PERFORMANCE IMPROVEMENTS FOR RISK BALANCED PORTFOLIOS

by

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Declaration

The work contained in this thesis is my own work unless otherwise stated.

Signature and date:

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Abstract

This paper focuses on the risk balanced portfolio approach. It comprises a careful consideration of the techniques that can be used for tactical portfolio tilts in order to improve portfolio performance. The emphasis is on usability of these techniques by the majority of investors, regardless of their asset preferences or risk appetite. After an overview of the topic, the macroeconomic regimes model is constructed for portfolio tilts. Then a topic is switched to assets covariances and correlations - a crucial step in portfolio construction. Hidden Markov, regression and ARIMA models are used in an attempt to predict future assets correlation values. The results are quantified and analysed.

Contents

1 Introduction

The dissertation concentrates on the improvements in trading strategies using risk-balanced portfolio techniques. These improvements are mostly implemented by defining and using the concept of regimes in the market and working with assets' covariances.

The aim of our approach is to examine both academical and industrial research papers, where the former is usually more rigorous and theoretical, while the latter is more applied and results driven. After choosing the methods most suitable for our purposes, we test and compare them all using the same metrics. Instead of simply replicating the methods that have been already developed, we combine (see a regression using inflation predictors in Section 4.4), improve (see an example of yield spread indicator in Section 2) and optimise them. Furthermore, note that some of the referenced papers have been written before the financial crisis. We, however, aim to apply all of our models to a different, post-crisis scenario and calibrate them using most recent datasets.

In order to study the common features and back-test our models properly we needed the data to satisfy several properties:

- it should exhibit behaviour typical of three asset classes: equities, commodities and bonds
- the time series should be long enough (should have long history)

Following this reasoning we focused on the US market data on equities', commodities' and bonds' indices. As a data source, we used Bloomberg L.P. private database to which we had access while working on this dissertation. The tickers that have been used can be found in Appendix A.3.

We start by introducing the main definitions and providing a brief overview of the literature on portfolio construction in Section 2. Next, we motivate the risk-balanced approach: construction of a portfolio based on the risks of its constituent parts. We test the risk-balanced approach in our framework, comparing it to other commonly used portfolios. The results agree with ones by Qian [33], showing improvements of the risk-balanced portfolio over most of other more traditional approaches in all the main metrics.

Next, in Section 3 we focus on the main problem: treating the risk-balanced portfolio as a benchmark we examine the literature with existent findings in order to choose and develop the most promising methods to improve portfolio performance. We discuss the ways to define regimes and evidence of their existence in financial markets. As shown by [4] and [3], such evidence comes from the observation of episodes of high volatility and exceptionally high correlations coinciding with bear markets.

For the purpose of regimes detection we focus on the macroeconomic data, surveys data, a number of approaches: Wicksellian approach, yield spread, etc. We focus on macroeconomic regimes to be able to tactically "tilt" portfolio defensively, when there are significant changes in inflationary climate or growth. A second approach is focused on changes in the covariance matrix and is discussed in more details in the next section.

Section 4 focuses on the covariances and variances between asset classes. The covariance matrix construction is an essential step in creation of the risk-balanced portfolio and is a topic worth a paper on its own. However, we briefly overview the progress that has been made in the field of covariance matrices estimation by Ledoit in [24], Kwan in [23], Pourahmadi in [32] and others. After that, we switch to examining rolling correlations' time series and analyse them in an attempt to find factors that have an effect on these time series. For that purpose we analyse the data using QQ-plots and histograms, apply Hidden Markov Models in order to detect structural changes in covariances and fit the distribution parameters, using the Expectation–Maximisation algorithm. According to [21], the equities-bonds correlation is dependent on main macroeconomic variables. We use regression and ARIMA models in an attempt to forecast the correlation value.

We also backtest all of the constructed models and discuss the implementation of our models and the representation of the results. We choose some of the perspective strategies and summarise our suggestions for further research and for interpreting, developing and using these strategies.

We think that this paper will be of interest to institutional investors and asset managers in their research for "all-weather" portfolio allocation, that will diversify risk significantly, while maintaining competitive returns. We believe that such portfolio constructions might be underestimated now, under conditions of the longest bull market in history, but will be fully appreciated in the future.

2 Literature review and motivation of Risk balanced portfolio

In this section, we define terms that are essential for the concept of asset allocation and will be used throughout the paper. We introduce reader to the main ideas of Modern Portfolio Theory and Capital Asset Pricing Model, discuss its problems and limitations. We justify the risk-balanced method that is used as a benchmark for portfolio construction in our paper.

2.1 The development of Modern Portfolio Theory

2.1.1 Definitions and Concepts

In his book, Markowitz [27] mentions that diversification of investments was a common practice and people had an intuitive understanding of correlation between assets for long time before his famous paper was published. However, Markowitz in [26] was the first one to introduce precise tools to analyse portfolio by defining a risk and expected return. He determined the risk of a single asset, total risk of a portfolio and the idea of diversification with the aim to decrease that risk.

We start from defining a concept of return, taken from [31, Part 1, Slide 31]. Let $S(t)$ be the price of an asset on day t. We define the **simple return** on this asset as

$$R(t) := \frac{S(t) - S(t-1)}{S(t-1)}$$

and the **logarithmic return** as

$$r(t) := log(S(t)) - log(S(t-1))$$

Note, that logarithmic returns are approximately equal to simple returns for small values of R_t. The usage of one type or another depends on the context, from now on we will simply refer to **returns**. The MPT model represents a market of N assets by a random vector of returns

$$R = (R_1, ..., R_N)^{'}$$

We refer to the percentage of our wealth invested in asset i as **weight**, denoted by ω_i, for $i = 1, ..., N$. As a budget constraint, we have a requirement that the weights sum up to one: $\sum_{i=1}^{N} \omega_i = 1$. The **return of a portfolio** is then calculated as follows:

$$R_p := \sum_{i=1}^{N} R_i \omega_i = \omega^{'} R$$

where $\omega^{'}$ denotes the transpose of the column vector ω. It is assumed that there are $2N + \frac{N(N-1)}{2}$ known parameters in the model: expectation and standard deviation for every asset return and covariance (or correlation) between any pair of assets. To describe these parameters, we can introduce a **vector of expected returns** of assets m, and a symmetric, positive definite **covariance matrix** $\sum$:

$$m := (\mathbb{E}(R_1), ..., \mathbb{E}(R_N))^{'} = (m_1, ..., m_N)^{'},$$

$$\sum := \begin{pmatrix} \sigma_{11}^2 & \sigma_{12} & \dots & \sigma_{1N} \\ \sigma_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \sigma_{N1} & \dots & \dots & \sigma_{NN}^2 \end{pmatrix}$$

where σ_{ij} represents the covariance between returns of assets i and j, so σ_{ii}^2 is the variance of asset i. Then the **expected return of a portfolio** is simply

$$\mathbb{E}(R_p) := \mathbb{E}(\sum_{i=1}^{N} R_i\omega_i) = \sum_{i=1}^{N} \mathbb{E}(R_i)\omega_i = \sum_{i=1}^{N} m_i\omega_i = \omega^{'} m$$

We would also like to define **variance of a portfolio** here, as stated by Markowitz

$$\text{Var}(R_p) := \sum_{i=1}^{N}\sum_{j=1}^{N} \omega_i\omega_j\sigma_{ij} = \omega^{'} \sum \omega \tag{2.1}$$

In the MPT it is assumed that an investor is seeking to minimise the variance of their portfolio, while maximising the expected return (or keeping it at some level). This leads us to an optimisation problem

$$\begin{aligned} \underset{\omega}{\text{minimize}} \quad & \omega^{'} \sum \omega \\ \text{subject to} \quad & \omega^{'} m \geq \mu_{min}, \\ & \omega^{'} \mathbb{1} = 1. \end{aligned}$$

where by μ we denote the smallest accepted level of expected return and $\mathbb{1}$ is the column vector of size N with all entries equal to 1. Instead of reproducing an explicit solution of this problem, we explain why it can be solved and focus on its implications that are important for our research, referring an interested reader to [7, Sections 4.4.1, 5.5.3]. Note that the problem is convex and quadratic, we can write out the KKT conditions[1]

- $\sum \omega - \lambda m = 0; \quad \lambda \in \mathbb{R}$
- $\lambda \geq 0, \ \ \omega^{'} m \geq \mu_{min}, \ \ \omega^{'} \mathbb{1} = 1;$
- $\lambda(\omega^{'} m - \mu_{min}) = 0.$

Here, KKT are both necessary and sufficient conditions due to convexity and satisfied Slater's condition. There are two different cases, for $\omega^{'} m > \mu_{min}$ we compute $\omega_* = \frac{\sum^{-1} \mathbb{1}}{\mathbb{1}^{'} \sum^{-1} \mathbb{1}}$ that would be a solution if $\omega_*^{'} m \geq \mu_{min}$ is satisfied. Otherwise, we have a case when $\omega^{'} m = \mu_{min}$, solution to a problem then would be a linear combination of ω_* and ω_m, where $\omega_m = \frac{\sum^{-1} m}{\mathbb{1}^{'} \sum^{-1} m}$. Note, that ω_* minimises possible variance among all portfolios, since it is a solution to a following problem

$$\begin{aligned} \underset{\omega}{\text{minimize}} \quad & \omega^{'} \sum \omega \\ \text{subject to} \quad & \omega^{'} \mathbb{1} = 1. \end{aligned}$$

A portfolio with weights solving the problem above is called **global minimum variance portfolio**. ω_m is also minimising variance, but at a different level of expected return. It is referred to as **market portfolio**. Any minimum variance portfolio can be created as a linear combination of two other minimum variance portfolios, so they form a convex set. If a portfolio from this set has

[1]Karush–Kuhn–Tucker conditions, necessary conditions for a constrained local minimum

the expected return greater than the expected return of the global minimum variance portfolio, then the portfolio is an **efficient frontier portfolio**. A set of such portfolios form an **efficient frontier**, shown on Figure 1 below. Portfolios at the right end of the frontier have high expected returns coupled with high variance, while those at the left end have both low risk and low expected returns. An example of the former portfolio would be the one that is mostly invested in government bonds, which is suitable for risk-averse investors. An example of a risky portfolio would be the one containing equities or emerging market indices. Whatever the risk appetite of an investor is, they should never consider a portfolio which is not on the frontier according to Modern Portfolio Theory.

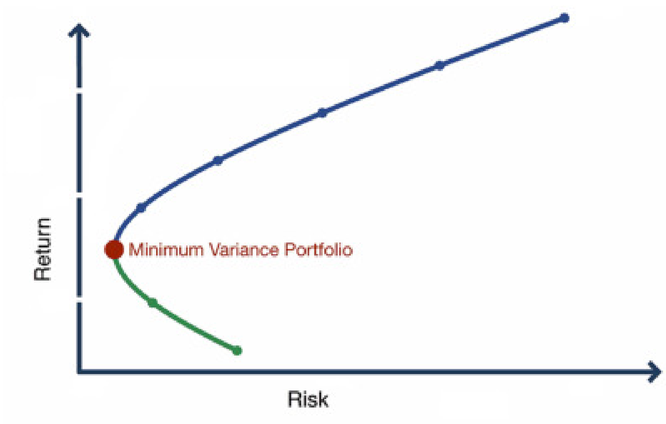

Figure 1: Efficient frontier. Any investor would want to choose the portfolio on the curved line. The red point represents global minimum variance portfolio. The curve includes green part if short-selling is allowed in the model.

Investors achieve that goal by the idea of diversification, holding a portfolio that consists of assets with small (or negative) covariance.

In order to describe investor's preferences, we introduce the **capital allocation line**(**CAL**) which represents return levels when considering a portfolio made of a risky and a hypothetical, risk-free asset. The slope of the capital allocation line represents the trade-off between return and risk. This leads to the following definition:

Definition 2.1 (Sharpe ratio)**.** An expected excess return (equivalently, risk premium) divided by the standard deviation of that excess return is called **Sharpe ratio**, denoted by

$$S_p := \frac{\mathbb{E}(R_p - R_b)}{\sqrt{Var[R_p - R_b]}} = \frac{\mathbb{E}(R_p - R_b)}{\sigma_p},$$

where R_p is a return of the portfolio, R_b is a return of the benchmark portfolio.

This index was described by Sharpe in [36], it is widely used in the industry to examine the performance of an investment. If we consider risky asset as our portfolio and risk-free asset as a benchmark, then Sharpe ratio becomes exactly the slope of the CAL.

2.1.2 Assumptions and Limitations

MPT brings important ideas about diversification and efficiency of a portfolio, however there are a lot of underlying assumptions that complicate its implementation in practice. We enumerate and discuss these assumptions after the following definition. A process is **covariance stationary** if it's mean does not depend on t and its covariance between any two random variables is the same.

Most commonly mentioned and restrictive assumptions of the Modern Portfolio Theory model are:

1. *Investors look only at first and second moments of returns and returns are assumed to be Gaussian.* This is far from reality due to stylised facts: the distribution of daily returns is assymetric and leptokurtic for the majority of assets. The assumption that investor's utility function is only dependent on return and variance has been studied in [16] with an application to utilities of Croatian institutional investors. It has been found in the paper that considering higher moments of portfolio return distribution is significant for actual utilities and the orderings. In [29] this assumption is claimed to be the main theoretic loophole that leads to severe underestimation of investment risk, which agrees with [38] studying Zagreb Stock Exchange. Portfolio structure changes considerably when the assumption is dropped and higher moments should be included in optimisation process. The model also does not consider alternative measures of risk such as Value-At-Risk and Expected Shortfall.

2. *All investors are rational and aim to maximise their utility with a given level of income or money. They are risk averse and their utility functions are symmetric and quadratic.* It means that a gain in utility when a portfolio value increases by 10 percent is equal to the loss when a portfolio decreases by 10 percent. These assumptions are logical and allow us to create simpler models. However, behavioural finance studies show that some of them are not satisfied. There is a good overview in [6] showing that investors are more sensitive to differences in probabilities at higher probability levels. They also discover "certainty effect" arising when people place much more weight on the outcomes that are certain relative to outcomes that are less probable. A phenomenon of loss aversion is justified experimentally in [5] supporting the asymmetry of the utility curve. Asset pricing framework is used to show that people are more sensitive to losses than to gains. This concept is called "narrow framing" and it is claimed in the paper that it could be used to refine the modelling of investor preferences.

3. *The markets are efficient, they absorb the information quickly and perfectly. Investors have full, free and correct information on the returns, their expectation and variance. Thus, all of them*

have access to the same information. In reality, this assumption is never true. In [10] it is shown that large institutional investors can dominate the market and affect prices of the indices and ETFs traded. They also can change the dependence structure of returns. Also, there were a lot of recorded cases of insider trading in history, as discussed in [13], with ongoing financial litigations. To avoid penalties and still be able to benefit from insider information, investors use alternative instruments, such as junk bonds. Furthermore, information spreads out with a certain finite speed and obeys certain laws (e.g. see Hawkes processes). A lot of data sets about production estimates in different countries, transaction volumes, etc. are costly and unavailable for most of the trading agents, but are used by a small number of asset management companies.

4. *Ideas of diversification assume that the true covariance matrix values are known. Returns are covariance stationary stochastic processes.* Both of these assumptions are wrong, we will show later in Section 4 that covariances between equities, bonds and commodities are constantly changing and their moments are changing as well. As we reduce the estimation window for sample covariances, the confidence interval increases and results become imprecise. Apart from this trade-off, the number of covariances to be estimated in the covariance matrix grows quadratically as the number of assets in a portfolio grows. Ledoit and Wolf in [24] state that sample covariance matrix should never be used for portfolio optimisation and propose shrinkage transformations to reduce the error.

5. Further assumptions include *no spread between bidding and asking prices, perfect market liquidity, no influence on market from politicians and no difference between short and long horizon views of investors.*

There are a number of assumptions in any theory and that should not be surprising. However, the implications of the limitations discussed above make it difficult to use MPT and CAPM in practice. In particular, introduction of skewness and kurtosis might lead to conflicting objective functions and non-convex optimisation problems that are not well behaved and are computationally hard.

It is difficult to accurately estimate all the parameters and research in [11] has proved that. Mean-variance portfolio optimisation was used together with various extensions of the model that have been developed in other papers with the purpose of reducing the estimation error. All the models have been tested on historical market data versus the naive $\frac{1}{N}$ portfolio, where the wealth is divided equally between N assets in a portfolio. The results were surprising: none of the models was consistently better than the $\frac{1}{N}$ portfolio in terms of annual rate, Sharpe ratio and other parameters. It was also shown that not less than 3000 months of data is needed for mean-variance model calibration for a portfolio with 25 assets. Otherwise, it will perform worse than the benchmark used. This suggests that a very simple rule can deliver equally good results in the long term and can be used as a benchmark. One more example of such a simple rule is a 60/40 portfolio, where 60% of the wealth is allocated to equities and the remaining 40% is allocated to bonds. This has been a traditional portfolio allocation for years. It assumes negative correlation between equities and bonds and provides solid portfolio performance during periods of economic growth. It is supposedly hedged against downturns due to the bonds component. Even though such allocation has been popular and performed reasonably, we will see later that it is absolutely disproportionate in terms of risk.

With this in mind, we propose another approach for portfolio construction in the next subsection. This approach maintains the benefits of diversification and outperforms many of the common benchmarks, including the ones discussed above.

2.2 Risk balanced portfolio construction

It has been discussed in the previous section that due to a lot of assumptions and the estimation error, it is impossible to target on practice the theoretically best combination of the variance and the portfolio mean. In reality, one often chooses a 60/40 or $1/N$ portfolio as a benchmark or as an investment strategy. The 60/40 portfolio is sometimes referred to as *"balanced"*. In practice, even though it looks fairly balanced, it is absolutely not diversified in terms of risk. We have defined a *variance of a portfolio* in (2.1). A quick calculation discovers that for a 60/40 portfolio the risk contribution from stocks is more than 90%, while contribution from bonds is very small - less than 10%. This phenomena happens mainly due to the difference between the two instruments in their variance. In order to balance risk contribution, one have to try to weight the assets accounting for the assets variances influence. Note, that from equation (2.1), we can deduce the participation of asset i in the portfolio risk

$$\sigma_{ith} := \frac{\sigma^T \sum M_i \sigma}{\sigma^T \sum \sigma}$$

where M_i is a matrix with a single non-zero entry. This entry is a 1 on the $i-th$ place on diagonal.

This leads to an optimisation problem that helps to find a set of weights equating all the risk contributions. Let a portfolio consist of N assets, then

$$\begin{aligned} arg \min_{\omega} \quad & \sum_{\mathrm{i=1}}^{\mathrm{N}} (\sigma_{\mathrm{ith}} - \frac{1}{\mathrm{N}})^2 \\ \text{subject to} \quad & \mathbf{0} \leq \omega \leq \mathbf{1}, \\ & \omega^{'} \mathbb{1} = 1. \end{aligned}$$

where $\mathbf{0}$ is an $n \times 1$ zero vector and $\mathbf{1}$ is an $n \times 1$ vector with all entries equal to one. The solution of this problem is a convex set. There are a lot of theoretical advancements for risk balanced portfolios.

These include closed form solutions, stability considerations and quick numerical algorithms. We refer an interested reader to a book by Roncalli [35], which is fully dedicated to risk balanced portfolios. A less mathematical read is Risk Parity Fundamentals book by Qian [33]. It is more narrative and economically based. Instead of discovering closed-form solutions, we are focusing on the practical aspects. The use of risk balanced portfolio will be motivated by comparing it with other portfolios with the help of a *backtest*. By **backtest** we define a simulation of a trading strategy with the means of historical data. Different strategies (or, in our case portfolios) will generate different results. Instead of concentrating solely on profit levels, we will judge the performance of a strategy be these important and widely used measures:

- **Compound Annual Growth Rate (CAGR)**. This measure is the mean annual growth rate of a portfolio over one year.
- **Sharpe ratio**. This is a very important measure and has been defined in 2.1.
- **Volatility**. A **sample volatility** estimate has to be defined. the formula is

$$\hat{\sigma}^2 := \frac{\sum_{i=1}^{N}(R_i - \bar{R})}{N-1}$$

  for N return observations. This number can be easily annualised.
- **Maximum Drawdown(MDD)**. An *MDD* is a risk measure showing the maximum portfolio loss from a peak to a trough. A peak is defined on a rolling basis (until the next peak is achieved).

In this section we use a long time period for a backtest (from year 1982 to 2017). However, we use slightly different periods later, since we need to reserve some time for an out-of-sample backtest, that is a backtest of a model that has been optimised or chosen in-sample. The four types of portfolio are tested: risk balanced, 60/40, $1/N$ and naive risk balanced. A **naive risk balanced** approach is similar to risk balanced, but all the covariances are not taken into account. Thus, the weight of each asset in a portfolio is defined as its variance over the sum of all variances. We are also using the **exponentially weighted moving average** technique to get the sample covariance estimates. All of the calculations are performed in the R software environment.

As it has been mentioned, we focus on the main asset types indices: bonds, stocks and commodities. Thus, we can ensure that the methods developed are likely to be widely applicable. The portfolio rebalancing procedure is performed such that we change the weight vector equating the risk contribution. The risk contribution changes from day to day due to changing returns and covariance values. This process is rather slow and that is why we perform the procedure monthly. Monthly returns are used in order to decrease the amount of noise. We also use total return index as if all the gains from the investments were re-invested again. An example showing how the weights of commodities, equities and bonds are dynamically changing through time is depicted on Figure 2. The results of a backtest procedure are on Table 1. We see that the RB portfolio is performing similarly to the naive RB, slightly outperforming it due to additional information in the covariance matrix. We also see that the Sharpe ratio of 1.41 is considerably higher that 1.07 and 1.01 values obtained with more classical strategies. We can also see a considerable improvement in both volatility and maximum drawdown level. Even though the 60/40 portfolio has a higher annualised return, this return comes with a considerably big increase in volatility. We can use *leverage* techniques to increase both the return and the variance of an RB portfolio, equating the returns of two portfolios. At that time, the risk-balanced portfolio will have a much smaller variance, which makes the alternatives inefficient. From now on we will use the RB portfolio approach as a default one, or, as a *benchmark*, comparing it to the other techniques that we develop to improve the performance measures even further.

	RB portfolio	60/40 portfolio	1/N portfolio	Naive RB portfolio
Years	1982–2017	1982–2017	1982–2017	1982–2017
CAGR	7.47	9.74	7.63	7.52
Sharpe	1.41	1.07	1.01	1.35
Volatility	5.22	9.1	7.58	5.5
MaxDD	-20.77	-30.57	-34.77	-23.04

Table 1: Backtest results of risk balanced portfolio and alternatives.

We hope we have shown the reader the importance of such approach for an investor, its importance in the light of financial crisis and good performance overall. We will discuss how we can focus on tactical tilts to improve its performance and what methods can be used to estimate covariances - an essential part in construction of a portfolio.

3 Regimes in financial world

There is no clear definition of what a regime in finance and financial markets is, but there is an intuitive interpretation. According to Oxford Dictionary, it is *the conditions under which a scientific or industrial process occurs.* More specifically, if we drop the assumption that there is no seasonality in financial data and that the time series are stationary, we can observe, define and detect periods in time when the data exhibits certain properties. If we can properly predict such periods in time, we can exploit them by changing the portfolio correspondingly.

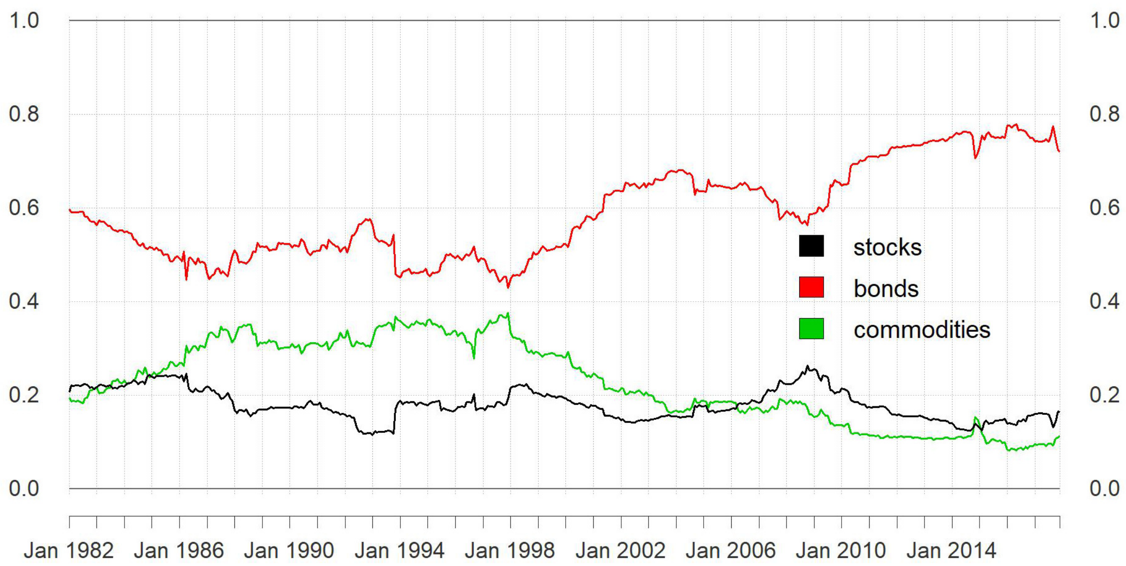

Figure 2: The weights of equities, bonds and commodities in an RB portfolio with monthly rebalancing.

3.1 Overview

3.1.1 Regime-based Markov models

The concept of different regimes in the market is not a recent one. Non-stationarity was discussed in 1989 by Hamilton [19]. He looked at real GNP figures in the US and observed non-stationarity of this time series. He proposed that the mean growth rate can occasionally change from one value to another. These changes or "shifts" are not observed directly. Instead, we can draw some conclusions observing irregularities in the time series changes. A non-linear time series model called *Markov switching model* is used for that purpose. It assumes different states and a switching mechanism between them. The mechanism usually follows an n–order Markov chain. It was shown in the paper that recession and growth states of the economy can be modelled using Markov switching model. The estimates of time periods when the cycle was changing are reported to be close to NBER macroeconomic data.

Hamilton's paper had an influence on the majority of further academic research in the field. A good overview of the models where this approach has been applied is given by Ang and Timmermann. According to [2], there is a potential to study returns, their means and volatilities, autocorrelations and covariance matrices using the regime-based method. It will allow to make better predictions as well as capture most of the stylised facts.

Remark 3.1. Most of the returns on actively traded financial assets demonstrate some statistical properties. These properties are called **stylised facts** [9], some of the main ones are enumerated here:

1. *Returns do not follow Gaussian distribution*
2. *Return distributions are heavy-tailed, slightly asymmetric*
3. *Returns are serially uncorrelated*
4. *Absolute returns are serially correlated*
5. *Volatility is clustered and highly persistent*
6. *Returns over longer periods exhibit aggregational Gaussianity*
7. *Extreme returns often coincide across multiple assets*

Apart from studying prediction of regimes, Ang and Timmermann are trying to understand what causes the regimes and suggest changes in economic policy and investor expectation as possible answers.

Research papers differ a lot in terms of practical applications as well. In [1] authors show the relationship between the business cycles and interest rates regimes, they apply Markov models to the interest rate data and compare the estimation results with one-regime benchmarks. They also mention complexities of using this approach and possible improvements. Including term spread and other additional information into the model helps to improve performance significantly. In another research [15], similar Markov switching model is applied to ex-post real rates and inflation. Apart from that, researchers conduct the sensitivity analysis and work on detecting the suitable number of regimes.

These and other academical papers improve the theoretical foundation of the topic discussed. Such developments are important as they provide closed form solutions for a lot of practical problems, improve the stability of results and lead to the development of more sophisticated models that can describe phenomena more precisely. However, we have seen a lot of industry papers and cannot underestimate their value. These papers are mostly focused on delivering practical results and describe models that could be used without delay. Following this reasoning, we continue our review focusing on these research papers and choosing the promising results to be critically assessed, improved and backtested. We acknowledge that the research topics fall mainly in one of the two categories in industry papers that we found. We discuss both of them below with a focus on macroeconomic regimes.

3.1.2 Market regimes

According to the stylised facts mentioned above, volatility levels of most asset types share some common properties. Particularly high or low levels of volatility tend to persist for a long time. This persistence or long memory property can be observed with the help of the *sample autocorrelation function (ACF)*.

Definition 3.2 (Autocorrelation function)**.** Under the assumption of stationarity, an **autocorrelation function (ACF)** of a time series $(X_t)_{t\in\mathbb{Z}}$ is a time series

$$\rho(k) := Cor[X_t, X_{t-k}], k = 0, 1, ... \tag{3.1}$$

where k is the *lag* of the time series $(X_t)_{t\in\mathbb{Z}}$. It is used to measure serial dependence.

Definition 3.3 (Sample autocorrelation function)**.** Given the observations of time series $(X_t)_{t\in\mathbb{Z}}$, we can calculate **sample autocorrelation function** as

$$\hat{\rho}(k) := \frac{\sum_{i=k+1}^{N}(x_i - \bar{x})(x_{i-k} - \bar{x})}{\sum_{i=1}^{N}(x_i - \bar{x})^2}, k = 0, 1, ..., N-1 \tag{3.2}$$

where $x_1, .., x_N$ are N observations, $\bar{x} := \frac{\sum_{i=1}^{N} x_i}{N}$ is their *sample mean.*

We want to demonstrate the volatility persistence with the help of S&P500 index. Using daily closing prices to calculate returns, we plot them (Figure 3) and plot the sample ACF of squared returns(Figure 4).

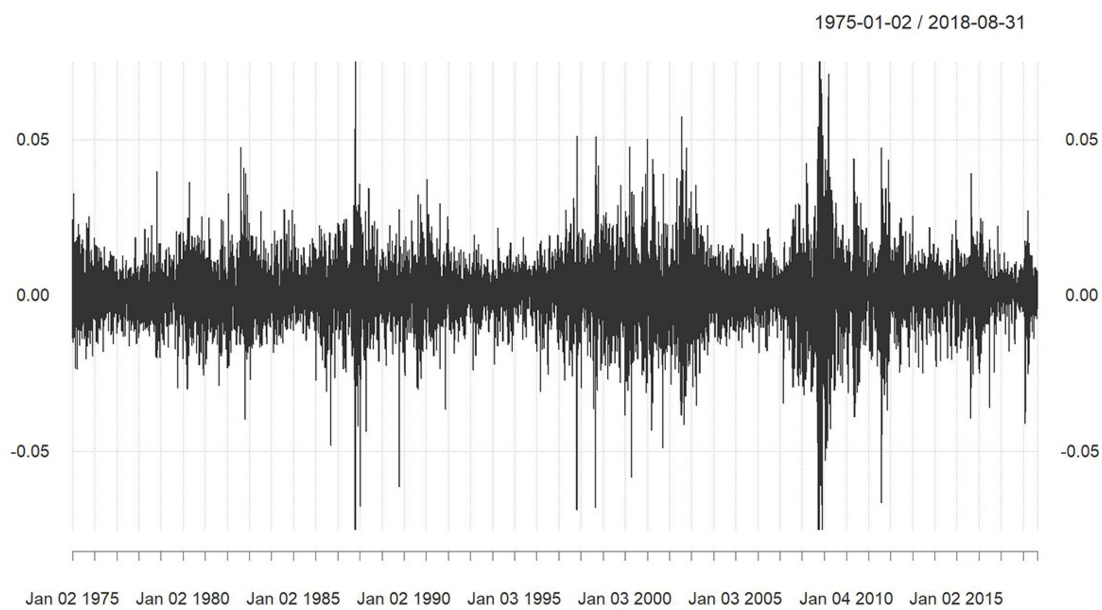

Figure 3: Daily returns of the S&P500 index, years 1975-2018.

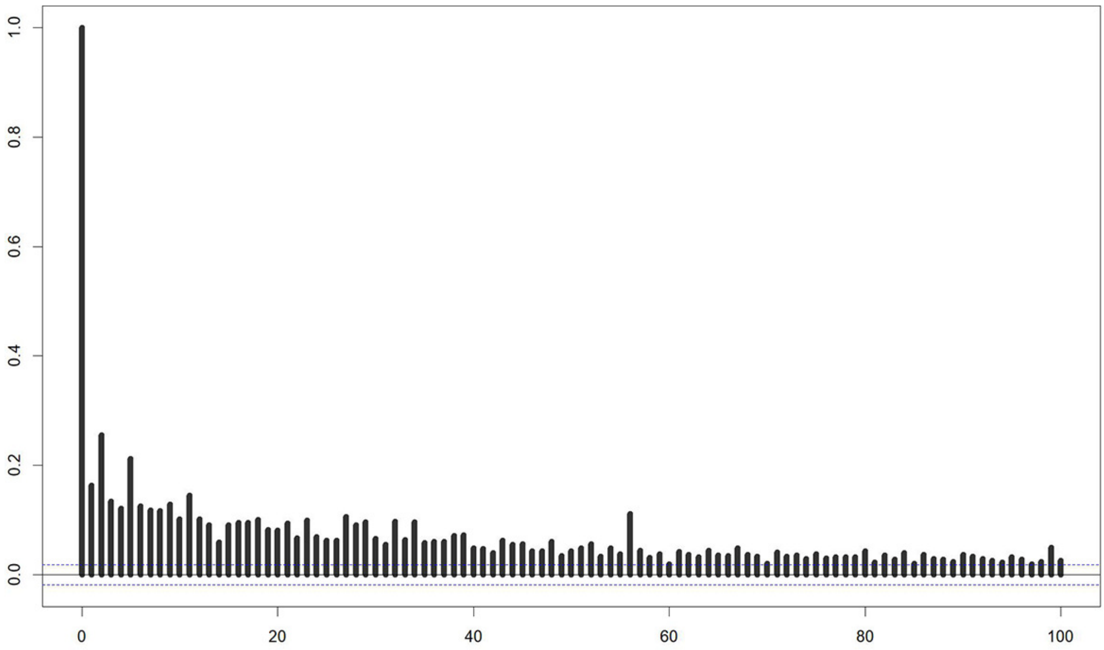

Figure 4: Sample autocorrelation function of squared returns of the S&P500 index, years 1975-2018. X-axis represents the number of lags, k.

We can observe volatility persistence on the plot of returns: high values and low values of returns are concentrated in particular places. Variance is a second moment, estimating the average of values deviations from the mean. Thus, by observing the sample ACF function of squared returns plotted we can make judgements about the volatility persistence. We see that for all of the lags $k = 1, 2, ..., 100$ the serial dependence is quite significant and diminishes very slowly (blue dotted lines represent 95% confidence interval).

The GJR-GARCH process, introduced by Glosten, Jagannathan and Runklein in 1993 [18], can be effectively used as a model for volatility. It comprises the famous GARCH model characteristics and explains the *leverage effect* introducing asymmetry in the model. This is done by including the categorical variable which is 1 when the previous return is negative and 0 otherwise. In effect, this model is looking at two different regimes defined from the observed market data.

Another popular approach that analyses market data to detect the current state is discussed in [4] and [3]. These research papers are concentrating on "Risk On" and "Risk Off" regimes. The former is characterised by a relatively stable environment in the market, usually associated with high economic growth rate. In this regime investors feel optimistic and prefer risky assets. Conversely, in a "Risk Off" regime the environment is unstable, there is a lot of uncertainty and increased volatility in the market. Investors prefer safer asset types, such as government bonds, cash and several stable currencies. Hidden Markov Models are proposed as a potential model to detect such regimes. We will explain them in more details later, in Section 4. Further proposed variables that will help to detect the regimes include inflation, economic growth and financial market turbulence. We will discuss the first two in details in Section 3.2. To understand the concept of financial market turbulence, we will require the following definition.

Definition 3.4 (Mahalanobis distance)**.** First introduced in [25], the **Mahalanobis distance** Δ is a measure between two populations. Given multivariate distribution and constant covariance matrix $\sum$, it is equal to

$$\Delta := \sqrt{(\mathbf{x}-\mathbf{y})\sum^{-1}(\mathbf{x}-\mathbf{y})'},$$

where $\mathbf{x}$ and $\mathbf{y}$ are vectors from the multivariate distribution.

In [8] Chow et al define **financial market turbulence** as squared *Mahalanobis distance* of the return time series, referring to periods in the market when volatility and correlation are particularly high. One can construct such a turbulent index and use fixed threshold level or optimise for the value of this level to differentiate turbulence periods from other times. This regime market model has been looked at in [14], [4] and [22]. Historical turbulence index has been able to detect most of the anomalistic events, crises and bubbles. Two separate covariance matrices can be estimated separately for high and low values of turbulence. The former matrix will then be used for portfolio rebalancing whenever the current market period is considered as turbulent.

We would like to note the idea of detecting different market regimes and discuss how this can be used in the context of macroeconomic changes. Next, in Section 3.2 we will discuss business cycles, inflation and economic growth. We will develop methods to define regimes within this framework and portfolio strategies relying on them. We will also check our strategies in practice.

3.2 Macroeconomic regimes

The **business** or **economic cycle** is usually defined as the movement of the *gross domestic product* in time. These movements are cyclical and in some periods are above the long-term trend, while in others - below it. A graphical representation of an economic cycle is given on Figure 5.

An economic **recovery** happens when the actual GDP growth rate is higher than the trend, economy is expanding and people's expectations are positive. As economic cycles are usually synchronised closely to capital market cycles, the recovery and economic boom are usually associated with the bull market and the "Risk On" regime that was discussed earlier. The mean growth rate of equities is usually high and the investors sustain it by using momentum trading strategies. As the peak level of actual growth is achieved, the inflation usually increases and weakens portfolios consisting of both bonds and equities. After that a period of **recession** follows which is often asso-

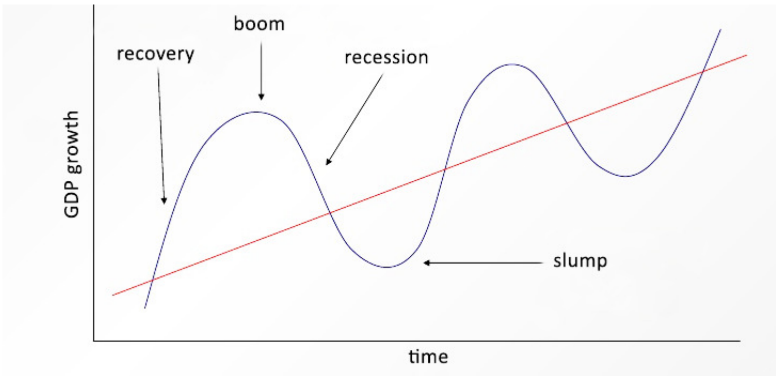

Figure 5: Economic cycle representation. The red line corresponds to long-term or trend GDP growth, the blue line corresponds to actual fluctuations of GDP.

ciated with bear market, increased volatility and the "Risk Off" regime. The majority of equities show a very low or negative rate of return during that period, a lot of investors switch to safer government bonds and cash, such as the Japanese yen.

The implication of these cycles on the portfolio cannot be underestimated. As we have seen in the previous paragraph, it is impossible to construct a portfolio that would benefit from every period of a cycle. Thus, we need to slightly change the portfolio constituents when the new period starts. These ideas were studied by Nawrocki and Carter in [30]. The authors note the difficulty of predicting economic cycles and its effect on equities and other asset classes. Looking at S&P 500 indices, they apply common statistical rules used by NBER[2] to define 4 periods in a cycle. After that, the best performing industry is found for every period. The results show that the underlying market structure is not stationary. The returns and volatilities are different under different cycle phases, as expected.

There are no clear and strict definitions of business cycle phases among researchers. Furthermore, it is difficult to predict when there will be a transition to the next cycle period due to a high number of parameters that affect the economy and market. These parameters are studied in books [28] and [20]. The key ones that can reflect the current state of an economy are the inflation rate and the economic growth rate. Close relationship between the inflation rate and a business cycle is explained in [28, Chapter 11, p.171]. The additional motivation to look at these variables in more detail comes from the nature of construction of a risk balanced portfolio. The portfolio consists of equities, commodities and bonds. The equities are expected to increase the overall performance during periods of high economic growth, while the bonds sustain the performance when the growth is low (this could be a period of recession). The commodities part of a portfolio

[2] the National Bureau of Economic Research

is more affected by inflation levels and hedges the portfolio against rising inflation, as discussed in Section 2.2. The idea then would be to increase the amount of risk allocated to equities when we are sure of their performance (high economic growth) and increase the amount of risk allocated to bonds when there is a recession period. We can do similar changes to the commodities part of the portfolio looking at the changes in inflation. Thus, if we accurately define economic growth and inflation and it would be possible to predict their current levels correctly, then our portfolio would benefit from most of the periods in a business cycle.

Our idea would be to partition the time into periods of particularly low and particularly high levels of inflation and economic growth. We would refer to these periods as different **macroeconomic regimes**. Note, that we look at inflation and growth as at two different, independent variables. In the long term they are the two main return drivers for asset classes and their combination gives us information about the environment for investment and defines which constituents of our portfolio can potentially outperform others. The idea has appeared in many industrial research papers, an example is [37], where the four macroeconomic regimes are defined and it is empirically shown that no single asset class dominates others in all the regimes. Both [14] and the previously mentioned paper [4] look at inflation and growth separately. One paper uses moving averages and another one uses Markov models to differentiate between high/low periods. The approach in paper [14] differs. Instead of using common variables to define and predict recessions and expansions, they use defined periods of recessions and expansions to calculate and compare values of inflation and yield spread in each period. On the contrary, we would like to use macroeconomic variables to detect a regime and change our portfolio accordingly. We introduce the following definition.

Definition 3.5 (Tactical asset allocation)**. Tactical asset allocation** (**TAA**) strategy is an active management portfolio strategy that shifts the percentage of wealth allocated to assets with the aim of taking advantage of changing market conditions.

We refer to the process of shifting the percentage of wealth allocated as **tactical portfolio tilt**. Based on the variables that will be introduced, we would like to create a rule that would tilt a portfolio accordingly.

When speaking about economic cycles, we make an assumption that inflation and growth variables are changing in time accordingly. It can be stated that these time series have similar patterns repeating with a certain frequency and this should be taken into account.

Definition 3.6 (Seasonality)**.** Time series is said to exhibit **seasonal behaviour** if it experiences regular and predictable changes in a precise amount of time, e.g. yearly or monthly. Time series is said to exhibit **cyclical behaviour** if it experiences regular and predictable changes with some periodicity that can vary.

To account for the fact that our data is cyclical we would require to know the length of a cycle. We can use this as the window length when calculating historical mean and other estimates. Thus, we will reduce seasonality risk and capture the long term trend of the data. According to NBER data on US business cycle expansions and contractions[3], there has been 11 cycles in years 1945-2009 and the average length of a cycle, calculated from peak to the following peak is 68.5 months. Thus, we can use a period of 69 months to calculate the historical average means and compare them with the latest observations of our variables.

There are different ways to calculate inflation which is commonly defined as annual percentage change in the price level. In many countries, including the US, a consumer price index (CPI) is often used. Similarly, there are GDP indices to aggregate the economic growth figures. As we are looking at the rate of change of these variables, we would be interested in **year over year(YOY)** method that compares absolute figures with the figures obtained a year ago. Without further delay, we proceed to Section 3.2.1 to describe the variables considered, models constructed and the results that were obtained.

3.2.1 Implementation and results

We will start explaining the construction of models by defining the variables that we used as inflation and economic growth predictors. As we concentrate on US market, the first variable that can be used is the CPI Index itself. We can use the most recent observations to analyse the current inflation level. However, it might me a good idea to consider predictions of an inflation rate for several reasons. First reason is that the inflation rate measures the change in prices over a long period of time that could be a quarter or one year. Together with the concept of *price stickiness*, this makes the estimation backward looking. The second reason is concerned with the lag of a variable and will be discussed later. For our purpose, we would like to have a forward looking measure that would allow to tilt portfolio accordingly before the changes take place. This leads us to considering indices that help to predict inflation. Among many, popular measures are:

- **Consumer confidence inflation rate expectation**. This indicator is provided by the Conference Board - a global, independent, non-profit research association. The indicator tracks a sentiment among households/consumers and is based on a survey. The target audience of a survey is a random sample of 3,000 US households.
- **Survey of Consumers**. The survey is conducted by the University of Michigan. Similarly to the previous one, the indicator tracks a sentiment among households/consumers. The median value is reported.

[3]data taken from http://www.nber.org/cycles.html

From now on we will refer to the three inflation indicators described above briefly as to "**CPI**", "**Conference Board**" and "**Michigan**" inflation indicators respectively. The data for all three indicators is reported monthly and starts from year 1991 as recorded in the database to which we had access. In addition, we would also like to mention two more additional indicators that are market based. The first one is the **one year ahead inflation swap rate** and the second is the **5-year, 5-year inflation forward rate**. It shows the expected annualised rate of inflation from year 5 to year 10. This data does not contain inflation figures themselves, but the changes in the values of these derivatives can be a good indication of inflation expectation changes. From now on we will refer to these indicators briefly as to "**Swap**" and "**Forward**" indicators respectively. The benefit of this data is its daily frequency. The drawback is that the data records start in years 2005 and 2004 respectively. These derivatives have only been actively traded in the last decade and earlier records are not available.

Proceeding similarly, we describe now the indicators that are used for economic growth prediction:

- **Gross domestic product** (**GDP**). A commonly used measure, it represents the market value of all goods and services produced in a year. The CPI indicator drawbacks discussed above are also applicable to this indicator. In addition, it is only reported quarterly. We use the US "GDP CURY" Index that is adjusted for inflation.
- **US Treasury yield spread**. A yield spread is the difference between yields on differing bonds of varying maturities. This indicator is the difference between a 10-year treasury bond yield and a 3-months treasury bond yield. It is a good approximation of the yield curve shape and, if interpreted properly, can be a good sign of recession or recovery in the economy. The information on yields is reported daily, thus, this indicator has daily frequency.
- **The Conference Board Leading Economic Index** (**LEI**). It is a composite average of ten components including *average weekly hours*, *unemployment figures*, *consumer expectations for business conditions* etc. Designed by the Conference Board, this index is supposed to predict booms and slumps in the business cycle in a more convincing manner than any individual component.
- **The Institute for Supply Management's Manufacturing Purchasing Managers Index** (**PMI**). The indicator relies on data compiled from purchasing and supply executives in the US and is based on a survey. It is calculated from the results of surveys on inventories, production etc.

The last two indicators are reported monthly and the time series have long history of records. From now on we will refer to these indicators briefly as to "**GDP**", "**Spread**", "**LEI**" and "**PMI**" growth indicators respectively.

All of the above quoted indicators are used as estimators of either inflation or economic growth. Thus, we would expect these indicators to correlate with one another. For an initial data exploration and future reference we report the correlation figures in Table 2. The inflation calculations are based on monthly frequences for the last 12 years and the results are presented in form of a correlation matrix in Table 2. For economic growth quarterly frequency and 53 years of data was used, the results are depicted in Table 3. We did not include the *Yield Spread* in the table, since this indicator is not directly representing the change in economic growth. We will comment on its meaning later.

indicator	CPI	Conf. Board	Michigan	Swap	Forward
CPI	1	-	-	-	-
Conf. Board	0.08	1	-	-	-
Michigan	0.51	-0.04	1	-	-
Swap	0.59	-0.02	0.25	1	-
Forward	0.22	0.43	0.60	0.04	1

Table 2: Sample correlation values of inflation indicators, years 2006–2017.

indicator	GDP	LEI	PMI
GDP	1	-	-
LEI	0.04	1	-
PMI	0.37	0.42	1

Table 3: Sample correlation values of growth indicators, years 1965–2017.

Definition 3.7 (Data smoothing)**.** The process of removing the noise from a data set, e.g time series, is called **smoothing**. It can be done in many ways, the most common ones are exponential smoothing and moving average.

It could be concluded that Michigan inflation indicator is highly correlated with CPI, Swap and Forward indicators, while CPI is less correlated with Forward indicator. Thus, Michigan indicator could be incorporating both market expectations and current values of inflation. However, we cannot conclude which indicator is more suitable because we are trying to capture the instantaneous inflation rate or the rate that would be in the market in the next month, once we rebalance and tilt our portfolio. Such rate is an unknown quantity and thus we cannot detect how successful the indicators are in predicting inflation. Similarly, PMI indicator is correlated with both GDP and

LEI indicator with coefficient around 0.40.

It is important to note, that the time series that we used is likely to be distorted in some way by the presence of noise. In order to reduce the noise we apply *moving average smoothing* techniques. We note, that the correlation between LEI and GDP indicators increases from 0.03 to 0.20 as we apply 6 months' moving average to the LEI indicator. This is done by changing each value of the time series to the arithmetic mean of its last 6 values:

$$y_t := \frac{\sum_{i=t-5}^{t} x_i}{6}. \tag{3.3}$$

The correlation coefficient between CPI and SWAP indicators increases from 0.59 to 0.79 as we apply 9 months' moving average to the SWAP indicator. After applying the moving average, each value will be affected by previous observations, which the main drawback of this technique.

Whenever the data is used as an input for a trading strategy, it is important to take into account the time constraints. We would like to define the **reporting lag** of a time series as the period of time that passes from the moment to which the observation relates to the moment when this observation is released. For instance, the CPI inflation figures for January are only released in the middle of February. If we decide to make decisions and rebalance the portfolio on the first day of each month, this indicator will be lagged for two months. We take that into account and correct CPI figures accordingly. Similarly, looking at the reporting dates, we have to impose a lag of 2 months and 1 month for LEI and PMI indicators respectively. The Michigan and Confidence Board indicators are not lagged, since they are released on the last day of each month and we can use these values on the next day. Swap and Forward indicators have daily frequency and, thus, not lagged as well. On the contrary, the GDP figures are only released 3 months after the quarter described. The figures are then revised and corrected twice: in 4 and 5 months time after the period referred.

The release lags are likely to affect the results in a negative way. This is a constraint that one has to impose when testing the strategy. Apart from incorporating these lags into our model, we would also want to test it without the lags. This setup will not be realistic, however, it allows to see the fundamental importance and usefulness of model inputs. The comparison then helps to understand whether the lags imposed change the performance significantly.

As discussed in Section 2, the portfolio is concentrated on US market and consists of diversified equities, bonds and commodities indices. The indices represent total return, so that all cash distributions are considered to be reinvested back into the index. The portfolio is rebalanced monthly, allowing us to follow the risk balanced approach while using monthly indicators. Converting the price data to monthly returns also reduces the noise compared to weekly or daily returns data. The backtesting period is chosen according to the available history of indicators used. We reserve a time period of five years for an out-of-sample backtest that will provide a way to test the assertions about the models that we choose.

Now we would like to explain the framework for the portfolio tactical tilts. We have already discussed the independence of economic growth and inflation. The combination of two can be represented graphically as follows: let economic growth be the x-axis and inflation be the y-axis. Then, we will get a well-known **four quadrant model**. Instead of concentrating on the nominal values of variables, we can imagine the upper half-plane as the one where inflation is relatively high, while the bottom half-plane as the one with relatively low inflation. We proceed similarly with economic growth. In order to quantify what *relatively high* means, we introduce the following definition.

Definition 3.8 (Z-score)**.** Let $x_1, x_2, ..., x_N$ be a set of observations of a random variable X with mean μ and variance σ^2. We define the **z-score** of an observation x_i to be

$$z_i := \frac{x_i - \mu}{\sigma}$$

If the population mean μ and variance σ^2 are not known, the sample estimates are used instead. As has been discussed above, a window of 69 previous observations in the monthly time series can be taken. This is the average length of one cycle and this period can be used to find the sample mean $\hat{\mu}$ and the sample variance $\hat{\sigma}$, avoiding seasonality risk. If this transformation is applied to every value of time series x_t, the formula becomes

$$z_t := \frac{x_t - \hat{\mu}_{t-1}}{\hat{\sigma}_{t-1}}, \tag{3.4}$$

where $\hat{\mu}_{t-1}$ is the sample rolling mean and $\hat{\sigma}_{t-1}$ is the sample rolling variance.

The result of such a transformation can be seen on Figure 6, where the three monthly inflation time series (*CPI*, *Conf. Board* and *Michigan*) were transformed using the method described above. The two horizontal red lines represent the threshold levels of ± 1 sample rolling standard deviation $\hat{\sigma}_t$. This is a parameter in the model and can be changed. However, such a deviation is considered to be a significant change and is widely used in the industry.

From now on it can be defined when the period is inflationary and when the economic growth is relatively high in the quadrant model. A graphical representation of this setup is depicted on Figure 7. In literature this model is often explained to have only four distinct regimes (corresponding to four quadrants). This makes the model intuitively simple, but does rarely work in

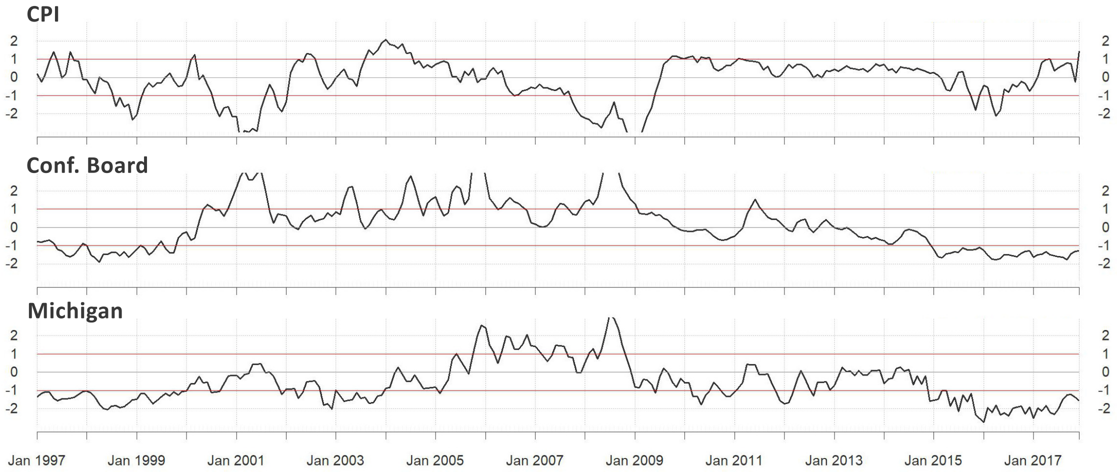

Figure 6: An example of z-score graphs (3 inflation indicators).

practice. We note the presence of noise in any of all the macroeconomic estimators used. Thus, we only consider the signal to be present if the deviation of an indicator is significant from its long term trend. In other words, the period is considered to have high economic growth if the z-score transformed time series of an economic growth estimator shows value $z_t \geq 1$ at time t. On the contrary, the period is considered to have low economic growth if the value is $z_t \leq -1$. These rules allow to turn a z-score transformed time series into a discrete signal. The range of this signal is a set $\Omega = \{-1, 0, +1\}$. Applied to the inflation time series, it represents a neutral regime and a low/high inflation regime. The rule is similar for the economic growth time series as well. An example of such transformation can be seen on Figure 8.

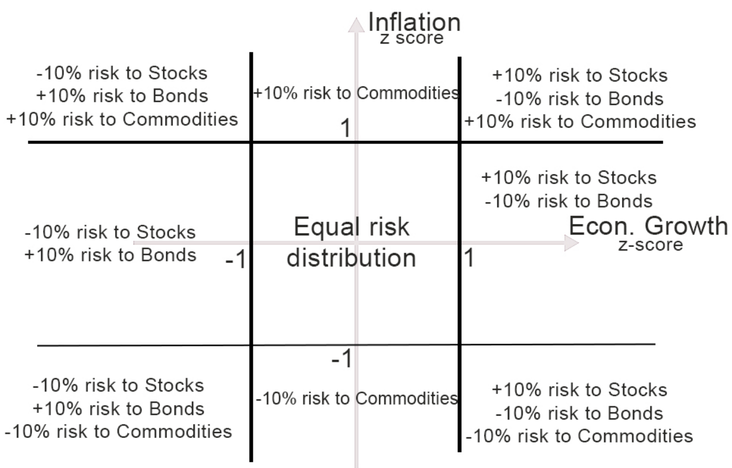

Figure 7: The inflation-growth quadrants representing 9 regimes.

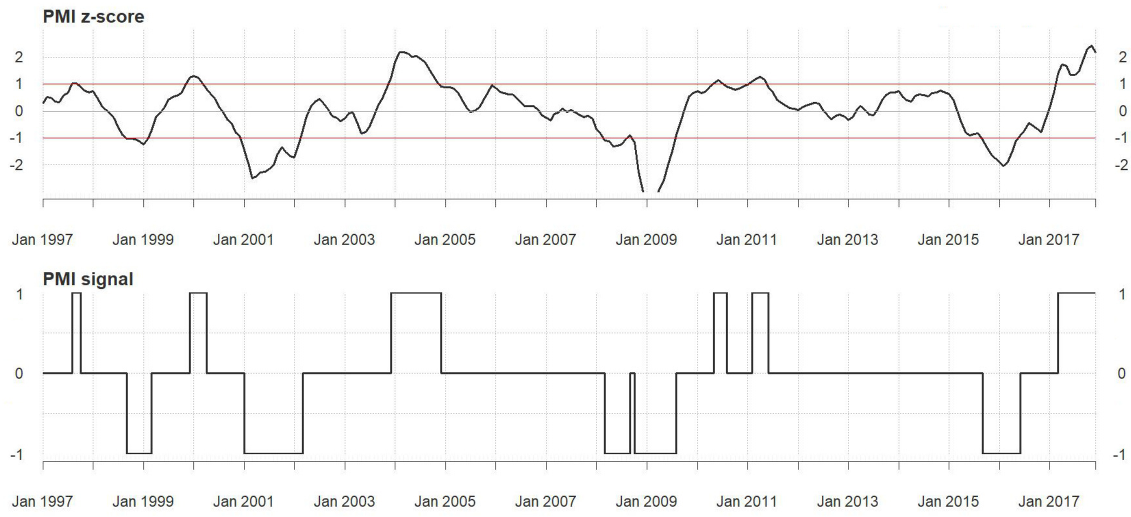

Figure 8: An example of a z-score transfered to a signal.

Applying the same rule to the y-axis (representing inflation), we get the model with 9 different situations or regimes. For each of these regimes a vector of weights ω has to be defined. Instead of creating a rule for ω, one can create a rule for the risk vector r and use the optimisation framework to turn one into another. We leave the risk allocation to each of the asset classes as $\frac{1}{3}$ if the regime is neutral. However, we tilt the portfolio by allocating more risk to the commodities and less risk to other asset types if there is inflationary pressure. The reason for that is inflation being the main return driver for commodities. Similarly, we slightly decrease the commodities risk if the regime shows low inflation. We proceed similarly with bonds and equities using the economic growth reasoning. An intuitive and visual explanation of these tilts is given on Figure 7. The exact values of the risk vector r for each of 9 regimes can be found via optimisation once an objective function is found (for example, it could be maximisation of P&L or Sharpe ratio). However, since for every change in the vector in any of the 9 regimes the behaviour of portfolio changes, the number of parameters to be optimised is high. Such optimisation is both computationally difficult and it introduces the risk of overfitting. In this paper we concentrate on fundamental usefulness of the strategies introduced and their relative performance. Thus, we re-allocate approximately 10% of risk to/from each commodity that is supposed to be affected in each of the regimes. We also perform further backtests in which we increase the magnitude of tilts to test the stability of models performance and observe the magnitude and direction of parameters changes.

The combination of inflation and growth signals defines a specific regime. All such combinations are represented in Table 4. The *Swap* and the *Forward* inflation indicators have limited usability because the data does not have history long enough to properly backtest them. However, these indicators should contain market information regarding inflation expectations and we would encourage their usage in the future.

Model №1	CPI + GDP
Model №2	Conf. Board + GDP
Model №3	Michigan + GDP
Model №4	CPI + LEI
Model №5	Conf. Board + LEI
Model №6	Michigan + LEI
Model №7	CPI + PMI
Model №8	Conf. Board + PMI
Model №9	Michigan + PMI
Model №10	CPI + Spread
Model №11	Conf. Board + Spread
Model №12	Michigan + Spread

Table 4: The 12 models constructed by combining various inflation and economic growth signals.

The yield spread indicator is transformed to a signal in a slightly different way to others. After transforming the daily time series to the monthly one and applying the z-score transformation, we introduce the following asymmetric model: if the z-score shows value $z_t \geq 1$ at time t, there is strong evidence of economic growth expectations. If, on the contrary, the z-score shows value $z_t \leq -1$ at time t, the regime is switched to neutral. Finally, the last rule that is applied examines the spread itself. If the spread value is negative, the negative signal is switched on. We have introduced this model to account for the fact that the shape of the yield curve can be changing and flattening within one of the economic cycle periods. However, the inverted shape of the curve is a strong indicator of instability and recession.

Introducing the rules for a tactical tilt of a portfolio affects the monthly rebalancing procedure. An example of weight distribution can be seen on Figure 9. The graph above represents the weights distribution of asset classes for a risk balanced portfolio without tilts. The graph below represents the weights distribution for Model №9, where the risk is re-distributed due to inflation(*Michigan*) and growth(*PMI*) signals. The results of the backtest of all the models are presented in Table 5 and in Appendix A.1. On the first one the P&L graph is presented and one can see the performance of a portfolio under different models through the whole period used for the backtest (what years?). On the second one can see all the relevant parameters that were defined in Section 2.2. The results show no significant difference between most of the models in terms of Sharpe ratio, Maximum Drawdown and other parameters. An exception might be the models №3, №8 and №9.

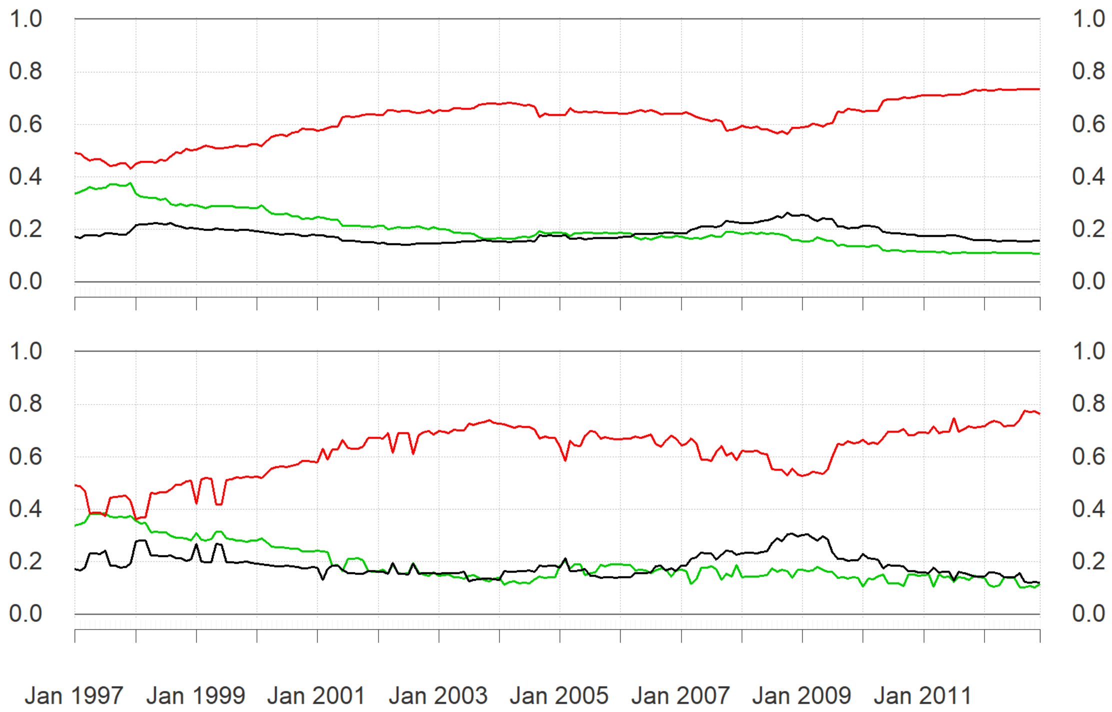

Figure 9: A comparison of the weights of equities, bonds and commodities in an RB portfolio with monthly rebalancing. Tilted by signals from Model 6 (below) and not tilted(above).

These models performed slightly better and we have chosen them for an out-of-sample backtest.

	Model №1	Model №2	Model №3	Model №4	Model №5	Model №6
Years	1997-2012	1997-2012	1997-2012	1997-2012	1997-2012	1997-2012
CAGR	4.69	4.97	4.99	4.72	4.85	4.86
Sharpe	0.84	0.89	0.92	0.82	0.84	0.87
Volatility	5.66	5.60	5.44	5.83	5.80	5.64
MaxDD	-22.71	-20.94	-18.79	-24.52	-23.42	-21.8

	Model №7	Model №8	Model №9	Model №10	Model №11	Model №12
Years	1997-2012	1997-2012	1997-2012	1997-2012	1997-2012	1997-2012
CAGR	4.87	5.11	5.13	4.81	4.94	4.94
Sharpe	0.85	0.90	0.93	0.81	0.83	0.85
Volatility	5.76	5.7	5.52	6.02	5.99	5.84
MaxDD	-23.38	-21.73	-19.64	-24.58	-23.48	-22.06

Table 5: Performance results of the 12 models constructed by combining various inflation and economic growth signals.

It has been mentioned briefly that we have also conducted the same backtesting procedure increasing the risk percentage used for portfolio tilts. The motivation is to observe whether the promising models perform consistently better or deviate to the other side. That could suggest some thoughts on the stability of these models. Furthermore, we perform one more backtest for our models without any lags imposed on the indicators. This might be useful to see the significance of the imposed lag impact on performance. All these results can be found in Appendix A.1.

3.3 Wicksellian approach

The Swedish economist Knut Wicksell introduced in his book the idea of two interest rates at any point in time: the natural rate of interest and the prevailing rate in the market [39, Chapter 8, p.102]. In this section the theory is inspected in details and used to create a model that will produce inflation and economic growth signals. We partially follow the survey conducted by Gave in [17] when detecting various macroeconomic periods using the Wicksellian approach. Wicksell argued that the natural rate is unobservable and can be defined as the return on the capital. The market rate is observable, it is the rate at which the economy grows. The important idea is that these two rates should coincide in the long term. Thus, if there is a deviation between two rates, monetary consequences in the shape of changes in prices take place. If the money (or market) rate is above, natural rate prices would rise. If market rate is below, they would fall.

We approximate cost of money or the return on capital by the Baa bond yield index. We want to subtract from it a general, long trend inflation value. For that we take the year over year CPI inflation rate and smooth it for 10 years. Now we need to compare this natural rate with the economy growth rate or GDP. For that we calculate the spread (see Figure 10)

$$Wick.Spread := BondYield - GDP.$$

If this spread is negative, according to Wicksellian's theory an inflationary pressure is expected in order to bring the spread value back to zero. A positive value of spread means a low-inflation signal. For the economic growth signal we use the idea from [17]. Looking at the same spread, we choose a threshold value h, e.g. $h = 0.025$. This is a parameter in the model. If the value of a spread crosses the threshold h, then an economy has to stop and recession is inevitable - a signal of low growth. Similarly, if the spread value drops below $-h$, then this is a sign of a forthcoming recovery and a signal of high growth. Otherwise, if the value of a spread lies inside the interval $(-h, h)$, we cannot expect any change of the economy (neutral regime).

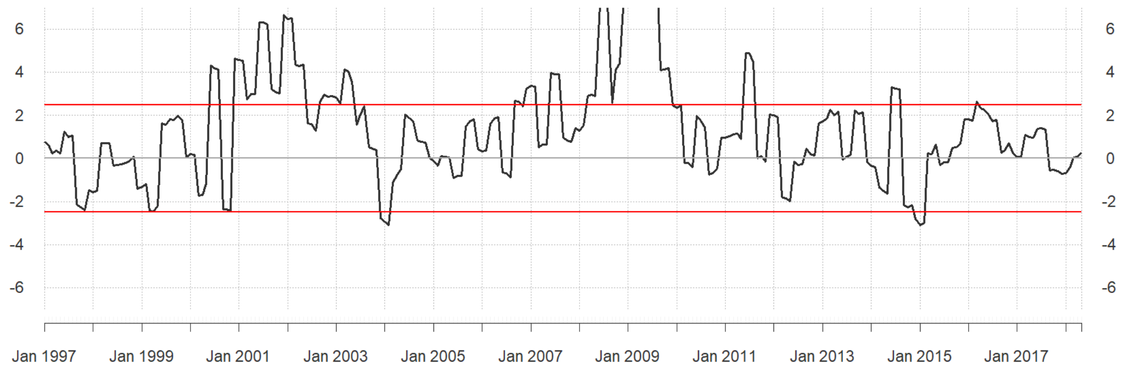

Figure 10: The Wicksellian spread. Red lines represent threshold values of ±2.5%.

3.3.1 Implementation and results

We proceed in a very similar way to setup described in Section 3.2.1. The portfolio consists of the same instruments and is rebalanced monthly. We take into account the release lag of the Wicksellian spread and perform a backtest both with and without such lags. The risk weights are also tilted more for additional backtest. The results are presented in Table 6. They show a remarkable improvement in most of the metrics for a normal lagged indicator, compared to the benchmark. An attempt to see the separate contributions of growth and inflation signals has been made. A remarkable conclusion is that there is a considerable effect from inflation, while the economic growth showed worse results. However, their combination shows the best portfolio improvement. These results are also presented in Table 6.

	RB portfolio	lagged	not lagged	heavy	growth only	inflation only
Years	1997–2012	1997–2012	1997–2012	1997–2012	1997–2012	1997–2012
CAGR	4.90	5.11	4.94	4.73	4.65	5.08
Sharpe	0.89	0.95	0.94	0.94	0.91	0.94
Volatility	5.51	5.40	5.25	5.01	0.86	5.43
MaxDD	-20.77	-18.54	-18.59	-11.71	-16.23	-16.1

Table 6: Performance results of the models based on Wicksellian spread. An RB portfolio(benchmark) is included for comparison.

3.4 Out-of-sample results

We have chosen the three best performing models from 12 initial models. We have also added a Wicksellian spread as a promising indicator. We test these strategies together with the risk balanced benchmark in order to see how they perform in another, out-of-sample period of time.

This also gives more confidence that the models are not overfitted or chosen by the accident. The models performance does not look better that benchmark. The only model that shows a slight improvement in all the metrics is th Wicksellian one. This model can be of great interest to investors and we encourage interested readers to continue our research in that field. However, we now proceed to the next section, where we discuss covariances and correlations.

	RB portfolio	Model №3	Model №8	Model №9	Wicksellian
Years	2012-2017	2012-2017	2012-2017	2012-2017	2012-2017
CAGR	2.22	2.10	1.93	2.14	2.35
Sharpe	0.86	0.82	0.75	0.83	0.91
Volatility	2.56	2.55	2.56	2.57	2.55
MaxDD	-3.36	-3.25	-3.49	-3.05	-2.77

Table 7: The four models chosen for out-of-sample backtest. Compared with RB portfolio (benchmark).

4 Covariances and correlations

4.1 Overview and importance

It has been shown in Section 2.2 that the essential step in construction of a risk balanced portfolio is the covariance matrix. It is also used in the MPT portfolio construction, and the problem of estimating the elements of this matrix has been discussed in Section 2.1.2. An imprecise estimation might lead to misspecified portfolio weights, increased risk and, as a consequence, decline in Sharpe ratio and other parameters. A quick overview of the developments in that field of estimation will be given first. Next, we will start analysing some of the properties relevant to the data we used. We will then implement and compare a couple of models to predict future covariance values.

Definition 4.1 (The population correlation coefficient)**.** Let X and Y be two random variables with expectations μ_X, μ_Y and standard deviations σ_X, σ_Y respectively. Then their **Pearson correlation coefficient** is defined by

$$\rho_{XY} := \frac{cov(X,Y)}{\sigma_X \sigma_Y} = \frac{\mathbb{E}[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X \sigma_Y}$$

where we implicitly defined $cov(X,Y)$, the covariance between X and Y.

Since the population coefficient is often unknown, we need the following estimation:

Definition 4.2 (Sample correlation coefficient)**.** Let $x_1, x_2, ..., x_N$ be a set of observations of a random variable X with mean μ_X and standard deviation σ_X. Similarly, let $y_1, y_2, ..., y_N$ be a set of observations of a random variable Y with mean μ_Y and standard deviation σ_Y. Let $\bar{x}$ and $\bar{y}$ be the *sample means* of random variables X and Y, as defined in (3.2). Then their **sample correlation coefficient** r_{XY} is defined as

$$r_{XY} := \frac{\Sigma_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\Sigma_{i=1}^{n}(x_i - \bar{x})^2 \Sigma_{i=1}^{n}(y_i - \bar{y})^2}} \tag{4.1}$$

In this chapter we would like to study *sample correlations* instead of looking at exponentially weighted estimators. Using sample estimates, we concentrate on the techniques used to predict future covariance values. Under the assumption of *covariance stationarity*, we can take the returns observations of two assets and use equation (4.1) to calculate the estimate. The problem is that as the number of assets in a portfolio increases, the number of coefficients to be estimated increases quadratically. Furthermore, if the number of observations is high, then the stationarity assumption is challenged. If the number of observations is low, then the matrix might not be invertible and the error increases. If we start looking at the data of higher frequency, e.g. daily returns, then this will add even more noise to the estimates.

Ledoit and Wolf claim in [24] that sample covariance matrix contains an estimation error and should never be used. Authors propose a matrix transformation called **shrinkage**. This procedure is carefully described in [23] by Kwan. The main aims of using this method are to ensure that the resulting matrix is invertible and the estimation error is reduced. It is done by choosing a **target matrix** which is usually an *identity matrix* or *diagonal matrix* with sample variances on the diagonal. It could also be a matrix with the same sample correlation estimates and sample variances on the diagonal, such matrix is called the *constant correlation target*. When the target is chosen, a convex combination of sample covariance matrix and the target is used. The scalar $\lambda \in [0, 1]$ in such convex combination is called **shrinkage intensity** and is optimised.

Further methods of covariance matrices estimation are described in [32]. They include *factor* and *principal component models*, *constant correlation model*, *Cholesky decomposition*, *LASSO type penalised approach* and *elementwise shrinkage*. The elementwise shrinkage method is subdivided to *banding*, *tapering* and *thresholding* methods and each of the methods looks at matrix elements individually, thus, not taking into account the matrix structure. Some companies such as BARRA and APT propose their estimators of covariance matrices based on their own factor models. An important point is made by Disatnik in [12]. Author shows that there is no statistical difference in using shrinkage estimators or more complicated methods. Instead, one can simply choose suitable *target matrix* and find an arithmetic average of such matrix and the sample estimate. That is, to take the intensity of $\lambda = 0.5$ instead of optimising for the best value. Furthermore, the estimation procedures become more relevant as the number of assets in a portfolio increases. Thus, we note that these techniques should always be considered by investor, yet for the purposes of this paper they are less relevant. For completeness, we have backtested the strategy using the arithmetic sum of a diagonal matrix and sample covariance matrix. The results were not significantly different from the original strategy in all the parameters. Instead of focusing on estimation techniques, we discuss sample correlations of the equities, bonds and commodities indices in the next subsection and study predictive models in subsequent sections.

4.2 Sample correlation analysis

In order to see the evolution of sample correlation values in time, we need to use rolling window technique, similar to the one used in (3.4). To calculate the correlation time series for equities and bonds, one has to take returns time series for both assets. The two parameters that one have to define for the purpose of calculation are:

1. the number of previous observations taken or the *length of the window*
2. the number of observations by which the window will move each time or *step*

Firstly, we note that using daily returns data is inappropriate for our purposes. These type of returns contain high amount of noise which will make sample estimate imprecise. However, in order to get sufficient number of observations for future analysis, we change the returns time series to weekly frequencies.

To see how number of observations affect the precision of estimated value, we use the example of equities and bonds correlation. For each size of the rolling window, a corresponding sample correlation time series is constructed. Then for each value in this time series a confidence interval is found. The lengths of confidence intervals are averaged across all the values in a time series. In such a way we construct the average confidence interval length for different lengths of the rolling window. A resulting graph for 95% and 80% confidence intervals can be seen on Figure 11. It can be seen that as we increase the number of observation, the confidence interval decreases. This happens due to the fact that sample correlation is an *unbiased* estimator of correlation. However, the rate by which the interval decreases for each additional observation, diminishes. Thus, even for 40 observations the 80% confidence interval width remains quite big(around 0.3) and 95% confidence interval is around 0.45. As we increase the number of observations, the *covariance stationarity* assumption cannot longer hold. This complication is more serious since we are trying to capture the idea of a constantly changing, rolling correlation. Ideally we would only want to take the most recent observations for calculations. Thus, we will take a relatively small amount of 26 observations that is equivalent to 6 months and will have to compromise the error. We would also move the rolling window by one observation (one week). Ideally, we would want these windows to be disjoint, however it would decrease the number of observations in more than 20 times. The resulting time series for 3 sample correlations are depicted on Figure 12. We can see that all the correlations are very dynamic and their estimation and prediction cannot be underestimated by a portfolio manager.

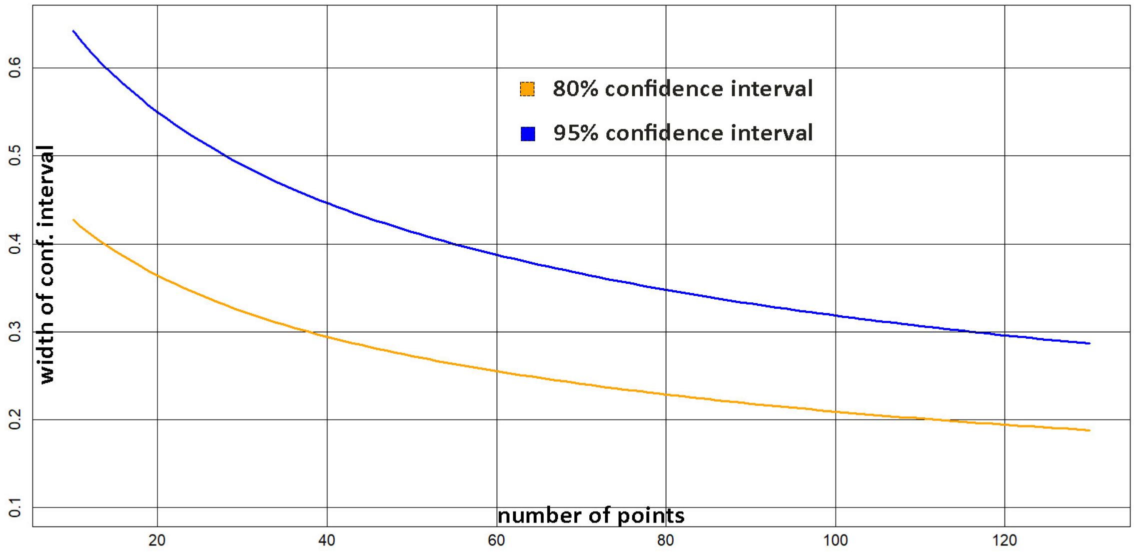

Figure 11: The dependence of the average width of confidence interval on the number of observations for sample correlation coefficient.

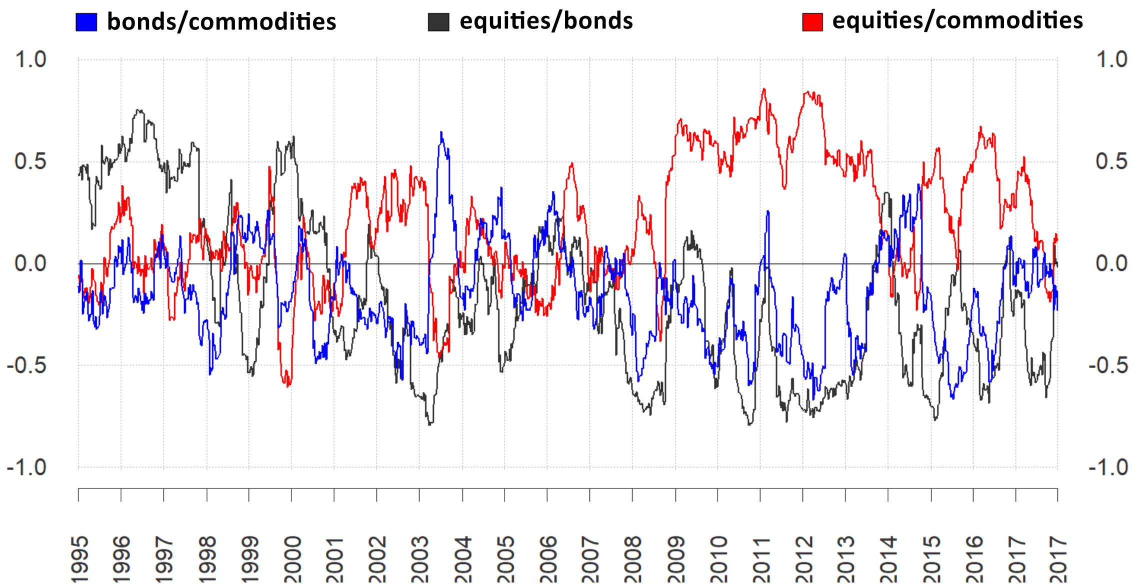

Figure 12: The dynamics of sample correlation coefficients for three constituents of an RB portfolio.

From now on, we would also concentrate on the correlation between equities and bonds. Firstly, this correlation is of great importance for any portfolio manager, since these two instruments are often assumed to be negatively correlated and play an important part in portfolio diversification.

Secondly, all the results that we get can be equivalently applied to other assets.

An important question to answer is how the correlation time series evolve, whether the observations come from a common distribution and what distribution they come from. We would be interested whether there are any observations similar to the stylised facts of returns. The empirical observations $x_1, x_2, ..., x_N$ can be compared with normal distribution using the **quantile-quantile** plot, the results are presented on Figure 13. The parameters of normal distribution are chosen in such a way that the quantile-quantile plot passes through the first and third quantiles of the "theoretical" line. Firstly, we observe that the distribution does not have fat tails. On the contrary, we see the plot deviating to the right from a line on the top and to the left on the bottom. Secondly, the plot poorly fits the line and the distribution might not be unimodal. We check these assumptions by plotting the histogram on the same Figure 13. The shape of the histogram does not look unimodal and does not resemble any of the commonly used distribution families. This might be a result of a changing market environment and presence of several regimes. We study this possibility further in the next subsection with the help of Hidden Markov Models.

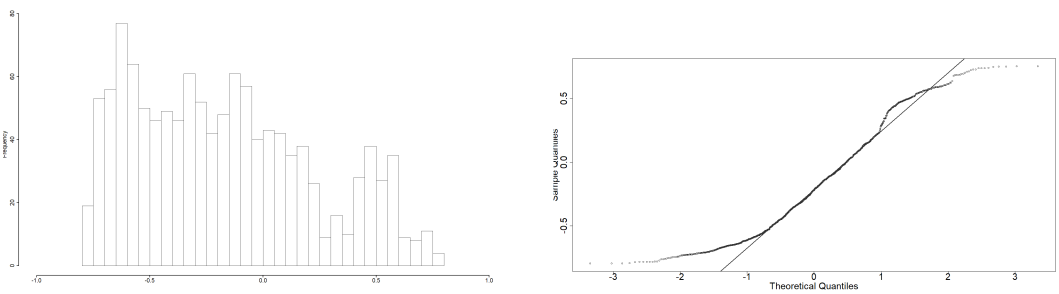

Figure 13: Histogram and QQ plot for equities/bonds sample correlation observations.

Prior to concluding the discussion in this subsection, we would like to mention that there are different ways to define a correlation between two random variables, mainly: Pearson, Kendall and Spearman. While Pearson correlation measures a linear association between two variables, Spearman correlation depicts any monotonic relationships between them. We use Pearson coefficient for several reasons. Firstly, it is the most common measure and is often used by practitioners in the industry. Secondly, the results that we obtained using Spearman sample correlation were remarkably similar to the Pearson estimates. Finally, the calculation of Spearman correlation is based on ranks and there is no clear relationship between the covariance value and Spearman correlation.

4.3 Hidden Markov Model

The idea of **Hidden Markov Models** (**HMM**) was proposed and developed more than 40 years ago by Baum et al. This statistical model is based on the *Markov model*, a model that is characterised by *states*, *transition probabilities* between these states and the *Markov property* at any point in time. The *Markov property* assumes that future states depend only on the current state, not on the events that occurred before it. In HMM model the states are unobserved or "hidden". This setup shows resemblance to the regimes models and is thoroughly used by researchers. A good tutorial on the usage of this model is given in [34]. Instead of describing the setup in full, we will mention the main aspects and the practical results.

As has been mentioned already, the regimes in the market are unobservable and can be represented by hidden states of the HMM model. However, the observations from one regime or another are visible, these are the realisations of correlation time series. Since each regime has a probability distribution, the realised sequence of observations gives some information about the state sequence. To identify the state sequence and the parameters of the model, a statistical iterative process has to be used. There is no closed form solution for the problem, however, the model can be fitted by the maximum likelihood estimation. An example of an iterative method is an **expectation–maximisation** algorithm. The algorithm requires the specification of the number of regimes and the probability distribution of each regime. These are the parameters of the model. The result of an algorithm is a **posterior probabilities** time series for each of the regimes defined, the parameters of these regimes that include the mean and variance for each regime distribution and the *transition matrix* with estimated probabilities of transiting or staying in the same regime. Note, that a combination of two or three normal distributions can give a rise to a wide number of probability density functions, depending on the parameters of each distribution. We applied the expectation–maximisation algorithm to the sample correlation time series, specifying two regimes with Gaussian distribution, the results are presented on Figure 14. The posterior probabilities are close to the extreme values of 0 and 1 respectively for most of the days and can be easily transformed to the state sequence using a fixed threshold level as a decision rule. Given the last observation in a state sequence and the estimated transition probabilities, we predict the next value in correlation time series as follows:

1. Using the pseudo-random number generator, we obtain an observation of a random variable X with standard uniform distribution $U(0,1)$. Using this number we can construct in an obvious way a rule to decide whether the state is switched or not.

2. Given a state, we then generate a random number from Gaussian distribution with the estimated mean and variance.

Using this setup, we can perform a backtest by choosing a period of time, re-fitting the HMM model for each new observation and comparing the one week ahead predictions with true values of correlation time series. The results of such test are given in Section 4.6. It should be mentioned

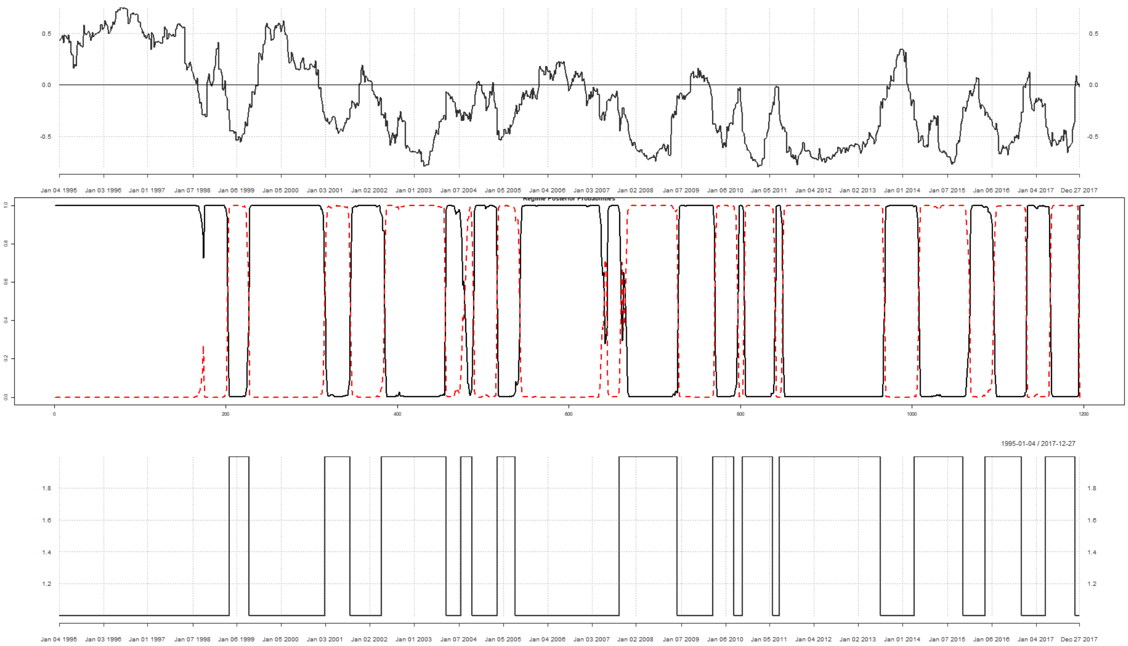

Figure 14: From top to bottom: sample correlation time series, posterior probabilities obtained by fitting the HMM model and the signal obtained from posterior probabilities.

that a similar procedure can be separately performed for other sample correlation time series. Therefore, the method could be used for the prediction of a correlation matrix and, thus, the portfolio covariance matrix.

4.4 Regression Model

The regression analysis helps to estimate the relationship between a *dependent variable* and *independent variables*(*predictors*). Such a model could be used in an attempt to detect how the correlation between two assets, being a dependent variable, changes when some other macroeconomic variables change. Moreover, by estimating the coefficients of a regression model, we can evaluate the sensitivity of the correlation to the changes of independent variables.

Regression models are frequently used in financial research. In this section we will partly replicate the research conducted by Johnson et al. in [21]. However, we will not only critically approach this paper, but also use different estimates, predictors and time periods for our purposes. The papers mentions the importance of the equities-bonds correlation estimate for portfolio construction and admits its dynamic behaviour. The researchers then construct a sample correlation time series in a similar manner to the process described in Section 4.2. Similarly to our model, they take bonds and equities indices to capture general features of correlation between these two types of assets.

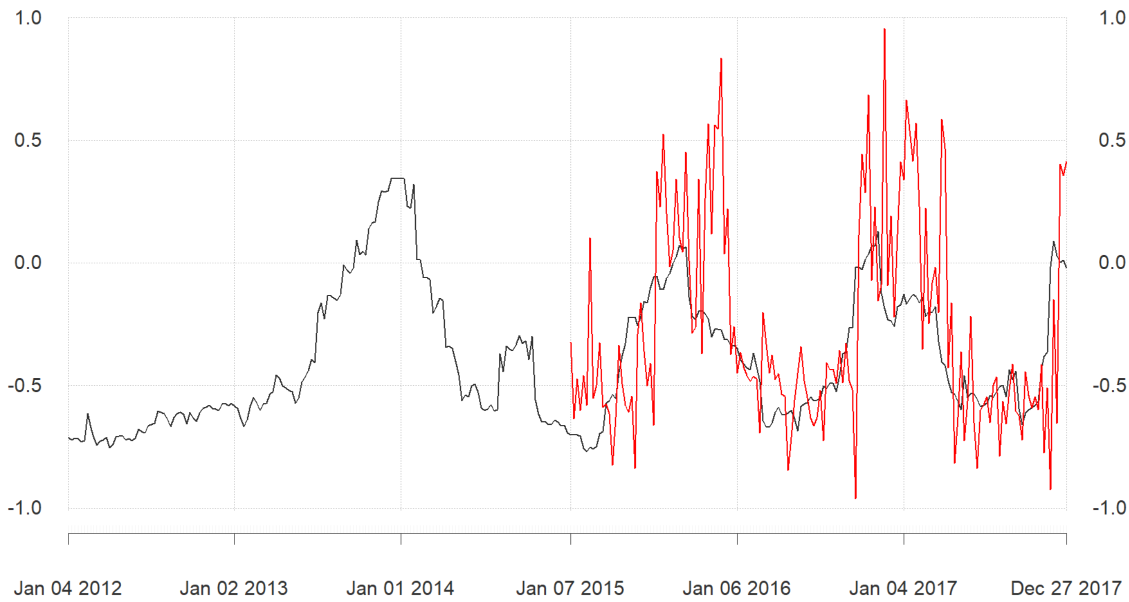

Figure 15: HMM model with 2 states is used to obtain one-week ahead predictions of correlation(in red). Observed values of correlation are in black.

They develop two regression models with the purpose of forecasting. It is claimed that among factors that affect correlation figures, there is inflation and real interest rates, which can be explained by economic reasoning. However, we believe that there are too many assumptions and complications in a conducted research. Firstly, instead of using daily returns, the researchers use monthly figures and, as a result, they only get yearly estimated correlation time series. This frequency is too low for our purposes and we continue using weekly returns observations, compromising between the error and number of observations. Secondly and most importantly, the researchers use 12 values of assets returns to calculate the correlation estimate. This number of observations is definitely too small, it is seen on Figure 11 that for such an estimate the 80% confidence interval is likely to have the approximate length of 0.45. Moreover, this assumes that the covariance does not change significantly in the period of one year and we have shown previously that this is not the case. Thus, we presume that the results based on such estimates cannot be of any practical use.

The problem of using inflation and real interest rates in a regression model is that both variables require inflation figures. Inflation figures, however, are often quoted monthly. Thus, instead of using CPI figures, as it was done in the research, we can only use Swap and Forward indicators defined in Section 3.2.1. These indicators are supposed to reflect the expectations about inflation and can be used as an inflation predictor variable in regression. However, these indicators cannot be used to calculate the real interest rates. For that calculation the best option is to linearly interpolate monthly CPI indicator or use the latest value known (interpolation is used for fitting the linear regression model, while the latest value has to be used for prediction).

Since there are only two independent variables in our model, we concentrate on the linear regression model and do not consider Lasso penalisation and other regression models. The model is represented by

$$\hat{\rho}(t) = \alpha_0 + \alpha_1 \pi(t) + \alpha_2 r(t) + \varepsilon(t), \tag{4.2}$$

where $\hat{\rho}(t)$ is the observation from the sample correlation time series at time t, $\pi(t)$ and $r(t)$ denote inflation and real interest rates at time t respectively, $\varepsilon(t)$ is an error term and α_0, α_1, α_2 are regression parameters.

The real interest rates time series is the average three-months US government bond yield over 6 months, adjusted for inflation. Both the Swap and the Forward indicators are smoothed using 3 months moving average technique as described in (3.3). Also, we impose a lag of 1 week on the predictor variables in order to use the most recent values for the next week prediction. The process described below is conducted twice: once for the Swap indicator and once for the Forward indicator. We use the historical observations of sample correlation and apply the method of the Least Squares to fit the model and obtain estimators for the parameters α_0, α_1 and α_2. The estimated parameters show that there is significant relationship between independent variables and sample correlation values. On the Figure 16 we plot the sample correlation time series, the fitted values of correlation and the 95% confidence interval for these fitted values using the Forward indicator.

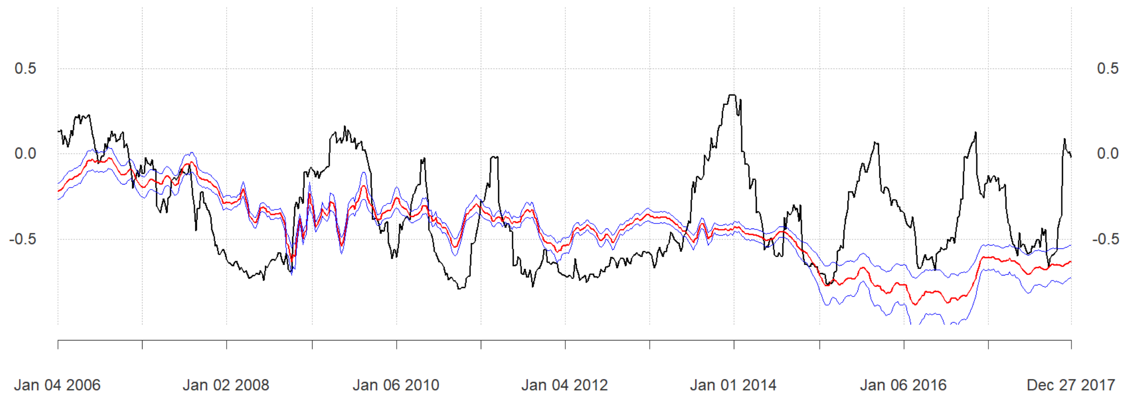

Figure 16: The red line represents fitted by regression values of correlation (Forward inflation indicator is used). The blue lines represent 95% confidence interval for each fitted value.

Next, for the years 2015-2017 the predictor variables are used to get the estimate of correlation for the next week. The model is re-fitted every week to obtain better estimates. The numerical values are presented and analysed in Section 4.6.

One should keep in mind the limitations of our approach. The regression model described above is not attempting to predict true correlation figures. Furthermore, it is not attempting to predict the values of exponentially weighted sample correlation values that are often used as an industry standard. Instead, it tries to predict the sample correlation time series constructed in Section 4.2. Thus, one cannot judge whether the predicted time series is better or worse than the realised sample correlation. Since the true value of correlation always remains unknown, one can only try to check the quality of prediction by performing a portfolio backtest using the predicted values. However, this procedure is likely to be affected by other unobservables and subject to error.

4.5 ARIMA Model

An **autoregressive integrated moving average** (**ARIMA**) model is a generalisation of an **autoregressive moving average** (**ARMA**) model. This model is specified by three parameters: p, d and q. This model is often used to forecast the non-stationary time series. It is done by differencing the time series and the integrated part d is responsible for that. The autoregressive part p is used to account for the autocorrelation of time series, while the moving average part q makes the regression error a linear combination of error.

We explore the sample correlation time series that was constructed in Section 4.2 to see whether ARIMA is a good model to use. Next, we find the parameters for ARIMA model and use its predictive power.

Firstly, we have decomposed a time series to eliminate seasonality. Thus, the original time series became a sum of a seasonal, trend and cycle component. A cycle component is the one that we continue using to fit the model - it is depicted on Figure 18. Secondly, one should check the time series for stationarity. To do that, we perform the augmented Dickey-Fuller (ADF) test. The null hypothesis in this test is that the time series is non-stationary. The low p-value suggests that we can reject null-hypothesis and the time series is stationary. We can proceed by differencing the time series, though it is not necessary. Subtracting from each other the two consecutive values of a time series, we will get a differenced sequence. For all values of t we do the following

$$Y_t := X_t - X_{t-1}$$

The augmented Dickey-Fuller test on the differenced time series shows its stationarity. Thus, we can take the value $d = 1$ or $d = 0$ for the integrated parameter. In order to determine parameters p and q, we plot the sample ACF and partial ACF for the differenced time series using (3.2). We observe certain spikes on the plots and note the number of lags, so that we can try these values later for parameters p and q. Using R environment, we fit the ARIMA model trying several values for parameters p and q. Examining ACF and PACF plots for model residuals and information criterions, we find that ARIMA(2,1,2) model should be fitted. The ACF and PACF plots for model residuals are presented on Figure 17. Next, for the years 2015-2017 we use the model to get a prediction for the next week. We re-fit the model every week to obtain better estimates, but leave the parameter values unchanged. The numerical results are presented in a table in Section 4.6.

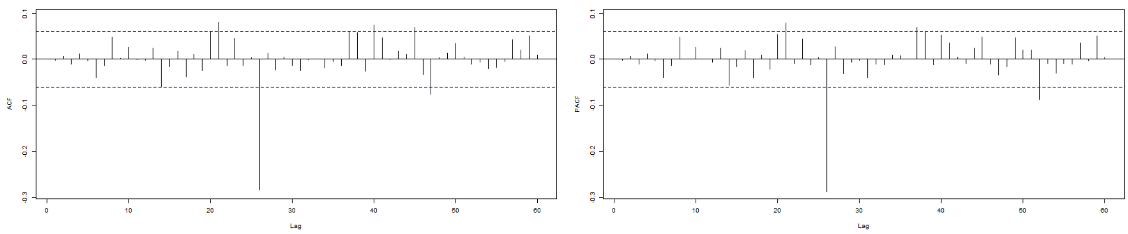

Figure 17: ACF and PACF plots for the residuals of the fitted ARIMA model.

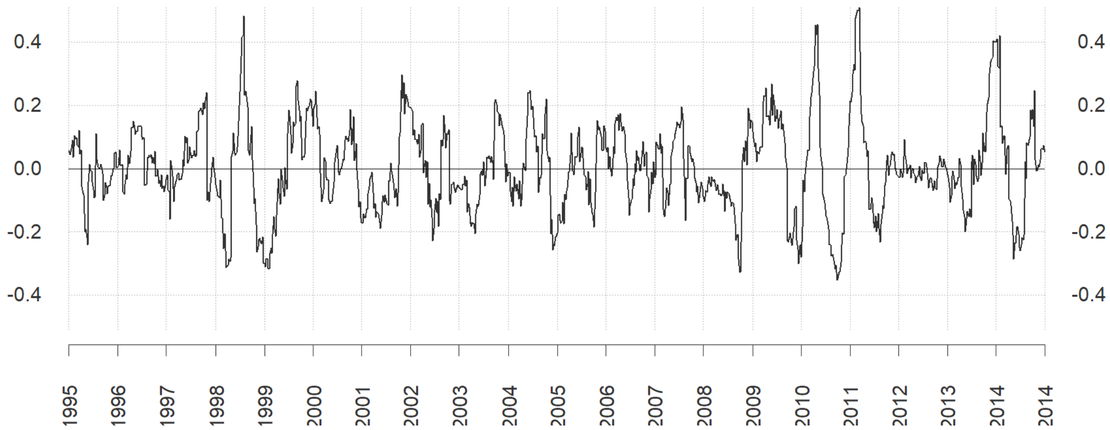

Figure 18: A cycle component of the decomposed correlation time series. The process is performed to eliminate seasonality from the time series and apply ARIMA model.

4.6 Models comparison

In this section we compare the models discussed and developed in Section 4. We have analysed the possible ways to estimate a covariance between two assets. Ideally, one wants to capture the short-term correlation that is relevant specifically to the present moment in time. However, it is impossible to do in practice. We have constructed the sample correlation time series using the rolling window to get its values. The same period has been chosen for models backtesting procedure. The models were predicting the values of correlation in the way it would be done on practice. The predicted values are then compared with realised values of sample correlation. For that purpose we have calculated the sum of squared errors for each of the models, the results are given in Table 4.6. We see that this model predicts closer and could be used in a similar way by investors. However, note that this is a relative measure and it cannot be used to confidently conclude which model performs better.

Models	HMM	ARIMA	Regression with Forward	Regression with Swap
Sum of squared errors	19.95204	9.141255	11.65137	9.869906

5 Conclusion

In this paper we have reviewed the MPT portfolio theory that has influenced most of the portfolio theories developed after it. We have explained the idea of risk balanced portfolio approach and have proved on practice its benefits. We have also conducted a substantial research on the regimes in finance and other models that can be used for tactical portfolio tilts with the aim of improving the RB portfolio performance furthermore. We have developed several models based on macroeconomic regimes. Some of the models have performed reasonably well during in-sample period, however when compared with the RB benchmark portfolio out-of-sample, they did not prove their performance edge. The potential problems might be poorly chosen indicators or a different estimation time horizon. However, a Wicksellian model has shown very nice performance in all the metrics out-of-sample and should be developed and improved further. It is important to note that a combination of economic and growth signals produced the best results.

We have also reviewed the work that has been done on covariance matrices estimation. Unfortunately, it is impossible to accurately estimate the value of a short term correlation due to statistics limitations. However, we have seen some evidence of rapid changes of these correlation values. It is difficult to test one's model for correlation, since there is no true value of correlation known to be compared to. However, we developed procedures to accommodate three different models with the aim of sample correlation prediction. According to the squared error test, the best model is the ARIMA model and regression model with Swap coefficient. However, one should always keep in mind that these results can only give information about the relative models performance.

For all of the interested readers of our paper, we would suggest exploring the success if Wicksellian model in more details. We would also advise to apply the HMM model in the same or different environment to estimate the volatility of the correlation coefficient itself. Risk balanced portfolio is a promptly developing field and the models that are created in such a process might prove to be useful in all parts of finance and other spheres.

A Appendix

A.1 Extra results from the backtest in Section 3.2.1.

The results of a backtest of 12 models **without** any **imposed lags** are presented in Table 8. The results with **increased tilt weights** are presented in Table 9.

	Model №1	Model №2	Model №3	Model №4	Model №5	Model №6
Years	1997-2012	1997-2012	1997-2012	1997-2012	1997-2012	1997-2012
CAGR	4.28	5.17	5.07	4.32	4.70	4.48
Sharpe	0.70	0.87	0.90	0.71	0.74	0.75
Volatility	6.24	5.97	5.63	6.23	6.45	6.12
MaxDD	-28.1	-20.94	-17.21	-28.43	-27.16	-24.24

	Model №7	Model №8	Model №9	Model №10	Model №11	Model №12
Years	1997-2012	1997-2012	1997-2012	1997-2012	1997-2012	1997-2012
CAGR	4.41	5.32	5.19	4.88	5.26	5.25
Sharpe	0.70	0.86	0.89	0.73	0.77	0.79
Volatility	6.42	6.25	5.90	6.88	6.94	6.71
MaxDD	-28.8	-23.16	-18.96	-28.81	-27.39	-24.92

Table 8: The 12 models constructed by combining various inflation and economic growth signals

	Model №1	Model №2	Model №3	Model №4	Model №5	Model №6
Years	1997-2012	1997-2012	1997-2012	1997-2012	1997-2012	1997-2012
CAGR	5.17	4.86	4.99	4.94	4.73	4.78
Sharpe	0.93	0.87	0.92	0.86	0.82	0.85
Volatility	5.55	5.59	5.45	5.76	5.82	5.65
MaxDD	-20.94	-21.19	-18.75	-23.36	-23.60	-21.62

	Model №7	Model №8	Model №9	Model №10	Model №11	Model №12
Years	1997-2012	1997-2012	1997-2012	1997-2012	1997-2012	1997-2012
CAGR	5.26	5.01	5.13	5.01	4.84	4.94
Sharpe	0.94	0.88	0.93	0.85	0.82	0.85
Volatility	5.63	5.70	5.53	5.95	5.98	5.85
MaxDD	-21.61	-21.86	-19.45	-23.48	-23.70	-22.03

Table 9: The 12 models are constructed by combining various inflation and economic growth signals

A.2 Packages used

As it was mentioned in the paper, all the calculations were performed using the R Project for Statistical Computing, *https://www.r-project.org/*.

A list of used packages:

– *tseries* package;

– *xts* package;

– *forecast* package;

– *ecm* package;

– *SIT* package;

– *quantmod* package;

– *depmixS4* package.

A.3 Data Tickers

The list of Bloomberg L.P. tickers used:

"SPX index", "NDDLWI Index", "BCOM Index", "SPGSLETR Index", "LUATTRUU Index", "LEGATRUH Index", "USGG10YR Index", "USGG3M Index", "GDP CURY Index", "LEI CHNG", "NAPMPMI Index", "CPI YOY Index", "FWISUS55 Index", "CONSP5MD Index", "USSWIT1 CMPN Curncy", "CONCINFL Index".

References

[1] A. Ang and G. Bekaert. Regime switches in interest rates. *Journal of Business & Economic Statistics*, 20(2):163–182, 2002.

[2] A. Ang and A. Timmermann. Regime changes and financial markets. *Annual Review of Financial Economics*, 4(1):313–337, 2012.

[3] C. Audrey and M. Zita. Risk on, risk off in a multifactor world. *MSCI Applied research*, 2012.

[4] G. Bamania and Y. Lustig. Risk on, risk off? using a regime change model to support investment decisions. *AXA IM*, 2013.

[5] N. Barberis and M. Huang. Mental accounting, loss aversion, and individual stock returns. *The Journal of Finance*, 56(4):1247–1292, 2001.

[6] N. Barberis and R. Thaler. A survey of behavioral finance. *Handbook of the Economics of Finance*, 1(1):1053–1128, 2003.

[7] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

[8] G. Chow, E. Jacquier, M. Kritzman, and K. Lowry. Optimal portfolios in good times and bad. *Financial Analysts Journal*, pages 65–73, 1999.

[9] R. Cont. Empirical properties of asset returns: stylized facts and statistical issues. *Quantitative Finance*, 1:223–236, 2001.

[10] R. Cont and L. Wagalath. Institutional investors and the dependence structure of asset returns. *International Journal of Theoretical and Applied Finance*, 19(02):1650010, 2016.

[11] V. DeMiguel, L. Garlappi, and R. Uppal. Optimal versus naive diversification: How inefficient is the 1/n portfolio strategy? *The review of Financial studies*, 22(5):1915–1953, 2007.

[12] D. Disatnik and S. Benninga. Shrinking the covariance matrix—simpler is better. *Journal of Portfolio Management*, 33(4):56–63, 2007.

[13] A. Doffou. Insider trading: a review of theory and empirical work. *Journal of Accounting and Finance Research*, 11(1), 2003.

[14] E. J. Flint, F. Chikurunhe, and A. J. Seymour. Regime-based tactical allocation for equity factors and balanced portfolios. 2017.

[15] R. Garcia, P. Perron, et al. An analysis of the real interest rate under regime shifts. *The Review of Economics and Statistics*, 78(1):111–125, 1996.

[16] M. Gardijan and T. Škrinjarić. Estimating investor preferences towards portfolio return distribution in investment funds. *Croatian Operational Research Review*, 6(1):1–16, 2015.

[17] C. Gave. Four quadrants: a Wicksellian analysis. Technical report, Gavekal Research, 2016.

[18] L. R. Glosten, R. Jagannathan, and D. Runkle. On the relationship between garch and symmetric stable process: Finding the source of fat tails in data. *Journal of Finance*, 48:1779–1802, 1993.

[19] J. D. Hamilton. A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica: Journal of the Econometric Society*, pages 357–384, 1989.

[20] L. H. Hunt. *Dynamics of Forecasting Financial Cycles: Theory, Technique and Implementation.* JAI Press, Greenwich, CT, 1976.

[21] N. Johnson, V. Naik, S. Page, N. Pedersen, and S. Sapra. The stock-bond correlation. *Pimco Quantitative Research (Working paper)*, 2013.

[22] M. Kritzman and Y. Li. Skulls, financial turbulence, and risk management. *Financial Analysts Journal*, 66(5):30–41, 2010.

[23] C. C. Kwan. An introduction to shrinkage estimation of the covariance matrix: A pedagogic illustration. *Spreadsheets in Education (eJSiE)*, 4(3):6, 2011.

[24] O. Ledoit and M. Wolf. Honey, I shrunk the sample covariance matrix. *The Journal of Portfolio Management*, 40(4):110–119, 2004.

[25] P. C. Mahalanobis. On the generalized distance in statistics. National Institute of Science of India, 1936.

[26] H. Markowitz. Portfolio selection. *The journal of finance*, 7(1):77–91, 1952.

[27] H. Markowitz. The early history of portfolio theory: 1600-1960. *Financial Analysts Journal*, 55(4):5–16, 1999.

[28] G. H. Moore. *Business cycles, inflation, and forecasting.* National Bureau of Economic Research, 1983.

[29] B. Naqvi, N. Mirza, W. Azeem Naqvi, and S. Rizvi. Portfolio optimisation with higher moments of risk at the pakistan stock exchange. *Economic research-Ekonomska istraživanja*, 30(1):1594–1610, 2017.

[30] D. Nawrocki and W. Carter. Phase of the business cycle and portfolio management. Technical report, Working Paper 1997, Villanova University, 1997.

[31] M. Pakkanen. *Lecture Notes in Quantitative Risk Management*. Imperial College London, 2017-2018.

[32] M. Pourahmadi. Covariance estimation: The GLM and regularization perspectives. *Statistical Science*, 26(3):369–387, 2011.

[33] E. E. Qian. *Risk Parity Fundamentals*. CRC Press, 2016.

[34] L. R. Rabiner. A tutorial on hidden markov models and selected applications in speech recognition. *Proceedings of the IEEE*, 77(2):257–286, 1989.

[35] T. Roncalli. *Introduction to risk parity and budgeting*. CRC Press, 2013.

[36] W. F. Sharpe. Mutual fund performance. *The Journal of business*, 39(1):119–138, 1966.

[37] A. Z. Sheikh and J. Sun. Regime change: Implications of macroeconomic shifts on asset class and portfolio performance. *Journal of Investing*, 21(3):36, 2012.

[38] T. Škrinjarić. Portfolio selection with higher moments and application on zagreb stock exchange. *Zagreb International Review of Economics & Business*, 16(1):65–78, 2013.

[39] K. Wicksell. *Interest and prices*. Ludwig von Mises Institute, 1965.