,i) \end{aligned} \tag{2.2.21}$$

Equation (2.2.21) can be rewrite as

$$\begin{aligned} a_t(t) &= -Q\tilde{a}(t) \\ a_t(t) &= (a_t(t,1), a_t(t,2), \cdots, a_t(t,d))^T \\ g_t(t) &= -Qg(t) - b(t) \\ g(t) &= (g(t,1), g(t,2), \cdots, g(t,d))^T \end{aligned} \tag{2.2.22}$$

where

$$b(t) = (b(t,1), b(t,2), \cdots, b(t,d))^T b(t,i) = \frac{1}{2}(\frac{\mu_i - r_i}{\sigma_i})^2 \tilde{a}(t,i) + r_i\tilde{a}(t,i) \tag{2.2.23}$$

Then we can conclude that the optimal value function $V(t,x,i)$ has a solution given by

$$V(t,x,i) = \tilde{a}(t,i)lnx + g(t,i),\ \forall(t,x,i) \in [0,T]x[0,\infty[x\mathbb{D}$$

where $\tilde{a}(t) = \exp\left[Q(T-t)\right] \cdot 1$ and $g(t) = \exp\left(-Qt\right) \cdot \int_t^T \exp\left(Qs\right)b(s)ds$.

From the above two examples, we know that there exists a closed-form solution to the HJB equation for power and logarithmic utility functions by guessing the form of the solution. However, anything beyond that, when we are dealing with general utilities, the HJB equation would become more and more complex of non-linear SDEs, which cannot be solved by guessing the solution form. Thus, in the next part of the paper, we will introduce a new method — Dual Control Method to help us deal with the genearl utilities.

Chapter 3

Utility Maximization with Dual Control Method

3.1 Basic Theory

In this section, we will introduce the basic theory of dual control method and its application on the two markets we have discussed in the previous section.

The core of the Dual control theory is to use the Legendre transformation to transform the non-linear HJB equation into a linear partial differential equation, and to transform the original space optimization problem into a dual space pricing problem, thus simplifying the calculation.

In mathematics and physics, Legendre transformation is a kind of involution transformation, which is named after French mathematician Adrien Marie Legendre. In classical dynamics, it is often used to derive the Hamiltonian dynamical system from the Lagrangian dynamical system, which is widely used in thermodynamics.

In the late 1980s, Karatzas et al. (1987) [20] and Cox Huang (1989) [21] among with others, realized that for optimal investment and consumption, Dual control theory could delve deeper into the nature of the problem and provide a more powerful solution. For the Merton problem (see the general form of Merton (1969) [22]), the Dual control theory can be used to prove the form of the optimal solution.

Since then, Dual control theory has been widely applied to the optimal investment problem and has come up with a standard solving solution step:

Step 1: Try to solve the analytic solution of the original problem

Step 2: If Step 1 fails, find the dual problem of the original problem;

Step 3: Try to solve the analytical solution of the dual problem, and through the inverse transformation, find the analytic solution of the original problem;

Step 4: If Step 3 fails, find the numerical solution of the dual problem by inverse transformation.

3.2 Dual Control in General Market

Let us first use the general market introduced in Section 2.1 as an example to illustrate the Dual control theory.
Recall that in the optimization process, the value function has the following dynamic programming principles:

$$u(t,x) = sup_{\pi} E[u(t+h, X_{t+h})|X_t = x]$$

Using the Ito principle and the dynamic programming principle, we get the following HJB equation (2.1.17) and optimal allocation policy (2.1.16):

$$\begin{cases} \frac{\partial V(t,x)}{\partial t} + rxV_x - \frac{1}{2}\theta^2 \frac{V_x^2}{V_{xx}} = 0 \\ V(T,x) = U(x) \end{cases}$$

$$\pi^*_{t,x} = -\frac{\theta}{\sigma}\frac{V_x}{xV_{xx}}$$

The HJB equation is a nonlinear partial differential equation, and its numerical solution is very complex. The analytic solution only exists for the CRRA (power or logarithmic) utility functions. So we consider simplifying the HJB equation by using the Legendre transformation.

Define a positive stochastic process Y such that $X^\pi Y$ is a super martingale for any feasible wealth process X^π, and assume Y satisfies the following SDE:

$$dY_t = Y_t(\alpha(t)dt + \beta(t)dW(t)) \tag{3.2.1}$$

with initial condition $Y_0 = y$, where α and β are some stochastic processes. We need to choose α and β such that $X^\pi Y$ is a super martingale for all admissible control process $\pi \in \mathcal{R}$.
By applying Ito's lemma, we have

$$\begin{aligned} d(X^\pi(t)Y(t)) &= X^\pi(t)dY(t) + Y(t)dX^\pi(t) + d[X^\pi, Y]_t \\ &= X(t)Y(t)[(r + \alpha(t) + \pi(t)\sigma(\theta + \beta(t)))dt + (\pi(t)\sigma + \beta(t))dW(t)] \\ X^\pi(0)Y(0) &= xy \end{aligned}$$

$X^\pi Y$ is a super martingale if and only if

$$\alpha(t) + r + \pi(t)\theta\sigma + \pi(t)\beta(t)\sigma \leq 0$$

for all $\pi \in \mathcal{R}, t \in [0, T]$.
According to the assumption, we have for all $t \in [0, T]$

$$\begin{aligned} \theta + \beta(t) &= 0 \\ r + \alpha(t) &\leq 0 \end{aligned}$$

Define the dual utility function as:

$$\tilde{U}(y) = sup_{x\geq 0}(U(x) - xy) \tag{3.2.2}$$

which is decreasing, convex, and continuous for all y.

The dual utility function can be derived as the following steps:

1. Solve $U'(x) - y = 0$ for x in terms of y
2. Get the maximum point of $\tilde{V}(y)$ as $I(y) = (U')^{-1}(y)$
3. Compute $\tilde{U}(y) = U(I(y)) - yI(y)$

Moreover, we can define the dual function as:

$$\tilde{V}(t, y) = E[\tilde{U}(Y(T))|Y(t) = y] \tag{3.2.3}$$

By the Feynman-Kac theorem, we know that the dual-valued function $\tilde{V}$ (3.2.3) satisfies the following linear partial differential

$$\begin{cases} \frac{\partial \tilde{V}}{\partial t} - ry\tilde{V}_y + \frac{1}{2}\theta^2 y^2 \tilde{V}_{yy} = 0, \\ \tilde{V}(T, y) = \tilde{U}(y) \end{cases} \tag{3.2.4}$$

This equation (3.2.4) is a typical Black-Scholes equation, which is easy to solve and calculate. We have derived the SDE for the dual process Y and the corresponding dual minimization problem, that is, the dual process is given by

$$dY = -rYdt - \theta YdW$$

with the solution of Y with inital value $Y_0 = y_t$ given by

$$Y_t = y_0 \exp\left[(-r - \frac{1}{2}\theta^2)(T-t) - \theta(W_T - W_t)\right]$$

The following verifies that the inverse of the dual-valued function is the primal-valued function. Define the inverse transformation function of dual-valued function as

$$\bar{V}(t,x) = inf_{y\geq 0}(\tilde{V}(t,y) + xy) \tag{3.2.5}$$

Now, we have a primal problem of $sup_\pi E[U(X_T)]$ and a dual problem of $inf_y(E[V(Y_T)] + xy)$. From the definition of the dual function $\tilde{V}$, we have a inequality such that primal problem is bounded above by the dual problem:

$$sup_\pi E[U(X_T)] \leq inf_{y\geq 0}(E[\tilde{V}(Y(T))] + xy) \tag{3.2.6}$$

Now we can find a representation of the classical solution to the HJB equation (2.1.17) via two simple convex dual operations and solution of a linear PDE, all are relatively easy to perform.

For $0 \leq t < T$, since $\tilde{V}(\cdot)$ is C^∞, the minimum point is obtained by solving

$$\frac{\partial}{\partial y}\tilde{V}(t,y) + x = 0 \tag{3.2.7}$$

Since $\tilde{V}(\cdot)$ is strictly convex, so $\tilde{V}_y(t,\cdot)$ is strictly increasing. Then for every $x > 0$, there exists unique y solving (3.2.7) denoted by $y = y(t,x)$. We have

$$\bar{V}(t,x) = \tilde{V}(t, y(t,x)) + xy(t,x) \tag{3.2.8}$$

From the chain rule we have the following derivative relations

$$\begin{aligned} \bar{V}_t &= \tilde{V}_t, \\ \bar{V}_x &= y, \\ \tilde{V}_y &= -x, \\ \tilde{V}_{yy} &= -\frac{1}{\bar{V}_{xx}} \end{aligned} \tag{3.2.9}$$

Substituting (3.2.9) into (3.2.4), we would get

$$\begin{cases} \frac{\partial \bar{V}(t,x)}{\partial t} + rx\bar{V}_x - \frac{1}{2}\theta^2\frac{\bar{V}_x^2}{\bar{V}_{xx}} = 0, \\ \bar{V}(T,x) = U(x) \end{cases} \tag{3.2.10}$$

which coincides with equation (2.1.17), thus we have

$$V(t,x) = \bar{V}(t,x) = inf_{y\geq 0}(\tilde{V}(t,y) + xy) \tag{3.2.11}$$

Therefore, the primal value function of the primitive space can be obtained by solving the dual value function in the dual space and then inverse transformation. The partial differential equation in the dual space is a linear partial differential equation, which greatly simplifies the calculation.

With the dual control method, we can find the optimal value function u, optimal control π^* and optimal wealth process X^* in the following way:
Find optimal value function u using $\bar{V}(t,x) = \tilde{V}(t, y(t,x)) + xy(t,x)$, then optimal control π^* is:

$$\pi^*(t,x) = \frac{\theta}{\sigma}\frac{1}{x}y(t,x)\tilde{V}_{yy}(t, y(t,x)) \tag{3.2.12}$$

and optimal wealth X^* by solving SDE $dX_t = rX_t dt + \pi_t X_T((\mu - r)dt + \sigma dW_t)$.
We have shown that $\bar{V}(t,x)$, defined by (3.2.8), is a classical solution to the HJB equation (2.1.17) and satisfies the terminal condition $\bar{V}(T,x) = U(x)$.

There are closed-form classical solutions to the HJB equation for CRRA utilities.

Example 3.2.1 Power Utility

Assume power utility $U(x) = \frac{1}{p}x^p$ for $x > 0$, where $p < 1$ and $p \neq 0$. In this case, the dual problem can be written as

$$\bar{V}(t,x) = inf_{y\geq 0}(\tilde{V}(t,y) + xy)$$

The dual function of U is given by

$$\tilde{U}(y) = sup_{x>0}(\frac{x^p}{p} - xy)$$

Follow the procedures we mentioned above, we first get the derivative with respect to x of $\frac{x^p}{p} - xy$ and set it equal to be 0, we have

$$x^{p-1} - y = 0, \quad x = y^{\frac{1}{p-1}}$$

Then the dual utility function can be written as

$$\tilde{U}(y) = -\frac{1}{q}y^q, \quad \text{with } q = \frac{p}{p-1}$$

Recall that the dual value function is give by

$$\begin{aligned}
\tilde{V}(t,y) &= E[\tilde{U}(Y(T))|Y_t = y] \quad \text{where Y satisfies SDE (3.2.1)} \\
&= \tilde{U}(y)\exp\left[(\frac{1}{2}q(q-1)\theta^2 - qr)(T-t)\right] \\
\text{where } \theta &= \frac{\mu - r}{\sigma}
\end{aligned}$$

Then we solve the equation

$$\frac{\partial}{\partial y}\tilde{V}(t,y) + x = 0$$

$$x - y^{q-1}\exp\left[(\frac{1}{2}q(q-1)\theta^2 - qr)(T-t)\right] = 0$$

We would get $y^* = y(t,x)$ as

$$y^* = x^{\frac{1}{q-1}}\exp\left[(\frac{1}{2}q\theta^2 + \frac{qr}{q-1})(T-t)\right]$$

Substituting it into (3.2.8), we get

$$\begin{aligned}
V(t,x) = \bar{V}(t,x) &= \tilde{V}(t,y^*) + xy^* \\
&= U(x)\exp\left[p(r + \frac{1}{2}\theta^2\frac{1}{1-p})(T-t)\right]
\end{aligned}$$

Recall that the maximum of the Hamiltonian in the HJB equation is achieved at (2.1.16) $\pi^*(t,x) = -\frac{\theta}{\sigma}\frac{V_x(t,x)}{xV_{xx}(t,x)}$, by calculation we have

$$V_x = x^{p-1}\exp\left[p(r + \frac{1}{2}\theta^2\frac{1}{1-p})(T-t)\right] \text{ and } V_{xx} = (p-1)x^{p-2}\exp\left[p(r + \frac{1}{2}\theta^2\frac{1}{1-p})(T-t)\right]$$

Then we have

$$\pi^*(t,x) = \frac{\theta}{(1-p)\sigma}$$

Substituting $\pi^*(t,x)$ into the wealth process equation of X_t^π (2.1.2), we have the following linear SDE

$$dX_t = X_t((r + \frac{\theta^2}{1-p})dt + \frac{\theta}{1-p}dW_t)$$

There is a strong solution to the SDE above, called Merton wealth process, detail in [22]. The optimal wealth process is given by

$$X_t = x \exp\left[(r + \frac{\theta^2}{1-p} - \frac{1}{2}\frac{\theta^2}{(1-p)^2})t + \frac{\theta}{1-p}W_t\right]$$

Example 3.2.2 Logarithmic utility
Assume logarithmic utility $U(x) = \ln x$ for $x > 0$. Similarly as the previous example, we first drive the dual utility function as the following:

$$\frac{\partial}{\partial x}\ln x - xy = 0 \Rightarrow \frac{1}{x} - y = 0 \Rightarrow x = \frac{1}{y} \quad \tilde{U}(y) = -\ln y - 1$$

Then dual value function is give by

$$\begin{aligned}\tilde{V}(t,y) &= E[\tilde{U}(Y(T))|Y_t = y] \quad \text{where Y satisfies SDE (3.2.1)}\\ &= \tilde{U}(y) + (r + \frac{1}{2}\theta^2)(T-t)\\ &\text{where } \theta = \frac{\mu - r}{\sigma}\end{aligned}$$

Then we solve the equation

$$\frac{\partial}{\partial y}\tilde{V}(t,y) + x = 0$$

$$x - \frac{1}{y} = 0$$

We would get $y^* = y(t,x)$ as

$$y^* = \frac{1}{x}$$

Substituting it into (3.2.8), we get

$$\begin{aligned}V(t,x) = \bar{V}(t,x) &= \tilde{V}(t,y^*) + xy^*\\ &= U(x) + (r + \frac{1}{2}\theta^2)(T-t)\end{aligned}$$

Recall that the maximum of the Hamiltonian in the HJB equation is achieved at (2.1.16) $\pi^*(t,x) = -\frac{\theta}{\sigma}\frac{V_x(t,x)}{xV_{xx}(t,x)}$, by calculation we have

$$V_x = \frac{1}{x} \text{ and } V_{xx} = -\frac{1}{x^2}$$

Then we have

$$\pi^*(t,x) = \frac{\theta}{\sigma}$$

Substituting $\pi^*(t,x)$ into the wealth process equation of X_t^π (2.1.2), we have the following linear SDE

$$dX_t = X_t[(r + \theta^2)dt + \theta dW_t]$$

There is a strong solution to the SDE above, called Merton wealth process, detail in [22]. The optimal wealth process is given by

$$X_t = x \exp\left[(r + \theta^2 - \frac{1}{2}\theta^2)t + \theta W_t\right]$$

Compared with Example 2.1.1 and Example 2.1.2, we get the exact same solution of optimal wealth process by using dual HJB method.

Example 3.2.3 A non-HARA utility Consider the following non-HARA utility function from Bian and Zheng (2015) [7]:

$$U(x) = \frac{1}{3}H(x)^{-3} + H(x)^{-1} + xH(x)$$

for $x > 0$, where

$$H(x) = (\frac{2}{-1+\sqrt{1+4x}})^{1/2}$$

The dual function of U is defined by

$$\tilde{U}(y) = sup_{x>0}(\frac{1}{3}H(x)^{-3} + H(x)^{-1} + xH(x) - xy)$$

Taking derivative with respect to x of $\frac{1}{3}H(x)^{-3} + H(x)^{-1} + xH(x) - xy$ and set it equal to be 0, we have

$$y = -H(x)^{-4}H'(x) - H(x)^{-2}H'(x) + H(x) + xH'(x)$$
$$0 = -H(x)^{-4}H'(x) - H(x)^{-2}H'(x) + xH'(x)$$
$$y = H(x)$$

Then the dual function can be written as

$$\tilde{U}(y) = \frac{1}{3}y^{-3} + y^{-1}$$

And the dual value function is given by

$$\tilde{V}(t,y) = E[\tilde{U}(Y(T))|Y(t) = y]$$

where Y satisfies the SDE (3.2.1), and we have the solution to this SDE as

$$Y_T = y_t \exp[(-r - \frac{1}{2}\theta^2)(T-t) - \theta(W_T - W_t)]$$

Substituting Y_T into the dual value function $\tilde{V}$, we would get

$$\begin{aligned}
\tilde{V}(t,y) &= E[\frac{1}{3}y^{-3}\exp(3(r+\frac{\theta^2}{2})(T-t) + 3\theta(W_T - W_t)) + y^{-1}\exp((r+\frac{\theta^2}{2})(T-t) + \theta(W_T - W_t))] \\
&= \frac{1}{3}y^{-3}\exp(3(r+\frac{\theta^2}{2})(T-t))E[\exp(3\theta(W_T - W_t))] + y^{-1}\exp((r+\frac{\theta^2}{2})(T-t))E[\exp(\theta(W_T - W_t))] \\
&= \frac{1}{3}y^{-3}\exp(3(r+\frac{\theta^2}{2})(T-t))\exp(\frac{9\theta^2(T-t)}{2}) + y^{-1}\exp((r+\frac{\theta^2}{2})(T-t))\exp(\frac{\theta^2(T-t)}{2}) \\
&= \frac{1}{3}y^{-3}\exp((3r+6\theta^2)(T-t)) + y^{-1}\exp((r+\theta^2)(T-t))
\end{aligned}$$

Then we need to solve the equation $\frac{\partial}{\partial y}\tilde{V}(t,y) + x = 0$. We have the following equation:

$$\begin{aligned}
0 &= xy^4 - \exp[(3r+6\theta^2)(T-t)] - y^2\exp[(r+\theta^2)(T-t)] \\
y^2 &= \frac{\exp[(r+\theta^2)(T-t)] + \sqrt{\exp[(2r+2\theta^2)(T-t)] + 4x\exp[(3r+6\theta^2)(T-t)]}}{2x} \\
y^* = y(t,x) &= \frac{1}{\sqrt{2x}}[\exp((r+\theta^2)(T-t)) + \sqrt{\exp((2r+2\theta^2)(T-t)) + 4x\exp((3r+6\theta^2)(T-t))}]^{1/2}
\end{aligned}$$

Recall the solution is given in the form (3.2.8),

$$\begin{aligned}
V(t,x) &= \tilde{V}(t,y^*)) + xy* \\
&= \frac{1}{3}y^{*-3}\exp((3r+6\theta^2)(T-t)) + y^{*-1}\exp((r+\theta^2)(T-t)) + xy^*
\end{aligned}$$

Recall that the maximum of the Hamiltonian in the HJB equation is achieved at (2.1.16) $\pi^*(t,x) = -\frac{\theta}{\sigma}\frac{V_x(t,x)}{xV_{xx}(t,x)}$, by calculation we have

$$V_x = y^* \text{ and } V_{xx} = -\frac{1}{\tilde{V}_{yy}}$$

Then we have

$$\pi^*(t,x) = \frac{\theta}{\sigma}\frac{4y^{*-4}\exp((3r+6\theta^2)(T-t)) + 2y^{*-2}\exp((r+\theta^2)(T-t))}{x}$$

Substituting $\pi^*(t,x)$ into the wealth process equation of X_t^π (2.1.2), we have the following linear SDE

$$\begin{aligned} dX_t^{\pi^*} &= [\theta^2(4y^{*-4}\exp((3r+6\theta^2)(T-t)) + 2y^{*-2}\exp((r+\theta^2)(T-t))) + rX_t^{\pi^*}]dt \\ &+ \theta(4y^{*-4}\exp((3r+6\theta^2)(T-t)) + 2y^{*-2}\exp((r+\theta^2)(T-t)))dW_t \end{aligned}$$

We can rewrite the above equation in terms of Y_t and recall that Y_t has a solution as $Y_t = y^*\exp((-r-\frac{1}{2}\theta^2)(T-t) - \theta(W_T - W_t))$, then we have

$$\begin{aligned} X_t^{\pi^*} &= Y_t^{-4}\exp((3r+6\theta^2)(T-t)) - Y_t^{-2}\exp((r+\theta^2)(T-t)) \\ &= y^{*-4}\exp((3r+6\theta^2)T + (r-4\theta^2)t + 4\theta W_t) + y^{*-2}\exp((r+\theta^2)T + rt + 2\theta W_t) \end{aligned}$$

Now, we have shown that with Dual Control Method, we can deal with general utilities under the general market, and we will be able to find a closed form solution for them. However, when we extend our market into a regime-switching model, things get more complex, and only using dual control method cannot help us solve the problem anymore.

3.3 Dual Control in Regime-Switching Market

In this section, we extend the model into regime-switching market, and then illustrate the dual control method under this market.
Similarly as before, the utility function U is continuous, increasing, and concave (but not necessarily strictly increasing and concave), and $U(0) = 0$. The dual function of U is defined by:

$$\tilde{U}(y) = sup_{x\geq 0}(U(x) - xy) \tag{3.3.1}$$

for $y \geq 0$ satisfies $\tilde{U}(\infty) = 0$. Also, $\tilde{U}(y)$ is a continuous, decreasing and concave function on $[0,\infty)$ and has $U(\infty) = 0$, details in Bian and Zheng (2015) [7].
Define a dual process Y_t as

$$dY(t) = Y(t)(-r(t)dt - \theta(t)dW(t) + CdM(t)) \tag{3.3.2}$$

where C is a constant row vector in $\mathbb{R}^d$, W is a Brownian motion, and M_t is the martingale defined as before in (5.1.1). The solution equation (3.3.2) at time T, with initial condition $Y_t = y$, can be written as

$$\begin{aligned} &Y(t) = yexp(A_{t,T}) \\ &\text{where} A_{t,T} = -\int_t^T (r_u + CQ'\alpha_u + \frac{1}{2}\theta_u^2)du - \int_t^T \theta_u dW_u + \sum_{t<s} ln(1 + C(\alpha_s - \alpha_{s_-})) \end{aligned} \tag{3.3.3}$$

Y_T defined as in (3.3.3) for all $T > t$ satisfies the SDE (3.3.2) can be shown by the Ito's formula for semi-martingales.
The dual value function is defined by

$$\tilde{V}(t,y,i) = E_{t,y,i}[\tilde{U}(Y(T))] \tag{3.3.4}$$

Since Y_t satisfies SDE (3.3.2), and $\tilde{U}(\cdot)$ is a decreasing concave function, then we say that for $y > 0$ and fixed t, i, function $\tilde{V}(t,y,i)$ is also a decreasing concave function. Then for fixed t, i, we denote by $W(t,x,i)$ as the dual function of $\tilde{V}(t,y,i)$ given by

$$W(t,x,i) = inf_{y>0}(\tilde{V}(t,y,i) + xy) \tag{3.3.5}$$

3.4 Recent research on analytical solutions for Utility maximization

When we are dealing with general market, it is easy for us to find an analytical solution for utility maximization. When $U()$ is a CRRA utility, one may solve this problem by guessing the form of the solution; moreover, even when $U(\cdot)$ is not a CRRA utility, we can apply dual control method with the help of HJB equation to find the solution. Bian and Zheng (2015) [7] use the dual control method to prove that there exists a classical solution for HJB equation and the optimal value function is a conjugate function of the optimal dual value function.

However, when we extend the market into a regime-switching market, we no longer can find a analytical solution. When $U(\cdot)$ is a CRRA utility function, we can find the exact solution by trail and error in the form of $V(t,x,i) = U(x)a(t,i)$ (power) or $V(t,x,i) = a(t,i)U(x) + g(t,i)$ (log). When $U(\cdot)$ is a general utility, it is difficult to solve the HJB equation (2.2.9) since it is a system of fully nonlinear PDEs, which is impossible to solve by hand. Fu et al.(2014) [18] define a functional operator $\mathcal{M}$ given in (2.2.6) and a sequence of functions $\{H_n\}$ by $H_{n+1} = \mathcal{M}H_n$ and show that H_n converges to the value function V as n tends to ∞. This algorithm of Fu et al. (2014) [18] helps to find an approximate value of the dual value function by reducing the system of fully coupled non-linear PDEs to a system of decoupled nonlinear PDEs. This method works for solving non-HARA utilities; however, it is computational complexity as we have to solve a stochastic control problem using the iteration through H_n.

By Ma et al. (2017) [13], he introduced an efficient Monte Carlo method based on the dual control framework to find the lower and upper bounds of the value function for general utilities and show that these bounds are tight and can be improved with increased number of the dual control variables. We will illustrate in this paper how the algorithm is worked and apply it using numerically tests of power and non-HARA utilities.

Chapter 4

Utility Maximization with Dual Control Monte-Carlo Method

4.1 Basic theory of Monte Carlo method

The modern Monte Carlo method was first invented by Stanislaw Ulam in 1940. He was working on a nuclear weapons program at Los Alamos, after Ulam's breakthrough, John von Neumann realized the importance of the approach and ran the Monte Carlo simulation through computer programming. At the beginning of the simulation, they used real random numbers from nature, but the programming ran very slowly. Then von Neumann developed a method for generating pseudo random numbers, the middle-square method. Although it was criticized as primitive, von Neumann realized that it was faster than his simulations.

The Monte Carlo method is the core algorithm of the Manhattan Project. Because of its fast and high efficiency, it overcomes the disadvantage of the computing tool at that time. In the 1950s, it was used by researchers at the Los Alamos National Laboratory to develop calculations for the hydrogen bomb, and was popularized in fields as diverse as physics, physical chemistry and operations research. During this period, the RAND Corporation and the US Air Force has funded the research of Monte Carlo method and applied it to a wider range of fields.

Monte Carlo method was first introduced to finance in a 1964 Harvard Business Review article by Hertz (1964) [23]. In the article, he applies Monte Carlo method to corporate finance, and analyzes risk management issues in capital investment. In 1977, Boyle (1977) [24] first introduced Monte Carlo method to the pricing of Financial derivatives in the Journal of Financial Economics.

Nowadays, Monte Carlo method is used in all areas of the financial markets.

Financial analysts analyze net present value (NPV) expectations, volatility and sensitivity by simulating random cash flows, and screen for investments with NPV greater than 0.

In derivatives, Monte Carlo method is used to calculate the price of options. The price of the option at the initial time is the price of the option at the initial time by averaging and discounting all the rewards in each of the possible paths of the underlying asset through the Monte Carlo method and following the rules in each of the paths.

In the field of fixed income securities and their derivatives, Monte Carlo method is used to simulate the change of interest rates over time. Using Monte Carlo simulations, you can analyze the return curve of bonds and price bond derivatives.

In terms of personal finance, Monte Carlo can simulate an individual's portfolio, and calculate whether at the time of retirement, it meets the expected pension requirements; and therefore, decide whether to take on more risk or put more money into savings at the moment.

The Monte Carlo method, widely used in insurance, marketing, and disaster simulation models, can efficiently simulate events such as earthquakes and hurricanes that lack sufficient data to estimate risk using traditional methods.

In reinsurance, Monte Carlo method can simulate the complex reinsurance structure of an insurance company, analyze the effect of net risk compensation on the insured assets, the effect of risk transfer on reinsurance, and so on.

In the case of investment linked insurance, the Monte Carlo method can simulate the pricing of insurance products under complex rules, which can be used as a reference for the trading of insurance policies. At the same time, it can optimize the asset allocation, calculate the investment strategy, and guarantee the value preservation and increment of the insurance fund.

Monte Carlo method is an important method of stochastic simulation that yields numerical results by repeatedly trying random sampling. The main idea is to calculate the definite problem by stochastic method.

For example, when calculating the volume of a set, the volume is connected with the probability. All the possible results are generated by the method of random sample generation, and the volume of the set is estimated by counting the samples falling into the set.

The theoretical foundation of Monte Carlo is law of large number (SLLN) and the central limit theorem (CLT). The law of large numbers guarantees that the Monte Carlo estimate will converge to the true value as the number of tests increases. The Central Limit Theorem provides an estimate of the degree of similarity of errors after a limited number of trials. We state these results without proofs.

Theorem 4.1.1 (SLLN). *Let $(X_n, n \geq 1)$ be a sequence of independent and identically distributed (iid) random variables with values in $\mathcal{R}^d$. Assume $E|X_1| < \infty$. For $N \geq 1$, denote sample mean of $(X_1, \cdots, X_N)$ by*

$$\bar{S}_N = \frac{1}{N}\sum_{i=1}^{N} X_i$$

Then

$$lim_{N\to\infty}\bar{S}_N = E[X_1], \quad P\text{-}a.s.$$

Theorem 4.1.2 (CLT). *Let $(X_n, n \geq 1)$ be a sequence of real valued iid random variables. Assume $\mu = E[X_1]$ and $\sigma^2 = Var(X_1) < \infty$. Consider normalized sequence of $\bar{S}_N$ with mean $E[\bar{S}_N] = \mu$ and variance $Var(\bar{S}_N) = \frac{1}{N}\sigma^2$:*

$$Y_N = \frac{\bar{S}_N - E[\bar{S}_N]}{\sqrt{var(\bar{S}_N)}} = \frac{\sqrt{N}}{\sigma}(\frac{1}{N}\sum_{i=1}^{N} X_i - \mu)$$

The sequence Y_N converges in law to $N(0,1)$, namely

$$lim_{N\to\infty}P(Y_N \leq x) = \Phi(x) \text{ for all } x \in \mathcal{R}$$

Take solving definite integral as an example

$$\alpha = \int_0^1 f(x)dx$$

We can think of this integral as an expectation $E[f(U)]$, where U is uniformly distributed over $[0,1]$. Suppose we can generate n i.i.d $[0,1]$ uniformly distributed sequence of numbers $U_1, U_2, \cdots$ by some algorithms (such as linear congruence algorithm), then by averaging the values of the f function at these n random points, we get the Monte Carlo estimate of the integral

$$\hat{\alpha}_n = \frac{1}{n}\sum_{i=1}^{n} f(U_i)$$

If function $f(\cdot)$ is integrable on $[0,1]$, then by the weak law of large numbers, when $n \to \infty$, $\hat{\alpha}_n$ converges in probability to α. Moreover, if $f(\cdot)$ is square-integrable, we define

$$\sigma_f^2 = \int_0^1 (f(x) - \alpha)^2 dx$$

then using Monte Carlo, $\hat{\alpha}_n - \alpha$ has a Normal distribution with mean equal to 0 and standard deviation equals to $\frac{\sigma_f}{\sqrt{n}}$. As n grows larger, the estimated results are going to get better and better. Since α is unknown, such that σ_f is also unknown, which can be estimated as

$$s_f = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(f(U_i) - \hat{\alpha} - n)^2}$$

The form of the standard deviation $\frac{\sigma_f}{\sqrt{n}}$ is a central feature of the Monte Carlo method. To cut the margin of the error in half, you need to increase the selection by four times, and to increase it by one decimal place, you need to increase it by 100 times. The general calculation method of trapezoid formula is defined as

$$\alpha \approx \frac{f(0) + f(1)}{2n} + \frac{1}{n}\sum_{i=1}^{n-1} f(\frac{i}{n})$$

For a double differentiable function, the order of convergence is $O(n^{-2})$. Thus, in a one dimensional integral problem, the Monte Carlo method does not have an advantage. For higher dimensional integral, however, Monte Carlo method can effectively deal with the curse of dimensionality.

For a integral has dimension higher than $[0, 1]^d$, after n Monte Carlo tests, the standard error is still $\frac{\sigma_f}{\sqrt{n}}$, which will not increase as the dimension increases. Instead, when the trapezoidal formula is used for the numerical integration, for a continuous double differentiable function, a d-dimensional integral has the order of convergence $O(n^{-\frac{2}{d}})$. All the definite numerical integration methods have the problem that the order of convergence decreases as the dimension increases. As a result, Monte Carlo method is very attractive when dealing with high dimensional integrals.

In the following sections, we will use Monte Carlo simulation to calculate the optimal investment return and the optimal investment strategy for the optimal investment problem in the regime-switching market, based on the Dual control theory and the extension of the market.

4.2 Dual Control Monte Carlo in Regime-Switching Market

The next theorem is the main idea behind this method of tight bounds for utility maximization. Detail see in Ma et al. (2017) [13].

Theorem 4.2.1 (Tight Bounds). *Let $W(t, x, i)$ be given by equation (3.3.5) and S_1 and S_2 be sets of vectors C satisfying $|C_i| < \frac{1}{2}$ for $i \in \mathbb{D}$. Then the optimal value function $V(t, x, i)$ defined in equation (2.2.5) satisfies*

$$V(t, x, i) \leq inf_{C \in S_1} W(t, x, i) \tag{4.2.1}$$

Furthermore, suppose $\tilde{V}(t, y, i)$ given by equation (3.3.4) is twice continuously differentiable and strictly convex for $y > 0$ and fixed t and i. Let $y = y^*(t, x, i, C)$ written as y^* be the solution of

$$\frac{\partial \tilde{V}(t, y, i)}{\partial y} + x = 0 \tag{4.2.2}$$

Let the feedback control $\bar{\pi}_i(t, x)$ be defined by

$$\bar{\pi}(t, x, i) = \frac{\theta_i}{\sigma_i}\frac{y^*}{x}\frac{\partial^2}{\partial y^2}\tilde{V}(t, y^*, i) \tag{4.2.3}$$

for $t \in [0, T]$ and $x > 0$. Let $\bar{X}$ be the unique strong solution of SDE (2.2.3) with control process $\pi(t) = \bar{\pi}_{\alpha(t)}(t, \bar{X}_t)$. Let $\bar{W}(t, x, i)$ be defined by

$$\bar{W}(t, x, i) = E_{t,x,i}[U(\bar{X}(T))] \tag{4.2.4}$$

The optimal value function $V(t,x,i)$ satisfies

$$V(t,x,i) \geq sup_{C \in S_2} \bar{W}(t,x,i) \tag{4.2.5}$$

4.2.1 Construction of upper and lower bounds algorithm

By applying this theorem, the Monte-Carlo method can be used to find the tight lower and upper bounds. To compute the tight lower bound, one key step is to find a solution of (4.2.2). We assume that the utility U is strictly concave and satisfies Inada's condition of $U'(0) = \infty, U'(\infty) = 0$, which implies that $\tilde{U}$ is continuously differentiable and $\tilde{U}(0) = -\infty, \tilde{U}(\infty) = 0$. From equation (3.3.4), the pathwise differentiation method gives

$$\frac{\partial \tilde{V}(t,y,i)}{\partial y} = \frac{1}{y} E_{t,y,i}[Y(T)\tilde{U}'(Y(T))] \tag{4.2.6}$$

which is $-\infty$ as $y \to 0$, and 0 as $y \to \infty$. Choosing a sufficiently small y and a sufficiently large y such taht the expression $\frac{\partial \tilde{V}(t,y,i)}{\partial y} + x$ has opposite signs for $x > 0$. Then We can then use the bisection method to find the solution to equation (4.2.2).

The Markov chain process α can be generated in a standard procedure as follows. Assume the MCP is at state i. Generate two independent standard uniform variables ξ_1 and ξ_2, define

$$\tau_i = -\frac{1}{q_{ii}} ln \xi_1$$

where $q_{ii} = \sum_{j \neq i} q_{ij}$ is the intensity rate of MCP jumping from state i to some other state (not decided yet). Then τ_i is the first jump time of MCP from state i. To decide which state it jumps to, divide interval [0, 1] by d 1 subintervals, with length of $q_{ij}/q_{ii} \quad for \quad j \neq i$. If ξ_2 is realized in the jth subinterval, the MCP has jumped to state j at time τ_i. Repeat these steps to generate a sample path of MCP on the interval [0, T].

The dual control variable C in (3.3.2) must satisfy $1 + C(\alpha(t) - \alpha(t-)) > 0$ for $t \in [0,T]$ in (3.3.3). This sufficient condition is to ensures that $|C_i| < \frac{1}{2}$. In numerical implementation, one may be chosen randomly by specifying a distribution for C which ensures $|C_i| < \frac{1}{2}$ and then generating samples C by simulation, or deterministically by specifying particular values such as fixed grid points on a d-dimensional hyper-cube satisfying $|C_i| < \frac{1}{2}$. By doing this, we might be having a tighter lower and upper bounds.

Monte-Carlo method for computing the tight upper bound:

Step 1: Sample d independent uniform variables C i in [0.4 , 0.4] , which are components of a vector C .

Step 2: Generate M_1 sample paths of Brownian motion W and MCP α, which are used to compute Y_T with $Y_0 = y$ and the average derivatives:

$$\frac{\partial \tilde{V}(0,y,i)}{\partial y} \approx \frac{1}{y} \frac{1}{M_1} \sum_{l=1}^{M_1} Y(T)\tilde{U}'(Y(T))$$

Step 3: Use the bisection method to solve equation (4.2.2)

$$\frac{\partial \tilde{V}(t,y,i)}{\partial y} + x = 0$$

and get the solution $y \approx y^*$

Step 4: Compute the upper bound

$$W_i(0,x) \approx \tilde{V}(0,y^*,i) + xy^*$$

Step 5: Repeat Steps 1–4 N_1 times and then compute the tight upper bound $inf_{C \in S_1} W(0,x,i)$

Monte-Carlo method for computing the tight lower bound:

Step 1: Sample d independent uniform variables C i in [0.4 , 0.4] , which are components of a vector C.

Step 2: Generate M_2 sample paths of Brownian motion W and MCP α, which are used to find the control process $\bar{\pi}$ given by equation (4.2.3)

$$\bar{\pi}(t, x, i) = \frac{\theta_i}{\sigma_i} \frac{y^*}{x} \frac{\partial^2}{\partial y^2} \tilde{V}(t, y^*, i)$$

and the wealth process $\bar{X}$ given by (2.2.3)

$$d\bar{X}(t) = \bar{X}(t)(r(t)dt + \bar{\pi}(t)\sigma(t)(\theta(t)dt + dW(t))), \quad 0 \leq t \leq T$$

Step 3: Compute the lower bound

$$\bar{W}(0, x, i) \approx \frac{1}{M_2} \sum_{l=1}^{M_2} U(\bar{X}(T))$$

Step 4: Repeat Steps 1–3 N_2 times and then compute the tight lower bound $sup_{C \in S_2} \bar{W}(0, x, i)$.

We will illustrate this algorithm using numerical examples of power and non-HARA utilities in the later section.

Remark 4.2.2. This method studied by Ma et al. (2017) [13] broadens the application scope of stochastic control theory, and provides a fast and effective numerical method for calculating the utility function of investment income, return and investment strategy without analytic solution in the state transition market, solved a class of problems that could not be optimized before. Inspired by this work, we now consider a new situation where the market is partially informed.

Most papers focus on frictionless markets, while in practice due to information asymmetry, trading strategy constraints, such as short selling, borrowing and endowments et al., the optimal controls under perfect market assumption may not be realized. In the next sectoin, we will extend our problem into the regime-switching markets with partial information where the only information available to investors is the asset prices, while the return processes are unknown in advance. Based on this model, we will try to solve the same utility maximization problem.

Chapter 5

Utility Maximization in Regime-Switching Markets with Partial Information

The market model is the same regime-switching models as previous section, and the utility maximization problem is the same. However, we are dealing with incomplete information, the only information available to investors is the asset prices, while the return processes are unknown in advance. In order to caputre those unavailable information, we followed by the paper from Bjok et al. [25], and Zhu & Zheng (2021) [14] which points out to model the return process as a finite state Markov chain that leads to Wonham filter.

5.1 Market Model Set Up

Similarly as before, we consider a fixed time period $[0, T]$ and the stochastic basis $(\Omega, \mathbb{F}, \mathcal{F}, \mathcal{P})$ for financial markets, where the filtration $\mathcal{F} = \{\mathbb{F}_t\}_{0 \leq t \leq T}$ satisfies the usual conditions. We model the market by a continuous-time stationary Markov chain process (MCP) by $\{\alpha(t)\}$.

We identify the state space of $\{\alpha(t)\}$ as a finite set of unit vectors $\mathbb{E}\{e_1, e_2, \cdots, e_d\}$ where $e_i \in \mathbb{R}^d$ is a column vectors with unity in the ith position and zeros elsewhere, $i = 1, \cdots, d$. Denote by $\mathbf{Q} = (q_{ij})_{dxd}$ the generator of the Marov chain $\{\alpha(t)\}$ with $q_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^{d} q_{ij} = 0$ for each $i \in \mathbb{D}\{1, \cdots, d\}$. Then the MCP $\{\alpha(t)\}$ has a semi-martingale representation

$$\alpha_t = \alpha_0 + \int_0^t \mathbf{Q}' \alpha_v dv + M_t, \quad 0 \leq t \leq T \tag{5.1.1}$$

where $\mathbf{Q}'$ is the transpose of $\mathbf{Q}$, M_t is an $\mathbb{R}^d$-valued martingale with respect to the filtration generated by $\{\alpha(t)\}$.

We consider we consider a market consisting of one risk-free bond account whose price is denoted by $S_0(t)$ and one risky security with prices denoted by $S(t)$

$$\begin{aligned} dS_0(t) &= r(t)S_0(t)dt, \quad S_0(0) = 1, \quad t \in [0, T] \\ dS(t) &= S(t)(\mu(t)dt + \sigma(t)dW(t)) \end{aligned} \tag{5.1.2}$$

where $\{W(t), t \in [0, T]\}$ is a $\mathbb{R}$-valued standard Brownian motion. Also, $r(t) = r\alpha(t), \mu(t) = \mu\alpha(t), \sigma(t) = \sigma\alpha(t)$, and $r = (r_1, \cdots, r_d), \mu = (\mu_1, \cdots, \mu_d), \sigma_{dxd} = (\sigma_1, \cdots, \sigma_d)$ are vectors of risk-free interest rates, return rates and volatility rates respectively, with r_i, μ_i, σ_i being the rate in regime i. Denote by $\theta = (\theta_1, \cdots, \theta_d)$ the market prices of risk with $\theta_i = \frac{\mu_i - r_i}{\sigma_i}$ for $i \in \mathbb{D}\{1, \cdots, d\}$.

Remark 5.1.1. We assume $\sigma(t)$ are uniformly bounded by $\mathcal{F}_t^S$-progressively measurable processes on $\Omega \times [0, T]$. We also assumes that there exists $k \in \mathbb{R}^+$ such that

$$z^T \sigma(t) \sigma^T(t) z \geq k|z|^2$$

for all $(z, w, t) \in \mathbb{R} \times \Omega \times [0, T]$. This ensures that the metrics of $\sigma(t), \sigma(t)^T$ are invertible and uniformly bounded.

We introduce the following notations:

$$\mathcal{H}^1(0,T;\mathbb{R}) \triangleq \{v : \Omega x[0,T] \to \mathbb{R} | v \in \mathcal{F}^S, E[\int_0^T |v(t)|^2 dt] <]\infty\}$$
$$\mathcal{H}^2(0,T;\mathbb{R}) \triangleq \{\xi : \Omega x[0,T] \to \mathbb{R} | \xi \in \mathcal{F}^S, E[\int_0^T |\xi(t)|^2 dt] <]\infty\}$$

Define a self-financing trading strategy as $\pi = (\pi(t))_{t\in[0,T]}$, is an $\mathcal{F}_t$-adapted process such that $\int_0^T \pi_t^2 dt < \infty$ a.s. and π is proportional portfolio process. Additionally, the set of admissible portfolio strategies is given by

$$\mathcal{A} = \{\pi \in \mathcal{H}^2(0,T;\mathbb{R}) : \pi(t) \in K \quad for\ t \in [0,T]a.e.\}$$

where $K \subseteq \mathbb{R}$ is a closed convex set containing 0.

Some examples of K are discussed in Sass (2007) [26], including $K = \mathbb{R}^n$, $K = \mathbb{R}^+$, and so on.

We denote X_t^π as the investor's wealth at time t, is given as

$$\begin{aligned} dX^\pi(t) &= X^\pi(t)\{[r(t) + \pi^T(t)(\mu(t) - r(t)1)]dt + \pi^T(t)\sigma(t)dW(t)\}, \quad 0 \le t \le T \\ X^\pi(0) &= x \end{aligned} \tag{5.1.3}$$

For initial capital $x > 0$, $\mathcal{A}_K(x)$ denotes the class of K-admissible trading strategies π. Define the value of the expected utility maximization problem as

$$V(x) \triangleq sup_{\pi\in\mathcal{A}_k(x)} E[U(X^\pi(T))] \tag{5.1.4}$$

As we mentioned before, the return processes are unknown in advance. By involving partial information, the utility maximization problem becomes a non-Markov, setting, such that the Hamilton-Jacobi-Bellman (HJB) eqaution of dynamic programming method invalid. To reduce the partially observed problem to an equivalent problem under full information, we define the innovation process $\hat{V} = (\hat{V}(t))_{t\in[0,T]}$,

$$\hat{V}(t) = \int_0^t \sigma^{-1}(u)(\mu(u) - \hat{\mu}(u))du + W(t) \tag{5.1.5}$$

where $\hat{\mu}(t) = E[\mu(t)|\mathcal{F}_T^S]$ is the filter for $\mu(t)$.
Then we would have the following representatives:

$$\begin{aligned} dS(t) &= Diag(S(t)(\hat{\mu}(t)dt + \sigma(t)d\hat{V}(t))) \\ dX^\pi(t) &= X^\pi(t)\{[r(t) + \pi^T(t)(\hat{\mu}(t) - r(t)1)]dt + \pi^T(t)\sigma(t)d\hat{V}(t)\} \end{aligned} \tag{5.1.6}$$

As we are discussing the regime-switching model, we would have the continuous time Markov chain, under which the Wonham filter would be described by a stochastic differential equation, see Bjork et al. (2021) [25]. In this case, the dynamics of asset prices would be given by

$$dS(t) = Diag(S(t)(\mu(H(t))dt + \sigma(t)d\hat{V}(t))) \tag{5.1.7}$$

where we assumte that $\mu(H(t)) = (\mu^1(H(t))), \cdots, \mu^N(H(t))^T$ and $\mu_i(H(t)) = \mu H(t)$, where $H(t)$ is a stationary, irreducible, continuous time Markov chain independent of W with state space $\{e_1, \cdots, e_N\}$, e_k is the k-th unit vector in $\mathbb{R}^d$. The columns of $\mu \in \mathcal{R}^{NxN}$ denote the N possible states of $\mu(t)$ and $H(t)$ has the rate matrix $Q = (q_{ij})$. The filter $\hat{H}(t) = E[H(t)|\mathcal{F}_t^S]$ is given as:

$$d\hat{H}(t) = Q^T\hat{H}(t)dt + (Diag(\hat{H}(t)) - \hat{H}(t)\hat{H}(t)^T)(\sigma(t)^{-1}\mu)^T d\hat{V}(t) \tag{5.1.8}$$

where $\hat{V}(t)$ is the innovation process given in (5.1.5). The initial condition $H(0)$ of process H is assumed to be independent of W, and $\mu(h)$ is assumed to be deterministic with respect to h. Based on the new filter $\hat{H}(t)$ defined in (5.1.8), we would rewrite the wealth process as below:

$$dX^{\pi}(t) = X^{\pi}(t)\{[r(t) + \pi^T(t)(\hat{\mu}_{\hat{H}}(t) - r(t))]dt + \pi^T(t)\sigma(t)d\hat{V}(t)\} \tag{5.1.9}$$

Under this transformation, the original partially observed problem has been reduced to a related problem as in the full information. After solving the reformulated completely observed problem, the partially observable case would be discussed by embedding the filtering equations for unobservable processes. As discussed in Bjork et al. [25], for general hidden Markov models, the infinite dimensional state space problem of the Kolmogorov backward equation would make it impossible to give explicit solutions of optimal control. However, we can still derive the corresponding Primal HJB equation and explore the transformed optimal control problem using the equation.

We define the value functions using the new innovation process and Markov process

$$\begin{aligned} V^{\pi}(t,x,h) &= E_{t,x,h}[U(X^{\pi}(T))] \quad \text{for all } \pi \in \mathbb{R} \\ J(t,x,h) &= sup_{\pi \in \mathcal{A}_k(x)} V^{\pi}(t,x,h) \end{aligned} \tag{5.1.10}$$

In the following sections, we will discuss different method for the transformed optimal control problem, and explore the analytical or numerical solution to the problem.

5.2 Stochastic Control Method

Applying the stochastic control approach, We can derive the HJB equation as following. Recall that the HJB equation under regime switching model has the following form given in equation 2.2.9

$$\frac{\partial V(\cdot,\cdot,i)}{\partial t} + r_i x \frac{\partial V(\cdot,\cdot,i)}{\partial x} - \frac{1}{2}\theta_i^2 (\frac{\partial V(\cdot,\cdot,i)}{x})^2 / \frac{\partial^2 V(\cdot,\cdot,i)}{\partial x^2} + \sum_{j=1}^{d} q_{ij} V(\cdot,\cdot,j) = 0, \quad i \in \mathbb{D}$$

Then it follows as:

$$\begin{aligned} 0 = J_t + sup_{\pi \in \mathcal{K}} \{ & x(r(t) + \pi(t)^T(h - r(t)\cdot \mathbf{1}))J_x + \frac{1}{2}x^2\pi(t)^T\sigma(t)\sigma(t)^T\pi J_{xx} + \hat{\mu}_{\hat{H}}^T(t,h)J_h \\ & + x\pi(t)^T\sigma(t)\sigma(t)^T J_{xh} + \frac{1}{2}Tr[\sigma(t)\sigma(t)^T J_{hh}]\} \end{aligned} \tag{5.2.1}$$

where the boundary condition $J(T,x,h) = U(x)$.

Equation (5.2.1) is a nonlinear PDE with control constraint, which is in general difficult to solve, even numerically. There is one important special case in which the nonlinear PDE (5.2.1) can be simplified into a semi-linear PDE, and the solution may have a representation in terms of the solution of a BSDE. When $U(\cdot)$ is a CRRA utility, the optimal value function $J(t,x,h)$ can be solved it with the help of the HJB equation (5.2.1).

Example 5.2.1 Power utility: $U(x) = \frac{1}{p}x^p$ for $x > 0$, where $p < 1$ and $p \neq 0$.
Then the value function $J(\cdot)$ has a solution given by

$$J(t,x,h) = \frac{1}{p}x^p g(t,h)^{1-p} \tag{5.2.2}$$

We first compute the derivatives of J followed from equation (5.2.2):

$$
\begin{aligned}
J_t &= \frac{1}{p}x^p(1-p)g^{-p}g_t \\
J_x &= x^{p-1}g^{1-p} \\
J_{xx} &= (p-1)x^{p-2}g^{1-p} \\
J_h &= \frac{1}{p}x^p(1-p)g^{-p}g_h \\
J_{xh} &= x^{p-1}(1-p)g^{-p}g_h \\
J_{hh} &= \frac{1}{p}x^p(1-p)(g^{-p}g_{hh} - pg^{-p-1}g_hg_h)
\end{aligned} \tag{5.2.3}
$$

Substituting equation (5.2.3) into the HJB (5.2.1), we assume that $g \geq 0$ and the coefficients of the linear parabolic differential equation for g are polynomials in H, the function g is sufficiently differentiable given by [27], we obtain that g has to satisfy the partial differential equation given by

$$
\begin{aligned}
0 &= \frac{1}{p}x^p(1-p)g^{-p}g_t + sup_{\pi\in\mathcal{K}}\{x^pg^{1-p}[r(t) + \pi(t)^T(\hat{\mu}_{\hat{H}} - r(t)\cdot\mathbf{1})] \\
&+ \frac{1}{2}x^p(p-1)g^{1-p}\pi(t)^T\sigma(t)\sigma(t)^T\pi(t) + \hat{\mu}_{\hat{H}}(t,h)\frac{1}{p}x^p(1-p)g^{-p}g_h \\
&+ x^p(1-p)g^{-p}g_h\pi(t)^T\sigma(t)\sigma(t)^T \\
&+ \frac{1}{2}Tr[\sigma(t)\sigma(t)^T\frac{1}{p}x^p(1-p)(g^{-p}g_{hh} - pg^{-p-1}g_hg_h)]\} \\
0 &= g_t + \hat{\mu}_{\hat{H}}(t,h)g_h + \frac{1}{2}Tr[\sigma(t)\sigma(t)^Tg_{hh}] - G(t,h,g,\sigma(t)g_h) \quad \text{where} \\
G(\cdot) &= sup_{\pi\in\mathcal{K}}\{g_x[r(t) + \pi(t)^T(\hat{\mu}_{\hat{H}} - r(t)\cdot\mathbf{1})] \\
&+ \frac{1}{2}x^2\pi(t)^T\sigma(t)\sigma(t)^T\pi(t)g_{xx} + x\pi(t)^T\sigma(t)\sigma(t)^Tg_{xh}\}
\end{aligned} \tag{5.2.4}
$$

with $g(T,h) = 1$ for all $h \in \mathcal{A}$.
Moreover, by [28], if there exists a classical solution f of (5.2.4), then f has a BSDE representation in the form of $f(t,h) = Y^{t,h}(t)$ and (Y,Z) is the solution of the following BSDE:

$$
\begin{cases}
-dY(s) = G(s,\hat{H},Y(s),Z(s)ds - Z(s)d\hat{v}(s)), t \leq s \leq T \\
Y(T) = 1
\end{cases} \tag{5.2.5}
$$

and $\hat{H}(s)$ satisfies (5.1.8) with initial condition $\hat{H}(0) = h$.
The Feynman-Kac formula yields a stochastic representation of g given by:

$$
g(t,h) = E[exp(r(t)\frac{p}{1-p}(T-t) + \int_t^T \frac{p}{2(1-p)^2}(\hat{\mu}^TZ(s) - r(t))^2\sigma^{-1}(\sigma^{-1})^Tds)|Z(t) = h] \tag{5.2.6}
$$

where the stochastic process $(Z(t)) \in \mathbb{R}^d$ is a solution of the SDE

$$
\begin{aligned}
dZ(t)^k &= a_k(Z(t))dt + b_k(Z(t))d\hat{V}(t) \\
a_k(h) &= \hat{\mu}_{\hat{H}}(t,h) + \frac{p}{1-p}\hat{H}(\hat{\mu} - \hat{\mu}^T\hat{H})(\hat{\mu}^T\hat{H} - r(t))(\sigma(t)\sigma(t)^T)^{-1} \\
b_k(h) &= (\sigma(t)\sigma(t)^T)^{-1}(\hat{\mu} - \hat{\mu}^T\hat{H})^T(\hat{\mu} - \hat{\mu}^T\hat{H})\hat{H}\hat{H}^T
\end{aligned} \tag{5.2.7}
$$

The the maximum point of the optimal portfolio strategy $\pi^* = (\pi_t^*) \in \mathcal{A}$ is well-defined and given by

$$
\pi_t^* = \frac{1}{1-p}(\hat{\mu}^T\hat{H} - r(t))(\sigma(t)\sigma^T(t))^{-1} + \frac{\hat{H}(\hat{\mu} - \hat{\mu}^T\hat{H})g_h(t,h)}{g(t,h)}(\sigma(t)\sigma^T(t))^{-1} \tag{5.2.8}
$$

Example 5.2.2 Logarithmic utility: Recall that log utility is defined as $U(x) = lnx$ for $x > 0$, then the value function $V(t, x, i)$ has a solution given by

$$V(t, x, h) = log(x) + f(t, h) \tag{5.2.9}$$

Similarly as before, we first compute the derivatives of V followed from equation (5.2.9):

$$\begin{aligned} V_t &= f_t \\ V_x &= \frac{1}{x} \\ V_{xx} &= -\frac{1}{x^2} \\ V_h &= f_h \\ V_{xh} &= 0 \\ V_{hh} &= f_{hh} \end{aligned} \tag{5.2.10}$$

Substituting equation (5.2.10) into the HJB (5.2.1), we get that

$$0 = f_t + \hat{\mu}_{\hat{H}}(t, h) f_h + \frac{1}{2} Tr[\sigma(t)\sigma(t)^T f_{hh}] + sup_{\pi \in \mathcal{K}}\{r(t) + \pi(t)^T(\hat{H} - r(t) \cdot \mathbf{1})\} \tag{5.2.11}$$

We assume that $f \geq 0$ and after some algebra with f is sufficiently differentiable, $f(\cdot)$ is given by

$$\begin{aligned} f(t, h) &= E_{t,h}[\int_t^T r_s + (\hat{\mu}_s - r_s)\pi_s - \frac{1}{2}\pi_s^T \sigma_s \sigma_s^T \pi_s ds + \int_t^T \sigma_s \pi_s d\hat{V}_S] \\ &= E_{t,h}[\int_t^T r_s + (\hat{\mu}_s - r_s)\pi_s - \frac{1}{2}\pi_s^T \sigma_s \sigma_s^T{}_s \pi_s ds] \end{aligned} \tag{5.2.12}$$

Note that we need $\pi_s \in [-M, M]$ in order to have $E[\int_0^T \pi_s d\hat{V}_s] = 0$ and that $_\pi$ does not depend on x. The the maximum point of the optimal portfolio strategy $\pi^* = (\pi_t^*) \in \mathcal{A}$ is well-defined and given by

$$\pi_t^* = (\hat{\mu} - r(t))(\sigma(t)^(\sigma(t)^T)^{-1} \tag{5.2.13}$$

However, those above two solutions are only based on trail and error method. They are only a guessing form of the solution and are too computationally complex. It does not really help us to solve the optimal problem. In the next section, we will introduce the dual control method under this market and try to solve the problem.

5.3 Dual control Method

The dual control method is the same as before, we denote the dual function of U by $\tilde{U}(\cdot)$ such that it is a continuous, decreasing and convex function on $(0, \infty)$, and it is defined as

$$\tilde{U}(y) = sup_{x>0}(U(x) - xy)$$

The dual process is given as:

$$\begin{aligned} &dY^{(y,v)}(t) = -Y^{(y,v)}\{[r(t) + \delta_K(v(t))]dt + (\sigma^{-1}(t)(\hat{\mu}(t) - r(t) \cdot \mathbf{1} + v(t)))^T d\hat{V}(t), \quad 0 \leq t \leq T \\ &Y^{(y,v)}(0) = y \end{aligned} \tag{5.3.1}$$

where $\delta_K()$ is the suport function of the set $-K$, defined by $\delta_K(z) = sup_{\pi \in K}\{-\pi^T z\}, z \in \mathbb{R}$, and the dual control process $v \in \mathcal{D}$,defined by

$$\mathcal{D} = \{v = \Omega \times [0, T] \to \mathbb{R} | v \in \mathcal{F}^s \int_0^T [\delta_K(v(t)) + |v(t)^2|]dt < \infty \ a.s.\} \tag{5.3.2}$$

The dual value function is given by

$$\tilde{V}(t,x,h) = inf_{(y,v)\in(0,\infty)\times\mathcal{D}}(xy + E_{t,y,v,h}[\tilde{U}(Y^{y,v}(T))]) \tag{5.3.3}$$

Recall that the primal and dual value functions satisfy the following weak duality relation:

$$\mathbb{E}_{\pi\in K}[U(X^{\pi}(T))] \leq sup_{\pi\in K}\mathbb{E}[U(X^{\pi})(T)] \leq inf_{y,v}(\mathbb{E}_{y,v,h}[\tilde{U}(Y(T))] + xy) \tag{5.3.4}$$

Any $(y^*, v^*) \in (0,\infty)\times\mathcal{D}$ satisfying

$$x_0 y^* + E[\tilde{U}(Y^{(y^*,v^*)}(T))] = \tilde{V}$$

is called the **optimal dual control** and the corresponding $Y^{(y^*,v^*)}$ is called the **optimal dual process**.

Fix y, the dual value function becomes

$$\tilde{J}(t,y,h) = inf_{v\in\mathcal{D}}E_{t,v,y,h}[\tilde{U}(Y^{(y,v)}(T))] \tag{5.3.5}$$

Suppose K is a closed convex cone as we defined previously, which gives $\delta_K(v) = 0$ for $v \in \tilde{K}$ and ∞ otherwise, where $\tilde{K} = \{v : v^T\pi \geq 0, \forall \pi \in K\}$ is the positive polar cone of K for the corresponding optimal dual control problem.

By using the dynamic programming principle (DPP) and applying Ito's Lemma, $\tilde{J}$ satisfies the following HJB equation:

$$\begin{aligned} 0 = inf_{v\in\tilde{K}}\{&\tilde{J}_t - r(t)y\tilde{J}_y + \frac{1}{2}y^2[\sigma(t)^{-1}(h - r(t)\cdot\mathbf{1} + v(t))]^T[\sigma(t)^{-1}(h - r(t)\cdot\mathbf{1} + v(t))]\tilde{J}_{yy} \\ &+ \tilde{J}_h^T(\delta_K(v(t))(\hat{H} - h)) - y[\sigma(t)^{-1}(h - r(t)\cdot\mathbf{1} + v(t))]^T\sigma_t\sigma(t)^T\tilde{J}_{yh} + \frac{1}{2}Tr[\sigma(t)\sigma(t)^T\tilde{J}_{hh}]\} \end{aligned} \tag{5.3.6}$$

where $\tilde{K} = \{v : v^T\pi \geq 0, \forall\pi \in \mathcal{K}\}$ is the positive polar cone of K; and $\tilde{J}(T,y,h) = y$.

Remark 5.3.1. For general utilities with closed convex constraint case, one can write the HJB equation but is not possible to find an exact solution even for power utility due to we only have partial information. The Monte Carlo method we have introduced in the previous chapter can help to solve this problem numerically; however, we need to solve equation (4.2.2) $\frac{\partial\tilde{V}(t,y,i)}{\partial y} + x = 0$ which is too computationally complex.

Based on our constrained optimization problems, another method applies. Li & Zheng (2018) [29] study the constrained portfolio optimization problem for maximizing expected utility from terminal wealth/consumption and construct the necessary and sufficient conditions for both primal constraint problem and dual one in terms of forward and backward stochastic differential equations(FBSDEs). In the next section, we will show that with the help of a systems of FBSDEs, we would be able to solve the partially informed optimal problem numerically. This idea was first introduced by Zhu & Zheng (2021) [14].

Chapter 6

Stochastic Maximum Principle for Utility Maximization

6.1 Background Information

In this chapter, we still focus on the utility maximization problem under the regime-switching market with partial information. In the previous chapter, we have introduced two other methods to help solve the optimal problem. The stochastic control is wedded to the Hamilton-Jacobi-Bellman equation and the requirement of an underlying Markov state process. Using this method, we can derive a guessed solution to the power utility maximization problem; however, this form of solution is too computationally complex, which does not actually solve the problem. Then we use the stochastic dual control method which we introduced in Chapter 3 to solve the optimal investment problem. The spirit of this approach is to suitably embed the constrained problem in an appropriate family of unconstrained ones and find a member of this family for which the corresponding optimal policy obeys the constrains. However, despite the evident power of this approach, it is nevertheless true that obtaining the corresponding dual problem remains a challenge as it often involves clever experimentation and subsequently show to work as desire. In the previous chapter, we can only use this method to derive the corresponding HJB equation (5.3.6), but it is impossible to solve this equation as consists of a systems of nonlinea PDEs.

In this chapter, we follow the approach given in Li & Zheng (2018) [29]. We use the machinery of stochastic maximum principle and Backward Stochastic Differential Equations (BSDEs) to derive the necessary and sufficient conditions of the primal and dual problems separately. Then we progress to a stochastic approach to simultaneously characterise the necessary and sufficient optimality conditions for dual problems as systems of Forward and Backward Stochastic Differential Equations (FBSDEs) coupled with static optimality conditions. This formulation allows us to characterize the primal optimal control as a function of the adjoint processes coming from the dual FBSDEs. Along with the Monte Carlo Method, we will be able to find an approximate solution to the optimal problem under this regime-switching partial informed markets.

6.2 Necessary and Sufficient Condition for the Primal Problems

In this section, we followed by Li & Zheng (2018) [29], we will first derive the necessary and sufficient optimality conditions for the primal problems and use these two conditions to extend to the dual problems.

From the stochastic maximum principle (SMP), we first introduce an admissilbe control $\pi \in \mathcal{A}$ and a solution X^π of the SDE (5.1.9), the associated adjoint equation is the following linear BSDE in the unknown processes $P_1 \in \mathcal{H}^2(0,T;\mathbb{R})$ and $Q_1 \in \mathcal{H}^2(0,T;\mathbb{R})$:

$$\begin{cases} dP_1(t) = -\{[r(t) + \pi^T(t)(\hat{\mu}_{\hat{H}}(t) - r(t))]P_1(t) + Q_1^T(t)\sigma^T(t)\pi(t)\}dt + Q_1^T(t)d\hat{V}(t) \\ P_1(T) = -U'(X^\pi(T)) \end{cases} \tag{6.2.1}$$

We now state the necessary and sufficient optimality conditions for the primal problem.

Theorem 6.2.1 (Necessary & Sufficient condition for the primal problem). *Let $\hat{\pi} \in \mathcal{A}$ and the strictly positive adapted process $X^{\hat{\pi}}$ satisfy the SDE (5.1.9), then there exists a unique solution $(\hat{P}_1, \hat{Q}_1)$ to the adjoint BSDE (6.2.1). Moreover, $\hat{\pi}$ is optimal for the primal problem if and only if the solution $(X^{\hat{\pi}}, \hat{P}_1, \hat{Q}_1)$ of FBSDE:*

$$\begin{cases} dX^{\hat{\pi}}(t) = X^{\hat{\pi}}(t)\{[r(t) + \hat{\pi}^T(t)(\hat{\mu}_{\hat{H}}(t) - r(t))]dt + \hat{\pi}^T(t)\sigma(t)d\hat{V}(t)\}, \\ X^{\hat{\pi}}(0) = x_0, \\ d\hat{P}_1(t) = -\{[r(t) + \hat{\pi}^T(t)(\hat{\mu}_{\hat{H}}(t) - r(t))]\hat{P}_1(t) + \hat{Q}_1^T(t)\sigma^T(t)\hat{\pi}(t)\}dt + \hat{Q}_1^T(t)d\hat{V}(t) \\ \hat{P}_1(T) = -U'(X^{\hat{\pi}}(T)) \end{cases} \tag{6.2.2}$$

and satisfies

$$-X^{\hat{\pi}}(t)\sigma(t)[\hat{P}_1(t)(\hat{\mu}_{\hat{H}}(t) - r(t))\sigma^{-1}(t) + \hat{Q}_1(t)] \in N_K(\hat{\pi}(t)) \ for\ \forall t \in [0,T], \mathbb{P} - a.s. \tag{6.2.3}$$

where $N_K(x)$ is the normal cone to the closed convex set K at $x \in K$, defined as

$$N_K(x) = \{y \in \mathbb{R} : \forall x^* \in K, y^T(x^* - x) \leq 0\}$$

From this theorem and the dual control method, we will in the next section introduce the necessary and sufficient condition for the dual problems.

6.3 Necessary & Sufficient Condition for the Dual Problems

Next we address the dual problem. To establish the existence of an optimal solution, we impose the following condition:

According to the optimal dual control method, there exists some $(\hat{y}, \hat{v}) \in (0, \infty) \times \mathcal{D}$ such that $\tilde{V} = x_0\hat{y} + E[\tilde{U}(Y^{(\hat{y},\hat{v})}(T))]$. Then given some admissible control $(\hat{y}, \hat{v}) \in (0, \infty) \times \mathcal{D}$ with the state dual process $Y^{(y,v)}$ that solves the SDE (5.3.1) and $E[\tilde{U}(Y^{(y,v)}(T))^2] < \infty$, the associated adjoint equation for the dual problem is the following linear BSDE in the unknown processes $\hat{P}_2 \in \mathcal{H}^2(0,T;\mathbb{R})$ and $\hat{Q}_2 \in \mathcal{H}^2(0,T;\mathbb{R})$:

$$\begin{cases} d\hat{P}_2(t) = \{[r(t) + \delta_K(v(t))]^T\hat{P}_2(t) + \hat{Q}_2^T(t)[(\hat{\mu}_{\hat{H}}(t) - r(t))\sigma^{-1}(t) + \sigma^{-1}(t)v(t)]\}dt + \hat{Q}_2^T(t)d\hat{V}(t), \\ \hat{P}_2(T) = -\tilde{U}'(Y^{(\hat{y},\hat{v})}(T)) \end{cases} \tag{6.3.1}$$

Since $\hat{P}_2Y^{\hat{y},\hat{v}}$ is a margtingale, we can find $\hat{P}_2(t), 0 \leq t \leq T$ from the relation

$$\hat{P}_2Y^{\hat{y},\hat{v}}(t) = E[\hat{P}_2Y^{\hat{y},\hat{v}}(T)|\mathcal{F}_t] = -E[Y^{\hat{y},\hat{v}}(T)\tilde{U}'(Y^{\hat{y},\hat{v}}(T))|\mathcal{F}_t] \tag{6.3.2}$$

Then we derive the necessary and sufficient conditions of optimality of the dual problem.

Theorem 6.3.1 (Necessary & sufficient condition for optimality of the dual problem). *Let $(y, v) \in (0, \infty) \times \mathcal{D}$ and $Y^{(y,v)}$ be the corresponding state dual process satisfying the SDE (5.3.1) with*

$E[\tilde{U}(Y^{(y,v)}(T))^2] < \infty$. Then the random variable $Y^{(y,v)}(T)\tilde{U}'(Y^{y,v}(T))$ is square integrable and there exists a solution $(\hat{P}_2, \hat{Q}_2)$ to the adjoint BSDE (6.3.1).
Let $(\tilde{y}, \tilde{v}) \in (0,\infty)\mathcal{D}$. Then $(\tilde{y}, \tilde{v})$ is optimal for the dual problem if and only if the solution $(Y^{(\tilde{y},\tilde{v})}, \hat{P}, \hat{Q})$ of FBSDE

$$\begin{cases} dY^{(\tilde{y},\tilde{v})}(t) = -Y^{(\tilde{y},\tilde{v})}(t)\{[r(t)+\delta_K(\tilde{v}(t))]dt + [\sigma^{-1}(t)(\hat{\mu}_{\hat{H}}(t) - r(t)\cdot \mathbf{1} + \tilde{v}(t))]^T d\hat{V}(t)\} \\ Y^{(\tilde{y},\tilde{v})}(0) = \tilde{y} \\ d\hat{P}_2(t) = \{[r(t)\hat{P}_2(t) + \hat{Q}_2^{\ T}\sigma^{-1}(t)(\hat{\mu}_{\hat{H}}(t) - r(t)\cdot \mathbf{1}]\}dt + \hat{Q}_2^{\ T}(t)d\hat{V}(t) \\ \hat{P}_2(T) = -\tilde{U}'(Y^{(y,v)}(T)) \end{cases} \tag{6.3.3}$$

satisfies the following conditions

$$\begin{cases} \hat{P}_2(0) = x_0 \\ \hat{P}_2^{\ -1}(t)[\sigma^T(t)]^{-1}\hat{Q}_2(t) \in K \\ \hat{P}_2(t)\delta_k(\tilde{v}(t)) + \hat{Q}_2^{\ T}(t)\sigma^{-1}(t)\tilde{v}(t) = 0, \quad \forall t \in [0,T]\mathbb{P}a.s. \end{cases} \tag{6.3.4}$$

We can now state the dynamic relations of the optimal portfolio and wealth processes of the primal problem and the adjoint processes of the dual problem and vice versa.

Theorem 6.3.2 (From dual problem to primal problem)**.** *Suppose that $(\hat{y}, \hat{v})$ is the optimal for the dual problem. Let $(Y^{(\tilde{y},\tilde{v})}, \hat{P}, \hat{Q})$ be the associated process that solves FBSDE (6.3.3) and satisfies condition (6.3.4). Define*

$$\hat{\pi}(t) = \frac{\sigma(t)^{-1}\hat{Q}_2(t)}{\hat{P}_2(t)}, \ t \in [0,T] \tag{6.3.5}$$

Then $\hat{\pi}$ is teh optimal control for the primal problem with initial wealth x_0. The optimal wealth process and associated adjoint process are given by

$$\begin{cases} X^{\hat{\pi}}(t) = \hat{P}_2(t), \\ \hat{P}_1(t) = -Y^{(\hat{y},\hat{v})}(t), \\ \hat{Q}_1(t) = Y^{(\hat{y},\hat{v})}[\sigma^{-1}(t)\hat{v}(t) + \sigma^{-1}(t)(\hat{\mu}_{\hat{H}}(t) - r(t)\cdot \mathbf{1})] \end{cases} \tag{6.3.6}$$

Then provided by the necessary and sufficient conditions, we will be able to transfer the original optimal problem into a system of FBSDEs and solve for the optimal control.

Example 6.3.1 Power Utility Let $U(x) = \frac{1}{p}x^p$ for $x > 0$, where $p < 1$ and $p \neq 0$. The dual function is defined as $\tilde{U}(y) = -\frac{1}{q}y^q$ for $q = \frac{p}{p-1}$. Without any constraint under partial information, we assume K is the whole space, which leads to $v(t) = 0 \forall t$. Using the method of Li & Zheng (2018) [29] and the previous two theorems, we illustrate this example.

In order to find the optimal $\hat{\pi}$, we will be needing to solve the adjoint BSDE involving $\hat{P}_2(t)$ and $\hat{Q}_2(t)$ as follows:

$$\begin{aligned} d\hat{P}_2(t) &= \{[r(t)\hat{P}_2(t) + \hat{Q}_2^{\ T}\sigma^{-1}(t)(\hat{\mu}(\hat{H}) - r(t)\cdot \mathbf{1}]\}dt + \hat{Q}_2^{\ T}(t)d\hat{V}(t) \\ \hat{P}_2(T) &= -\tilde{U}'(Y^{(y,v)}(T)) \\ d\hat{H}(t) &= Q^T\hat{H}(t)dt + (Diag(\hat{H}(t)) - \hat{H}(t)\hat{H}(t)^T)(\sigma(t)^{-1}\mu)^T d\hat{V}(t) \end{aligned}$$

where $\hat{H}$ has the following SDE (5.1.8).

The above BSDE is a systems of nonlinear PDEs. Since the innovation process $\hat{V}(t)$ and the Wonham filter $\hat{H}$ is driven by the same Brownian Motion as the wealth process X^{π}, it is difficult to solve the optimal problem.

Remark 6.3.3. If we are dealing with linear Stochastic differential equations under Kalman filtering of OU processes, see Zhu & Zheng (2021) [14] for details. We will be able to use all of the above methods, including stochastic control, dual control and stochastic maximum principle to solve the optimal problem. Moreover, all the methods can be applied for other general cases. However when we change the model into a regime-switching market, even for specific CRRA utility function under full space, it is almost unlikely to give the optimal control explicitly, not to mention adding closed convex constraints. The regime-switching market with partial information assumption makes those method to be invalid. Additionally, the derivation of π under this case requires the corresponding dual control v^* and certain functions of y, both of which are difficult to be given explicitly.

6.4 FBSDEs Monte Carlo Tight Bounds

For utilities in regime-switching markets with partial information, one can write the HJB equation (both primal and dual), but it is not possible to find an analytical solution even for power utility due to the double driven Brownian Motion which complex the whole equation into a nonlinear SDEs. Then we show that it is possible to use the dual FBSDEs to find upper and lower bounds for the value function, even works for general utilities.

Recall that the primal and dual value functions satisfy the following weak duality relation:

$$\mathbb{E}_{\pi\in K}[U(X^{\pi}(T))] \leq sup_{\pi\in K}\mathbb{E}[U(X^{\pi})(T)] \leq inf_{y,v}(\mathbb{E}_{y,v,h}[\tilde{U}(Y(T))] + xy) \tag{6.4.1}$$

The inequalities show that the dual formulation gives an upper bound for the primal value function, instead of focusing on the exact controls, we try to explore the tight lower and upper bounds for the value function of general cases. In what follows, we will focus on finding tight bounds by Stochastic Maximum Principle(SMP).

Using Theorem (6.3.1), (y, v) is an optimal dual control if and only if $(Y^{(y,v)}, \hat{P}, \hat{Q})$ satisfying equation (6.3.3) and equation (6.3.4). Denote by $\pi(t) = \hat{P}^{-1}[\sigma^T(t)]^{-1}\hat{Q}(t)$ from Theorem (6.3.2) and we can rewrite equation (6.3.4) as

$$\begin{cases} \hat{P}(0) = x_0, \\ \pi(t) \in K, \\ \delta_K(v(t)) + \pi^T(t)v(t) = 0, \quad \forall t \in [0,T]\mathbb{P}a.s. \end{cases} \tag{6.4.2}$$

Then we can write the dual FBSDE system of (6.3.3) and (6.3.4) as the following:

$$\begin{cases} dY(t) = -Y(t)[r(t)] + \delta_K(v(t))]dt - Y(t)[\sigma^{-1}(t)(\hat{H}(t) - r(t)\cdot\mathbf{1}) + v(t)]^T d\hat{V}(t) \\ d\hat{P}(t) = \hat{P}(t)[r(t) + \pi^T(t)(\hat{H}(t) - r(t)\cdot\mathbf{1})]dt + \hat{P}(t)\pi^T(t)\sigma(t)d\hat{V}t \\ d\hat{H}(t) = Q^T\hat{H}(t)dt + (Diag(\hat{H}(t) - \hat{H}(t)\hat{H}(t)^T)(\sigma^{-1}(t)\mu)^T d\hat{V}(t) \\ Y(0) = 1, P(0) = x_0, \hat{H}(0) = h_0, \pi(t) \in K, \end{cases} \tag{6.4.3}$$

and

$$E[w|\hat{P}(T) + \tilde{U}^T(yY(T))|^2 + (1-w)\int_0^T [\delta_K(v(t)) + \pi(t)v(t)]dt] = 0 \tag{6.4.4}$$

with

$$min_{y,\pi,v}E[w|\hat{P}(T) + \tilde{U}^T(yY(T))|^2 + (1-w)\int_0^T [\delta_K(v(t)) + \pi(t)v(t)]dt] \text{ subject to equation (6.4.3)} \tag{6.4.5}$$

where $w \in (0,1)$ is a given constant, and δ_K is the support function of $-K$ mentioned in (5.3.1).

Remark 6.4.1. Note that equation (6.4.3) is a forward controlled SDE system with state variables $Y, \hat{P}, \hat{H}$ and control variables π, v, and equation (6.4.5) is a standard control problem with an additional decision variable $y > 0$. If we can manage to find (y, π, v) that makes the objective function zero, then we have solve equation (6.3.3) and (6.3.4). The key advantage of equation (6.4.5) over the dual FBSDE system (6.3.3) and (6.3.4) is that (6.4.5) is an optimal control problem with known optimization techniques, which can help solve the problem.

6.4.1 Construction of Tight Upper and Lower Bounds Algorithm

The dual FBSDE system of (6.3.3) and (6.3.4) is difficult to find its solution. We may only be able to find (y, π, v) that makes the objective function close enough to zero, which is not the exact solution to FBSDE system. However, (y, π, v) and $(Y, \hat{P})$ may provide with us a lower and upper bound for the system as

$$\begin{aligned} \text{Lower Bound} &= E[U(\hat{P}(t))] \\ \text{Upper Bound} &= x_0 y + E[\tilde{U}(Y(t))] \end{aligned} \tag{6.4.6}$$

The algorithm is quit the same as in the full information part (Dual Control Monte Carlo Method), and it goes as the following:

Step 1: Dived the interval $[0, T]$ into n subintervals with grid points $t_i = ih, i = 0, 1, \cdots, n$ and step size $h = \frac{T}{n}$.

Step 2: On each interval $[t_i, t_{i+1}), i = 0, 1, \cdots, n-1$, choose constant controls π_i and v_i that are $\mathcal{F}_{t_i}$ measurable.

Step 3: Iterate through equation (6.4.3) by using a discrete time controlled system with $Y(t_i)$ denoted by Y_i as

$$\begin{cases} Y_{i+1} = Y_i - Y_i[r_i + \delta_K(v_i)]h - Y_i[\sigma_i^{-1}(\hat{H}_i - r_i \cdot \mathbf{1} + v_i)]^T(\hat{V}_{i+1} - \hat{V}_i) \\ \hat{P}_{i+1} = \hat{P}_i + \hat{P}_i[r_i + \pi_i^T(\hat{H}_i - r_i \cdot \mathbf{1})]h + \hat{P}_i\pi_i^T\sigma_i(\hat{V}_{i+1} - \hat{v}_i) \\ \hat{H}_{i+1} = \hat{H}_i + Q^T\hat{H}_i h + (Diag(\hat{H}_i) - \hat{H}_i\hat{H}_i^T)(\hat{\sigma}_i^{-1}\hat{\mu}_i)(\hat{V}_{i+1} - \hat{V}_i) \\ Y_0 = 1, P_0 = x_0, \hat{H}_0 = h_0, \pi_i \in K, i = 0, \cdots, n-1 \end{cases} \tag{6.4.7}$$

where $\hat{V}_{i+1} - \hat{V}_i, i = 0, 1, \cdots, n-1$ are independent Normal random variables of $N(0, h)$, usually, we would generate n independent standard normal random variables $N(0, 1)$ denoted by $Z_{i+1}, i = 0, 1, \cdots, n-1$ to replace $\hat{V}_{i+1} - \hat{V}_i$ with $\sqrt{h}Z_{i+1}$.

Step 4: Generate M sample paths of MCP α, which are used to compute different vector paths of risk-free interest rates, return rates and volatility rates.

Step 5: Repeat Step 1 to 3 M times and take average of M copies of $|\hat{P}_n + \tilde{U}'(yY_n)|^2$, and we would get an approximate value for the following equation

$$min_{y,(\pi_i, v_i)_{i=0}^{n-1}} E[w|\hat{P}_n + \tilde{U}'(yYn)|^2 + (1-w)h\sum_{i=0}^{n-1}[\delta_K(v_i) + \pi_i v_i]] \quad \text{subject to (6.4.7)} \tag{6.4.8}$$

Then with a total of $2n + 1$ variables we would be able to minimize the objective function (6.4.8) by finding the (y, π, v).

Step 6: The lower bound is given by $LB = E[U(\hat{P}_n)]$ and the upper bound by $UB = x_0 y + E[\tilde{U}(Y_n)]$.

Chapter 7

Numerical Examples

In this section, we will use the numerical methods we have introduced in the previous chapters to compute the lower and upper bounds for optimal problem under regime-switching markets with full and partial information. At the end of the chapter, we will compare the results from the two scenarios.

7.1 Full Information Case

When dealing with the regime-switching markets with full information, we use the Dual control Monte Carlo method to compute the tight bounds. Detailed algorithm can be found in Chapter 4.2.

7.1.1 Power Utility

In this example, we assume that $U(x)$ is a power utility given by

$$U(x) = \frac{1}{p}x^p, \quad 0 < p < 1$$

and the dual function is given by

$$\tilde{U}(y) = -\frac{1}{q}y^q, \quad q = \frac{p}{p-1}$$

Follow the method we discussed in Chapter 4.2, we need to solve the equation (4.2.2) $\frac{\partial \tilde{V}(t,y,i)}{\partial y} + x = 0$. Given the SDE $Y(T) = y_t exp(A_{t,T})$ in (3.3.3), we have

$$\tilde{V}(t, y, i) = -\frac{1}{q}y^q \beta(t, i),$$
$$\beta(t, i) = E_{t,i}[exp(qA_{t,T})]$$

By solving equation (4.2.2), we have the following equation

$$y_i^* = (\frac{x}{\beta(t, i)})^{p-1}$$

Then follow the last step of the algorithm, we have the tight upper bound as:

$$UB = inf_{C \in \mathcal{S}_1} W(t, x, i) = inf_{C \in \mathcal{S}_1}\{-\frac{1}{q}(y_i^*)^q \beta(t, i) + xy_i^*\}$$

To calculate the lower bound, we first calculate the optimal control $\bar{\pi}(t, x, i)$ given by equation (4.2.3) $\bar{\pi}(t, x, i) = \frac{\theta_i}{\sigma_i}\frac{y^*}{x}\frac{\partial^2}{\partial y^2}\tilde{V}(t, y^*, i)$, by direct computation, we have

$$\bar{\pi}(t, x, i) = \frac{\theta(t, i)}{\sigma(t, i)}\frac{1}{1-p}$$

Substituting this $\bar{\pi}$ into the wealth process X^{π} and solve we get

$$X^{\bar{\pi}}(T) = x_t \exp[\int_t^T (r_s + \frac{1-2p}{2(1-p)^2}\theta_s^2)ds + \frac{\theta_s}{1-p}dW_s]$$

Follow the last step of the algorithm, we have the tight lower bound as:

$$LB = sup_{C \in \mathcal{S}_2}\{\bar{W}(t,x,i) = U(x)E_{t,i}E_{t,i}[\exp[\int_t^T (pr_s + \frac{p}{2(1-p)}\theta_s^2)ds]]\}$$

For numerical solution, we consider the following set ups.

For power utility, we take $p = \frac{1}{2}$, the intial wealth at time $t = 0$ is $x_0 = 1$, the total investment period $T = 1$, and the number of simulations is $M_1 = M_2 = 10^7$.
We consider a 2-state Markov chain process with generating matrix

$$Q = \begin{pmatrix} -a & a \\ b & -b \end{pmatrix}$$

where a,b are positive constants from the uniform distribution on interval [0.1, 2], which means the transition of one state to another can be slow (average once every 10 years) or fast (average twice a year) or anything in between. The states are a "growth economy"(state 1) and a "recession economy"(state 2). We generate 5 sample sets of a, b for robustness. The riskless interest rates, return, and volatility rates of risky asset are given by

$$r = (0.05, 0.01), \quad \mu = (0.13, 0.07), \quad \sigma = (0.2, 0.3)$$

The comparisons are carried out for the cases of $C = 0$ and additional control vector C (sampling $N_1 = 50$ times for computing the tight upper bound and $N_2 = 0$ times for computing the tight lower bound). In the test, the benchmark value is the primal value explicitly given by equation (2.2.13)

$$V(t,x,i) = a(t,i)\frac{x^p}{p}, \quad \forall (t,x,i) \in [0,T]x[0,\infty[x\mathbb{D}$$

where $a(t) = exp[-(\Lambda - Q)(T-t)] \cdot 1$ and $\lambda(i) = \frac{1}{2}(\frac{\mu_i - r_i}{\sigma_i})^2 \frac{p}{p-1} - r_i p$ for $i \in \mathbb{DD}$, given by equation (2.2.18).

			No Control C		With Control C	
a,b	α_0	Benchmark	LB	UB	LB	UB
1.873029	1	2.3099	2.3095	2.3103	2.3098	2.3100
1.779791	2	2.2248	2.2243	2.2250	2.2247	2.2249
1.675422	1	2.3529	2.3516	2.3531	2.3528	2.3530
0.512917	2	2.0003	1.9998	2.0004	2.0003	2.0004
0.669115	1	2.2937	2.2930	2.2941	2.2937	2.2940
0.477458	2	2.2270	2.2264	2.2276	2.2268	2.2271
0.356473	1	2.0868	2.0864	2.0877	2.0866	2.0869
0.972562	2	2.0521	2.0518	2.0524	2.0521	2.0522
1.742312	1	2.1799	2.1796	2.1804	2.1795	2.1804
1.100127	2	2.1270	2.1263	2.1274	1.1270	2.1271
			diff	rel-diff(%)	diff	rel-diff(%)
		mean	0.000951	0.043366	0.0003	0.012313
		std	0.000301	0.013041	0.00022	0.010152

Table 7.1: Numerical results for Power utility with Full information

The numerics in Table (7.1) show that the dual-control Monte- Carlo method generates correct and tight lower and upper bounds on the primal value. From the table, we can see that the use of control vector C can tighten the gap between the lower and upper bounds. We have also listed statistics in Table (7.1), including the mean and the standard deviation of absolute and relative difference between the lower and upper bounds. The relative difference is defined as $\frac{UB-LB}{LB} * 100$. It is clear that the gap between the tight lower and upper bounds is very small given the mean and standard deviation difference between them.

7.1.2 A non-HARA utility

In this example, we assume that $U(x)$ is a non-HARA utility given by Bian and Zheng (2015) [7]:

$$U(x) = \frac{1}{3}H(x)^{-3} + H(x)^{-1} + xH(x)$$

for $x > 0$, where

$$H(x) = (\frac{2}{-1+\sqrt{1+4x}})^{1/2}$$

The dual function of U is defined by (from previous Example 3.2.3)

$$\tilde{U}(y) = \frac{1}{3}y^{-3} + y^{-1}$$

Given the SDE $Y(T) = y_t exp(A_{t,T})$ in (3.3.3), we have

$$\begin{aligned}\tilde{V}(t,y,i) &= \frac{1}{3}y^{-3}D(t,i) + y^{-1}F(t,i),\\ D(t,i) &= E_{t,i}[\exp{(-3A_{t,T})}],\\ F(t,i) &= E_{t,i}[\exp{(-A_{t,T})}]\end{aligned}$$

By solving equation (4.2.2), we have the following equation

$$y_i^* = \sqrt{\frac{1}{2x}(F(t,i) + \sqrt{F^2(t,i) + 4xD(t,i)})}$$

The tight upper bound is given by

$$UB = inf_{C\in\mathcal{S}_1}\{\frac{1}{3}(y^*)^{-3}D(t,i) + (y^*)^{-1}F(t,i) + xy^*\}$$

Then we calculate the optimal control $\bar{\pi}(t,x,i)$ given by

$$\bar{\pi}(t,x,i) = \frac{2\theta_i}{\sigma_i}\frac{1}{x}[(y^*)^{-4}D(t,i) + x]$$

Substituting this $\bar{\pi}$ into the wealth process X^{π} and solve we get

$$X^{\bar{\pi}}(t+\Delta t) = (1 + r(t)\Delta t)X^{\bar{\pi}}(t) + X^{\bar{\pi}}(t)\bar{\pi}(t)\sigma(t)(\theta(t)\Delta(t) + W_{\Delta t})$$

Follow the last step of the algorithm, we have the tight lower bound as:

$$LB = sup_{C\in\mathcal{S}_2}E_{t,x,i}[U(X^{\bar{\pi}}(T))]$$

For numerical solution, we consider the following set ups.

Basically we follow the same set up in the previous example with the intial wealth at time $t = 0$ is $x_0 = 1$, the total investment period $T = 1$, and the number of simulations is $M_1 = M_2 = 10^7$. We consider a 2-state Markov chain process with generating matrix

$$Q = \begin{pmatrix} -a & a \\ b & -b \end{pmatrix}$$

where $a = b = 0.5$. The riskless interest rates, return, and volatility rates of risky asset are given by

$$r = (0.05, 0.01), \quad \mu = (0.13, 0.07), \quad \sigma = (0.2, 0.3)$$

	No Control C				With Control C			
α_0	LB	UB	diff	rel-diff(%)	LB	UB	diff	rel-diff(%)
1	2.5220	2.5288	0.0068	0.2696	2.5220	2.5243	0.0023	0.0912
2	2.4923	2.4967	0.0044	0.1765	2.4923	2.4935	0.0012	0.0481

Table 7.2: Numerical results for non-HARA utility with Full information

The comparisons are carried out for the cases of $C = 0$ and additional control vector C (sampling $N_1 = 50$ times for computing the tight upper bound and $N_2 = 50$ times for computing the tight lower bound). Since under regime-switching model, there is no closed-form formula for this non-HARA utility, thus we do not have any benchmark to compare with. However, with the previous power utility example, we believe that our algorithm works well for this utility as well.

The numerics in Table (7.2) show that the dual-control Monte- Carlo method generates correct and tight lower and upper bounds on the primal value. We have also listed statistics in the table, the relative difference between the lower and upper bounds show that the gap between the lower and upper bounds are small.

Clearly from both examples, we can see that the dual control Monte Carlo method has provide with us an accurate approximation of the true optimal value function. Moreover, with the extrac control vector C, the gap between the lower and upper bounds have been tighten. Thus, we can conclude that this method is a good approximation, and we will use those results as an benchmark to compare with the results in partial information.

7.2 Partial Information

When dealing with the regime-switching markets with partial information, we use the FBSDE Monte Carlo method to compute the tight bounds. Detailed algorithm can be found in Chapter 6.4. For simplicity, we assume the market has riskless asset and one risky asset and r, σ are constants. We first consider the case where K is the whole space (i.e. $K = \mathbb{R}$).

Since K is the whole space, then $\tilde{K} = \{0\}$ and $v_i = 0$ for all i, and we consider controls π_i in the following form: $\pi_i = a + b\hat{H}_i$ with a, b being constants. Then we can rewrite discrete system (6.4.7) as the following:

$$\begin{cases} Y_{i+1} = Y_i - Y_i rh - Y_i[\sigma^{-1}(\hat{H}_i - r\cdot \mathbf{1})]^T\sqrt{h}Z_{i+1} \\ \hat{P}_{i+1} = \hat{P}_i + \hat{P}_i[r + \pi_i^T(\hat{H}_i - r\cdot\mathbf{1})]h + \hat{P}_i\pi_i^T\sigma\sqrt{h}Z_{i+1} \\ \hat{H}_{i+1} = \hat{H}_i + Q^T\hat{H}_i h + (Diag(\hat{H}_i) - \hat{H}_i\hat{H}_i^T)(\hat{\sigma}^{-1}\hat{\mu}_i)\sqrt{h}Z_{i+1} \\ Y_0 = 1, P_0 = x_0, \hat{H}_0 = h_0, \\ \pi_i = a + b\hat{H}_i \end{cases} \tag{7.2.1}$$

with

$$min_{y,a,b}E[|\hat{P}_n + \tilde{U}'(yYn)|^2] \quad \text{subject to (7.2.1)} \tag{7.2.2}$$

The lower bound is given by $LB = E[U(\hat{P}_n)]$ and the upper bound by $UB = x_0 y + E[\tilde{U}(Y_n)]$.

7.2.1 Power Utility

The utility function is power utility given by

$$U(x) = \frac{1}{p}x^p, \quad 0 < p < 1$$

The dual function is given by

$$\tilde{U}(y) = -\frac{1}{q}y^q, \quad q = \frac{p}{p-1}$$

and the derivative of dual function with respect to y is given by

$$\tilde{U}'(y) = -y^{q-1}$$

For numerical results, we follow the same numerical set-ups in previous section. We consider a 2-state Markov chain process with generating matrix

$$Q = \begin{pmatrix} -c & c \\ d & -d \end{pmatrix}$$

where c,d are positive constants from the uniform distribution on interval [0.1, 2]. We generate 6 samples of c and d to illustrate the robustness. The riskless interest rates, return, and volatility rates of risky asset are given by

$$r = (0.05, 0.01), \quad \mu_0 = (0.13, 0.07), \quad \sigma = (0.2, 0.3)$$

along with $x_0 = 10, n = 10, T = 1,$ such that step size$h = \frac{T}{n} = 0.1$, the number of simulations is $M = 10^7$. Define the initial guess of h_0 as below:

$$h_0 = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}$$

Moreover, we will varying through h_0 and σ_0 to see how the tight bounds would change.

c,d	α_0	h_0	LB	UB	rel-diff(%)
1.873029	1	0.05	2.1244	2.1474	1.0826
1.779791	2	0.05	2.0280	2.0487	1.0207
1.675422	1	0.05	2.2991	2.3224	1.0134
0.512917	2	0.05	2.2556	2.2806	1.1084
0.669115	1	0.1	2.2220	2.2540	1.4402
0.477458	2	0.1	2.1036	2.1286	1.1884
0.356473	1	0.1	2.3750	2.4038	1.2126
0.972562	2	0.1	2.2797	2.3098	1.3208
1.742312	1	0.2	2.4030	2.4390	1.4981
1.100127	2	0.2	2.3232	2.3622	1.6787
1.742312	1	0.2	2.3843	2.4293	1.8873
1.100127	2	0.2	2.2369	2.2789	1.8776
				mean	1.3607
				std	0.3164

Table 7.3: Numerical results for Power utility with Partial information by Varying h_0

Tables (7.3) and (7.4) indicate that FBSDEs Monte Carlo method gives reasonable ranges for optimal values. The results show that by using dual method, we can always give a range for the value function and generate tight lower and upper bounds. From the tables we can see that as the h_0 increases, the value of the tight bounds seem to be increasing as well, along with the relative difference between them. Similar situation happens when we vary through σ_0. However, based on the relative difference, we still can conclude that this method provides with us an accurate tight bounds for the optimal value function.

7.2.2 A non-HARA Utility

In this example, we assume that $U(x)$ is a non-HARA utility given by Bian and Zheng (2015) [7]:

$$U(x) = \frac{1}{3}H(x)^{-3} + H(x)^{-1} + xH(x)$$

for $x > 0$, where

$$H(x) = \left(\frac{2}{-1 + \sqrt{1 + 4x}}\right)^{1/2}$$

c,d	α_0	σ_0	LB	UB	rel-diff(%)
1.873029	1	(0.1,0.2)	2.1645	2.1875	1.0626
1.779791	2	(0.1,0.2)	2.0277	2.0484	1.0209
1.675422	1	(0.1,0.2)	2.0475	2.0725	1.2210
0.512917	2	(0.1,0.2)	2.0263	2.0496	1.1499
0.669115	1	(0.2,0.3)	2.3361	2.3611	1.0702
0.477458	2	(0.2,0.3)	2.2906	2.3226	1.3970
0.356473	1	(0.2,0.3)	2.3177	2.3465	1.2426
0.972562	2	(0.2,0.3)	2.1652	2.1952	1.3906
1.742312	1	(0.3,0.4)	2.4949	2.5339	1.5632
1.100127	2	(0.3,0.4)	2.3505	2.3865	1.5316
1.742312	1	(0.3,0.4)	2.4591	2.5041	1.8299
1.100127	2	(0.3,0.4)	2.3976	2.4396	1.7517
				mean	1.3526
				std	0.2716

Table 7.4: Numerical results for Power utility with Partial information by Varying σ_0

The dual function of U is defined by (from previous Example 3.2.3)

$$\tilde{U}(y) = \frac{1}{3}y^{-3} + y^{-1}$$

thus the derivative of the dual function with respect to y is given by:

$$\tilde{U}'(y) = -y^{-4} - y^{-2}$$

For numerical results, we follow the same numerical set-ups in the power utility case, with generating matrix Q with $c = d = 0.5$. Also with an initial guess of h_0 as

$$h_0 = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}$$

and vary through h_0 and σ_0 to see the change of the results.

α_0	h_0	LB	UB	rel-diff(%)
1	0.05	2.2957	2.3211	1.1064
2	0.05	2.1420	2.1690	1.2133
1	0.1	2.3864	2.4184	1.3409
2	0.1	2.2127	2.2439	1.4078
1	0.2	2.4038	2.4384	1.4373
2	0.2	2.2354	2.2722	1.6536
			mean	1.3599
			std	0.1901

Table 7.5: Numerical results for non-HARA utility with Partial information by Varying h_0

Tables (7.5) and (7.6) indicate that FBSDEs Monte Carlo method gives reasonable ranges for optimal values. From the tables we can see that as the h_0 increases, the value of the tight bounds seem to be increasing as well, along with the relative difference between them. Similar situation happens when we vary through σ_0. However, based on the relative difference, we still can conclude that this method provides with us an accurate tight bounds for the optimal value function.

7.3 Comparison between the model

Theoretically, despite of the utility function we choose, the value of the optimal control function under full information should always be greater than the values under partial information. When

α_0	σ_0	LB	UB	rel-diff(%)
1	(0.1,0.2)	2.1955	2.2215	1.1843
2	(0.1,0.2)	2.1955	2.2215	1.1843
1	(0.2,0.3)	2.2337	2.2591	1.1371
2	(0.2,0.3)	2.2209	2.2521	1.4026
1	(0.3,0.4)	2.4368	2.4713	1.4178
2	(0.3,0.4)	2.3407	2.3775	1.5721
			mean	1.3455
			std	0.1609

Table 7.6: Numerical results for non-HARA utility with Partial information by Varying σ_0

dealing with partial information, we need to make assumptions and approximation such that we can transfer this incomplete dataset into a complete dataset. In our case, we use the Wonham filter to transform our partially informed model setting into a fully informed model. By doing this, some of the true values are not observed and will be lost. Thus suggesting the value of the optimal control function will be smaller.

In practice, when we use the dual control based Monte Carlo method to find the tight upper and lower bounds for the optimal value function. The numerical results from both fully and partially informed model are quit close. In addition, the tight upper and lower bounds in the partially informed model is slightly less than the values in the fully informed model. This result consists with what we have discussed above in theoretical. By comparing the results from the two model, and see from the relative difference between the tight upper and lower bounds, when dealing with full information market, we tends to have a more accurate tight bounds compare to the values in the partially informed market. This also makes sense, even when we are usign approximation method to calculate the optimal value functions, with full information in hand, we still tends to have a better approximation as we do not deal with the loss of information. However, we can still conclude that this Monte Carlo method for computing an approximate value for optimal value function is quit accurate for both full and partial information markets.

Chapter 8

Conclusion

In this paper, we study the dual control approach for the optimal asset allocation problem in the general market, and extend it into a continuous-time regime-switching market. Under the fully informed market, we followed a Monte Carlo algorithm to compute the tight lower and upper bounds of the value function based on a system of fully coupled nonlinear partial differential HJB equations. Then we focus on the portfolio optimization problem with limited market information under full space. By transforming partial information problems into observable cases using Wonham filter principle, the optimal controls are explored with the help of stochastic control, dual control and dual FBSDEs respectively. Applying the Monte Carlo method on the dual FBSDEs, we can find the tight lower and upper bounds of the primal objective value function. To show the method is reliable and accurate, we illustrate with some numerical tests for power and non-HARA utility functions under both fully and partially informed markets. The results indicate that under both markets this method can provide tight bounds. We can therefore find the approximate value function and its corresponding control strategies numerically for general utility functions in a regime switching Black-Scholes model with both full and partial information.

Since in our paper, we assume we are dealing with a full space, and we do not consider any constraints. In future studies, we can study for some constrained cases. Some examples of K are discussed in Sass (2007) [26], including $K = \mathbb{R}^n$, $K = \mathbb{R}^+$, and so on.

Bibliography

[1] Harry M Markowitz. *Portfolio selection*. Yale university press, 1968.

[2] Robert C Merton. An analytic derivation of the efficient portfolio frontier. *Journal of financial and quantitative analysis*, 7(4):1851–1872, 1972.

[3] William F Sharpe. Capital asset prices: A theory of market equilibrium under conditions of risk. *The journal of finance*, 19(3):425–442, 1964.

[4] John Lintner. The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. In *Stochastic optimization models in finance*, pages 131–155. Elsevier, 1975.

[5] Jan Mossin. Equilibrium in a capital asset market. *Econometrica: Journal of the econometric society*, pages 768–783, 1966.

[6] Robert C Merton. Optimum consumption and portfolio rules in a continuous-time model. In *Stochastic optimization models in finance*, pages 621–661. Elsevier, 1975.

[7] Baojun Bian and Harry Zheng. Turnpike property and convergence rate for an investment model with general utility functions. *Journal of Economic Dynamics and Control*, 51:28–49, 2015.

[8] James D Hamilton. A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica: Journal of the econometric society*, pages 357–384, 1989.

[9] Qing Zhang, G Yin, and RH Liu. A near-optimal selling rule for a two-time-scale market model. *Multiscale Modeling & Simulation*, 4(1):172–193, 2005.

[10] G Yin, Qing Zhang, F Liu, RH Liu, and Y Cheng. Stock liquidation via stochastic approximation using nasdaq daily and intra-day data. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, 16(1):217–236, 2006.

[11] Toshiki Honda. Optimal portfolio choice for unobservable and regime-switching mean returns. *Journal of Economic Dynamics and Control*, 28(1):45–78, 2003.

[12] Jörn Sass and Ulrich G Haussmann. Optimizing the terminal wealth under partial information: The drift process as a continuous time markov chain. *Finance and Stochastics*, 8(4):553–577, 2004.

[13] Jingtang Ma, Wenyuan Li, and Harry Zheng. Dual control monte-carlo method for tight bounds of value function in regime switching utility maximization. *European Journal of Operational Research*, 262(3):851–862, 2017.

[14] Harry Dongmei Zhu, Zheng. Tight bounds for constrained utility maximization with drift uncertainty. 2021.

[15] Richard Bellman and Robert E Kalaba. *Dynamic programming and modern control theory*, volume 81. Citeseer, 1965.

[16] Harry Zheng. Introduction to dual control method for utility maximization. 2019.

[17] Ioannis Karatzas, Steven E Shreve, I Karatzas, and Steven E Shreve. *Methods of mathematical finance*, volume 39. Springer, 1998.

[18] Jun Fu, Jiaqin Wei, and Hailiang Yang. Portfolio optimization in a regime-switching market with derivatives. *European Journal of Operational Research*, 233(1):184–192, 2014.

[19] Alan F Karr et al. J. jacod and an shiryaev, limit theorems for stochastic processes. *Bulletin (New Series) of the American Mathematical Society*, 21(2):369–374, 1989.

[20] Ioannis Karatzas, John P Lehoczky, and Steven E Shreve. Optimal portfolio and consumption decisions for a “small investor” on a finite horizon. *SIAM journal on control and optimization*, 25(6):1557–1586, 1987.

[21] John C Cox and Chi-fu Huang. Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of economic theory*, 49(1):33–83, 1989.

[22] Robert C Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *The review of Economics and Statistics*, pages 247–257, 1969.

[23] David B Hertz. Risk analysis in capital investment. *Harvard business review*, 42:95–106, 1964.

[24] Phelim P Boyle. Options: A monte carlo approach. *Journal of financial economics*, 4(3):323–338, 1977.

[25] Tomas Björk, Mark HA Davis, and Camilla Landén. Optimal investment under partial information. *Mathematical Methods of Operations Research*, 71(2):371–399, 2010.

[26] Jörn Sass. Utility maximization with convex constraints and partial information. *Acta Applicandae Mathematicae*, 97(1):221–238, 2007.

[27] Peter E Kloeden and Eckhard Platen. Stochastic differential equations. In *Numerical Solution of Stochastic Differential Equations*, pages 103–160. Springer, 1992.

[28] Huyên Pham. *Continuous-time stochastic control and optimization with financial applications*, volume 61. Springer Science & Business Media, 2009.

[29] Yusong Li and Harry Zheng. Dynamic convex duality in constrained utility maximization. *Stochastics*, 90(8):1145–1169, 2018.

[30] Ankur Ankan and Abinash Panda. *Hands-on Markov models with python: Implement probabilistic models for learning complex data sequences using the Python ecosystem*. Packt Publishing Ltd, 2018.

[31] Renato Feres. Notes for math 450 continuous-time markov chains and stochastic simulation. *Disponıvel em¡ http://www. math. wustl. edu/ feres/Math450Lect05. pdf*, 2007.

[32] Ulrich Rieder and Nicole Bäuerle. Portfolio optimization with unobservable markov-modulated drift process. *Journal of Applied Probability*, 42(2):362–378, 2005.

[33] Baojun Bian, Sheng Miao, and Harry Zheng. Smooth value functions for a class of nonsmooth utility maximization problems. *SIAM Journal on Financial Mathematics*, 2(1):727–747, 2011.

[34] Jakša Cvitanić and Ioannis Karatzas. Convex duality in constrained portfolio optimization. *The Annals of Applied Probability*, pages 767–818, 1992.

[35] Chantal Labbé and Andrew J Heunis. Conjugate duality in problems of constrained utility maximization. *Stochastics: An International Journal of Probability and Stochastics Processes*, 81(6):545–565, 2009.

LI_JIAROU_01891595

GRADEMARK REPORT

FINAL GRADE

/0

GENERAL COMMENTS

Instructor

PAGE 1

PAGE 2

PAGE 3

PAGE 4

PAGE 5

PAGE 6

PAGE 7

PAGE 8

PAGE 9

PAGE 10

PAGE 11

PAGE 12

PAGE 13

PAGE 14

PAGE 15

PAGE 16

PAGE 17

PAGE 18

PAGE 19

PAGE 20

PAGE 21

PAGE 22

PAGE 23

PAGE 24

PAGE 25

PAGE 26

PAGE 27

PAGE 28

PAGE 29

PAGE 30

PAGE 31

PAGE 32

PAGE 33

PAGE 34

PAGE 35

PAGE 36

PAGE 37

PAGE 38

PAGE 39

PAGE 40

PAGE 41

PAGE 42

PAGE 43

PAGE 44

PAGE 45

PAGE 46

PAGE 47

PAGE 48

PAGE 49

PAGE 50

PAGE 51

PAGE 52

PAGE 53

PAGE 54