# LONGEVITY RISK: AN INTENSITY-BASED **APPROACH**

by

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## Declaration

I declare that the work contained in this thesis is my own work unless otherwise stated.

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#### Abstract

The systematical underestimation of the past life expectancy improvements and the stochastic nature of mortality, makes longevity a high profile risk. The development of a mature longevity market, where investors can manage their exposure to longevity risk, is of growing interest. This paper describes a modeling framework for mortality, by defining the death arrival of an individual as the first jump of a doubly stochastic Poisson process. A two-factor Gaussian process with age-dependent drift and diffusion is used to model the stochastic intensity and is calibrated to U.K. mortality data. The model captures the effects of the whole mortality surface and allows imperfect correlations between generations. Finally, we define and describe longevity-linked instruments, with linear and non-linear payoffs, that can be used to hedge the longevity risk from a portfolio of pensions or annuities.

**Key words:** stochastic mortality, longevity risk, doubly stochastic Poisson process, reduced-form intensity models, two-factor Gaussian process, longevity swaps, longevity caps/floors, longevity swaptions.

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## 1 Introduction

#### 1.1 Motivation

In the last few decades, the human life expectancy has experienced significant improvements, especially in the developed countries. In the United Kingdom, the life expectancy of a new born has improved by 17.92% since 1950, according to the Mortality Human Database (HMD) [24, U.K Life expectancy at birth table]. During the fifties the life expectancy of a newborn was approximately seventy years, but now in the twenty-tens has reached the eighty years. Many factors have contributed to these improvements, such as the better lifestyle, the new technologies and progress in medicine, the better education, etc. Although this is good news for humanity, it has to deal with the new challenges arose from people live longer. A major impact is on pensions, as the risk of savings not be enough to cover retirement is increasing. The risk that people live longer than expected, does not affect only individuals. Pension funds, annuity providers, the government and Defined Benefits (DB) pension schemes are some the of the institutions exposed to such risk, with their liabilities related to retirement increasing. In simple terms, they have to pay the guaranteed income to the oldies longer than expected. Moreover, this effect is magnified due to the low interest rate environment, as pension related investments are mostly into fixed income markets.

Many studies have shown that life expectancy evolves in stochastic fashion. In recent literature, many stochastic models have been introduced to describe the evolution of mortality. In order to show that mortality is best modeled as a stochastic process and to capture its trends over the years, we plot the survival functions and death curves based on the U.K. total population life table taken from the HMD. The life tables consider a hypothetical population of 100000 each year and describe its observed mortality experience. In figure 1 (left), we show the number of survivors  $\ell_x$  at each age x from birth until the age of 110, observed at the beginning of each decade from the thirties to the twenty-tens. We notice that the survivor numbers are improving through the years i.e. the shape of the curve becomes more rectangular. As described by E. Pitacco (2002) [38], this effect is known as rectangularization and the reason for its occurrence is that deaths increasingly concentrated around the mode. In other words, the volatility of deaths around the mode is decreasing. This is illustrated as well in figure 1 (right), where we plot the number of deaths  $d_x$  with respect to age x. Also, notice that the mode of the number of deaths has moved to the right, meaning that the age that most deaths occur tends to be higher in recent years. This phenomenon is known as expansion, in the actuarial literature.

Before we get into details about the modeling framework for both mortality and financial markets, it is important to understand the risks enclosed into the portfolios of pension funds and annuity providers. They are typically exposed to two major sources of uncertainty. The first source are the risks related to the financial markets. The most important is the interest rate risk, as most

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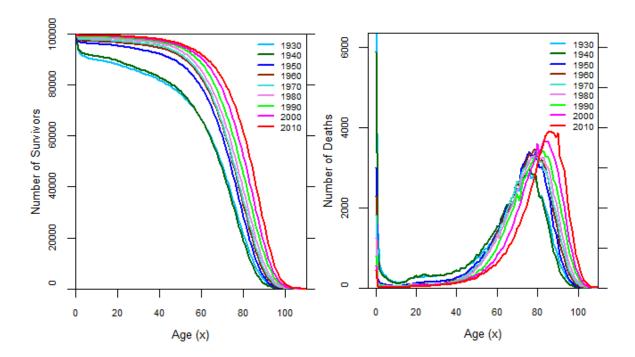


Figure 1: Number of survivors  $\ell_x$  (left) and number of deaths  $d_x$  (right) at each age x from birth until the age of 110, observed at the beginning of each decade from the thirties to twenty-tens. Data corresponds to U.K life table of total population obtained from HMD. From the curves on the left figure we can see the phenomenon of rectangularization and from the curves on the right the expansion

of the assets on their balance sheet are fixed income securities. The market for interest rates is well developed, with many liquid products available in the market, that can be used for hedging and managing this risk. The second source of uncertainty that affects their portfolios is mortality risk, which will be our primarily focus in this paper. There are two sources of mortality risk that affecting a portfolio of pensions or annuities; the unsystematic (idiosyncratic) and systematic (longevity) risk. As described by E. Biffis, M. Denuit and P. Devolder (2010) [6], the former is the risk related to the randomness of death occurrences in a portfolio of insureds. In other terms, is the volatility around the expected number of deaths in a portfolio and by considering a large portfolio of insureds, this risk is diversifiable. On the other hand, the systematic or longevity risk cannot diversify by considering a large portfolio, since it is a trend risk rather a volatile one, and affects the portfolio in the same direction. In particular, this risk associated with the randomness of survival probabilities and as the mortality tends to improve in the past few years, we can say (in layman's term) that the longevity risk is the chance that people live longer than expected.

In recent years, longevity risk became a high profile risk. The main reasons are that longevity risk has been systematically underestimated and the current environment of low interest rates. In addition, under the new regulatory framework for insurance sector in European Union, Solvency

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II, pension funds and insurers are obligated to reserve longevity risk (solvency) capital (see e.g. J.S. Li and A. Luo (2012) [31] and J. Loeys et. al. (2007) [34]). Many financial institutions, corporations and the government who fund retirements of the working population, are seeking to hedge the longevity risk, holding on their portfolios. The insurance and reinsurance sectors have a risk appetite to take a part of the longevity risk, in order to offset the long position they have in term assurances. However, they do not have sufficient capital to absorb all of the longevity risk volume (see e.g G. Jones (2013) [27]). According to the Global Pesnions Assets Study (2016) by Willis Tower Watson, only the U.K. pension market accounts for approximately £3trn.

A possible candidate, big enough to absorb the large volume of longevity risk, is the capital market. In the past few years, there has been an increasing interest from the capital markets to issue longevity-linked securities, such as the mortality and longevity bonds issued by Swiss Re in 2003 and European Investment Bank in 2004, respectively. In addition, mortality indexes were announced by Credit Suisse, Goldman Sachs, J.P. Morgan and few others. Recently, another family of longevity-linked instruments are traded; the Longevity Swaps. Although most of the transactions were customized over-the-counter (bespoke) longevity swaps, there were and few more standardized based on a mortality index. Also, the weak correlation between mortality and financial market, it makes the longevity-linked instruments even more attractive for all kind of investors - hedgers, speculators and arbitragers, who are willing to diversify their portfolios.

The need for adequate pricing and reserving methodologies for longevity-linked products is of a main concern. Despite the fact that so much effort has devoted in academic and industry research, there is no accepted framework for mortality risk modeling yet. In order to incorporate with the mortality improvements, there was a shift form the classical actuarial deterministic methods and the projected mortality tables. Most of the research now is based on continuous-time stochastic mortality models that take into account the effects on the mortality due to age, calendar year and generation. Also, a popular approach in the literature considers the time of death of an individual as the first jump time of a doubly stochastic Poisson process.

The aim of this paper is to define a continuous-time mortality framework, based on reduced-form intensity models. Moreover, under this framework, we choose a suitable model for the stochastic mortality intensity and calibrate it to the data of U.K. males total population. Finally, we define and give pricing methodology for longevity-linked securities, with linear and non-linear payoffs.

## 1.2 Literature review

Actuaries used to work under the assumption of deterministic mortality, using life tables or deterministic models, such as the Gompertz-Makeham law of mortality. In order to capture the mortality improvements, the projected mortality tables were adopted, providing forecasts of the future mortality at different ages and calendar years. During the last few decades, the improve-

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ments in mortality rates and equivalently in survival probabilities, appear to occur in a stochastic fashion and be systematically underestimated. Many papers analyzed the historical mortality trends (see e.g. E. Pitacco [39]). As the traditional actuarial deterministic models seems to be inadequate for capturing this improvements, many researchers turn their attention to stochastic models in order to capture the uncertainty in the survival probabilities. The key paper by R.D. Lee and L.R. Carter (1992) [30], introduced the Lee-Carter model. It is a discrete-time model for mortality projections and it was widely used not only in academia, but also in the industry. Briefly, the model describes a linear relationship between the logarithm of central death rates far a specific age and a function of two variables; the initial age x and time t, given by

$$\log(m_{x,t}) = \alpha_x + b_x k_t + e_{x,t}, \qquad t = 1, 2, \dots, n \qquad x = 1, 2, \dots, \omega,$$

where  $m_{x,t}$  is the central death rate for age x and time t (more details on following sections),  $e_{x,t}$  is a random error, and  $\alpha_x$ ,  $b_x$  and  $k_t$  are the set of parameters to be estimated. Following Lee-Carter, many other discrete-time models were introduced, for example the two-factor model by A.J.G. Cairns et. al. (2006b) [15].

The need for closed-form solutions for the insurance and mortality-linked products, led to the consideration of continuous-time stochastic models. Many models were proposed to capture the evolution of mortality for single generation. Most of them are based on the same ideas used in short rate models. For example, A.J.G. Cairns et. al. (2006) [14] described a modeling framework of the term-structure of mortality, very similar to the one used in interest-rate models. A number of researchers exploit similarities between mortality risk and credit risk. P. Artzner and F. Delbean [1] were the first to introduce the reduced-form intensity models, commonly used in credit risk, for insurance applications. This framework defines the death times as the first jump of a doubly stochastic Poisson process. This approach is followed in our paper. E. Luciano and E. Vigna (2008) [35] modeled the stochastic intensity of the doubly stochastic Poisson process using affine processes, such as the Ornstein-Uhlenbeck process with and without jumps and the Feller process. In 2005, E. Biffis [4] considered a two-factor affine process for the mortality intensity, in order to capture the effects of different generations, an important feature that mortality models need to have. P. Jevtic, E.Luciano and E.Vigna (2012) [26] described a framework that takes into account the cohort (generation) effect and allows for non-perfect correlations between different generations.

The aim is the development of the longevity market, where investors can hedge and manage the longevity exposure of their portfolios, or to take long position in longevity risk, for example to diversify their portfolios. Many longevity-linked instruments have been proposed and some of them were traded. For example, the longevity bonds introduced by D. Blake and W. Burrows (2001) [9], issued by the European Investment Bank with the BNP Paribas as a structurer in 2004. Now, longevity swaps, introduced by K. Dowd (2003) [19], have received growing attention. M.C. Fung, K. Ignatieva and M. Sherris (2015) [21] and M.M. Boyer and L. Stentoft (2013) [10] considered the

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hedge effectiveness of longevity-linked derivatives, where the former used a hypothetical portfolio of annuities with exposure to longevity risk.

Finally, E.Biffis et. al. (2014) [5] introduced the counterparty risk in the longevty swap transactions. They examined the impact of the bilateral default risk and collateral rules on the marking-to-market of longevity swaps. In our paper, we will assume that there is no credit risk.

#### 1.3 Research structure

The rest of the paper is organized as follows. In Section 2, we describe the mortality market framework. We follow the reduced-form intensity approach, borrowed from credit risk modeling. We apply the results of Jeanblanc and Rutoswski (2000) [25] with the default risk being the time of death of an individual. First, we briefly describe the theory of enlargement of filtrations and stopping times, as they play a crucial role in our analysis. Then, we define the hazard processes and their consistency within the actuarial literature. In section 2.3, we prove the equivalence between the hazard process and the single-jump process associated with the death of an individual, and then we define it as the first-jump of a doubly stochastic Poisson process. Most of the definitions and theorems used in this section borrowed from P. Brémaud (1981) [11, Chapter 2, pages 18-55] and P. Protter (2004) [39]. Finally, section 2.4.3 deals with the behavior of the stochastic mortality under an equivalent change of measure.

Section 3 provides a brief description of the financial market. In section 4, we start by considering the available mortality and longevity data, along with the available mortality indexes. Then, we move into specification of a model for the mortality intensity process. A two-factor Gaussian model is used, with age-dependent drift and diffusion terms. This model was introduced by M.C. Fung et. al. (2015) [21], but also used previously to model short-rate dynamics in interest rate markets (see e.g. D. Brigo and F. Mercurio (2007)[13, Section 4.2.1, page 143]. Then, we estimate the model parameters using least square estimations and the differential evolution algorithm. Finally in this section, we simulate the paths of mortality intensity and the death curves using Monte Carlo simulations.

In section 5, we describe some financial instruments with underlying the realized survival rate, with linear and non-linear payoffs, that can be used to hedge and manage the longevity risk from pension and annuity portfolio. We provide their definitions and their pricing methodology. A key assumption used, is the independence between financial and mortality market. Finally, in last section 6 we conclude with a brief summary. Additional useful materials, such as definitions, tables and figures are provided in the Appendix A and B.

## 2 Mortality market framework

In this section, we will setup up the framework to describe the uncertainty linked to mortality. Our aim is to characterize the time of death of an individual as the first jump time of a doubly stochastic Poisson process. We borrow the reduce-form intensity approach from credit risk modeling, under which the default time is modeled as a totally inaccessible stopping time. We first give a brief description of the theory of the enlargement of filtrations and the stopping times. We give the definitions of the hazard processes and the stochastic intensities, and show the equivalence between them and the single-jump default process. Moreover, we look at some important criteria that reasonable mortality intensities must follow. Finally, we study the behavior of the stochastic intensity under an equivalent change of measure.

## 2.1 Information flow and enlargement of filtration

Filtrations are a fundamental feature for the characterization of stochastic processes and play an important role in financial mathematics. They represent the evolution of information or knowledge with time. An important concept that the reduced-form intensity models are using is the theory of enlargement of filtrations.

#### 2.1.1 Filtrations and usual hypothesis

Let us recall the definition of the filtration and the important class of filtrations that satisfy the usual hypotheses.

**Definition 2.1** (Filtration). Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. A filtration on  $(\Omega, \mathcal{F}, \mathbb{P})$  is a non-decreasing family  $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$  of sub- $\sigma$ -fields of  $\mathcal{F}$  i.e for each t,  $\mathcal{F}_t$  included in  $\mathcal{F}$  and for each  $0 \leq s \leq t$ ,  $\mathcal{F}_s$  is a subset of  $\mathcal{F}_t$ . The probability space endowed with such filtration  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ , is called filtered probability space.

We say that a stochastic process  $(X_t)_{t\geq 0}$  is adapted to the filtration  $\mathbb{F}$ , if for each non-negative t,  $X_t$  is  $\mathcal{F}_t$ -measurable. A stochastic process is always adapted to its natural filtration,  $\mathcal{F}_t^X := \sigma(X_s; s \leq t)$ ; the smallest filtration to which X is adapted. Also, we say that a stochastic process is progressively measurable, if  $X_s$  is  $\mathcal{F}_t$ -measurable, for all  $0 \leq s \leq t$ . In order to be able to apply the general results of stochastic calculus, we will assume that all the filtrations that we will use satisfy the usual hypotheses.

**Definition 2.2** (Usual Hypotheses). The filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  satisfies the usual hypotheses if

- it is complete i.e.  $\mathcal{F}_0$  contains all the  $\mathbb{P}$ -null sets
- the filtration is right continuous, i.e. for any  $t \geq 0$ ,  $\mathcal{F}_t = \mathcal{F}_{t+}$ , with  $\mathcal{F}_{t+} = \bigcap_{s > t} \mathcal{F}_s$

#### 2.1.2 Stopping times and enlargement of filtration

In a default-free market, we usually given a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  endowed with the filtration  $\mathbb{F}$ , which contains information of the behavior of the risk factors up to and including time t. For example, in the Black-Scholes market framework,  $\mathbb{F}$  contains information of the stocks' prices dynamics. Now, if we let another filtration  $\mathbb{G} := (\mathcal{G})_{t \leq 0}$  such that  $\mathcal{F}_t$  is a proper subset of  $\mathcal{G}_t$  for all non-negative t, an important question to ask, is how an  $\mathcal{F}_t$ -martingale behave when is considered under the enlarged filtration  $\mathbb{G}$ . The theory of the enlargement of filtration provide the tools to answer this question.

First, let us consider the two types of enlargement; the *initial* and the *progressive*. The former is used to model insider trading, where the market participants do not have the same information but they invest in the same financial market. In this case, new information arrive at the initial point in time. In the case of the *progressive enlargement*, new information arrive progressively with time. In other words, if  $\mathcal{H}_t$  denote the new information at time t, the enlarged filtration at time t is given by  $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$ . It is an important tool used to compute the intensity of the default time under the reduced-form models, in credit-risk framework. We will apply a similar approach to the mortality risk modeling, where the default time will be the time of death of an individual or equivalently its residual lifetime. We will consider the default time  $\tau$  to be a stopping time in the enlarged filtration, so let us recall the definition of the stopping time.

**Definition 2.3** (Stopping time). Let  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  be a filtered probability space. The random variable  $\tau : \Omega \to [0, \infty]$  is a stopping time if the event  $\{\tau \leq t\}$  belong to  $\mathcal{F}_t$  for all non-negative t.

Similarly with the credit risk models, we will consider the stopping time  $\tau$  to be *totally inaccessible*. So, let us give the definitions of two of the important types of stopping times, borrowed from P. Protter (2004) [39, Section 3.2, page 103]. The classification of the stopping times is based on whether is possible to predict the time of their occurrence.

**Definition 2.4** (Predictable Stopping time). A stopping time  $\tau$  is *predictable* if there exists an increasing sequence of stopping times  $(\tau_n)_{n\geq 1}$  such that  $\lim_{n\to\infty} \tau_n = \tau$  a.s. and  $\tau_n < \tau$  whenever  $\tau > 0$ , for all n. The sequence  $(\tau_n)_{n\geq 1}$  is said to announce  $\tau$ .

**Definition 2.5** (Totally Inaccessible Stopping time). A stopping time  $\tau$  is totally inaccessible if for every predictable stopping time  $\sigma$ ,

$$\mathbb{P}\{\omega : \tau(\omega) = \sigma(\omega) < \infty\} = 0$$

**Remark 2.6.** A totally inaccessible stopping time can never be announced by an increasing sequence of predictable stopping times. Intuitively, it occurs with total surprise.

Let us consider the default process  $(H_t)_{t\geq 0}$  given be  $H_t = \mathbf{1}_{\{\tau\leq t\}}$  and its natural filtration  $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ . The idea is to enlarged the initial filtration  $\mathbb{F}$  with the default process  $(H_t)_{t\geq 0}$ , such

that the new filtration  $\mathbb{G}$  to be smallest filtration with  $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$  and  $\tau$  be a stopping time with respect to  $\mathbb{G}$ .

Remark 2.7. The default process  $(H_t)_{t\geq 0}$  is  $\mathcal{G}_t$ -adapted, but it is not necessarily  $\mathcal{F}_t$ -adapted, and the  $\mathcal{G}_t$ -stopping time  $\tau$  is not necessarily an  $\mathcal{F}_t$ -stopping time.

Let us consider an important property of the enlarged filtration  $\mathbb{G}$ , that will be used to show that the default process admits an intensity. One may refer to P. Protter (2004) [39, Lemma, page 370] for the proof of the following lemma.

**Lemma 2.8.** If A is a  $\mathcal{G}_t$ -predictable process then there exists an  $\mathcal{F}_t$ -predictable process  $\Lambda$  such that  $A = \Lambda$  on  $[0, \tau]$ 

In general, an  $\mathcal{F}_t$ -martingale is not a  $\mathcal{G}_t$ -martingale. In mathematical finance is important to consider the cases where the martingale property is preserved in the enlarged filtration. We do not want the dynamics of an asset to change when we move to the new filtration, in order to avoid arbitrage opportunities. So, let us consider the **H**-hypothesis, as it referred to by P. Bremaud and M. Yor [12, Theorem 3, page 284], and some important equivalent forms.

**Definition 2.9** (H-hypothesis). Every  $\mathcal{F}_t$ -square integrable martingale is a  $\mathcal{G}_t$ -square integrable martingale.

**Remark 2.10.** Under the **H**-hypothesis, an  $\mathcal{F}_t$ -Brownian motion preserved its property in the  $\mathbb{G}$  filtration.

**Theorem 2.11.** Let  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  be a sub-filtration of  $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ . The following statements are equivalent:

- 1. The **H**-hypothesis is satisfied.
- 2. For all non-negative t, the  $\sigma$ -fields  $\mathcal{G}_t$  and  $\mathcal{F}_{\infty}$  are conditionally independent given  $\mathcal{F}_t$ .
- 3. For all non-negative t,  $\mathbb{P}(\tau \leq t | \mathcal{F}_{\infty}) = \mathbb{P}(\tau \leq t | \mathcal{F}_{t})$

*Proof.* For the proof we refer the reader to M. Jeanblanc and M. Rutkowski (2000) [25]. □

Remark 2.12. The intuition behind the third expression in theorem 2.11 is that the conditional probability of default before t, depends only on the evolution of the risk factors up to time t. In the case of mortality risk, the conditional probability of death before t, depends only on the evolution of mortality up to time t, and not on the whole path.

## 2.2 Hazard process

Now that we have the concept of enlargement of filtrations, we will start building the continuoustime mortality modeling framework. We borrowed this setup from credit risk modeling, using the

intensity based approach, with default to be the death of an individual. For the rest of this section, we use the techniques borrowed from M. Jeanblanc and M. Rutkowski (2000) [25] and P. Artzner and F. Delbean (1995) [1], which was the first paper that applied the intensity models' framework to insurance context. Our goal is to show the link between hazard process and doubly stochastic Poisson processes.

#### 2.2.1 Residual lifetime and hazard process

Let  $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{F})$  be a filtered probability space satisfying the usual hypotheses. Let (x) denote an individual life aged x at time t = 0, where x is a non-negative real number. The filtration  $\mathbb{G}$  is the progressive enlarged filtration  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ , where  $\mathbb{F} = (\mathcal{F}_t)_{t \leq 0}$  holds information of the mortality dynamics and other financial factors. In other words, we can write  $\mathbb{F}$  as  $\mathbb{F} = \mathbb{F}^{Int.} \vee \mathbb{F}^{Mort.}$ , the natural filtrations of the interest rate and mortality dynamics. In our case, we will have the natural filtration of two Brownian Motions. The filtration  $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$  is the natural filtration of the default process given by  $H_t = \mathbb{I}_{\{\tau \leq t\}}$  and it holds information whether death occurred or not up to and including time t.

We consider an insured (x) and we model his/her residual lifetime/death time  $\tau$  as a  $\mathcal{G}_t$ -stopping time. For any non-negative t, we consider the process  $F_t := \mathbb{P}(\tau \leq t | \mathcal{F}_t)$ , the distribution function of  $\tau$ , conditional on the information up to and including time t. We assume that  $F_t$  is a cadlag process with  $F_t < 1$  and  $F_0 = 0$ .

**Lemma 2.13.** Let  $F = (F_t)_{t \le 0}$  be a càdlàg process, defined as  $F_t := \mathbb{P}(\tau \le t | \mathcal{F}_t)$ , with  $F_0 = 0$  and  $F_t < 1$  for all non-negative t. Then F is a bounded, non-negative  $\mathcal{F}_t$ -submartingale.

*Proof.* It is easy to see that F is bounded and non-negative since it defined as a probability. Now, since for all  $0 \le t \le s$  the event  $\{\tau \le t\}$  is a subset of  $\{\tau \le s\}$ , we have

$$\mathbb{E}^{\mathbb{P}}[F_s|\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[\mathbb{P}(\tau \le s|\mathcal{F}_s)|\mathcal{F}_t] = \mathbb{P}(\tau \le s|\mathcal{F}_t) \ge \mathbb{P}(\tau \le t|\mathcal{F}_t) = F_t.$$

The second equality follows from the tower property of conditional expectations.  $\Box$ 

Also, from the process F we can define the survival process of the stopping time  $\tau$  conditional on the  $\mathbb{F}$ -filtration by  $S_t := 1 - F_t = \mathbb{P}(\tau \ge t | \mathcal{F}_t)$ . Intuitively, gives the probability that (x) survives the next t years conditional on the evolution of mortality up to and including t. Following the same arguments in the proof above, we can show that  $S = (S_t)_{t \le 0}$  is a bounded, non-negative  $\mathcal{F}_t$ -supermartingale.

The use of hazard functions is a familiar concept to actuaries. In the traditional actuarial framework, the deterministic hazard function is used to calculate the survival probabilities i.e.  $S(t) = e^{-A(t)} \text{ where } A(t) := \int_0^t \mu(u) du \text{ is the cumulative hazard function and } \mu(\cdot) \text{ is called } hazard function or the force of mortality. Here we provide a more general definition of the hazard function$ 

borrowed from M. Jeanblanc and M. Rutkowski (2000) [25, Definition 3.1, page 18], where it can be time-varying deterministic or a stochastic process. Then in the following lemma we consider the evaluation of the survival probability with respect to G-filtration and we follow the proof from E. Biffis, M. Denuit and P. Devolder (2010) [6, Proposition A.1, page 30]. This is an important result for the valuation of defaultable contingent claims.

**Definition 2.14** (Hazard Process). Let  $F_t := \mathbb{P}(\tau \leq t | \mathcal{F}_t)$  and assume that  $F_0 = 0$  and  $F_t < 1$ , for all non-negative t. The  $\mathbb{F}$ -hazard process  $\Gamma$ , of the stopping time  $\tau$ , is  $\mathcal{F}_t$ -adapted and is defined as  $S_t = 1 - F_t = e^{-\Gamma_t}$ , or equivalently as  $\Gamma_t = -\log(1 - F_t) = -\log(S_t)$  for all non-negative t.

**Lemma 2.15.** For every bounded,  $\mathcal{G}_t$ -measurable contingent claim Y and every non-negative  $t \leq T$ , we have

$$\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\{\tau > T\}}Y|\mathcal{G}_t] = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\{\tau > T\}}Y|\mathcal{F}_t]}{\mathbb{P}(\tau > t|\mathcal{F}_t)} = \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\{\tau > T\}}Y|\mathcal{F}_t]. \tag{2.1}$$

The survival probability of the stopping time  $\tau$  conditional on the  $\mathbb{G}$ -filtration is given by

$$\mathbb{P}(\tau > T | \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{P}(\tau > T | \mathcal{F}_t)}{\mathbb{P}(\tau > t | \mathcal{F}_t)} = \mathbb{1}_{\{\tau > t\}} \mathbb{E}^{\mathbb{P}}[e^{\Gamma_t - \Gamma_T} | \mathcal{F}_t]$$
(2.2)

Proof. In the current information structure, we have that  $\mathcal{G}_t$  and  $\mathcal{F}_t$  agree on  $\{\tau > t\}$  i.e. for all  $A \in \mathcal{G}_t$  there exist an event  $B \in \mathcal{F}_t$  such that  $A \cap \{\tau > t\} = B \cap \{\tau > t\}$ . Also, we can easily see that  $\mathbb{1}_{\{\tau > T\}} = \mathbb{1}_{\{\tau > t\}} \mathbb{1}_{\{\tau > T\}}$ . Starting form the left hand side of equation (2.1), for arbitrary  $A \in \mathcal{G}_t$  we have

$$\begin{split} \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\{\tau > t\}}\mathbb{1}_{\{A\}}Y] &= \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\{\tau > T\}}\mathbb{1}_{\{A \cap \{\tau > t\}\}}Y] = \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\{\tau > T\}}\mathbb{1}_{\{B \cap \{\tau > t\}\}}Y] \\ &= \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\{\tau > T\}}\mathbb{1}_{\{B \cap \{\tau > t\}\}}Y] = \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{\{\tau > T\}}\mathbb{1}_{\{B\}}Y], \end{split}$$

where  $B \in \mathcal{F}_t$ . Taking now the right hand side of equation (2.1) and for arbitrary  $A \in \mathcal{G}_t$  we have

$$\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\tau>t\}}\mathbb{1}_{\{A\}}\frac{\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\tau>T\}}Y|\mathcal{F}_{t}\right]}{\mathbb{P}(\tau>t|\mathcal{F}_{t})}\right] = \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\tau>t\}}\mathbb{1}_{\{B\}}\frac{\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\tau>t\}}\mathbb{1}_{\{\tau>T\}}Y|\mathcal{F}_{t}\right]}{\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_{t}\right]}\right] \\
= \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{B\}}\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\tau>T\}}Y|\mathcal{F}_{t}\right]\right] \\
= \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{B\}}\mathbb{1}_{\{\tau>T\}}Y\right],$$

where  $B \in \mathcal{F}_t$ . Since both sides are equal, this completes the proof. Now, to prove the expression in equation (2.2), we let Y := 1. Then we have

$$\mathbb{P}(\tau > T | \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{P}(\tau > T | \mathcal{F}_t)}{\mathbb{P}(\tau > t | \mathcal{F}_t)} = \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E}^{\mathbb{P}} [\mathbb{1}_{\{\tau > T\}} | \mathcal{F}_t] \\
= \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E}^{\mathbb{P}} [\mathbb{E}^{\mathbb{P}} [\mathbb{1}_{\{\tau > T\}} | \mathcal{F}_T] | \mathcal{F}_t] = \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E}^{\mathbb{P}} [\mathbb{P}(\tau > T | \mathcal{F}_T) | \mathcal{F}_t] \\
= \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E}^{\mathbb{P}} [e^{-\Gamma_T} | \mathcal{F}_t] \\
= \mathbb{1}_{\{\tau > t\}} \mathbb{E}^{\mathbb{P}} [e^{\Gamma_t - \Gamma_T} | \mathcal{F}_t]$$

Let us assume that the  $\mathbb{F}$ -hazard function  $\Gamma = (\Gamma_t)_{t\geq 0}$  is absolutely continuous. Then, we can write  $\Gamma_t = \int_0^t \gamma_u du$  where the process  $\gamma$  is called the intensity of the stopping time  $\tau$  and it is progressively measurable with respect to the  $\mathbb{F}$ -filtration. The results in the Lemma (2.15) can extend as follows.

Corollary 2.16. Let the  $\mathbb{F}$ -hazard process  $\Gamma$  be absolutely continuous, then for every non-negative  $t \leq T$  we have

$$\mathbb{P}(\tau > T | \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{P}(\tau > T | \mathcal{F}_t)}{\mathbb{P}(\tau > t | \mathcal{F}_t)} = \mathbb{1}_{\{\tau > t\}} \mathbb{E}^{\mathbb{P}}[e^{\Gamma_t - \Gamma_T} | \mathcal{F}_t] = \mathbb{1}_{\{\tau > t\}} \mathbb{E}^{\mathbb{P}}[e^{\int_0^t \gamma_u du - \int_0^T \gamma_u du} | \mathcal{F}_t]$$
$$= \mathbb{1}_{\{\tau > t\}} \mathbb{E}^{\mathbb{P}}[e^{-\int_t^T \gamma_u du} | \mathcal{F}_t].$$

#### 2.2.2 Consistency with the actuarial framework

Now, let us consider a special case where it leads to the standard actuarial framework, where the intensity of the stopping time  $\tau$  is time-varying and deterministic. Assume that the information available at time t is only if the individual (x) died before t, and if he/she died before t the exact time of death. Also, assume that the interest rate is constant. In terms of filtrations, we translate these information as  $\mathbb{G} = \mathbb{H}$  i.e the only filtration is the natural filtration of the default process up to and including t. Let  $F(t) = \mathbb{P}(\tau \leq t)$  the cumulative distribution of  $\tau$  and  $S(t) = 1 - F(t) = \mathbb{P}(\tau > t)$  the survival probability. In addition, assume that F is differentiable.

**Proposition 2.17** (Force of mortality). In this special case, the intensity function or force of mortality of  $\tau$  is given by

$$\gamma(t) = \frac{d}{dt}\Gamma(t) = \frac{f(t)}{S(t)},\tag{2.3}$$

for all non-negative t, where f(t) = F'(t) is the density function of  $\tau$ .

*Proof.* For all  $t \geq 0$ ,

$$\gamma(t) = \lim_{dx\downarrow 0} \frac{1}{dx} \mathbb{P}(\tau \le t + dx | \tau \ge t) = \lim_{dx\downarrow 0} \frac{1}{dx} \frac{\mathbb{P}(t \le \tau \le t + dx)}{\mathbb{P}(t \ge \tau)}$$
$$= \frac{1}{S(t)} \lim_{dx\downarrow 0} \left( \frac{F(t + dx) - F(t)}{dx} \right)$$
$$= \frac{1}{S(t)} \frac{d}{dt} F(t) = \frac{f(t)}{S(t)},$$

where the last equality follows from the fact that f is right-continuous.

In the following proposition, is show that we can write S(t) in terms of hazard function and this coincide with the Corollary 2.16, given that the individual (x) is alive at t.

**Proposition 2.18.** For all  $t \geq 0$ ,

$$S(t) = e^{-\Gamma(t)} = e^{-\int_0^t \gamma(u)du}.$$

*Proof.* Assume that f is a continuous function. Also, using F(t) = 1 - S(t) and Proposition 2.17, we have

$$f(t) = F'(t) = -S'(t) \Rightarrow S'(t) = -f(t) = -S(t)\gamma(t)$$

Then, we need to solve the first-order ordinary differential equation

$$S'(t) = -S(t)\gamma(t) \qquad t > 0$$

with initial condition S(0) = 1 - F(0) = 1

$$\frac{S'(t)}{S(t)} = -\gamma(t) \Rightarrow \frac{d}{dt} \log(S(t)) = -\gamma(t) \Rightarrow \log(S(t)) - \log(S(0)) = -\int_0^t \gamma(u) du$$

$$\Rightarrow \log(S(t)) = -\int_0^t \gamma(u) du$$

$$\Rightarrow S(t) = e^{-\int_0^t \gamma(u) du} = e^{-\Gamma(t)}$$

Thus, in this special case where  $\mathbb{G} = \mathbb{H}$ , the hazard process theory coincides with the traditional actuarial framework. Before we move to the case where the intensity is a stochastic process, lets take a closer look to the intuition behind the hazard process. For small h > 0,

$$f(t)h = S(t)\gamma(t)h \approx \mathbb{P}(t \le \tau \le t + h)$$

 $\approx$  the proportion of individuals (x) who will fail between time t and t+h

and

$$\gamma(t)h \approx \mathbb{P}(\tau \in [t, t+h)|\tau > t, \mathcal{G}_t)$$

 $\approx$  proportion of individuals (x), amongst those who survive to time t, who fail within the next h units of time

## 2.3 Stochastic intensity

Now, our aim is to show the equivalence between the hazard process and the single-jump process associated with the death time of an individuals (x). We will study the  $\mathbb{F}$ -intensity process, associated with the stopping time  $\tau$ , using the theory of intensity models and point processes. We follow closely the theory given by P. Bremaud (1981) [11] and P. Protter (2004) [39].

#### 2.3.1 Point processes and intensity

Let us start be providing the formal definition of a *point process* and some of its important properties.

**Definition 2.19** (Point Process). A point process or counting process  $N = (N_t)_{t\geq 0}$  is a stochastic process that can be described via a sequence  $\{T_0, T_1, \ldots\}$  of random times taking values in  $[0, \infty]$  with  $T_0 = 0$  and  $T_n < T_{n+1}$  whenever  $T_n < \infty$ , such that

$$N_t = \begin{cases} n, & \text{if } t \in [T_n, T_{n+1}) \\ +\infty, & \text{if } t \ge \lim_{n \uparrow +\infty} T_n \end{cases}$$

#### Remark 2.20.

- The paths of N are right-continuous and piecewise constant, starting at zero, and its jumps are upward jumps of magnitude one.
- $T_n$  denotes the *n*-th jump time of N
- The point process is nonexplosive iff  $T_{\infty} = \lim_{n \uparrow \infty} T_n = +\infty$

An appropriate definition of the *stochastic intensity* is given by P. Bremaud (1981) [11, D7 Definition, page 27]. Here, we will provide the following proposition as the definition of intensity instead.

**Proposition 2.21.** Let  $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$  be a probability space endowed to the enlarged filtration  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$  and N be a nonexplosive point process adapted to the  $\mathbb{G}$ -filtration. Let  $\lambda$  be a non-negative  $\mathcal{F}_t$ -predictable process, such that for all non-negative t,  $\mathbb{E}^{\mathbb{P}}[\int_0^t \lambda_u du] < \infty$ . Then the following statements are equivalent:

- 1.  $\lambda$  is the intensity of N
- 2. for all non-negative t,  $N_t \int_0^t \lambda_u du$  is a  $\mathcal{G}_t$ -martingale

**Remark 2.22.** We refer the reader to P. Bremaud (1981) [11, T12 Theorem, page 31] for the proof of uniqueness of predictable intensities.

Now, recall the residual lifetime  $\tau$  of an individual (x), which is a  $\mathcal{G}_t$ -stopping time, and consider it as the first jump of a nonexplosive point process N i.e.  $\tau := T_1$ . We say that  $\tau$  admits an intensity if the point process does. So, let us recall the default process  $N_t = \mathbb{1}_{\{\tau \leq t\}}$  for all non-negative t which is corresponds to the counting process associated with  $\tau$ . Also, we assume that the  $\mathcal{G}_t$ -stopping time  $\tau$  is totally inaccessible (see Definition. 2.5).

**Proposition 2.23.** The counting process N associated with the  $\mathcal{G}_t$ -stopping time  $\tau$ , defined by  $N_t = \mathbb{1}_{\{\tau \leq t\}}$  for all non-negative t, is right-continuous, bounded, increasing  $\mathcal{G}_t$ -submartingale.

*Proof.* From the definition of N we can easily see that it is right-continuous, bounded above by 1 and below by 0, and it is increasing. Hence, for all  $0 \le t \le s$  we have that  $N_t \le N_s$ . Taking the expectation on both sides, conditional on the information up to and including t, we have,

$$N_t = \mathbb{E}^{\mathbb{P}}[N_t|\mathcal{G}_t] \leq \mathbb{E}^{\mathbb{P}}[N_s|\mathcal{G}_t].$$

Thus, the process N is a  $\mathcal{G}_t$ -submartinagale.

Since N is a submartinagale, we can use the Doob-Meyer decomposition to write N as the sum of a martingale and an increasing predictable process. It is important to point out that N satisfies the conditions under which the Doob-Meyer decomposition can be applied i.e N is of Class D (see appendix A.1).

Now, let us state a special case of the Doob-Meyer decomposition theorem, where the sub-martingale assume to have jumps at totally inaccessible stopping times. For the proof of the following theorem, we refer the reader to P. Protter (2004) [39, Theorem 8, page 111].

**Theorem 2.24** (Doob-Meyer Decomposition: Case of Totally Inaccessible Jumps). Let N be a càdlàg supermartingale of Class D with  $N_0 = 0$ , and such that all jumps of N occur at totally inaccessible stopping times. Then there exists a unique, increasing, continuous, predictable process  $A = (A_t)_{t \geq 0}$  with  $A_0 = 0$  such that N - A is a uniformly integrable martingale.

**Remark 2.25.** The importance of the Doob-Meyer decomposition in the case of totally inaccessible stopping times, is that the compensator A of N is continuous, which it is not generally true in the general case.

Now, applying Theorem 2.24 to the default process  $N=(N_t)_{t\geq 0}$ , implies that N has a unique, increasing, continuous,  $\mathcal{G}_t$ -predictable compensator  $A=(A_t)_{t\geq 0}$  such that N-A is a  $\mathcal{G}_t$ -martingale. From the definition of N, we have that  $N_t$  is constant for all t greater than  $\tau$ . This implies that  $A_t$  must also be constant for all t greater than  $\tau$ , such that  $A_t = A_{t \wedge \tau}$ , where  $A_{t \wedge \tau}$  denotes the process A stopped at  $\tau$ .

#### 2.3.2 Martingale hazard process

Our aims in this section is to show that the default process admits an  $\mathcal{F}_t$ -predictable intensity and the equivalence between the  $\mathbb{F}$ -hazard process and the single-jump process. In particular, we are going to show that under some assumptions, the compensator of the default process N is equal to the hazard process  $\Gamma$ . So, our next step is to show that there exist a  $\mathcal{F}_t$ -predictable process equals the  $\mathcal{G}_t$ -predictable compensator A of N. In other words, there exist a unique, increasing, continuous  $\mathcal{F}_t$ -predictable process  $\Lambda$  such that  $A_t = A_{t \wedge \tau} = \Lambda_{t \wedge \tau}$ , for all non-negative t. Using the property of the enlarged filtration given by Lemma 2.8, we can easily show that there exist an  $\mathcal{F}_t$ -predictable process  $\Lambda$  such that  $A_t = A_{t \wedge \tau} = \Lambda_{t \wedge \tau}$ , for all non-negative t. We call the process  $\Lambda$ 

martingale hazard process, as it defined in M. Jeanblanc and M. Rutkowski (2000) [25, Definition 4.3, page 33].

**Definition 2.26** (Martingale Hazard Process). An increasing, continuous  $\mathcal{F}_t$ -predictable process  $\Lambda$  is called  $\mathbb{F}$ -martingale hazard process of the stopping time  $\tau$  if and only if  $\Lambda_0 = 0$  and the process M given by  $M_t = N_t - \Lambda_{t \wedge \tau}$  is a  $\mathcal{G}_t$ -martingale.

We will assume from now on that the  $\mathcal{F}_t$ -predictable process  $\Lambda$  is absolutely continuous with respect to the Lebesgue measure. This implies that there exists a  $\mathcal{F}_t$ -predictable process  $\lambda = (\lambda_t)_{t\geq 0}$  such that  $\Lambda_{t\wedge\tau} = \int_0^{t\wedge\tau} \lambda_s ds$ . Thus, in the Definition 2.26 the process M given by  $M_t = N_t - \int_0^{t\wedge\tau} \lambda_s ds$  is a  $\mathcal{G}_t$ -martingale and from Proposition 2.21,  $\lambda$  the stochastic intensity of N.

#### 2.3.3 Equivalence between hazard process $\Gamma$ and single-jump point process

Let us now consider the relationship between the  $\mathbb{F}$ -hazard process  $\Gamma$  and the  $\mathbb{F}$ -martingale hazard process  $\Lambda$ . A natural question to be asked, is whether the two processes coincide. Let us assume that the  $\mathbb{F}$ -hazard process  $\Gamma$  is well defined and recall the  $\mathbb{H}$ -hypothesis (see Definition 2.9). As proved by M. Jeanblanc and M. Rutkowski (2000) [25, Section 4.2, page 36] the answer to this question is positive under some conditions. Following their analysis we summarise the results in the following proposition. As the proof of this proposition is very technical, it is beyond of our scope and we refer the reader to M. Jeanblanc and M. Rutkowski (2000) [25, Section 4.2, page 36] or E. Biffis, M. Denuit and P. Devolder (2010) [6, Appendix A, page 24] for more details about the proof.

**Proposition 2.27.** Let  $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$  be a probability space endowed to the enlarged filtration  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$  and  $\tau$  be a stopping time with respect to  $\mathbb{G}$ . Assume that:

- a) the **H**-hypothesis holds,
- b) the process F given by  $F_t = \mathbb{P}(\tau \leq t | \mathcal{F}_t)$  is continuous.

Then, the  $\mathbb{F}$ -hazard process  $\Gamma$  equals the  $\mathbb{F}$ -martingale hazard process  $\Lambda$  i.e. for all non-negative t,  $A_t = \Lambda_{t \wedge \tau} = \Gamma_{t \wedge \tau}$  and the process  $N - \Gamma$  is a  $\mathcal{G}_t$ -martingale.

#### Remark 2.28.

 We can consider a more general case by replacing the assumption a) in the proposition above with "F is an increasing process". Under the H-hypothesis, and in particular from the equivalent form

$$\mathbb{P}(\tau \le t | \mathcal{F}_{\infty}) = \mathbb{P}(\tau \le t | \mathcal{F}_t)$$

for all non-negative t (see Theorem 2.11), implies that F is increasing.

- 2. Beyond the technicality importance of the assumption b), it is also important for our theory to be consistent with classical actuarial theory. In particular, it ensures that the survival functions are smooth.
- 3. Intuitively, the above proposition gives us the freedom to base our mortality modeling framework solely on single-jump processes associated with the death of an individual (x).

Let us recall Lemma 2.15, which provides an important result for the valuation of contingent claims. In addition, if the assumptions in the Proposition 2.27 holds, we can extend Lemma 2.15 as follows.

Corollary 2.29. Under the assumptions in the Proposition 2.27, the survival probability of the stopping time  $\tau$  conditional on the  $\mathbb{G}$ -filtration is given by

$$\begin{split} \mathbb{P}(\tau > T | \mathcal{G}_t) &= \mathbf{1}_{\{\tau > t\}} \frac{\mathbb{P}(\tau > T | \mathcal{F}_t)}{\mathbb{P}(\tau > t | \mathcal{F}_t)} = \mathbf{1}_{\{\tau > t\}} \mathbb{E}^{\mathbb{P}}[e^{\Gamma_t - \Gamma_T} | \mathcal{F}_t] = \mathbf{1}_{\{\tau > t\}} \mathbb{E}^{\mathbb{P}}[e^{\Lambda_t - \Lambda_T} | \mathcal{F}_t] \\ &= \mathbf{1}_{\{\tau > t\}} \mathbb{E}^{\mathbb{P}}[e^{\int_0^t \lambda_u du - \int_0^T \lambda_u du} | \mathcal{F}_t] \\ &= \mathbf{1}_{\{\tau > t\}} \mathbb{E}^{\mathbb{P}}[e^{-\int_t^T \lambda_u du} | \mathcal{F}_t], \end{split}$$

where  $\Gamma$  is the  $\mathbb{F}$ -hazard process and  $\lambda$  is the stochastic intensity of  $\tau$ . Here, we also assumed that  $\Gamma = \Lambda$  is absolutely continuous with respect to Lebesgue measure.

#### 2.4 From point process to doubly stochastic Poisson process

In the previous section, we proved that we can describe the default time  $\tau$  as the jump time of a single-jump process. Here, we will go one step further and characterize  $\tau$  as the first jump time of a doubly stochastic Poisson process. The definition of the Poisson process is given in the Appendix A. We are going to use the *doubly stochastic Poisson processes* to model the residual lifetime of an individual (x), in a similar way that are used to model default in credit risk framework.

#### 2.4.1 Doubly stochastic Poisson process

Let assume for a moment that the intensity process  $\lambda$  is deterministic i.e.  $\lambda_t = \lambda(t)$  for all non-negative t. The following theorem by S.Watanabe [42] allows us to characterize the point process N as a Poisson process. It can be seen as the equivalence to the Levý's Characterization Theorem of Brownian motion.

**Theorem 2.30** (Watanabe's Characterization of Poisson Process). Let  $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$  be a probability space endowed to the enlarged filtration  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$  and N be a nonexplosive point process adapted to the  $\mathbb{G}$ -filtration. Also, let  $\lambda(t)$  be a locally integrable non-negative measurable function. Assume that the process given by  $N_t - \int_0^t \lambda(u) du$  for all non-negative t is a  $\mathcal{G}_t$ -martingale. Then,  $N_t$  is a  $\mathcal{G}_t$ -Poisson process with intensity  $\lambda(t)$ .

In our modelling framework, we consider the intensity to be time-varying and stochastic. So, we assume that the default process N is a doubly stochastic process. The term doubly stochastic first used by Cox in 1955 (see [17]) and represent the processes with Poisson structure but with time-varying, stochastic intensity. The intuition behind these processes is that the probability of default is itself random. So, if we condition on any path of the intensity i.e.  $\lambda_u(\omega)$  for  $u \in [0,t]$  and  $\omega \in \Omega$ , then the default process N become a Poisson process with intensity  $\int_0^t \lambda_u(\omega) du$ .

Now, before we discuss some of the important constructive results that will be useful for mortality modelling, let us consider a more formal definition of the *doubly stochastic Poisson process*, through a fundamental characterization theorem. We refer the reader to P. Bremaud [11, Theorem T4, page 25] for the proof.

**Theorem 2.31** (Characterization of Doubly Stochastic Poisson Processes). Let  $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$  be a probability space endowed to the enlarged filtration  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$  and N be a nonexplosive point process adapted to the  $\mathbb{G}$ -filtration. Let  $\lambda = (\lambda_t)_{t\geq 0}$  be a non-negative  $\mathcal{F}_t$ -progressively measurable process such that for every t,  $\Lambda_t := \int_0^t \lambda_u du < \infty$ ,  $\mathbb{P}$  a.s. The process N is an inhomogeneous Poisson process with stochastic intensity  $\lambda$  if for every positive  $\mathcal{F}_t$ -predictable process  $\phi = (\phi_t)_{t\geq 0}$  the following equality is satisfied:

$$\mathbb{E}^{\mathbb{P}}\bigg[\int_{0}^{\infty}\phi_{u}dN_{u}\bigg]=\mathbb{E}^{\mathbb{P}}\bigg[\int_{0}^{\infty}\phi_{u}\lambda_{u}du\bigg].$$

Now, we follow the same framework as in D. Brigo and F. Mercurio (2007) [13, Section 22, page 757], but in our case the default is the death of an individual (x). An important property of the Poisson processes is the distribution of the time interval between two consecutive jumps. Let us start with the time-homogeneous deterministic case first, before we consider the doubly stochastic processes. Assume that  $\tau_1, \tau_2, \tau_3 \dots$  are the first, second etc. jump times of the time-homogeneous Poisson process with intensity  $\lambda$ . Then, the time intervals between any two consecutive jumps i.e  $\tau_1, \tau_2 - \tau_1, \ldots$ , are independent and identically distributed random variables and follows an exponential distribution with parameter  $\lambda$ . Equivalently,  $\lambda \tau_1, \lambda(\tau_2 - \tau_1), \ldots$  are independent and identically distributed standard exponential random variables. As we are interested only in the first jump time, we will focus on the exponential random variable  $\tau := \tau_1$ . The time-inhomogeneous Poisson process N with deterministic intensity  $\gamma(t)$  can be written as  $N_t = M_{\Gamma(t)}$ , where M is the standard Poisson process with intensity  $\lambda := 1$  and  $\Gamma$  denote the hazard function of  $\tau$  (see Section 2.2). In other words, the first jump time of N is  $\tau$  if and only if the first jump time of M is  $\Gamma(\tau)$ . In this case, the time intervals between two consecutive jumps have the same property as before, except that are not identically distributed. Thus, the first jump time  $\Gamma(\tau)$  of M is exponentially distributed with parameter 1. As in D. Brigo and F. Mercurio (2007) [13, Appendix C.6.1, page 913] we can write  $\xi := \Gamma(\tau)$ , so that  $\xi$  is a standard exponential random variable.

Thus, the jump time  $\tau$  of the inhomogeneous Poisson process N can be written as

$$\tau = \Gamma^{-1}(\xi)$$

where  $\xi$  is exponential random variable with mean 1.

For the doubly stochastic processes we have that  $\tau = \Lambda^{-1}(\xi)$ , where  $\Lambda$  is the  $\mathbb{F}$ -martingale hazard process. This follows from the definition of doubly stochastic Poisson processes, since if we condition on  $\mathbb{F}$  we have a Poisson structure. From the Definition 2.26, we have that  $\Lambda$  is an  $\mathcal{F}_t$ -predictable process and  $\lambda = (\lambda_t)_{t\geq 0}$  is the stochastic intensity such that  $\Lambda_{t\wedge\tau} = \int_0^{t\wedge\tau} \lambda_s ds$ . Note that in this case we have stochasticity in both  $\xi$  and  $\lambda$ . Finally, let us calculate two important probabilities under the doubly stochastic process.

#### Proposition 2.32.

- 1. The probability of an individual (x) die in a small time interval dt, conditional on the fact that she/he is still alive and on the intensity path so far, is approximately  $\lambda_t dt$
- 2. The survival probability is given by

$$\mathbb{P}(\tau \ge t) = \mathbb{E}^{\mathbb{P}} \left[ e^{-\int_0^t \lambda_u du} \right].$$

Proof.

1.

$$\mathbb{P}(\tau \in [t, t + dt) | \tau \geq t, \mathcal{F}_t) = \frac{\mathbb{P}(t \leq \tau < t + dt) | \mathcal{F}_t)}{\mathbb{P}(\tau \geq t | \mathcal{F}_t)} = \frac{\mathbb{P}(\tau \leq t + dt | \mathcal{F}_t) - \mathbb{P}(\tau \leq t | \mathcal{F}_t)}{\mathbb{P}(\tau \geq t | \mathcal{F}_t)}$$

$$= \frac{\mathbb{P}(\tau \leq t + dt | \mathcal{F}_t) - \mathbb{P}(\tau \leq t | \mathcal{F}_t)}{\mathbb{P}(\tau \geq t | \mathcal{F}_t)}$$

$$= \frac{\mathbb{P}(\Lambda(\tau) \leq \Lambda(t + dt) | \mathcal{F}_t) - \mathbb{P}(\Lambda(\tau) \leq \Lambda(t) | \mathcal{F}_t)}{\mathbb{P}(\Lambda(\tau) \geq \Lambda(t) | \mathcal{F}_t)}$$

$$= \frac{\mathbb{P}(\xi \leq \Lambda(t + dt) | \mathcal{F}_t) - \mathbb{P}(\xi \leq \Lambda(t) | \mathcal{F}_t)}{\mathbb{P}(\xi \geq \Lambda(t) | \mathcal{F}_t)}$$

$$= \frac{1 - e^{-\Lambda(t + dt)} - 1 + e^{-\Lambda(t)}}{e^{-\Lambda(t)}} = \frac{e^{-\Lambda(t)} - e^{-\Lambda(t + dt)}}{e^{-\Lambda(t)}}$$

$$= \frac{e^{-\int_0^t \lambda_u du} - e^{-\int_0^{t + dt} \lambda_u du}}{e^{-\int_0^t \lambda_u du}} = \frac{e^{-\int_0^t \lambda_u du} (1 - e^{-\int_t^{t + dt} \lambda_u du})}{e^{-\int_0^t \lambda_u du}}$$

$$= 1 - e^{-\int_t^{t + dt} \lambda_u du}$$

$$\approx \int_t^{t + dt} \lambda_u du \approx \lambda_t dt$$

We used the fact that  $\xi := \Lambda(\tau)$  is standard exponential random variable and  $\Lambda(t)$  is  $\mathcal{F}_{t}$ predictable process. Also, the two approximations at the end are good for small exponent
and small "dt", respectively.

2.

$$\mathbb{P}(\tau \ge t) = \mathbb{P}(\Lambda(\tau) \ge \Lambda(t)) = \mathbb{P}\left(\xi \ge \int_0^t \lambda_u du\right) = \mathbb{E}^{\mathbb{P}}\left[\mathbb{P}\left(\xi \ge \int_0^t \lambda_u du \middle| \mathcal{F}_t\right)\right]$$
$$= \mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^t \lambda_u du}\right]$$

Here, we used again the fact that  $\xi := \Lambda(\tau)$  is standard exponential random variable and  $\Lambda(t)$  is  $\mathcal{F}_t$ -predictable process.

#### Remark 2.33.

- i) From the second part of Proposition 2.32, we can see that the survival probability has the same form as the zero-coupon bond pricing formula in a short rate model. In the place of the stochastic short rate, we have the stochastic mortality rate  $\lambda$ . We will consider this in more details in the next section.
- ii) The random variable  $\xi$  is independent of the filtration  $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$  i.e. from all other sources of randomness. So, given all the information about the other sources, we can not have complete information of default process. This is consistent with the assumption that the default time is totally inaccessible.
- iii) The fact that  $\xi$  is an external source of randomness makes the model incomplete.

#### 2.4.2 Dynamics of the stochastic process $\lambda$

So far, we have seen how the doubly stochastic Poisson processes with stochastic intensity  $\lambda$  can be used to model the residual lifetime of an individual (x). Let us now focus on the stochastic intensity  $\lambda$  and consider some of the important criteria, beyond analytical tractability, that the reasonable mortality models must follow. In our model, the stochastic intensity corresponds to the force of mortality of an individual (x) and we will use the notation  $\lambda_x(t)$  throughout the paper. It is important that mortality models keep the force of mortality positive, since there is no reasonable intuition behind negative values. The dynamics of the model should also be consistent with the historical data, as well as the demographic and actuarial theory. For example, mortality rates based on life tables, normally follow a modest increase between ages 20 to 40 and then a steeper steady increase after the age of 40.

**Remark 2.34.** In actuarial terminology, the probability that an individual (x) dies before the age x + 1 is called *mortality rate* and is denoted by  $q_x$ . Figure 2 (left) shows the mortality rates based on the 2013 U.K. total population life table [24]. In order to capture the main features of the mortality, we plot the mortality rates on a logarithmic scale (see figure 2 (right)).

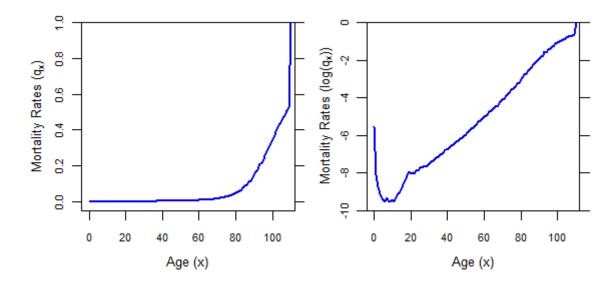


Figure 2: Mortality rates  $q_x$  based on the 2013 U.K. total population life table (left) and their logarithmic scale (right). The logarithmic scale can capture the high parenital mortality and the accident hump.

Another important thing to notice is the mortality rates between birth and the age 20. The mortality rates curve is inverted, as we can see from the figure 2 (right). Perinatal mortality  $q_0$  i.e. mortality rates immediately after birth are high with steep decline the following year. This can be explained by problems arising during the pregnancy or the birth. Interestingly, there is a large increase in mortality rates between ages 10 and 20. In actuarial terminology this observation is commonly known as accident hump, since most of the deaths during these ages are caused by accidents. As we are interested in managing and hedging the longevity risk that pension funds and annuity providers face, we will focus on the mortality evolution for older ages, around and above retirement, since these are the ages where longevity risk has the most financial impact.

Finally, the mean-reverting property it seems to be less desirable in mortality than interest rate models. From the figure 2, we observe that the mortality rates at older ages are increasing exponentially. Moreover, as pointed out in A.J.G. Cairns, D. Blake and K. Dowd (2006) [14, Section 1.5, page 10], the inclusion of mean reversion would mean that if the mortality improvements have been faster than anticipated in the past then the potential for further mortality improvements will be significantly reduced in the future. The development of new cures, the emergence of new diseases and how they will affect mortality in the future is really hard to predict. These give us some evidence against the strong mean-reverting property in the stochastic mortality models.

#### 2.4.3 Stochastic intensity under change of measure

In this section, we will study the behavior of the stochastic intensity under an equivalent change of measure. When we are dealing with interest rate models, we do not need to consider the dynamics under the real-world probability measure  $\mathbb{P}$ . The market for interest rate products is well developed, and to calibrate our models and use them for pricing, we only need to consider the dynamics under the risk-neutral measure  $\mathbb{Q}$ . This is not the case when we are dealing with longevity derivatives. As the longevity market is not fully developed yet and there is a limited number of market products with underlying the mortality, we will work under the real world measure in order to estimate the parameters of the mortality models. In particular, we are going to fit our model to the historical data, such as observed life tables of the reference population, and such tables are created using statistics i.e under  $\mathbb{P}$  measure.

Our aim is to show if there exist a stochastic intensity and how its dynamics behave under the equivalent martingale measure  $\mathbb{Q}$ . As in the previous setup, let  $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$  be a probability space endowed to the enlarged filtration  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ . Also, we assume that the filtration  $\mathbb{F}$  is generated by a d-dimensional Brownian motion W. Here, we will follow the approach of E. Biffis, M. Denuit and P. Devolder (2010) [6, Section 3.1, pages 7-9] and summarize the results into the following proposition.

**Proposition 2.35.** Let N be a doubly stochastic process on  $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$  with  $\mathcal{F}_t$ -predictable stochastic intensity  $\lambda$ . Fix a time horizon T>0, for the probability measure  $\mathbb{Q}$  equivalent to  $\mathbb{P}$ , the Radon-Nikodỳm process  $\eta=(\eta_t)_{t\geq 0}$  is given by  $\eta_t=\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{G}_t}=\mathbb{E}^{\mathbb{P}}[\eta_T|\mathcal{G}_t]$ , where  $\eta_T$  is a strictly positive  $\mathcal{G}_T$ -measurable random variable with  $\mathbb{E}^{\mathbb{P}}[\eta_T]=1$ . Assume that the decomposition  $\eta=\eta'\eta''$  holds, where

$$\eta' = exp\left(-\int_0^t \zeta_s dW_s - \frac{1}{2} \int_0^t ||\zeta_s||^2 ds\right)$$
 (2.4)

and

$$\eta'' = (1 + \phi_{\tau} \mathbb{1}_{\{\tau \le t\}}) exp\left(-\int_{0}^{t \wedge \tau} \phi_{s} d\Lambda_{s}\right), \tag{2.5}$$

where  $\zeta$  and  $\phi$  are  $\mathcal{G}_t$ -predictable processes, with  $\int_0^T \phi_t \lambda_t dt < \infty$  almost surely and  $\phi > -1$ . Then,

- i)  $\tilde{W} = W + \int_0^t \zeta_s ds$  is a G-Brownian motion under  $\mathbb{Q}$ ,
- ii) the process  $\tilde{M}$  given by  $\tilde{M}_t = M_t \int_0^{t \wedge \tau} \phi_s d\Lambda_s = N_t \int_0^{t \wedge \tau} (1 + \phi_s) d\Lambda_s$  is a  $\mathcal{G}_t$ -martingale under  $\mathbb{Q}$ ,
- iii) the martingale hazard process of  $\tau$  is  $\tilde{\Lambda} = \int_0^t (1+\phi_s)d\Lambda_s$

*Proof.* For a complete proof of the proposition and the decomposition of  $\eta$  we refer the read to E. Biffis, M. Denuit and P. Devolder (2010) [6, Appendix B, page 24].

In other words, the above proposition says that if the stopping time  $\tau$  has a stochastic intensity  $\lambda$  under the real world probability measure, then under the equivalent probability measure  $\mathbb{Q}$  the stochastic intensity is given by  $(1 + \phi)\lambda$ .

Remark 2.36. In the mortality market setup where the lifetime  $\tau$  is the first jump of a doubly stochastic Poisson process, one can argue that the  $\mathcal{G}_t$ -predictable process  $\phi$  corresponds to the market price of the unsystematic mortality risk,, whereas the  $\mathcal{G}_t$ -predictable process  $\zeta$  reflect the market price of longevity risk.

## 3 Financial Market

In this section, we will give a brief description of the financial market. More precisely, we will focus on the money-market account and the short rate processes.

## 3.1 Money-market and equivalent martingale measure

Consider the probability space  $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$  endowed to the enlarged filtration  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ . The subfiltration  $\mathbb{F}$  contains all the information of the default-free risk factors, such as the interest rate dynamics. In the financial market, we assume that there exist a money-market account B, where  $B_t$  denotes the value of a bank account at time  $t \geq 0$ . The money-market account satisfies the stochastic differential equation

$$dB_t = r_t B_t dt, \qquad B_0 = 1, \tag{3.1}$$

where the process  $r=(r_t)_{t\geq 0}$  denotes the instantaneous risk-free rate or short rate. We assume that the market is arbitrage free, such that there exist an equivalent martingale measure  $\mathbb{Q}$ , called the risk-neutral propability measure. Under  $\mathbb{Q}$ , the discounted price of any tradeable asset is a martingale. An  $\mathcal{F}_T$ -measurable and square-integrable contingent claim H, with maturity T, has a unique price  $V_t$  at time t, given by

$$V_t = \mathbb{E}^{\mathbb{Q}} \left[ exp(-\int_t^T r_u du) H_T | \mathcal{F}_t \right]. \tag{3.2}$$

**Example 3.1.** The value at time t < T, of a zero-coupon bond with maturity T, is given by

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp(-\int_{t}^{t} r_{u} du) H_{T} | \mathcal{F}_{t}\right] = \mathbb{E}^{\mathbb{Q}}\left[\exp(-\int_{t}^{T} r_{u} du) | \mathcal{F}_{t}\right] = \mathbb{E}^{\mathbb{Q}}\left[D(t,T) | \mathcal{F}_{t}\right], \tag{3.3}$$

where  $D(t,T) = \exp(-\int_t^T r_u du)$  is the discount factor. Recall that the payoff of a zero-coupon bond is 1 unit of currency at maturity T i.e.  $H_T = 1$ .

## 3.2 Short-Rate dynamics

Now, let us specify the stochastic dynamics of the short rate process  $r = (r_t)_{t\geq 0}$ . Typically, under the risk-neutral probability measure the short rate satisfies an Itó diffusion i.e satisfies the stochastic differential equation

$$dr_t = \alpha(t, r_t)dt + \sigma(t, r_t)dW_t, \tag{3.4}$$

where W is an  $\mathbb{F}$ -Brownian motion, where the drift b and the diffusion coefficient  $\sigma$  are assume to be Lipschitz continuous.

We assume that in the interest rate market, there are available liquid market data. In particular, we are given the term structure of interest rates and also market volatilities of a few vanilla options, such as Caps and Swaptions.

## 4 Data, model specification and parameters estimation

The first part of this section gives a brief description of the available mortality and longevity data. In section 4.2, we specify a continuous-time stochastic process for the mortality intensity. Then, we estimate the model parameters using the U.K. male observed life tables from the Human Mortality Database. Finally, using Monte Carlo simulations, we simulate the paths of the mortality intensity, the survival probabilities and death curves, under our model.

#### 4.1 Data

Before we continue with the model specification and parameters estimation, it is important to understand the available data for mortality and longevity risk. In contrast of the interest rate market, the longevity market is immature and incomplete, with lack of liquid market data. We will turn our attention to the actuarial tables, by first introducing the standard notation and the different types of tables available in the actuarial practice.

#### 4.1.1 Mortality and longevity Data

Actuaries have been using life tables to represent the probabilities of death of individuals at each age of a particular population. The classical actuarial life tables (or mortality tables) are based on the mortality experience of the whole population of a country and usually there are separate tables for males and females. Also, there are different tables for groups of people with particular characteristics, such as for smokers. The population considered in the life tables is not the actual population of the country, but a hypothetical single birth cohort consisting of  $\ell_x = 100000$  births. For the construction of the life tables, records of deaths during a period of time are considered, usually in a particular calendar year. Then, by using actuarial techniques, such as the Kaplan-Meier estimator, the survival probabilities for each age are calculated. Also, it is important to note that the data for the oldest ages are unreliable, so smoothing techniques are used to improved the quality of the mortality representation and techniques to end the life tables up to a certain age. Such tables are not suitable for quantify the longevity risk since mortality rates are changing over time, but they are only suitable for short-term mortality risk.

Before we introduce the longevity data that we will use to estimate the parameters of our model, let us consider the standard notation used in life tables and define some of important quantities. In the table 4.1.1 below, we summarize the entries of the life tables. Also, some basic relationships between these quantities are provided in the list below:

$$\bullet \ d_x = \ell_{x+1} - \ell_x,$$

$$q_x = \frac{\ell_{x+1}}{\ell_x},$$

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•  $L_x = \ell_x - 0.5d_x$  for  $x \in \mathbb{N}$  is the population at mid-year if we assume that the deaths occur uniformly throughout the year,

- $m_x = \frac{d_x}{L_x}$
- $q_r \approx 1 e^{-m_x}$ .

Entry	Description
$\ell_x$	Number of survivors at exact age $x$
$d_x$	Number of deaths between ages $x$ and $x + 1$
$q_x$	Probability that someone aged exactly $x$ will die before reaching age $x+1$
$m_x$	Central death rate between ages $x$ and $x + 1$
$L_x$	Average number of individuals alive at age $x$ during the year
$e_x$	Life expectancy at exact age $x$

Table 1: Standard notations and their descriptions that are used in life tables.

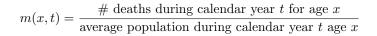
Now, in order to quantify the longevity risk, we also need to consider the evolution of the mortality rates in time. There are two types of tables that can be used to represent the level of the mortality rates at different periods of time; the period and cohort life tables. The period mortality tables represent the mortality experience of the entire population at a specific time period, for example a calendar year. On the other hand, cohort tables depict the mortality of a group of individuals from a given cohort, for example individuals that have been born in the same year, over their entire lifetime. Each type captures different effects on mortality. The cohort tables captures generational trends such as the improvements in the mortality experience of a generation due to a new vaccine. In contrast, the period tables best capture the changes of the mortality in a population. It is worth noting that the mortality rates present extreme fluctuations during periods of wars or pandemic diseases.

The classical actuarial tables underestimate the evolution of mortality. For example, these tables take into account that mortality is on average greater for older persons but they do not consider that mortality evolves with time. To solve this problem, insurance companies adopted the prospective life tables, which capture the mortality evolution in time. The prospective tables can be both period-based or cohort-based tables. Alternative way of dealing with mortality improvements are the dynamic mortality models such as the Lee-Carter, introduced by R.D. Lee and L.R. Carter (1992) [30] and the Cairns Blake Dowd [14] families of models. Our approach is the use of the doubly stochastic Poisson process with stochastic intensity follows a diffusion process.

For the estimation of the parameters of our model, we will use mortality data that are freely available at Human Mortality Database (HMD) [24]. According to the website, the Human Mortality Database (HMD) was launched in 2002 and it is a collaborative project involving research

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teams in the Department of Demography at the University of California, Berkeley (USA) and at the Max Planck Institute for Demographic Research (MPIDR) in Rostock (Germany). We will use the U.K. total population for males period life tables for the period 1922-2013 (downloaded on the 20th of August 2016). Our parameters estimation method is based on a discretized process, where the intensity is assumed to be constant over each aged and calendar year. It is approximated by the central death rates defined as follows:



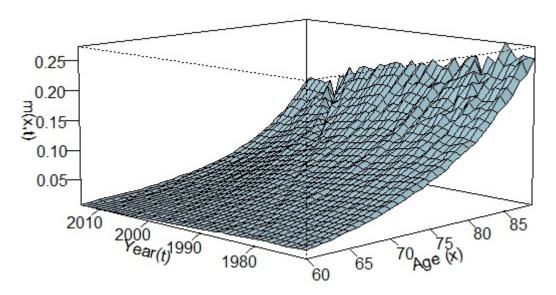


Figure 3: Central Death Rates m(x,t) for calendar years  $t=1970,1971,\ldots,2013$  and ages  $x=60,61,\ldots,90$ , observed from the LLMA index.

#### 4.1.2 Index based mortality data

As the interest for a liquid market for longevity and mortality-linked risks, where investors will effectively manage and transfer the longevity risk from their portfolios, has increased, various attempts have been made to create appropriate indexes. Some of the longevity indexes that launched since today are given in the following list:

- The Credit Suisse Longevity Index<sup>SM</sup> (CSLI), launched in 2006, is a standardized measure of the Expected Average Lifetime for a general population. It is based on government mortality and population statistics for the U.S. population. However, it is no longer actively marketed by Credit Suisse.
- Goldman Sachs Group Inc. released the QxX Index in January 2008, a longevity index that tracks the life expectancy of those who sell their insurance policies on the secondary market.

It had a short life of two years before the bank removed the plug because of lack of commercial activity.

- In March 2008, the Xpect Age and Cohort Indices were launched by Deutsche Börse. The
  index tracks certain cohorts on a monthly basis and gives information about their specific
  life expectancy.
- LifeMetrics indices developed by J.P. Morgan in 2007, but the ownership has been transfered to the Life and Longevity Market Association (LLMA). The current members of the LLMA are: AVIVA, AXA, Deutsche Bank, J.P. Morgan, Morgan Stanley, Prudential PLC and Swiss Re. The Index provides mortality rates and period life expectancy levels across various ages, by gender, for each national population covered. Currently the LLMA publishes index values for the U.S., England & Wales, the Netherlands and Germany. (see http://www.llma.org)

The aim of indices is to provide a method for measuring and managing longevity and mortality risk. Also, by using suitable indices to create standardized longevity-linked securities that will be traded in a liquid market, attracting all types of investors. An important drawback of indexed-based longevity securities is that they are subject to basis risk. The heterogeneity of the portfolios of different companies and different countries, implies that the longevity patterns that each portfolio experiences will be different for the population that the index may based on. It is important to note that the current available indices are based on observed rates and not on prices of traded securities.

#### 4.2 Model specification

In this section, we describe a continuous-time stochastic model for the mortality intensity. In particular, we choose the two-factor Gaussian model introduced by M.C. Fung, K. Ignatieva and M. Sherris (2015) [21]. We give a proof of the analytical formula for the survival probabilities and consider the dynamics of the model under the risk-adjusted measure.

## 4.2.1 Mortality model

In spite of the variety of models that have been proposed throughout the years, there is no accepted framework for mark-to-model longevity-linked derivatives. Starting from the discrete time model introduced in 1992 by R.D Lee and L.R. Carter (1992) [30], much research have be done, and several models have been introduced, both discrete and continuous time. More weight was given to time-homogeneous affine processes that are used to model the stochastic intensity under the doubly stochastic approach. For example, E.Lucano and E.Vigna (2008) [35] described the stochastic intensity  $\lambda_x(t)$  using Ornstein Uhlenbeck, with and without jumps, and Feller processes. In these models the mortality intensity captures the evolution of the mortality for a single generation at a

particular initial time. An important property that a mortality model should have, is to be able to describe the whole mortality surface. In other words, we want our model to capture the mortality dynamics for different generations as well. Another way to think of this, is in terms of survival probabilities. If we fix a horizon over which we consider the survival probability and an initial age, we want our model to capture the evolution over generations.

Another important criteria that a mortality model should have is the ability to capture the correlation between different cohorts, as the correlation between consecutive cohorts may be high but it is not perfect. A perfect correlation between generations means that the shocks to mortality surface is transmitted equally through generations, and this is not the case. This implies that we cannot just assume that each generation has its own mortality intensity, but we must also assume that the intensities of the different generations are correlated. Following the approach of P. Jevtic, E. Luciano and E. Vigna (2013) [26], first we will describe the cohort-based modeling framework and then we will consider a specific example.

Assume that each generation has its own mortality intensity and follows a additive multi-factor diffusion process

$$d\lambda_x(t) = \mathbf{1} \cdot d\mathbf{X}^x(t) \tag{4.1}$$

with,

$$d\mathbf{X}^{x}(t) = \boldsymbol{\mu}(\mathbf{X}^{x}(t))dt + \boldsymbol{\sigma}(\mathbf{X}^{x}(t))d\mathbf{W}^{x}(t)$$
(4.2)

where  $(W_1, W_2, \ldots, W_n)$  is an *n*-dimensional Brownian motion with instantaneous correlation matrix  $\boldsymbol{\rho}^x = \{\rho_{ij}^x\}_{1 \leq i,j \leq n}$ . Using the Cholesky Decomposition on the variance-covariance matrix of  $(W_1, W_2, \ldots, W_n)$ , we can decompose the vector  $\mathbf{W}^x$ , as follows

$$d\mathbf{W}^{x}(t) = \mathbf{D}^{x} d\mathbf{Z}^{x}(t) \tag{4.3}$$

where  $\mathbf{Z}(t) = (Z_1, Z_2, \dots, Z_n)$  is a vector of uncorrelated Brownian motions and  $\boldsymbol{\rho} = \mathbf{D}^x(\mathbf{D}^x)^T$ . If we assume that the uncorrelated risk factors  $(Z_1, Z_2, \dots, Z_n)$  are equal for all the generations, we have,

$$d\mathbf{X}^{x}(t) = \boldsymbol{\mu}(\mathbf{X}^{x}(t))dt + \boldsymbol{\sigma}(\mathbf{X}^{x}(t))\mathbf{D}^{x}d\mathbf{Z}(t). \tag{4.4}$$

**Remark 4.1.** The superscript x in X, W and D denotes the current age of the generation that is considered.

In this framework, we assume that we are given a filtered probability space  $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ , as described in section 2, and we consider the two-additive-factor Gaussian mortality model for the mortality intensity process  $\lambda_x$  of the generation with age x at time t=0. The dynamics of  $\lambda_x$  under the objective measure  $\mathbb{P}$  is given by

$$d\lambda_x(t) = dX_1^x(t) + dX_2^x(t), \tag{4.5}$$

where the processes  $(X_1^x(t))_{t\geq 0}$  and  $(X_2^x(t))_{t\geq 0}$  satisfy

$$dX_1^x(t) = \alpha_1 X_1^x(t)dt + \sigma_1 dW_1^x(t), \qquad X_1^x(0) = x_1, \tag{4.6}$$

$$dX_2^x(t) = (\alpha_2 x + \beta) X_2^x(t) dt + \sigma_2 e^{\gamma x} dW_2^x(t), \qquad X_2(0) = x_2^x, \tag{4.7}$$

where  $(W_1^x, W_2^x)$  is a two-dimensional Brownian motion with instantaneous correlation  $-1 \le \rho^x \le 1$ , i.e.

$$dW_1^x(t)dW_2^x(t) = \rho^x dt.$$

The parameters  $\{\alpha_1, \alpha_2, \beta, \sigma_1, \sigma_2, \gamma, x_1, x_2^x\}$  are positive constants and will be explained below. Performing a Cholesky decomposition on the variance-covariance matrix the of the pair  $(W_1^x, W_2^x)$ , we can also express the dynamics as follows:

$$dX_1(t) = \alpha_1 X_1(t)dt + \sigma_1 dZ_1(t), \qquad X_1(0) = x_1, \tag{4.8}$$

$$dX_2^x(t) = (\alpha_2 x + \beta) X_2^x(t) dt + \sigma_2 e^{\gamma x} (\rho^x dZ_1(t) + \sqrt{1 - (\rho^x)^2} dZ_2(t)), \qquad X_2(0) = x_2^x, \tag{4.9}$$

where  $(Z_1, Z_2)$  are independent Brownian motion same for all generations, such that

$$dW_1^x(t) = dZ_1(t) (4.10a)$$

$$dW_2^x(t) = \rho^x dZ_1(t) + \sqrt{1 - (\rho^x)^2} dZ_2(t)$$
(4.10b)

From the equations (4.8) and (4.9), one can notice that only the latter depend on the generation through the current age x. The first factor  $X_1$  of the two-factor Gaussian model is common for all generations and the parameters  $\{\alpha_1, \sigma_1\}$  are the drift and diffusion that describe the general trend of the mortality intensity. The model capture the increasing trend of volatilities at older initial ages using the exponential term in the diffusion part. As we can see from figure 6 the volatility increases significantly at older ages, and thus we expect the parameter  $\gamma$  must be positive in order to capture this property.

In addition, we are making the assumption that the correlation between each generation and the general trend is the same for all generations. In other words, the parameters  $\rho^x$  for all generations x, are equal to a common parameter  $\rho$ . As mentioned earlier, followed form the actuarial practice, the correlation between generations is high, but not perfect. The reason for making this assumption is to make the calibration of the model easier.

#### Remark 4.2.

1. From now on, we omit the superscript x for convenience.

- 2. The two-factor Gaussian model that we consider is analogous to G2++ two-factor model used by D.Brigo and F.Mercurio (2006) [7] to model the short rate dynamics in the interest rate framework. Moreover, it is analogous to the Hull-White two-factor model by Hull and White (1994) [23]. However, in our case we add another dimension, namely the generation.
- 3. The dynamics of the model given by the equations (4.8) (4.10b) are handy when we simulate the paths of the process in the following section, using Monte Carlo simulations.

#### 4.2.2 Analytical formula for survival probability

The main drawback of the two-factor Gaussian model (4.5) - (4.22) is that the intensity can take negative values with positive probability, due to its Gaussian distribution. Despite this fact, the Gaussian models are very useful due to their analytical tractability. In the interest rate framework, process for short rate with Gaussian distribution allows analytical formulas for Zero-Coupon bond prices, and also for interest rate derivatives with non-linear payoffs, such as European Options on Zero-Coupon Bonds and Interest Rate Caps/Floors. Similarly, the Gaussian distribution is very useful when we will consider longevity-linked securities with non-linear payoff, in the following section.

Now, let us proof the analytical formula for the survival probability, a fundamental quantity for the longevity market. One can easily notice that the survival probability is equivalent to the pricing equation of a default-free Zero-Coupon bond.

**Proposition 4.3.** We denote by  $S_{x+t}(t,T)$  the (T-t)-year expected survival probability of an individual aged x + t at time t, conditional on the filtration  $\mathbb{G}$ . Under the two-factor Gaussian process defined by equations (4.5)-  $(4.22, S_{x+t}(t,T))$  is given by

$$S_{x+t}(t,T) = \mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^t \lambda_u du} \middle| \mathcal{G}_t\right] = exp\left\{M + \frac{1}{2}V\right\}$$
(4.11)

where,

$$M = -\left(\frac{e^{\alpha_1(T-t)} - 1}{\alpha_1}\right) X_1(t) - \left(\frac{e^{\alpha_3(T-t)} - 1}{\alpha_3}\right) X_2(t)$$
 (4.12a)

and

$$\begin{split} V &= \frac{\sigma_1^2}{\alpha_1^2} \Big( T - t - \frac{2}{\alpha_1} e^{\alpha_1(T-t)} + \frac{1}{2\alpha_1} e^{2\alpha_1(T-t)} + \frac{3}{2\alpha_1} \Big) \\ &+ \frac{\sigma_3^2}{\alpha_3^2} \Big( T - t - \frac{2}{\alpha_3} e^{\alpha_3(T-t)} + \frac{1}{2\alpha_3} e^{2\alpha_3(T-t)} + \frac{3}{2\alpha_3} \Big) \\ &+ 2\rho \frac{\sigma_1 \sigma_3}{\alpha_1 \alpha_3} \Big( T - t - \frac{e^{\alpha_1(T-t)} - 1}{\alpha_1} - \frac{e^{\alpha_3(T-t)} - 1}{\alpha_3} + \frac{e^{(\alpha_1 + \alpha_3)(T-t)} - 1}{\alpha_1 + \alpha_3} \Big) \end{split} \tag{4.12b}$$

with  $\alpha_3 = \alpha_2 x + \beta$  and  $\sigma_3 = \sigma_2 e^{\gamma x}$ .

*Proof.* First, lets solve the equation (4.6) by considering the stochastic integration by parts

$$d(e^{-\alpha_1 t} X_1(t)) = e^{-\alpha_1 t} dX_1(t) + X_1(t) de^{-\alpha_1 t} + d\langle e^{-\alpha_1 t}, X_1 \rangle_t = e^{-\alpha_1 t} dX_1(t) - \alpha_1 e^{-\alpha_1 t} X_1(t) dt.$$

and substitute in the equation (4.6),

$$d(e^{-\alpha_1 t} X_1(t)) = e^{-\alpha_1 t} [\alpha_1 X_1(t) dt + \sigma_1 dW_1(t)] - \alpha_1 e^{-\alpha_1 t} X_1(t) dt = \sigma_1 e^{-\alpha_1 t} dW_1(t)$$

Now, taking the integral on both sides, we get

$$e^{-\alpha_1 u} X_1(u) - e^{-\alpha_1 t} X_1(t) = \sigma_1 \int_t^u e^{-\alpha_1 s} dW_1(s)$$

Dividing both side by  $e^{-\alpha_1 u}$ , we have the solution of (4.6), given by

$$X_1(u) = e^{\alpha_1(u-t)}X_1(t) + \sigma_1 \int_t^u e^{\alpha_1(u-s)}dW_1(s)$$
(4.13)

Now integrating both sides with respect to u on the interval [t, T], we obtain an integral form of  $X_1(t)$ ,

$$\int_{t}^{T} X_{1}(u) du = \int_{t}^{T} e^{\alpha_{1}(u-t)} X_{1}(t) du + \int_{t}^{T} \left( \sigma_{1} \int_{t}^{u} e^{\alpha_{1}(u-s)} dW_{1}(s) \right) du$$

$$= \left( \frac{e^{\alpha_{1}(T-t)} - 1}{\alpha_{1}} \right) X_{1}(t) du + \sigma_{1} \int_{t}^{T} \left( \int_{t}^{T} \mathbf{1}_{\{s \leq u\}} e^{\alpha_{1}(u-s)} dW_{1}(s) \right) du$$

Using Fubini's theorem, we can change the order of integration of the second term,

$$\int_{t}^{T} X_{1}(u) du = \left(\frac{e^{\alpha_{1}(T-t)} - 1}{\alpha_{1}}\right) X_{1}(t) du + \sigma_{1} \int_{t}^{T} \left(\int_{t}^{T} \mathbb{1}_{\{s \leq u\}} e^{\alpha_{1}(u-s)} du\right) dW_{1}(s) 
= \left(\frac{e^{\alpha_{1}(T-t)} - 1}{\alpha_{1}}\right) X_{1}(t) du + \sigma_{1} \int_{t}^{T} \left(\int_{s}^{T} e^{\alpha_{1}(u-s)} du\right) dW_{1}(s)$$

Computing the integral we get,

$$\int_{t}^{T} X_{1}(u)du = \left(\frac{e^{\alpha_{1}(T-t)} - 1}{\alpha_{1}}\right) X_{1}(t)du + \frac{\sigma_{1}}{\alpha_{1}} \int_{t}^{T} \left(e^{\alpha_{1}(T-s)} - 1\right) dW_{1}(s)$$
(4.14)

Lets consider equation (4.22), and let  $\alpha_3 = \alpha_2 x + \beta$  and  $\sigma_3 = \sigma_2 e^{\gamma x}$ . Following the same steps as above, we can obtain an integral form of  $X_2$ , given by

$$\int_{t}^{T} X_{2}(u)du = \left(\frac{e^{\alpha_{3}(T-t)} - 1}{\alpha_{3}}\right) X_{2}(t)du + \frac{\sigma_{3}}{\alpha_{3}} \int_{t}^{T} \left(e^{\alpha_{3}(T-s)} - 1\right) dW_{2}(s)$$
(4.15)

Both equations (4.14) and (4.15) are deterministic, except for the last integrals, which are Itó's integrals with deterministic integrands. This implies that equations (4.14) and ??eq:intX2) are Gaussian random variables. In addition, the sum of these two Gaussian random variables, is also a Gaussian random variable, i.e.

$$-\int_{t}^{T} \lambda_{x}(u)du \left| \lambda_{x}(t) = -\int_{t}^{T} \left( X_{1}(u) + X_{2}(u) \right) du \left| X_{1}(t), X_{2}(t) \right|$$
(4.16)

is a Gaussian random variable.

Finally, we need to calculate the mean M and the variance V of (4.16) and then use the moment generating function to obtain our result.

$$M = \mathbb{E}^{\mathbb{P}} \left[ -\int_{t}^{T} \left( X_{1}(u) + X_{2}(u) \right) du \, \middle| \, X_{1}(t), X_{2}(t) \right]$$

$$= \mathbb{E}^{\mathbb{P}} \left[ -\int_{t}^{T} X_{1}(u) du \, \middle| \, X_{1}(t) \right] + \mathbb{E}^{\mathbb{P}} \left[ -\int_{t}^{T} X_{2}(u) du \, \middle| \, X_{2}(t) \right]$$

$$= -\left( \frac{e^{\alpha_{1}(T-t)} - 1}{\alpha_{1}} \right) X_{1}(t) - \left( \frac{e^{\alpha_{3}(T-t)} - 1}{\alpha_{3}} \right) X_{2}(t)$$

$$\begin{aligned} \mathbf{V} &= \mathbb{V}ar^{\mathbb{P}} \bigg[ - \int_{t}^{T} \bigg( X_{1}(u) + X_{2}(u) \bigg) du \, \bigg| \, \mathcal{G}_{t} \bigg] \\ &= \mathbb{V}ar^{\mathbb{P}} \bigg[ \frac{\sigma_{1}}{\alpha_{1}} \int_{t}^{T} \bigg( e^{\alpha_{1}(T-s)} - 1 \bigg) dW_{1}(s) + \frac{\sigma_{3}}{\alpha_{3}} \int_{t}^{T} \bigg( e^{\alpha_{3}(T-s)} - 1 \bigg) dW_{2}(s) \, \bigg| \, \mathcal{G}_{t} \bigg] \end{aligned}$$

Using the formula of the variance of dependent random variable and the Itó's Isometry, we have

$$\begin{split} \mathbf{V} &= \frac{\sigma_1^2}{\alpha_1^2} \mathbb{V} ar^{\mathbb{P}} \bigg[ \int_t^T \Big( e^{\alpha_1(T-s)} - 1 \Big) dW_1(s) \bigg| \, \mathcal{G}_t \bigg] + \frac{\sigma_3^2}{\alpha_3^2} \mathbb{V} ar^{\mathbb{P}} \bigg[ \int_t^T \Big( e^{\alpha_3(T-s)} - 1 \Big) dW_2(s) \bigg| \, \mathcal{G}_t \bigg] \\ &+ 2 \frac{\sigma_1 \sigma_3}{\alpha_1 \alpha_3} \mathbb{C}ov^{\mathbb{P}} \bigg[ \int_t^T \Big( e^{\alpha_1(T-s)} - 1 \Big) dW_1(s), \int_t^T \Big( e^{\alpha_3(T-s)} - 1 \Big) dW_2(s) \bigg| \, \mathcal{G}_t \bigg] \\ &= \frac{\sigma_1^2}{\alpha_1^2} \int_t^T \Big( e^{\alpha_1(T-s)} - 1 \Big)^2 ds + \frac{\sigma_3^2}{\alpha_3^2} \int_t^T \Big( e^{\alpha_3(T-s)} - 1 \Big)^2 ds + 2\rho \frac{\sigma_1 \sigma_3}{\alpha_1 \alpha_3} \int_t^T \Big( e^{\alpha_1(T-s)} - 1 \Big) \Big( e^{\alpha_3(T-s)} - 1 \Big) ds \end{split}$$

Solving the integrals, the variance is given by

$$\begin{split} \mathbf{V} &= \frac{\sigma_1^2}{\alpha_1^2} \Big( T - t - \frac{2}{\alpha_1} e^{\alpha_1(T-t)} + \frac{1}{2\alpha_1} e^{2\alpha_1(T-t)} + \frac{3}{2\alpha_1} \Big) \\ &+ \frac{\sigma_3^2}{\alpha_3^2} \Big( T - t - \frac{2}{\alpha_3} e^{\alpha_3(T-t)} + \frac{1}{2\alpha_3} e^{2\alpha_3(T-t)} + \frac{3}{2\alpha_3} \Big) \\ &+ 2\rho \frac{\sigma_1 \sigma_3}{\alpha_1 \alpha_3} \Big( T - t - \frac{e^{\alpha_1(T-t)} - 1}{\alpha_1} - \frac{e^{\alpha_3(T-t)} - 1}{\alpha_3} + \frac{e^{(\alpha_1 + \alpha_3)(T-t)} - 1}{\alpha_1 + \alpha_3} \Big) \end{split}$$

Finally, using the moment generating function of the normal random variable Z,

$$\mathbb{E}\left[e^{tZ}\right] = \exp\left\{Mt + \frac{Vt^2}{2}\right\},\,$$

with t = 1, we have,

$$\mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^t \lambda_u du} \middle| \mathcal{G}_t\right] = \exp\left\{M + \frac{1}{2}V\right\}$$
(4.17)

with M and V as above.  $\Box$ 

### 4.2.3 Risk-adjustment and market price of longevity risk

As we discussed earlier in section 2.4.3, the market for longevity risk is incomplete and not fully developed yet. There aren't any liquid traded market products with underlying the survivorship.

Thus the contingent claims cannot be replicated by self-financing trading strategies. We assume that the market is arbitrage-free, and this implies that there exist an equivalent martingale measure, but it is not unique due to market incompleteness. We denote by  $\mathcal Q$  the set of equivalent martingale measures, such that trading a contingent claim  $H_T$  at a price  $\mathbb{E}^{\mathbb{Q}}[e^{\int_0^T r_t dt} H_T | \mathcal{G}_t]$  for  $\mathbb{Q} \in \mathcal{Q}$ , does not create any arbitrage opportunity. An important aspect about the pricing of contingent claims (in our case the longevity-linked securities) is to incorporate into pricing the cost of the hedge provider to accept the longevity risk from the counterparty that wants to eliminate it from its portfolio. In other word, we need to consider the dynamics of our model under the risk-adjusted measure and hence to specify the market price of longevity risk. Lets make this clear by considering the following example. Consider a security with payoff linked to the survivorship of a reference population and the hedge provider pays the hedger, if the observed survival probability is better than expected. The hedge provider will calculate the price of the security using the risk-adjusted measure, assuming a lower value of the mortality intensity and by extension higher value of the survival probability under the pricing measure i.e.  $S_{x+t}^{\mathbb{Q}}(t,T) > S_{x+t}(t,T)$  where  $S_{x+t}^{\mathbb{Q}}(t,T)$  denote the survival probability under the risk-adjusted measure. This assumption is necessary to reflect the compensation of the hedge provider for taking on the longevity risk.

Now, let us consider the behavior of the dynamics of the two-factor Gaussian model, introduced above, under the risk-adjusted measure. As described in section 2.4.3, the stochastic intensity  $\lambda$  of a doubly stochastic Poison process under an equivalent martingale measure becomes  $(1 + \phi)\lambda$  where  $\phi$  is a  $\mathcal{G}_t$ -predictable process. Also,  $\tilde{W} = W + \int_0^t \zeta_s ds$  is a  $\mathbb{G}$ -Brownian motion under  $\mathbb{Q}$ , where W is the Brownian motion under the real world probability measure  $\mathbb{P}$ . In this case the set  $\mathbb{Q}$  contains all the pairs  $(\phi, \zeta)$  for which the prices of the contingent claims do not induce arbitrage opportunities. As pointed out in remark 2.36, the process  $\phi$  corresponds to the market price of unsystematic mortality risk. The impact of the unsystematic mortality risk can be diversified by considering a large enough portfolio and hence we can assume that  $\phi = 0$ . In other words, we do not demand an excess return for this risk, only for the longevity risk. From this assumption and the proposition 2.35, it follows that the equivalent martingale measures  $\mathbb{Q}$  is defined by the Girsanov's theorem (see appedix A.3):

$$\eta_T = \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{G}_T} = \exp\left(-\int_0^t \zeta_s dW_s - \frac{1}{2} \int_0^t ||\zeta_s||^2 ds\right)$$
(4.18)

Now, recall the two-Gaussian model given by the equations (4.5) - (4.22). In order to retain the analytical tractability of the model under the two measures, we make the following assumptions. First, we do not required compensation from the first factor,  $X_1$ . Then, we assume that the market price of longevity risk process  $\zeta$  has the following functional form

$$\zeta(t) = \zeta X_2(t),$$

for all  $t \geq 0$  and for a constant  $\zeta$ . Note that the constant  $\zeta$  is also independent form the initial

age. This form is not unique and we choose it so that we have tractable dynamics under both measures. The same functional form was used by [13, Chapter 3, pages 60-61] to consider the objective measure dynamics of the Vasicek model. So, equation 4.19 becomes

$$\eta_T = \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{G}_T} = \exp\left(-\int_0^t \zeta X_2(s) dW_s - \frac{1}{2} \int_0^t \zeta^2 X_2^2(s) ds\right). \tag{4.19}$$

Under the risk-adjusted measure  $\mathbb{Q}(\zeta)$ , the process for  $\tilde{W}_1(t)$  and  $\tilde{W}_2(t)$  are given by

$$d\tilde{W}_1(t) = dW_1(t)$$
  
$$d\tilde{W}_2(t) = \zeta X_2(t)dt + dW_2(t),$$

and hence the dynamics of the model are given by

$$d\lambda_x(t) = dX_1(t) + dX_2(t), \tag{4.20}$$

where

$$dX_1(t) = \alpha_1 X_1(t)dt + \sigma_1 d\tilde{W}_1(t) \tag{4.21}$$

$$dX_2(t) = (\alpha_2 x + \beta - \zeta \sigma_2 e^{\gamma x}) X_2(t) dt + \sigma_2 e^{\gamma x} d\tilde{W}_2(t)$$

$$(4.22)$$

Remark 4.4. We denote the risk-adjusted measure  $\mathbb{Q}(\zeta)$  to emphasize that it is not unique and the constant  $\zeta$  can take different values. For a brief description about the potential factors that affect the size of the risk premium we refer the reader to [32, Section 3.5, page 12].

In the following corollary, we provide the analytical formula for the risk-adjusted survival probability.

Corollary 4.5. We denote by  $\tilde{S}_{x+t}(t,T)$  the (T-t)-year expected risk-adjusted survival probability of an individual aged x+t at time t, conditional on the filtration  $\mathbb{G}$ . Under the two-factor Gaussian process defined by equations 4.5- 4.22,  $\tilde{S}_{x+t}(t,T)$  is given by

$$\tilde{S}_{x+t}(t,T) = \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_0^t \lambda_u du} \, \middle| \, \mathcal{G}_t \right] = exp \left\{ \tilde{M} + \frac{1}{2} \tilde{V} \right\}$$
(4.23)

where,

$$\tilde{M} = -\left(\frac{e^{\alpha_1(T-t)} - 1}{\alpha_1}\right) X_1(t) - \left(\frac{e^{\alpha_3(T-t)} - 1}{\alpha_3}\right) X_2(t)$$
(4.24a)

and

$$\tilde{V} = \frac{\sigma_1^2}{\alpha_1^2} \left( T - t - \frac{2}{\alpha_1} e^{\alpha_1 (T - t)} + \frac{1}{2\alpha_1} e^{2\alpha_1 (T - t)} + \frac{3}{2\alpha_1} \right) 
+ \frac{\sigma_3^2}{\alpha_3^2} \left( T - t - \frac{2}{\alpha_3} e^{\alpha_3 (T - t)} + \frac{1}{2\alpha_3} e^{2\alpha_3 (T - t)} + \frac{3}{2\alpha_3} \right) 
+ 2\rho \frac{\sigma_1 \sigma_3}{\alpha_1 \alpha_3} \left( T - t - \frac{e^{\alpha_1 (T - t)} - 1}{\alpha_1} - \frac{e^{\alpha_3 (T - t)} - 1}{\alpha_3} + \frac{e^{(\alpha_1 + \alpha_3)(T - t)} - 1}{\alpha_1 + \alpha_3} \right)$$
(4.24b)

with  $\alpha_3 = \alpha_2 x + \beta - \zeta \sigma_2 e^{\gamma x}$  and  $\sigma_3 = \sigma_2 e^{\gamma x}$ .

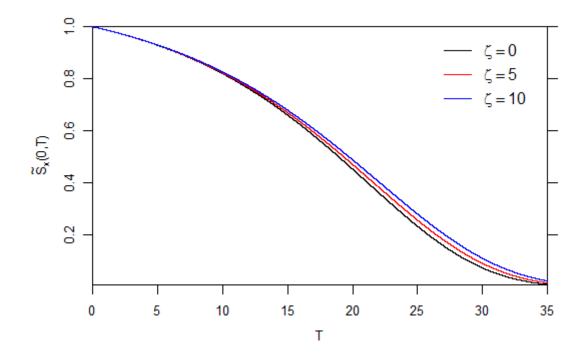


Figure 4: Risk-adjusted survival probability for different values of market price of longevity risk,  $\zeta \in \{0, 5, 10\}.$ 

In figure 4 we plot the risk-adjusted survival function for different values of the market price of longevity risk, using the analytical formulas from Proposition 4.3 and Corollary 4.5, and the estimated parameters from the following section 4.3. From the figure we can observe that higher market price of longevity risk  $\zeta$ , leads to higher survival probabilities. Also, given that  $\zeta$  is positive, the risk-adjusted survival probabilities are greater than the survival probabilities with no risk-adjustment. This result is consistent with the assumption made in the first paragraph in section 4.2.3.

As discussed previously, the longevity market is incomplete, with no available liquid, marked traded longevity products. Thus, the choice of risk adjusted measure is not unique. In our framework, we consider that it depends on the constant parameter  $\zeta$ , which can take a range of possible values. One way to estimate  $\zeta$ , is to consider the attitude towards risk in terms of a utility functions (see e.g. M.H.A Davis (1997) [18]). The most popular method in the literature so far to estimate  $\zeta$ , is to use the longevity bonds issued by European Investment Bank and BNP Paribas in 2004, but withdrawn a year later (see e.g. R. Meyicke and M. Sherris (2014) [37] and A.J.G. Cairns et. al. (2006b) [15]). More details for this approach will be considered in section 5.

### 4.3 Calibration

Our model aims to capture the stochastic trends in mortality at different ages across time, but also for different generations. In other words, it aims to describe the whole mortality surface. The ideal set of data for the calibration is cohort life tables, which represents the mortality of individuals from a given generation over the course of their lifetime. Unfortunately, there is a an important barrier for using cohort life tables; the requirement that all cohort members have died. According to J.R. Wilmoth et. al. (2007) [43, Section 2, pages 6-7] the cohort life tables are presented if there is at least one generation observed from birth until extinction (i.e all members of the cohort have died) and also, death rates are given in a cohort format if there are at least 30 consecutive calendar years of data for that cohort. For example the cohort life table for the generation 1951 (i.e. currently 65 years old) it would contain the mortality conditions of this generation until age 64. In practice, we may treat the older generations as "almost-extinct" and make some assumption on the future mortality of this cohort based on the death rate of the previous years. Another method to describe the cohort data in by LEXIS diagrams. As our goal is not to describe the actuarial and demographic methods for life tables construction, we refer the reader to J.R. Wilmoth et.al. (2007) [43] for an in-depth description of the methods used by Human Mortality Database to construct their life tables.

#### 4.3.1 Estimating the model parameters

Our approach will be based on the observed period life tables for UK males. We collect the data from Human Mortality Database [24], where we can find the period life tables with the year by year historical mortality experience of the the population under consideration, from 1922 to 2013. We will use the observed data form 1970 to 2013 and we will focus on the mortality evolution for ages ages 60 to 100 i.e around and above retirement, since these are the ages where longevity risk has the most financial impact. As we discussed in section 4.1.1 our estimation method is based on a discretized process, where the intensity is assumed to be constant over each aged and calendar year and it is approximated by the central death rates. The observed central death rates m(x,t) from our data are represented in figure 5.

Then, by taking the diagonals of the surface we have a good approximation of the mortality intensities of the reference population throughout their life. For the calibration of our model we will follow a similar procedure as M.C. Fung, K. Ignatieva and M. Sherris (2015) [21], and S. Wills and M. Sherris (2008) [40].

The first step is to estimate the diffusion coefficients and the correlation parameter  $\{\sigma_1, \sigma_2, \gamma, \rho\}$ . The analytical tractability obtained by the Gaussian distribution, allows us to calculate the variance

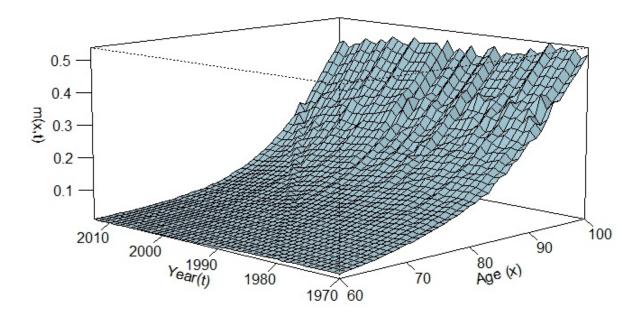


Figure 5: Central Death Rates m(x,t) for  $x=60,61,\ldots,100$  and  $t=1970,1971,\ldots,2013$  for UK males observed form HMD.

of the model explicitly. Let us consider the discretized version of our model, given by

$$\Delta \lambda_x = \Delta X_1 + \Delta X_2, \text{ with}$$

$$\Delta X_1 = \alpha_1 \Delta t + \sigma_1 \Delta W_1$$

$$\Delta X_2 = (\alpha_2 x + \beta) \Delta t + \sigma_2 e^{\gamma x} \Delta W_2.$$
(4.25)

Taking the variance we have,

$$Var(\Delta \lambda_x) = Var(\Delta X_1 + \Delta X_2) = Var(\sigma_1 \Delta W_1 + \sigma_2 e^{\gamma x} \Delta W_2)$$

$$= Var(\sigma_1 \sqrt{\Delta t} Z_1 + \sigma_2 e^{\gamma x} \sqrt{\Delta t} Z_2)$$

$$= (\sigma_1^2 + \sigma_2^2 e^{2\gamma x} + 2\sigma_1 \sigma_2 \rho e^{\gamma x}) \Delta t$$

$$(4.26)$$

The second equality follows from the Gaussian property of the Brownian motion and the third from the definition of variance of dependent random variables, since  $Z_1$  and  $Z_2$  are dependent normally distributed random variables with correlation  $\rho$ . Using the observed data, we can evaluate the sample variance of  $\Delta m(x,t)$ , where  $\Delta m(x,t)$  is the difference of central death rates, given by

$$\Delta m(x,t) = m(x+1, t+1) - m(x, t),$$

with  $x = 60, 61, \ldots, 99$  and  $t = 1970, 1971, \ldots, 2012$ . Note that we calcuate the difference between the diagonals of the surface in figure 5 in order to capture the cohort effects. We present the difference in central death rates in figure 6. As we can see, the differences increase for the older ages. This gives as evidence that the variance of our model increases with age and that explains the

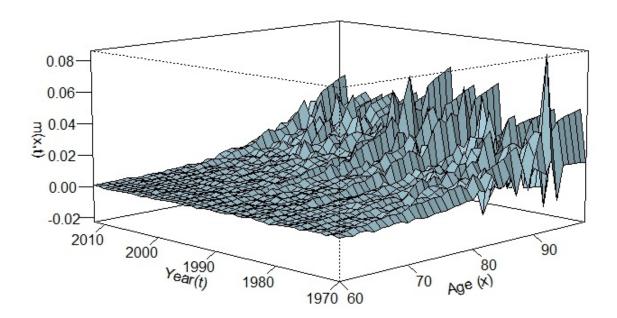


Figure 6: Difference of Central Death Rates  $\Delta m(x,t) = m(x+1,t+1) - m(x,t)$  for  $x = 60, 61, \ldots, 100$  and  $t = 1970, 1971, \ldots, 2013$  for UK males observed form HMD.

choice of the exponential term in the diffusion part of our model. Then, we estimate the diffusion parameters  $\{\sigma_1, \sigma_2, \gamma, \rho\}$  by fitting the model variance to the sample variance using least squares optimization. In mathematical terms, we minimize

$$\sum_{x=60,65...}^{100} (\mathbb{V}ar(\Delta \lambda_x | \sigma_1, \sigma_2, \gamma, \rho) - \mathbb{V}ar(\Delta m_x))^2$$
(4.27)

with respect to  $\{\sigma_1, \sigma_2, \gamma, \rho\}$ . The sample variance  $\mathbb{V}ar(\Delta m_x)$  is evaluated for each  $x = 60, 61, \ldots, 99$  over time i.e for a fix x in  $60, 61, \ldots, 99$  we calculate the sample variance of the observed data  $\{\Delta m(x,t); t = 1970, 1971, \ldots, 2012\}$ .

The next step, is to estimate the rest of the parameters that are associated with the drift of the diffusion process, namely  $\{\alpha_1, \alpha_2, x_1, x_2^x\}$ . For simplicity we will assume that our model captures only two generations, but one can easily generalize the following procedure to the whole mortality surface. We will fit our model to the current survival curve observed in 2013 (we use 2013 as our current time since the observed data from HMD is until 2013). The empirical survival probability with horizon T, can be calculated by

$$\hat{S}_x(0,T) = \prod_{i=1}^{T} (e^{-m(x+i-1,0)}). \tag{4.28}$$

This formula follows from the relationship between mortality rates and central death rates in section 4.1.1. We consider two generations with members currently 65 and 75 years old, respectively. Using the observed data for t=2013 and the equation (4.28), we calculate the empirical survival curves for this two generations. Then, we estimate the parameters  $\{\alpha_1, \alpha_2, x_1, x_2^{65}, x_2^{75}\}$  by fitting

the theoretical survival probabilities using the analytical formula from Proposition 4.3 to the empirical survival curves for ages 65 and 75 simultaneously. We can write this problem as a least squared optimization, with the following objective function

$$\sum_{x=65,75} \sum_{i=1}^{T_x} (S_x(0,i) - \hat{S}_x(0,i))^2, \tag{4.29}$$

to be minimized with respect to  $\{\alpha_1, \alpha_2, x_1, x_2^{65}, x_2^{75}\}$ , where  $T_{65} = 26$  and  $T_{75} = 16$ . Also,  $x_1 = X_1(0)$  and  $x_2^x = X_2^x(0)$ .

Parameter	Estimated Value
$\alpha_1$	0.028157
$\alpha_2$	0.000051
β	0.11584
$\overline{x_1}$	0.002472
$x_2^{65}$	0.008865
$x_2^{75}$	0.029171
$\sigma_1$	0.001162
$\sigma_2$	0.000001
$\gamma$	0.104240
ρ	-0.575273

Table 2: Estimated values of model parameters

Finally, in order to optimize the objective functions given by equations (4.27) and (4.29), the Differential Evolution algorithm was used. This method also used by P. Jevtic, E.Luciano and E.Vigna for similar application, but with different objective functions. Differential evolution is a global stochastic parallel direct search method introduced by R. Storn and K. Price in 1997 (see R. Storn and K. Price [41]). The algorithm is suitable for our data due to its capability of handling non-continuous, non-linear, non-differentiable and multidimensional objective functions. The algorithm is provided in the Appendix A.4 and for more details we refer the reader to R. Storn and K. Price (1997) [41]. The results of the optimization are presented in the table 4.3.1. First, we use the DE algorithm to estmate the diffusion parameters  $\{\sigma_1, \sigma_2, \gamma, \rho\}$ . We restrict our search space to

$$S^1 = \{ \sigma_1 \in [0, 1], \sigma_2 \in [0, 1], \gamma \in [0, 1], \rho \in [-1, 1] \},$$

and the optimization error is approximately  $1.05 \times 10^{-11}$ . Then, for the rest of the parameters, we restrict our search region to

$$\mathcal{S}^2 = \left\{ \alpha_1 \in [0, 1], \alpha_2 \in [0, 0.01], x_1 \in [0, 1], x_2^{65} \in [0, 1], x_2^{75} \in [0, 1] \right\},\$$

and the optimization error is approximately  $8.17 \times 10^{-5}$ . The reason for restricting  $\alpha_2$  below 0.01 is to avoid getting infinity values in the survival probability and in particular, in the variance equation (4.12b) for V.

#### 4.3.2 Simulations

Now, we check if the two-Gaussian model with the parameters estimated above produces reasonable paths for mortality intensity and survival curves, by using Monte Carlo simulations. In particular, we used the Euler and Milstein explicit scheme to simulate  $\lambda_x(t)$  (see e.g. D. Brigo and F. Mercurio (2007) [13, Section 22.7.3, page 797]). The dynamics of the model given by the equations (4.8) - (4.10b) used in the discretized scheme in order to capture the correlation between the two risk factors. In figures 7 and 8, we present the percentile of simulated paths for ages x = 65 and x = 75, respectively. As expected they follow an exponentially increasing trend at older ages. This result is consistent with our discussion in section 2.4.2. In addition, we observed that the volatility of mortality intensity for larger initial ages (i.e. older generations) is higher; a result that was expected as well.

An equally significant aspect, is the ability of the model to produce reasonable death curves. Some of the observed death curves from 1930 to 2010, were presented in figure 1. To this end, using the simulated paths above, we simulated the death curves, presented in figure 9. We have consider a population of 100000 of currently being age 65 years old and 75 years old, respectively. As all of the individuals come from the same cohort, we assuming as well that their death times have common mortality intensity  $\lambda_x(t)$ . To simulate the death times of each individual in the population i.e.  $\tau_i$  for  $i \in \{1, 2, ..., 100000\}$ , we recall the property of the first jump of the doubly stochastic Poisson process, as given in section 2.4.1. Namely, the first jump time is given by

$$\tau = \Lambda^{-1}(\xi),$$

where  $\xi$  is exponential random variable with rate equals to 1 and  $\Lambda^{-1}(\cdot)$  is the inverse of the  $\mathbb{F}$ -martingale hazard process  $\Lambda_{t\wedge\tau}=\int_0^{t\wedge\tau}\lambda_s ds$ . So, we define the first jump time as

$$\tau := \inf\{T : \xi \le \int_0^T \lambda_s ds\}. \tag{4.30}$$

So, the death times of the individuals in our hypothetical population can be simulated using the following steps:

- 1. Set a maximum age  $\omega$  and simulate the mortality intensity  $\lambda_x(t)$  for initial age x until age  $\omega$ .
- 2. Generate 100000 exponential random variables  $\xi_1, \xi_2, \dots, \xi_{100000}$ .
- 3. Using the equation 4.30 we calculate the death time of each individual. In case  $\xi_i > \int_0^T \lambda_s ds$ , then  $\tau_i = \omega x$ .

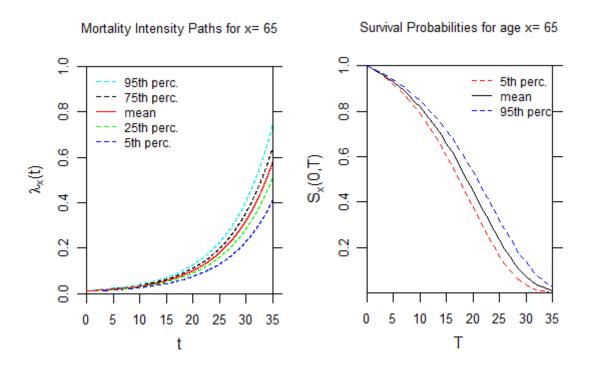


Figure 7: Mean of the simulated survival probabilities for U.K. males age x = 65 with the 5th and 95th percentile (on the right) and the mean path of the simulated mortality intensities with the 5th, 25th, 75th, and 95th percentiles for U.K. males age x = 65 (on the left).

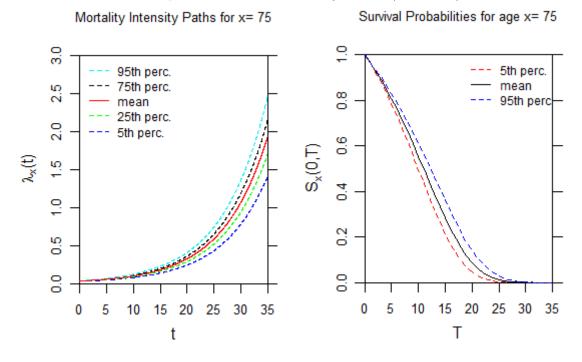


Figure 8: Mean of the simulated survival probabilities for U.K. males age x = 75 with the 5th and 95th percentile (on the right) and the mean path of the simulated mortality intensities with the 5th, 25th, 75th, and 95th percentiles for U.K. males age x = 75 (on the left).

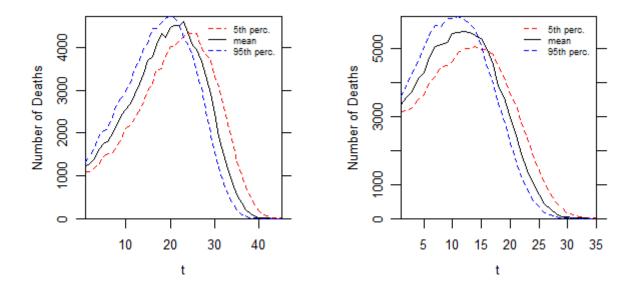


Figure 9: Simulated death curves for U.K males age x = 65 (left) and x = 75 (right), using the simulated paths from figures 7 and 8, respectively. We considered a hypothetical population of 100000 individuals with the same mortality intensity, with age 65 and 75, respectively, at t=0.

As we can see from figure 9, the two-factor Gaussian model produces reasonable death curves for both initial ages x = 65 and x = 75. Note that we plotted the simulated number of deaths occurred from the mean, 5th and 95th percentile of the simulated paths.

## 5 Longevity Derivatives

In this section, we describe some of the financial instruments with underlying the realized survival rate, that can be used to hedge and manage the longevity risk form a portfolio of pensions or annuities. Also, they can be used from someone wants to take exposure in longevity risk. For each instrument we will give the definition and the pricing method. First, we will consider contracts with linear payoffs and then we will introduce contracts with optionality in their payoffs i.e. non-linear payoffs. One can easily recognize the similarity of these products with the interest rate instruments. The main difference between this two markets is that the interest rate market is fully developed and liquid, in contrast with the longevity market. This implies that the products that could have been model-free in a liquid market, now they are model-dependent. Another important assumption is that longevity market is independent from the financial market. We borrowed many of the pricing techniques from D. Brigo and F. Mercurio (2007) [13] and rearranged them in order to fit to the longevity-linked instruments.

### 5.1 Linear payoffs

We now proceed to the definition of the longevity-linked instruments with linear payoff. We first consider the S-forward (Survivor-forward) contract, which allows the hedger to lock-in the survival rate between the inception of the contract (t=0) and the maturity T at a predetermined value K(T). These contracts and the longevity swaps that will be consider later in this section are first introduced by K.Dowd (2003) [19]. Here, we consider the definition of the S-forward contract given by Life and Longevity Markets Association (2010) [33].

**Definition 5.1** (S-forward). An S-forward contract is an agreement where one party promises to pay its counterparty an amount equal to the realized survival rate from the inception of the contract (t=0) until the maturity T i.e.  $\hat{S}_x(0,T) = e^{-\int_0^T \lambda_x(u)du}$ , based on an agreed population cohort. In return, the counterparty pays a fixed survival rate  $K \in (0,1)$ , agreed at the inception of the contract.

In the this contract, no money changes hands until the maturity T. The discounted payoff of the contract at inception t = 0 for the hedger (party that pays a fixed rate) is given by

$$V_0 = D(0, T)(e^{-\int_0^T \lambda_x(u)du} - K), \tag{5.1}$$

where  $D(0,T) = e^{-\int_0^T r_u du}$  is the discount factor. Taking the risk neutral expectation of the discounted payoff (5.1), we obtain the price of the hedger S-forward, as follows

$$SF(0,T;K) = \mathbb{E}^{\mathbb{Q}}[V_0 \mid \mathcal{F}_0] = P(0,T)(\tilde{S}_x(0,T) - K),$$
 (5.2)

where P(0,T) is the price at time t=0 of a zero-coupon bond with maturity T. Since the contract does not require an initial transaction to take place, the above equation (5.2) must be equals to

5.1 Linear payoffs 50

0. Hence, we can solve it with respect to K and find the fair value of the fixed survival rate of the S-forward. By doing this we get,

$$K = \tilde{S}_r(0,T)$$

In other terms, the fixed rate is equal to the risk-adjusted survival probability. If the longevity market was liquid, we could obtained  $\tilde{S}_x(0,T)$  from market data. Consequently, we need a model in order to calculate the risk-adjusted survival probability.

The above setup is based on standardized, index-based S-forwards and depends on the realized survival rate. Another type of such instruments, are the bespoke S-forwards, which are customized instruments and aim to much the mortality each hedger experience. They depend on the realized number of survivors in the reference population. Precisely, the discounted payoff of the bespoke contract at inception for the hedger is given by

$$V_0 = D(0,T)(\frac{n-N_t}{n} - K), (5.3)$$

where n is the total number of population at the inception of the contract and  $N_t$  is the number of deaths experienced in the population during [0,t]. It is easy to prove that  $\mathbb{E}^{\mathbb{Q}}[n-N_T] = n\tilde{S}_x(0,T)$ . Using this result, we can prove that the fixed survival rate is again given by the risk-adjusted survival probability, based on the mortality intensity of the current population. The main advantage of a bespoke over an indexed-based S-forwards, is that the latter may bearing basis risk. This is the mismatch of the cash flows of the S-forward and the liabilities of the portfolio, due to a difference in the mortality experience of the hedger's population and the index. Clearly, the employees having office based jobs have lower mortality rates than people working in the contractions industry.

A generalization of the S-forwards is the Longevity Swaps. The Longevity swaps can be seen as a portfolio of S-forwards with different maturities. Hence the price of this swap can be found by adding the prices of the S-forwards consisting the Longevity Swap. As with the S-forwards there are two different types; the bespoke and the indexed-based Longevity swaps. The following definition is based on the indexed-based type, but it can be easily rearrange for the bespoke. The longevity swaps based on the same idea as the Interest Rate Swaps (IRS), but with different underlying. So, our definition will be based on the definition of the IRS given by D. Brigo and F. Mercurio (2007) [13, Section 1.5, page 13].

**Definition 5.2** (Longevity Swaps). A Longevity Swap is a contract that exchanges payments between two differently indexed legs, starting from a future date. At every prespecified dates,  $T_{\alpha}, T_{\alpha+1}, \ldots, T_{\beta}$ , one party pays a fixed leg equal to  $NK(T_i)$ , where  $K(T_i) \in (0,1)$ , for  $T_i \in \{T_{\alpha}, T_{\alpha+1}, \ldots, T_{\beta}\}$ , is the fixed survival rate and N is the nominal value of the contract. The counterparty pays the floating amount  $N\hat{S}_x(0, T_i) = Ne^{-\int_0^{T_i} \lambda_x(u) du}$ , that depends on the realized survival rate observed during the period  $[0, T_i]$  for each  $T_i \in \{T_{\alpha}, T_{\alpha+1}, \ldots, T_{\beta}\}$ .

5.1 Linear payoffs 51

It is important to note that the fixed survival rate is different at each payment and depends on the payment date  $T_i$ , i.e. we have a series of fixed rates. As explained by Biffis, D. Blake, L. Pitotti and A.Sun (2014) [5, Section 2, page 9], the reason for having a series of fixed rates is that mortality increases with age. Since, the realized survival probabilities decreases at higher age, a fixed rate would be unfair to the hedger.

The discounted payoff at  $t < T_{\alpha}$  is

$$V_0 = \sum_{i=\alpha}^{\beta} \left[ D(0, T_i) \left( e^{-\int_0^{T_i} \lambda_x(u) du} - K(T_i) \right) \right]$$
 (5.4)

Since we treat each payment of the Longevity Swap as an S-forward with maturity the same as the payment date, the fixed rates are equal to the risk-adjusted survival probabilities i.e.  $K(T_i) = \tilde{S}_x(0, T_i)$  for each  $T_i \in \{T_\alpha, T_{\alpha+1}, \dots, T_\beta\}$ .

**Remark 5.3.** We will call Hedger Longevity Swap, the one that the holder is the party that wants to eliminate the longevity risk of its portfolio and hence paying the fixed survival rate.

In 2004, European Investment Bank issued a long-term longevity bond with the coupons linked to the survivor index of English and Welsh males aged 65 in 2003, provided by the Office for National Statistics (ONS). The bond issued in partnership with BNP Paribas, who act as a structurer, and Partner Re, who provide the longevity expertise and reinsurance capacity for the longevity bond. The maturity of the bond is 25 years. Although, a year later the bond has been withdrawn from the markets. The summary of the terms of the bond is provided in table B in Appendix B. More details can be found in D. Blake, A.J.G. Cairns and K. Dowd (2006) [8, Section 4] and Mark Azzopardi (BNP Paribas) (2005) [2]. As described by Cairns et. al. (2006b) [15] under some assumptions the initial price of the longevity bond is given by

$$V_0 = \sum_{T=1}^{25} P(0, T) e^{\delta T} \mathbb{E}^{\mathbb{P}} \left[ e^{-\int_0^T \lambda_x(u) du} \, | \, \mathcal{F}_0 \right]$$
 (5.5)

where  $\delta$  is a spread related to the risk premium of the bond. Also, to price the bond projected survival rates have been used.

Let us consider the price of the bond in our framework, under the two-factor Gaussian model. The discounted payoff at time t=0 is given by

$$\Pi_0 = \sum_{T=1}^{25} D(0,T) \left( e^{-\int_0^T \lambda_x(u) du} \right).$$

Taking the risk-adjusted expectation, the price of the bond is

$$V_0^{G2} = \sum_{T=1}^{25} P(0,T) \mathbb{E}^{\mathbb{Q}(\lambda)} \left[ e^{-\int_0^T \lambda_x(u) du} \, | \, \mathcal{F}_0 \right] = \sum_{T=1}^{25} P(0,T) \tilde{S}_x(0,T). \tag{5.6}$$

Recall from section 4.2.3, a possible way to estimate the market price of longevity risk is to choose  $\zeta$ , such that the equation (5.6) closely much (5.5) i.e.  $V_0^{G2}(\zeta) = V_0$ .

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### 5.2 Non-linear payoffs

The first longevity instruments with non-linear payoff that will be considered in this section, are the Longevity Caps and Longevity Floors. They are similar to the Longevity Swap contracts with the main difference to be the payment at each payment date is executed only if the cashflow is positive. This optionality makes the payoff of the contract non-linear. In the cap contracts, the holder (hedger) is the party that pays the fixed leg and receiving floating, and in the floor contract is the opposite. Here, we will focus on the Longevity Caps, but the same methods apply for the Longevity Floors.

The Longevity Caps can been seen as a portfolio of caplets, where each caplet has a payoffs given by  $\left(e^{-\int_0^T \lambda_x(u)du} - K(T)\right)^+$ , where T is the maturity and K(T) is the strike price. It is important to note that we have a different strike price for each caplet. As explain above in the case of the fixed survival rate of the Longevity Swaps, the strike price depends on the maturity T.

**Definition 5.4** (Caplet). The Caplet is a contract that gives the holder the right, but not the obligation, to enter into a transaction where he/she will pay a predetermined fixed rate K(T), at a future time T (the maturity), in exchange of a payment equals to the realized survival probability form t = 0, until the maturity T.

The discounted payoff at time t < T is given by

$$V_0 = D(0,T)(e^{-\int_0^T \lambda_x(u)du} - K(T))^+.$$
(5.7)

**Proposition 5.5.** The price of the Longevity Caplet, with maturity T and strike price K, under the two-factor Gaussian mortality intensity model is given by

$$Caplet(0; T, K, x) = P(0, T)\tilde{S}_x(0, T)\Phi(d_1) - KP(0, T)\Phi(d_2)$$
(5.8)

where  $d_1 = \frac{\log(\tilde{S}_x(0,T)/K) + 0.5\tilde{V}(0,T)}{\sqrt{\tilde{V}(0,T)}}$  and  $d_2 = \frac{\log(\tilde{S}_x(0,T)/K) - 0.5\tilde{V}(0,T)}{\sqrt{\tilde{V}(0,T)}}$ , and  $\tilde{V}$  is given by equation (4.24b).

*Proof.* Taking the expectation of the discounted payoff in equation (5.7), under the risk-adjusted measure

$$Caplet(0;T,K) = \mathbb{E}^{\mathbb{Q}}[V_0 \mid \mathcal{F}_0] = \mathbb{E}^{\mathbb{Q}}[(e^{-\int_0^T \lambda_x(u)du} - K)^+ \mid \mathcal{F}_0]$$
(5.9)

We have proved in Proposition 4.3 and Corollary 4.5 that  $Z = -\int_0^T \lambda_x(u) du$  is normally distributed

with mean  $\tilde{M}$  and variance  $\tilde{V}$  under the risk-adjusted measure. Using this result we can write

$$Caplet(0;T,K) = P(0,T) \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\tilde{V}}} \exp\left\{-\frac{1}{2} \left(\frac{z-\tilde{M}}{\sqrt{\tilde{V}}}\right)^{2}\right\} \left(e^{z}-K\right)^{+} dz$$

$$= P(0,T) \int_{\log K}^{+\infty} \frac{1}{\sqrt{2\pi\tilde{V}}} \exp\left\{-\frac{1}{2} \left(\frac{z-\tilde{M}}{\sqrt{\tilde{V}}}\right)^{2}\right\} e^{z} dz$$

$$+ P(0,T) \int_{\log K}^{+\infty} \frac{1}{\sqrt{2\pi\tilde{V}}} \exp\left\{-\frac{1}{2} \left(\frac{z-\tilde{M}}{\sqrt{\tilde{V}}}\right)^{2}\right\} K dz$$

$$= P(0,T) \int_{\log K}^{+\infty} \frac{1}{\sqrt{2\pi\tilde{V}}} \exp\left\{-\frac{1}{2} \left(\frac{z-\tilde{M}}{\sqrt{\tilde{V}}}\right)^{2}\right\} e^{z} dz$$

$$+ P(0,T) \int_{\log K}^{+\infty} \frac{1}{\sqrt{2\pi\tilde{V}}} \exp\left\{-\frac{1}{2} \left(\frac{z-\tilde{M}}{\sqrt{\tilde{V}}}\right)^{2}\right\} K dz$$

Using the substitution  $u = \frac{z - \tilde{M}}{\sqrt{\tilde{V}}}$ , we have

$$\begin{split} Caplet(0;T,K) &= P(0,T) \int_{\frac{\log K - \tilde{M}}{\sqrt{\tilde{V}}}}^{+\infty} \frac{1}{\sqrt{2\pi}} \mathrm{exp} \bigg\{ -\frac{1}{2} u^2 \bigg\} \mathrm{exp} \bigg\{ u \sqrt{\tilde{V}} + \tilde{M} \bigg\} du \\ &+ K P(0,T) \int_{\frac{\log K - \tilde{M}}{\sqrt{\tilde{V}}}}^{+\infty} \frac{1}{\sqrt{2\pi}} \mathrm{exp} \bigg\{ -\frac{1}{2} u^2 \bigg\} du \\ &= P(0,T) \int_{\frac{\log K - \tilde{M}}{\sqrt{\tilde{V}}}}^{+\infty} \frac{1}{\sqrt{2\pi}} \mathrm{exp} \bigg\{ -\frac{1}{2} u^2 \bigg\} \mathrm{exp} \bigg\{ u \sqrt{\tilde{V}} + \tilde{M} \bigg\} du \\ &+ K P(0,T) \Phi \bigg( -\frac{\log K - \tilde{M}}{\sqrt{\tilde{V}}} \bigg) \\ &= P(0,T) \mathrm{exp} (\frac{1}{2} \tilde{V} + \tilde{M}) \int_{\frac{\log K - \tilde{M}}{\sqrt{\tilde{V}}}}^{+\infty} \frac{1}{\sqrt{2\pi}} \mathrm{exp} \bigg\{ -\frac{1}{2} (u - \sqrt{\tilde{V}})^2 \bigg\} du \\ &+ K P(0,T) \Phi \bigg( -\frac{\log K - \tilde{M}}{\sqrt{\tilde{V}}} \bigg) \end{split}$$

Setting  $v = u - \sqrt{\tilde{V}}$ , we have

$$Caplet(0; T, K) = P(0, T) \exp(\frac{1}{2}\tilde{V} + \tilde{M})\Phi\left(-\left(\frac{\log K - \tilde{M}}{\sqrt{\tilde{V}}} - \sqrt{\tilde{V}}\right)\right)$$

$$+ KP(0, T)\Phi\left(-\frac{\log K - \tilde{M}}{\sqrt{\tilde{V}}}\right)$$
(5.10)

Now, recall that the risk-adjusted survival probability is given by

$$\tilde{S}_x(0,T) = \exp\left\{\tilde{M} + \frac{1}{2}\tilde{V}\right\}$$

Solving for  $\tilde{M}$  and substituting into 5.11, we obtain

$$Caplet(0;T,K) = P(0,T)\tilde{S}_x(0,T)\Phi\left(\frac{\log(\tilde{S}_x(0,t)/K) + 0.5\tilde{V}}{\sqrt{\tilde{V}}}\right)$$

$$+ KP(0,T)\Phi\left(\frac{\log(\tilde{S}_x(0,t)/K) - 0.5\tilde{V}}{\sqrt{\tilde{V}}}\right)$$

$$(5.11)$$

Setting 
$$d_1 = \frac{\log(\tilde{S}_x(0,T)/K) + 0.5\tilde{V}}{\sqrt{\tilde{V}}}$$
 and  $d_2 = \frac{\log(\tilde{S}_x(0,T)/K) - 0.5\tilde{V}}{\sqrt{\tilde{V}}}$  completes the proof.

5.2 Non-linear payoffs

**Remark 5.6.** In proposition 5.5 we consider the price of a Longevity Caplet at inception (t = 0). Now, if we consider the price at any time t < T, then the realized survival rate from inception until time t is know, since we condition on the information up to and including time t. Hence, we can write its expectation as follow

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(e^{-\int_{0}^{T}\lambda_{x}(u)du}\right)|\mathcal{G}_{t}\right] = \exp\left(e^{-\int_{0}^{t}\lambda_{x}(u)du}\right)\mathbb{E}^{\mathbb{Q}}\left[\exp\left(e^{-\int_{t}^{T}\lambda_{x}(u)du}\right)|\mathcal{G}_{t}\right].$$

We can denote the observed realized survival probability as  $\hat{S}_x(0,t)$ . The integral  $\left(-\int_t^T \lambda_x(u)du\right)$  is normally distributed with mean  $\tilde{M}(t,T)$  and variance  $\tilde{V}(t,T)$ , and that implies that the price of the caplet at any time t < T is given by

$$Caplet(t; T, K) = P(t, T)\hat{S}_{x}(0, t)\tilde{S}_{x+t}(t, T)\Phi(d_1) - KP(t, T)\Phi(d_2)$$
(5.12)

where  $d_1 = \frac{\log(\hat{S}_x(0,t)\tilde{S}_{x+t}(t,T)/K) + 0.5\tilde{V}(t,T)}{\sqrt{\tilde{V}(t,T)}}$  and  $d_2 = \frac{\log(\hat{S}_x(0,t)\tilde{S}_{x+t}(t,T)/K) - 0.5\tilde{V}(t,T)}{\sqrt{\tilde{V}(t,T)}}$ , and  $\tilde{M}$  and  $\tilde{V}$  is given by equations (4.24a) and (4.24b), respectively.

In the Appendix B, we plotted the surface of the Caplet Prices with respect to strike price (K) and time to maturity (T). Here, we assume that the interest rate is constant and equals to r = 4%.

The last longevity-linked derivative we will consider in this paper, is an option with underlying the Longevity Swap. We can call such options Longevity Swaptions, borrowing the name from the Interest Rate Swaptions. Let us give a formal definition first, and then analyze the pricing method of such instruments.

**Definition 5.7** (European Longevity Swaption). A European Hedger Longevity Swaption is a contract that gives the right but not the obligation to enter at a predetermined future time  $T_M < T_{\alpha}$  (the maturity of the swaption) into a Hedger Longevity Swap with payment dates  $T_{\alpha}, T_{\alpha+1}, \ldots, T_{\beta}$ .

We can write the discounted payoff by first considering the Longevity Swap price at the maturity  $T_M$  of the swaption, apply the positive-part operator and then discount it back at time  $t < T_M$ . The Longevity Swap price at time  $T_M$  is given by

$$V_{T_M} = \sum_{i=\alpha}^{\beta} \left\{ P(T_M, T_i) \left[ \hat{S}_x(0, T_M) \tilde{S}_{x+T_M}(T_M, T_i) - K(T_i) \right] \right\}.$$
 (5.13)

Hence, the discounted payoff of the swaption at time  $t < T_M$  is

$$\Pi_t = D(t, T_M) [V_{T_M}]^+.$$
 (5.14)

As the summation is inside the positive part operator, we can not decompose the payoff into a sum of other simpler products. Thus, for pricing the swaptions we need to consider the joint evolution of the survival rates involve into the payoff. As described by P. Collin-Dufrense and R.S Goldstein (2002) [16] (in the case of Interest Rate Swaptions, but it can easily adopted to the Longevity Swaptions), there is no closed-formed solutions for swaptions under the multi-factor affine models.

5.2 Non-linear payoffs 55

Since the survival probabilities are log-normally distributed, the payoff of the swaption can been described with probability density of the sum of log-normal random variables, which can not be calculated analytically. However, one can use Monte Carlo simulations or approximations (see eg. P. Collin-Dufrense and R.S Goldstein (2002) [16]) ,in order to calculate the swaption prices under the two-factor Gaussian process.

## 6 Conclusion

In the past few decades, life expectancy has been following an increasing trend, especially in developed countries. Their retirement systems is based on mortality forecasts in order to estimate retirement benefits. The systematical underestimation of life expectancy improvements, and in general the stochastic fashion of mortality, makes the longevity risk a high profile risk. The insurance and reinsurance sectors are not deep enough to fully absorb the longevity risk. Thus, the attention in the quest for hedging mechanism turned to the capital market and the aim became the development of a liquid longevity market. Capital market seems to be a reasonable place for the longevity risk from pension funds' and annuity providers' portfolios to be transferred to, due to its volume and the weak correlation between longevity and market risk. However, a developed market has not been reached yet. Although there were few attempts for issuing longevity products, there are still major challenges that need to be overcome. Mortality is not a traded asset and replicating strategies cannot be used for pricing. Thus, we have to deal with an an incomplete market where the risk-adjusted measure is not unique. In such a market with limited liquidity is hard to price the risk premium for holding longevity risk. The second major challenge is the accurate forecast of mortality rates. To this end, many models have been proposed, however there is no accepted framework vet for modeling mortality.

In this paper, we describe a possible framework, under which we are able to forecast morality rates and pricing longevity-linked securities. Identifying the similarities between longevity and credit risk, we choose to model mortality using the reduced-form intensity models, where the time of death of an individual is the first jump time of a doubly stochastic Poisson process. A major part of this paper is based on the theory behind the intensity models and tends to show the equivalence of using solely hazard functions, like in traditional actuarial framework, and the doubly stochastic Poisson framework. We use a stochastic intensity in order to capture the uncertainty in the survival probabilities, which is one of our main goal. In order to analyze and price longevity derivatives, we choose a simple two-factor Gaussian process to model the stochastic intensity. An important criterion that mortality models must have, is the ability to capture the characteristics of multiple generations. This adds another dimension and increases the complexity of the analysis. In this paper we made few assumptions in order to make the calculations easier and to concentrate on the main features, but one can easily relax these assumptions. For example, we consider non-perfect correlation between different cohorts but we assumed to be equal between every cohort. A major drawback for the two-factor Gaussian process is that it can take negative values with positive probability. Although its simplicity and this drawback, after simulating the paths, the survival probabilities and the death curves using Monte Carlo simulations, we get reasonable results.

The Gaussian property of the model enable us to find analytical formulas for the survival probabilities and for most of the longevity-linked derivatives. Under this model we define and give pricing methodologies for derivatives with linear and non-linear payoffs. Unfortunately, the hedging effectiveness of these instruments is not tested in this paper. Moreover, we assume a constant interest rate when pricing the derivatives. In equity market, where the maturity is not long-term, this can be seem as a reasonable assumption. In longevity market, we mostly consider long-term maturity products and the stochastic features of interest rates must be take into account.

Longevity risk is a complex risk, and its geographical variability complicates the issue of longevity-linked instruments, being market traded assets. Most of the transactions so far were bespoke longevity solutions, more customize to each investor in order to avoid basis risk. The issue of market traded solutions must be based on various indexes, and each must capture as close as possible the variations in mortality trends in different populations.

## References

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# A Appendix

## A.1 "Doob Class" (Class D)

The name "Class D" was given by P. A. Meyer in 1963. The following definition is taken from P. Protter (2004) [39, page 106].

**Definition A.1** (Class D). A càdlàg submartingale Z with  $Z_0 = 0$  is of Class D if the collection

 $\{Z_{\tau}: \tau \text{ a finite valued stopping time}\}$ 

is uniformly integrable.

### A.2 Poisson process

**Definition A.2** (Time-Inhomogeneous Poisson Process). We say that the process  $N = (N_t)_{t\geq 0}$  is a Poisson process with time-dependent intensity  $\lambda(t)$  if

- i)  $N_0 = 0$ ,
- ii)  $N_{t+s} N_s$  follows a Poisson distribution with parameter  $\int_s^t \lambda(u) du$ ,
- iii)  $N_t$  has independent increments.

### A.3 Girsanov's Theorem

**Theorem A.3** (Girsanov's Theorem [22]). Let W be an n-dimensional Brownian motion defined on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  and let  $\zeta$  be an n-dimensional measurable process adapted to  $\mathbb{F}$  and satisfies the Novikov's condition i.e.

$$\mathbb{E}\left(\exp\left\{\frac{1}{2}\int_0^T ||\zeta_s||^2 ds\right\}\right) < \infty.$$

Fix a time horizon T > 0 and define for all  $t \in [0, T]$ 

$$\eta_t := \exp\left\{-\int_0^t \zeta_s \cdot dW_s - \frac{1}{2} \int_0^t ||\zeta_s||^2 ds\right\}.$$

If  $\mathbb{E}[\eta_T] = 1$ , then  $\eta$  is an  $\mathcal{F}_t$ -martingale and

$$\tilde{W}_t = W_t + \int_0^t \zeta_s ds$$

is a Brownian motion under the  $\tilde{\mathbb{P}}$  measure, where

$$\tilde{\mathbb{P}}(A) := \int_{A} \eta_{T}(\omega) d\mathbb{P}(\omega),$$

for any  $A \in \mathcal{F}$ .

### A.4 Differential evolution algorithm

The Differential Evolution algorithm introduced by R. Storn and K. Price (1997) [41]. Following the original paper, the idea behind this algorithm is to start with a population  $\mathbf{X} \in \mathbb{R}^{D \times n_p}$ , where D is the number of parameters in our optimization problem and  $n_p$  the size of the population. This population is randomly initialized in our search space. In other words this population contains  $n_p$  potential solutions of the optimization problem. Then for each iteration  $n_I$  we proceed as follows. We set the current population matrix as the target matrix. For each member of the population i.e. for each column-vector  $\mathbf{X}_{..j}$  for  $j=1,2,\ldots,n_p$ , we do the following. We choose randomly three distinct column-vectors and also different from the current member. Let  $i_1, i_2, i_3$  with  $i_1 \neq i_2 \neq i_3 \neq j$  denote the index of such vectors. Then a new parameter vector is generated by adding the weighted difference between two population vectors, multiplied by a constant  $F \in [0, 2]$ , to a third vector, i.e.

$$\mathbf{X}_{.,j}^{new} = \mathbf{X}_{.,i_1} + F \times (\mathbf{X}_{.,i_2} - \mathbf{X}_{.,i_3})$$

This operation called "mutation". Then the trial vector  $\mathbf{x}^{trial} \in \mathbb{R}^D$  is created by mixing the mutated vector's parameters  $\mathbf{X}_{.,j}^{new}$  with the corresponding entry of the target matrix  $\mathbf{X}_{.,j}$ . This mixing called "crossover". The "crossover" is determined by the crossover constant  $CR \in [0,1]$  and it works as follows. For each element of the mutated vector  $\mathbf{X}_{.,j}^{new}$ , we generate a uniform random variable in [0,1]. If the value of the random variable generated is less than CR, then  $\mathbf{x}_i^{trial} = \mathbf{X}_{i,j}^{new}$ . If it is greater then  $\mathbf{x}_i^{trial} = \mathbf{X}_{i,j}$ . Then if the cost of the optimization is lower using  $\mathbf{x}^{trial}$ , rather than the cost with the target vector  $\mathbf{X}_{.,j}$ , we replace  $\mathbf{X}_{.,j}$  with  $\mathbf{x}^{trial}$  in the target matrix. We repeat this  $n_I$  times.

For the implementation of the Differential Evolution algorithm the function "DEoptim" was used in R, from the package DEoptim. We used the default parameters for crossover constant CR and F.

# B Tables and figures

Longevity Bond - Summary Term Sheet [2, Mark Azzopardi (BNP Paribas), 2005]	
Issuer	European Investment Bank
Security	£[550]m Longevity-linked EMTN
Index	Based on publicly available ONS data of English and
	Welsh mortality for a cohort of males aged $65$ in $2003$
Longevity Risk Period	Calendar years 2003 to 2027 inclusive
Maturity	25 years
Bond payoff	£50,000,000 * CSRt
CSRt	Cumulative Survival Rate (i.e. proportion of survivors) in
	the cohort at time t
Index Published	ONS Publication DH1 Mortality Statistics Table 8
Payment Frequency	Annual
Pricing	T.B.A.
Issue Date	T.B.A
Calculation Agent	BNP Paribas

Table 3: The table provides the summary terms of the EIB/BNP longevity bond issued in 2004 by European Investment Bank with BNP Paribas as the structurer.

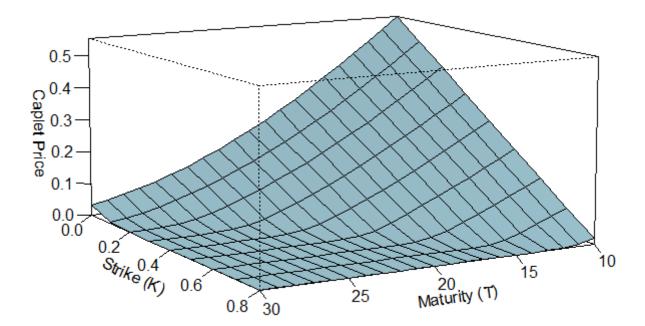


Figure 10: Caplet Prices under the two-factor Gaussian process with parameters given by table 4.3.1, with respect to Strike price (K) and Maturity (T). The prices obtained using the analytical of corollary 4.5, with constant interest rate r = 0.04 and market price of risk  $\zeta = 10$ .

## Mortality Intensity Paths for x= 65

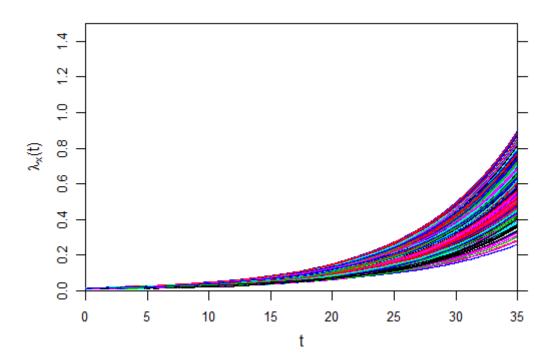


Figure 11: Simulated paths of the mortality intensity  $\lambda_{65}(t)$  of U.K. males age x=65 under the two-factor Gaussian model with parameters given in table 4.3.1, using Monte Carlo simulations.

### Mortality Intensity Paths for x= 75

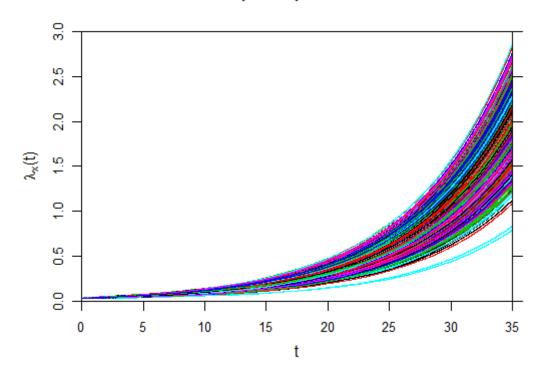


Figure 12: Simulated paths of the mortality intensity  $\lambda_{75}(t)$  of U.K. males age x=75 under the two-factor Gaussian model with parameters given in table 4.3.1, using Monte Carlo simulations.