Malliavin calculus applied to Monte Carlo methods in mathematical finance

by

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Declaration

The work contained in this thesis is my own work unless otherwise stated.

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Abstract

In this thesis, we show how the efficient Monte Carlo methods could be devised for finance by using the so-called Malliavin calculus. We propose a new method of computing Greeks for a wider class of options, the Malliavin weighted scheme, which was originally introduced by Fournié et al. in 1999. This approach is based on the Malliavin integration-by-parts formula on the Wiener space. Several numerical experiments with applications of Monte Carlo method are conducted, and our method is compared to the finite difference approximation to illustrate its efficiency. It is shown that the Malliavin weighted scheme significantly outperforms the finite difference method in the case of discontinuous payoff functionals, as expected.

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Contents

1 Introduction

1.1 Brief literature review about Malliavin calculus

In 1976, the celebrated paper "Stochastic calculus of variation and hypoelliptic operators" [1] written by Paul Malliavin was published, in which the author initiated the theory of stochastic calculus of variations (i.e., the so-called Malliavin calculus). In particular, to explore a probabilistic proof of Hörmander's "sum of squares" theorem is the original motivation of this theory. Moreover, it is an infinite-dimensional differential calculus for functions on a Wiener space, and Malliavin's technique has been further developed by Stroock, Bismut, Watanabe, and others. One of the important applications of this theory is to investigate the existence and regularity properties of smooth densities for solutions to stochastic differential equations (SDEs) (see e.g. Ikeda and Watanabe [4, 7]). The Malliavin calculus was developed in the context of a symmetric diffusion semigroup by Stroock [5, 6]. Furthermore, in [3], Bismut provided a direct method for proving Hörmander's theorem by applying the Malliavin integration-by-parts formula on the Wiener space.

Over the last few decades, Malliavin calculus has been applied to diverse fields. For example, consider the Heisenberg-Weyl algebra, Franz et al. [8] applied Malliavin calculus to quantum stochastic processes; by the use of Girsanov's theorem and the integration-by-parts formula of Malliavin calculus, they derived a diffusion equation served as the sufficient condition their Wigner densities need to satisfy. Besides, the convergence rate of the Euler discretization scheme on the solution to an SDE is discovered by Bally and Talay [9]. Furthermore, Malliavin calculus has been introduced a lot in economics and finance (e.g., [10–12]). In particular, Fournié et al. [13, 14] applied Malliavin calculus to calculate Greeks.

1.2 Why a new approach to evaluating Greeks?

Greeks are quantities representing partial derivatives of the estimated price of derivatives (e.g., options) with respect to various model parameters (risks), and are also called (risk) sensitivities. The growing emphasis on hedging and risk-management issues has indicated a greater need to efficiently compute Greeks, which are also very useful for the pricing of a product. Due to the fact that in general financial models (including stochastic volatility models), these risk ratios do not have explicit closed-form expressions, one often uses Monte Carlo methods to simulate the results (see [15–19]).

Various approaches have been used to estimate Greeks. First, finite difference method is a traditional way. Glynn [20] shown that under the central difference scheme and using common random variables, the optimal convergence rate is arbitrarily close to $n^{-1/2}$ in the number of observations required when the objective function is sufficiently smooth. Glasserman and Yao [21], and L'Ecuyer and Perron [22] all suggested that this rate is the best to be expected by Monte

Carlo simulation. The finite difference method is universally applicable; however, it may lead to a slow convergence rate and perform badly when the payoff function is discontinuous, e.g., in the case of digital, corridor, Asian or lookback options (see for example, [23]). We refer the reader to [24] for more details about finite difference method. To overcome this poor convergence rate, Broadie and Glasserman [25] proposed two different unbiased methods, a pathwise method and a likelihood ratio method, which can considerably increase the computational speed. Both methods require an interchange of a derivative and are exhibited to have remarkably better convergence than the finite difference method. However, the pathwise method cannot handle discontinuous (non-differentiable) payoffs, and the likelihood ratio method can only be useful when the density of the underlying model is explicitly known. We refer the reader to [26] for more details of the pathwise method, and [27, 28] for the likelihood ratio method.

To avoid the above limitations, Fournié et al. [13, 14] used an integration-by-parts formula to derive expressions of Greeks based on Malliavin calculus. They represented the Greeks as the expectation of the payoff function multiplied by a weight function

$$\text{Malliavin Greek} := \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_0^T r_s \mathrm{d}s} f(X_T) \cdot weight], \tag{1.1}$$

where $\mathbb{E}^{\mathbb{Q}}[\cdot]$ is the expectation under the risk-neutral measure $\mathbb{Q}$, X_t the underlying price at time t, f the payoff function (for instance, for European Call options, $f(X_T) =: (X_T - K)^+$), and r_s the risk-free interest rate. We note that the Malliavin weight is independent of the payoff function. We call this method the Malliavin weighted scheme, which is applicable to complicated and discontinuous payoffs. It has been shown that this method outperforms the finite difference method for discontinuous payoff functionals (e.g., in the case of digital, simple, double barrier and many other exotic options) (see [29–32]). Moreover, it performs well when we do not have an explicit knowledge of the underlying density. See section 3 for more details of Malliavin weighted scheme.

1.3 Aims and structure of the thesis

The main purpose of this thesis is to apply Malliavin calculus to compute Greeks and use Monte Carlo method to conduct numerical experiments to demonstrate the efficiency of our method.

The rest of this thesis is organised as follows. In Section 2, we briefly review the Malliavin calculus. Section 3 presents few related assumptions, and the detailed derivations of the Malliavin Delta and Gamma. We then apply the results derived to several models: classical Heston model, fractional Black-Scholes model, and finally the rough Heston model (needs further research). Closed-form Greeks, finite-difference Greeks and Malliavin Greeks are all derived, which are used in Section 4 to conduct numerical experiments and to illustrate their performance. Finally, we conclude the thesis in Section 5.

2 Malliavin calculus

In this section, we recall the basics of Malliavin calculus. The material covered here was taken from [10, 13, 29, 30, 34, 35].

Let $W = \{W_t, t \in [0,T]\}$ be a continuous Gaussian process with mean zero and covariance function $\mathbb{E}[W_t W_s] = R(t,s)$ such that $W_0 = 0$. In addition, W is defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{F}$ generated by W. Let H_1 be the first Wiener chaos, i.e., the closed Gaussian subspace of $L^2(\Omega)$ generated by W. As stated in [60, 61], we denote by $\mathcal{E}$ the set of step functions on $[0,T]$, and let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} \rangle_{\mathcal{H}} =: R(t,s).$$

Then the mapping $\mathbb{1}_{[0,t]} \mapsto W_t$ can be extended to a linear isometry between $\mathcal{H}$ and H_1 associated with W. We denote by $W(h)$ the image in H_1 of an element $h \in \mathcal{H}$.

Definition 2.1. A function $f : \mathbb{R}^d \to \mathbb{R}$ is of polynomial growth if there exist constants $C > 0$, $\beta > 0$ such that

$$|f(x)| \leq C(1 + |x|)^{\beta}, \quad \forall x \in \mathbb{R}^n. \tag{2.1}$$

We denote by $C_p^{\infty}(\mathbb{R}^n)$ the set of all infinitely continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and all of its partial derivatives have polynomial growth.

Let $\mathcal{S}$ designate the class of smooth and cylindrical random variables such that a random variable $F \in \mathcal{S}$ has the form

$$F = f(W(h_1), \cdots, W(h_n)), \tag{2.2}$$

where $f \in C_p^{\infty}(\mathbb{R}^n)$, $h_1, \cdots, h_n \in \mathcal{H}$, and $n \geq 1$.

2.1 Malliavin derivatives

Definition 2.2 (Malliavin derivative)**.** The derivative of a smooth and cylindrical random variable F of the form (2.2) is the $\mathcal{H}$-valued random variable given by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \cdots, W(h_n)) h_i(t), \tag{2.3}$$

where D is called the Malliavin derivative on $\mathcal{S}$.

Remark 2.1. *As stated in [30, page 15], since f has only polynomial growth, we have $DF \in L^2([0,T] \times \Omega)$. Sometimes we write* (2.3) *as*

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \cdots, W(h_n)) h_i(t). \tag{2.4}$$

Due to the fact that $h_i \in \mathcal{H}$ is only defined up to a set of Lebesgue measure 0, D_t is not really well-defined. We will use both D_t and D (defined with different domains and ranges) in this thesis.

We denote the domain of D in $L^p(\Omega)$ by the Sobolev space $\mathbb{D}^{1,p}$ for any $p \geq 1$, which means that $\mathbb{D}^{1,p}$ is the closure of $\mathcal{S}$ with respect to the norm

$$||F||_{1,p} = (\mathbb{E}[|F|^p] + \mathbb{E}[||DF||^p_{\mathcal{H}}])^{\frac{1}{p}}.$$

One can interpret $\mathbb{D}^{1,p}$ as an infinite-dimensional weighted Sobolev space. In particular, for $p = 2$, the space $\mathbb{D}^{1,2}$ is obviously a Hilbert space with the scalar product

$$\langle F, G\rangle_{1,2} = \mathbb{E}[FG] + \mathbb{E}[\langle DF, DG\rangle_{\mathcal{H}}].$$

Proposition 2.1 (chain rule)**.** *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives and $F = (F_1, \cdots, F_n)$ a random vector whose components belong to $\mathbb{D}^{1,2}$. Then $\phi(F) \in \mathbb{D}^{1,2}$ and*

$$D_t\phi(F) = \sum_{i=1}^{n} \frac{\partial \phi(F)}{\partial x_i} D_t F_i, \quad t \geq 0 \quad a.s.$$

Proof. The proof follows similar steps as in [37, page 97]. When ϕ is smooth, the result is consistent with the chain rule in classical analysis. If ϕ is not smooth, we define a non-negative function $\rho(x) := ce^{\frac{1}{x^2-1}}$ belonging to $C_0^\infty(\mathbb{R}^n)$ whose support is a unit ball with c chosen such that $\int_{\mathbb{R}^n} \rho(x)\mathrm{d}x = 1$, and take a sequence of regularisation kernels of the form $\rho_\varepsilon := \varepsilon^n \rho(\varepsilon x)$ to get a smooth approximation $\phi * \rho_\varepsilon$ for ϕ. Let us take the smooth approximations F^i_n of F^i, then we have

$$(\phi * \rho_\varepsilon) \circ F^i_n \to \phi * F^i \quad \text{as } \varepsilon \wedge n \to \infty \text{ in } L^p,$$

and the closeness of D implies

$$\begin{aligned}
&||D(\phi \circ F) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\phi(F)DF^i||_p \\
\leq &||D(\phi \circ F) - D((\phi * \rho_\varepsilon) \circ F)||_p + ||D((\phi * \rho_\varepsilon) \circ F) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\phi * \rho_\varepsilon)(F)DF^i_n||_p \\
&+ ||\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\phi * \rho_\varepsilon)(F)DF^i_n - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\phi(F)DF^i||_p \\
&\to 0.
\end{aligned}$$

□

Lemma 2.1. *Let $F_n, n \geq 1$ be a sequence of random variables in $\mathbb{D}^{1,2}$ that converges to F in $L^2(\Omega)$ and such that*

$$\sup_n \mathbb{E}[||DF_n||^2_{\mathcal{H}}] < \infty.$$

Then F belongs to $\mathbb{D}^{1,2}$, and the sequence of derivatives $\{DF_n, n \geq 1\}$ converges to DF in the weak topology of $L^2(\Omega; \mathcal{H})$.

Proof. see Nualart [35, page 28-29]. □

Proposition 2.2 (generalised chain rule)**.** *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a function such that*

$$|\phi(x) - \phi(y)| \leq K|x - y|$$

for any $x, y \in \mathbb{R}^n$. Suppose that $F = (F^1, \cdots, F^n)$ is a random vector whose components belong to the space $\mathbb{D}^{1,2}$. Then $\phi(F) \in \mathbb{D}^{1,2}$, and there exists a random vector $G = (G^1, \cdots, G^n)$ bounded by K such that

$$D(\phi(F)) = \sum_{i=1}^{n} G_i DF^i.$$

Proof. Here we follow the ideas in [35, page 29] and [37, page 98]. When the function ϕ is continuously differentiable, the result is consistent with that of Proposition 2.1 with $G_i = \partial_i \phi(F)$. We take the same modifier similar to that for Proposition 2.1, $\phi * \rho_\varepsilon$ converges to ϕ uniformly on compacts. Since

$$|\nabla(\phi * \rho_\varepsilon)| \leq K$$

for ε large enough, we have that the sequence $D((\phi * \rho_\varepsilon) \circ F)$ is bounded in $L^2(\Omega, \mathcal{F}, \mathbb{P}) \otimes H$. Hence, applying Lemma 2.1 gives $\phi \circ F \in \mathbb{D}^{1,2}$ and $D((\phi * \rho_\varepsilon) \circ F)$ converges to $D(\phi \circ F)$ weakly. On the other hand, $\nabla(\phi * \rho_{\varepsilon_k}) \circ F$ converges weakly to some $G \in \mathbb{R}^n, |G| < K$. Therefore, taking the weak limit in

$$D((\phi * \rho_\varepsilon) \circ F) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(F) DF^i$$

yields the result. □

Definition 2.3 (directional derivative)**.** Let $h \in L^2([0,T])$ be a deterministic function and consider functions in $C_0([0,T])$ of the form

$$\gamma_t = \int_0^t h_s \mathrm{d}s \quad \text{for} \quad t \in [0,T], \tag{2.5}$$

which are called directions. The set of all such directions in $L^2([0,T])$ is called Cameron-Martin Space. Note that the map $t \mapsto \gamma_t$ is continuous on $[0,T]$ and $\gamma_0 = 0$; hence, $\gamma \in C_0([0,T])$. Then consider a random variable $F : [0,T] \times \Omega \to \mathbb{R}$, the directional derivative of F at the point $\omega \in \Omega$ in the direction γ_t is defined by

$$(D_\gamma F)_t(\omega) = \lim_{\epsilon \to 0} \frac{F_t(\omega + \epsilon\gamma) - F_t(\omega)}{\epsilon}$$

if the limit exists in $L^2([0,T] \times \Omega)$.

Remark 2.2. *(see [36, page 14]) If F is differentiable, then for every $t \in [0,T]$, there exists a random variable $D_t F : [0,t] \times \Omega \to \mathbb{R}$ such that*

$$(D_\gamma F)_t(\omega) = \int_0^T (D_t F)_t(\omega) h_t \mathrm{d}t \tag{2.6}$$

for all $\omega \in \Omega$ and $\gamma \in C_0([0,T])$ of the form in Equation 2.5.

Proposition 2.3 (product rule). *If $F, G \in \mathbb{D}^{1,2}([0,T]\times\Omega)$, then $FG \in \mathbb{D}^{1,2}([0,T]\times\Omega)$ and*

$$D_t(FG) = (D_tF)G + F(D_tG).$$

In other words,

$$(D_t(FG))_t(\omega) = (D_tF)_t(\omega)G_t(\omega) + F_t(\omega)(D_tG)_t(\omega)$$

for all $(t,\omega) \in [0,T]\times\Omega$.

Proof. Here we apply the same method as in [36, page 17]. Let $\omega \in \Omega$ and γ_t be of the form (2.5). Then according to Definition 2.3, we have that

$$\begin{aligned}
(D_\gamma(FG))_t(\omega) &= \lim_{\epsilon\to 0} \frac{F_t(\omega+\epsilon\gamma)G_t(\omega+\epsilon\gamma) - F_t(\omega)G_t(\omega)}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{F_t(\omega+\epsilon\gamma)G_t(\omega+\epsilon\gamma) - F_t(\omega)G_t(\omega+\epsilon\gamma) + F_t(\omega)G_t(\omega+\epsilon\gamma) - F_t(\omega)G_t(\omega)}{\epsilon} \\
&= \lim_{\epsilon\to 0} \left(\frac{F_t(\omega+\epsilon\gamma) - F_t(\omega)}{\epsilon} G_t(\omega+\epsilon\gamma) + F_t(\omega)\frac{G_t(\omega+\epsilon\gamma) - G_t(\omega)}{\epsilon} \right) \\
&= (D_\gamma F_t(\omega))G_t(\omega) + F_t(\omega)(D_\gamma G_t(\omega)) \\
&= \left(\int_0^T (D_tF)_t(\omega)h_t \mathrm{d}t \right) \cdot G_t(\omega) + F_t(\omega) \left(\int_0^T (D_tG)_t(\omega)h_t \mathrm{d}t \right) \quad \text{(by Remark 2.2)} \\
&= \int_0^T [(D_tF)_t(\omega)G_t(\omega) + F_t(\omega)(D_tG)_t(\omega)]h_t \mathrm{d}t
\end{aligned}$$

and $LHS = \int_0^T (D_tFG)_t(\omega)h_t\mathrm{d}t$, which holds for any $h \in L^2([0,T])$. Thus, for all $(t,\omega) \in [0,T]\times\Omega$, we have

$$(D_tFG)_t(\omega) = (D_tF)_t(\omega)G_t(\omega) + F_t(\omega)(D_tG)_t(\omega),$$

that is,

$$D_t(FG) = (D_tF)G + F(D_tG).$$

□

Lemma 2.2. *Suppose that F is a smooth random variable and $h \in \mathcal{H}$. Then*

$$\mathbb{E}[\langle DF, h\rangle_{\mathcal{H}}] = \mathbb{E}[FW(h)]. \tag{2.7}$$

Proof. Here we follow the steps similar to [35, page 26]. It suffices to prove the result for $\mathcal{H}$ with $||h|| = 1$ since otherwise we can normalise Equation 2.7. Let us set $e_1, \cdots, e_n$ to be orthonormal elements of $\mathcal{H}$ satisfying $e_1 = h$. Also, consider F to be a smooth random variable of the form

$$F = f(W(e_1), \cdots, W(e_n)),$$

where $f \in C_p^\infty(\mathbb{R}^n)$. Moreover, let $\phi(x)$ denote the probability density function (PDF) of the standard normal distribution on $\mathbb{R}^n$, i.e.,

$$\phi(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left(-\frac{1}{2}\sum_{i=1}^n x_i^2 \right).$$

Then,

$$\begin{aligned}\mathbb{E}[\langle DF, h\rangle_{\mathcal{H}}] &= \int_{\mathbb{R}^n} \frac{\partial f(x)}{\partial x_1}\phi(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^n} f(x)x_1\phi(x)\mathrm{d}x \\ &= \mathbb{E}[FW(e_1)] \\ &= \mathbb{E}[FW(h)]. \end{aligned}$$

□

Proposition 2.4 (partial integration)**.** *Suppose that F and G are smooth random variables, and let $h \in \mathcal{H}$. Then it holds that*

$$\mathbb{E}[G\langle DF, h\rangle_{\mathcal{H}}] + \mathbb{E}[F\langle DG, h\rangle_{\mathcal{H}}] = \mathbb{E}[FGW(h)]. \tag{2.8}$$

Proof. Since F and G are smooth random variables, we have

$$\begin{aligned}\mathbb{E}[FGW(h)] &= \mathbb{E}[\langle D(FG), h\rangle_{\mathcal{H}}] \quad \text{(by Lemma 2.2)} \\ &= \mathbb{E}[\langle (DF)G + F(DG), h\rangle_{\mathcal{H}}] \quad \text{(by Proposition 2.3)} \\ &= \mathbb{E}[G\langle DF, h\rangle_{\mathcal{H}}] + \mathbb{E}[F\langle DG, h\rangle_{\mathcal{H}}]. \end{aligned}$$

□

Proposition 2.5. *The derivative operator D is a closable operator from $L^p(\Omega)$ to $L^p(\Omega; \mathcal{H})$.*

Proof. See Nualart [35, page 26]. □

2.2 Skorohod integral

Definition 2.4 (Skorohod integral)**.** We denote by δ the adjoint of the operator D, called the divergence operator. That is, δ is an unbounded operator on $L^2(\Omega; \mathcal{H})$ with values in $L^2(\Omega)$ such that the domain of δ, denoted by $Dom(\delta)$, is the set of $\mathcal{H}$-valued square integrable random variables $u \in L^2(\Omega; \mathcal{H})$ satisfying

$$|\mathbb{E}[\langle DF, u\rangle_{\mathcal{H}}]| = \mathbb{E}\left[\int_0^\infty D_t F u(t)\mathrm{d}t\right] \leq K(u)||F||_2, \tag{2.9}$$

for all $F \in \mathbb{D}^{1,2}$, where $K(u)$ is some constant depending on u but independent of F. If u belongs to $Dom(\delta)$, then the Skorohod integral, $\delta(u)$, is defined as the element of $L^2(\Omega)$ characterized by

$$\mathbb{E}[F\delta(u)] = \mathbb{E}[\langle DF, u\rangle_{\mathcal{H}}] \tag{2.10}$$

for any $F \in \mathbb{D}^{1,2}$. Equation 2.10 is called the dual relationship.

Remark 2.3. *(1) The elements of $Dom(\delta) \subset L^2([0,T] \times \Omega)$ are square integrable processes, and*

$$\delta(u) = \int_0^T u(t)\delta W_t,$$

which is called the Skorohod integral of the process u.

(2) The Skorohod integral can be considered as an extension of the Itô integral for non-adapted process (see Proposition 2.7 below for more details).

(3) The operator δ is closed.

(4) If $u \in Dom(\delta)$, then taking $F = 1$ in Equation 2.10 will lead to $\mathbb{E}[\delta(u)] = 0$. In addition, δ is a linear operator in $Dom(\delta)$.

If u is a simple $\mathcal{H}$-valued random variable of the form

$$u = \sum_{j=1}^{n} F_j h_j,$$

where $F_j \in \mathbb{D}^{1,2}$ and $h_j \in \mathcal{H}$, then from Proposition 2.4 we can deduce that $u \in Dom(\delta)$ and

$$\delta(u) = \sum_{j=1}^{n} (F_j W(h_j) - \langle DF_j, h_j \rangle_{\mathcal{H}}). \tag{2.11}$$

Remark 2.4. *In details, Equation 2.11 can be proved as follows: For an arbitrary $F \in \mathbb{D}^{1,2}$ we have*

$$\begin{aligned} \mathbb{E}[G\delta(u)] &= \mathbb{E}[\langle DG, u \rangle_{\mathcal{H}}] \quad \textit{(by Equation 2.10)} \\ &= \sum_{j=1}^{n} \mathbb{E}[F_j \langle DG, h_j \rangle_{\mathcal{H}}] \\ &= \sum_{j=1}^{n} \mathbb{E}[GF_j W(h_j) - \mathbb{E}[G\langle DF_j, h_j \rangle_{\mathcal{H}}]]. \quad \textit{(by Proposition 2.4)} \\ &\leq K(u)||G||_2. \end{aligned}$$

In Equation 2.11, the expression $F_j W(h_j) - \langle DF_j, h_j \rangle_{\mathcal{H}}$ is called the Wick product (see e.g., [10, 38]) of the random variables F_j and $W(h_j)$, which is denoted by $F_j \diamond W(h_j)$. Then, with this notation Equation 2.11 can be written as

$$\delta(u) = \sum_{j=1}^{n} F_j \diamond W(h_j).$$

Later, we will apply the following notation

$$\delta(u) = \int_0^T u_t \diamond \mathrm{d}W_t, \tag{2.12}$$

where u belongs to the domain of δ.

Proposition 2.6. *Let $F \in \mathbb{D}^{1,2}$. For all $u \in Dom(\delta)$ such that $\mathbb{E}[\int_0^T F^2 u_t^2 \mathrm{d}t] < \infty$, we have $Fu \in Dom(\delta)$ and*

$$\begin{aligned} \delta(Fu) &= F\delta(u) - \langle DF, u \rangle_{\mathcal{H}} \\ &= F\delta(u) - \int_0^T D_t F u_t \mathrm{d}t \end{aligned}$$

whenever the right hand side belongs to $L^2(\Omega)$*. In particular, if* u *is moreover adapted, we will have*

$$\delta(Fu) = F\int_0^T u_t \mathrm{d}W_t - \int_0^T D_t F u_t \mathrm{d}t. \tag{2.13}$$

Proof. Let $G \in \mathcal{S}_0$ be any smooth random variable. Then, we have

$$\begin{aligned}
\mathbb{E}[\langle DG, Fu\rangle_{\mathcal{H}}] &= \mathbb{E}[\langle FDG, u\rangle_{\mathcal{H}}] \\
&= \mathbb{E}[\langle D(FG) - GDF, u\rangle_{\mathcal{H}}] \quad \text{(by Proposition 2.3)} \\
&= \mathbb{E}[\langle D(FG), u\rangle_{\mathcal{H}} - \langle GDF, u\rangle_{\mathcal{H}}] \quad \text{(by linearity)} \\
&= \mathbb{E}[(F\delta(u) - \langle DF, u\rangle_{\mathcal{H}})G], \quad \text{(by Definition 2.4)}
\end{aligned}$$

which implies the results. □

Proposition 2.7. *Let* $(t,\omega) \in [0,T] \times \Omega$ *and let* $u \in L^2([0,T]\times\Omega)$ *be a stochastic process on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that*

$$\mathbb{E}\left[\int_0^T u^2(t,\omega)\mathrm{d}t\right] < \infty,$$

and suppose that $u(t,\omega)$ *is* $\mathcal{F}_t$*-adapted for* $t \in [0,T]$*. Then,* $u \in Dom(\delta)$ *and*

$$\delta(u) = \int_0^T u(t,\omega)\delta W_t = \int_0^T u(t,\omega)\mathrm{d}W_t \tag{2.14}$$

for all $(t,\omega) \in [0,T]\times\Omega$*.*

Proof. The proof follows from Proposition 2.6 by setting $F = 1$. □

Proposition 2.8. *Let* $u(s,\omega)$ *be a stochastic process such that*

$$\mathbb{E}\left[\int_0^T u^2(s,\omega)\mathrm{d}s\right] < \infty$$

and assume that $u(s,\cdot) \in \mathbb{D}^{1,2}$ *for all* $s \in [0,T]$*, that* $D_t u \in Dom(\delta)$ *for all* $t \in [0,T]$*, and that*

$$\mathbb{E}\left[\int_0^T (\delta(D_t u))^2 \mathrm{d}t\right] < \infty.$$

Then $\delta(u) \in \mathbb{D}^{1,2}$ *and*

$$D_t(\delta(u)) = u(t,\omega) + \int_0^T D_t u(s,\omega)\mathrm{d}W_s.$$

Assume in addition that $u(s,\omega)$ *is* $\mathcal{F}_s$*-adapted. Then*

$$D_t\left(\int_0^T u(s,\omega)\mathrm{d}W_s\right) = u(t,\omega) + \int_t^T D_t u(s,\omega)\mathrm{d}W_s. \tag{2.15}$$

Proof. See Øksendal [10, page 5.6-5.8]. □

2.3 Stochastic differential equations

Let $X_t \in \mathbb{R}^n$ be an Itô diffusion process with the dynamic:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \quad X_0 = x, \quad x \in \mathbb{R}^n, \tag{2.16}$$

where $\{W_t, 0 \leq t \leq T\}$ is a d-dimensional standard Brownian motion, and the coefficients b and σ represent the deterministic drift and diffusion (volatility) of our process, respectively. Its integral form is given by

$$X_t = x + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s. \tag{2.17}$$

Assumption 2.1. b and σ in (2.16) are continuously differentiable with bounded derivatives, and they also satisfy the following Lipschitz conditions:

$$\begin{aligned} |b(t,x)| + |\sigma(t,x)| &\leq K_1(1 + |x|), \\ |b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| &\leq K_2|x - y| \end{aligned} \tag{2.18}$$

for any $x, y \in \mathbb{R}^n$ and $t \in [0, T]$ with K_1, K_2 positive constants.

Since X_0 is independent of the σ-algebra $\sigma(W_s, s \geq 0)$ and $\mathbb{E}[|X_0|^2]$ is finite, Assumption 2.1 guarantees that the SDE (2.16) admits a unique strong continuous solution on $[0, T]$, adapted to the filtration $(\mathcal{F}_t^{X_0})_{t \in [0,T]}$ generated by X_0 and (W_t), such that $\mathbb{E}[\int_0^T |X_s|^2 \mathrm{d}s] < \infty$. Then $(X_t, t \in [0, T])$ belongs to $\mathbb{D}^{1,2}$.

Assumption 2.2. The diffusion matrix σ is uniformly elliptic, that is, there exists $\epsilon > 0$ such that for every $t \in [0, T]$ and $\xi, x \in \mathbb{R}^d$ with $\xi \neq 0$, we have $\xi^T \sigma^T(t, x)\sigma(t, x)\xi \geq \epsilon|\xi|^2$, where ξ^T and σ^T denote the transpose of ξ and σ, respectively.

Assumption 2.2 ensures that the process $\{\sigma^{-1}(t, X_t)Y_t, 0 \leq t \leq T\}$ belongs to $L^2([0, T] \times \Omega)$. Let $\{Y_t, 0 \leq t \leq T\}$ denote the first variation process associated to $\{X_t, 0 \leq t \leq T\}$, which is defined by the SDE:

$$\begin{aligned} \mathrm{d}Y_t &= b'(t, X_t)Y_t\mathrm{d}t + \sigma'(t, X_t)Y_t\mathrm{d}W_t, \\ Y_0 &= I_n, \end{aligned} \tag{2.19}$$

where I_n is the identity matrix of $\mathbb{R}^n$ and the prime denotes the derivatives with respect to the second variable. It is easy to see that the first variation process is the derivative of $(X_t)_{t \in [0,T]}$ with respect to its initial condition x, that is, $Y_t = \frac{\partial X_t}{\partial x}$.

Corollary 2.1. *Let $u(s, \omega)$ be an $\mathcal{F}_s$-adapted stochastic process and assume that $u(s, \cdot) \in \mathbb{D}^{1,2}$ for all s. Then $D_t u(s, \omega)$ is $\mathcal{F}_s$-adapted for all t and $D_t u(s, \omega) = 0$ for $s < t$.*

Proof. See Øksendal [10, page 5.4]. □

Proposition 2.9. *Let $u(t,\omega)$ be an $\mathcal{F}_t$-adapted process and let $0 \le t < T$. Then,*

$$D_t\left(\int_0^T u(s,\omega)\mathrm{d}s\right) = \int_t^T D_t u(s,\omega)\mathrm{d}s.$$

Proof. This is an immediate consequence of Corollary 2.1. □

Proposition 2.10. *Let $(X_t)_{t\in[0,T]}$ be the solution to the SDE* (2.16)*. Then the Malliavin derivative can be written as an expression of the first variation process $(Y_t)_{t\in[0,T]}$ as well as the diffusion (volatility) matrix σ:*

$$D_s X_t = Y_t Y_s^{-1}\sigma(s,X_s)\mathbb{1}_{\{s\le t\}} \quad a.s. \tag{2.20}$$

Proof. The proof is based on the idea exhibited in [30, page 23]. Assume that the solution X_t (see Equation 2.17) belongs to $\mathbb{D}^{1,2}$. Taking the Malliavin derivative on the both sides of (2.17) by Proposition 2.9 and 2.8, we have for $s < t$,

$$\begin{aligned} D_s X_t &= D_s\left(\int_0^t b(r,X_r)\mathrm{d}r\right) + D_s\left(\int_0^t \sigma(r,X_r)\mathrm{d}W_r\right) \\ &= \int_s^t D_s b(r,X_r)\mathrm{d}r + \sigma(s,X_s) + \int_s^t D_s\sigma(r,X_r)\mathrm{d}W_r. \end{aligned}$$

Then applying the chain rule (see Proposition 2.1), we obtain

$$D_s X_t = \int_s^t b'(r,X_r)D_s X_r\mathrm{d}r + \sigma(s,X_s) + \int_s^t \sigma'(r,X_r)D_s X_r\mathrm{d}W_r.$$

Fix r and set $Z_t := D_s X_t$ for $s < t$, we have SDE

$$\mathrm{d}Z_t = b'(t,X_t)Z_t\mathrm{d}t + \sigma'(t,X_t)Z_t\mathrm{d}W_t, \tag{2.21}$$

with initial condition $Z_s = \sigma(s,X_s)$. By Itô's formula, we can obtain the solution to (2.21):

$$Z_t = \sigma(s,X_s)\exp\left\{\int_s^t [b'(r,X_r) - \frac{1}{2}(\sigma'(r,X_r))^2]\mathrm{d}r + \int_s^t \sigma'(r,X_r)\mathrm{d}W_r\right\}. \tag{2.22}$$

Applying Itô's formula again to (2.19), we get the exact solution

$$Y_t = \exp\left\{\int_0^t [b'(u,X_u) - \frac{1}{2}(\sigma'(u,X_u))^2]\mathrm{d}u + \int_0^t \sigma'(u,X_u)\mathrm{d}W_u\right\}.$$

The solution to Equation 2.22 for any fundamental matrix Y_t with initial value y at time $t = s$ is $Y_t Y_s^{-1} y$. Note that $Z_t = D_s X_t$ is such a solution. Therefore,

$$D_s X_t = Y_t Y_s^{-1}\sigma(s,X_s)\mathbb{1}_{s\le t}.$$

□

Remark 2.5. *When σ is hypoelliptic, we have for $s \le t$,*

$$Y_t = D_s X_t Y_s \sigma^{-1}(s,X_s),$$

that is,

$$Y_t = \int_0^T \frac{Y_t}{T}\mathrm{d}s = \frac{1}{T}\int_0^T D_s X_t Y_s \sigma^{-1}(s,X_s)\mathrm{d}s.$$

3 Computation of Greeks

In this section we will be concerned with applications of the Malliavin calculus to finance; in particular, we calculate the price sensitivity, i.e., Greeks. We first introduce the so-called Malliavin weighted scheme, where we prove the integration-by-parts formula on the Wiener space, the Bismut-Elworthy-Li formula and the eqation for Gamma. Then, we apply the theory to the classical Heston model and the fractional Black-Scholes (fBS) model (driven by a fractional Brownian motion (fBM)). We derive the closed-form Greeks, finite-difference Greeks and Malliavin Greeks for these two models, respectively.

We now briefly summarise the Greeks we are going to investigate.

Greeks	Notation	Definition
Delta	Δ	$\frac{\partial C_0}{\partial S_0}$
Gamma	Γ	$\frac{\partial^2 C_0}{\partial S_0^2}$
Vega	$\mathcal{V}$	$\frac{\partial C_0}{\partial \sigma}$

Table 1 The Greeks for Call options at time 0 with Call price C_0, initial underlying stock price S_0 and volatility σ.

3.1 Malliavin weighted scheme

In this part, we introduce the Malliavin weighted scheme in more details. The key propositions and theorems stated here are known in the literature and they were taken from [13, 29, 39].

Recall the Malliavin Greek:

$$\text{Malliavin Greek} = \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_0^T r_s \mathrm{d}s} f(X_T) \cdot weight]$$

Assume that the payoff depends on a finite set of payment dates, $t_1, \cdots, t_n, n = 1, 2, \cdots$, with $t_0 = 0$ and $t_n = T$. Then, given the initial underlying price x, the price of the contingent claim is computed as follows:

$$P(x) = \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_0^T r_s \mathrm{d}s} f(X_{t_1}, \cdots, X_{t_n}) | X_0 = x]$$

Let F denote the discounted payoff, $F := \mathrm{e}^{-\int_0^T r_s \mathrm{d}s} f(X_{t_1}, \cdots, X_{t_n})$. Then

$$P(x) = \mathbb{E}^{\mathbb{Q}}[F | X_0 = x].$$

Let us write the weight function *weight* as a Skorohod integral, and call weight function generator w the Skorohod integrand:

$$weight = \delta(w)$$

We assume that the weight function is L^2-integrable, i.e.,

$$\mathbb{E}[weight^2]^{1/2} < \infty. \tag{3.1}$$

This ensures the existence of the Skorohod integral. To avoid the degeneracy of the integration-by-parts formula used when computing the Malliavin Greeks with the probability one, we introduce the set Λ_n defined by

$$\Lambda_n = \left\{ a \in L^2([0,T]) | \int_0^{t_i} a(t)\mathrm{d}t = 1, \quad \forall i = 1, \cdots, n \right\}. \tag{3.2}$$

Lemma 3.1. *[30, page 38] Let $a \in \Lambda_n$ and $X_{t_i} \in \mathbb{D}^{1,2}$. Then*

$$\int_0^T D_t X_{t_i} a(t) \sigma^{-1}(t, X_t) Y(t) \mathrm{d}t = Y(t_i), \quad i = 1, \cdots, n.$$

Proof. The lemma can be easily proved by applying Proposition 2.10:

$$\begin{aligned}
LHS &= \int_0^T D_t X_{t_i} a(t) \sigma^{-1}(t, X_t) Y(t) \mathrm{d}t \\
&= \int_0^T Y(t_i) Y(t)^{-1} \sigma(t, X_t) \mathbb{1}_{\{t \le t_i\}} a(t) \sigma^{-1}(t, X_t) Y(t) \mathrm{d}t \quad \text{(by Proposition 2.10)} \\
&= \int_0^T Y(t_i) Y(t)^{-1} Y(t) \sigma(t, X_t) \sigma^{-1}(t, X_t) a(t) \mathbb{1}_{\{t \le t_i\}} \mathrm{d}t \\
&= \int_0^T Y(t_i) a(t) \mathbb{1}_{\{t \le t_i\}} \mathrm{d}t \\
&= \int_0^{t_i} Y(t_i) a(t) \mathrm{d}t \\
&= Y(t_i) \int_0^{t_i} a(t) \mathrm{d}t \\
&= Y(t_i) \quad \text{(by the definition of set } \Lambda_n) \\
&= RHS
\end{aligned}$$

□

Proposition 3.1 (integration-by-parts formula)**.** *Let F and G be two random variables such that $F, G \in \mathbb{D}^{1,2}$. Consider a random variable $u(t,\omega)$ for fixed ω, $u(t,\cdot) \in \mathcal{H}$ such that $\langle DF, u \rangle_{\mathcal{H}} \neq 0$ a.s. and $Gu(\langle DF, u \rangle_{\mathcal{H}})^{-1} \in Dom(\delta)$. Then for any continuously differentiable function f of polynomial growth, we have*

$$\mathbb{E}[f'(F)G] = \mathbb{E}\left[f(F) \delta \left(\frac{Gu}{\langle DF, u \rangle_{\mathcal{H}}} \right) \right]. \tag{3.3}$$

Proof. Here we follow the steps similar to [30, page 24]. Applying the chain rule (see Proposition 2.1) gives

$$\langle Df(F), u \rangle_{\mathcal{H}} = \langle f'(F) DF, u \rangle_{\mathcal{H}} = f'(F) \langle DF, u \rangle_{\mathcal{H}}.$$

The condition $\langle DF, u \rangle_{\mathcal{H}} \neq 0$ implies $f'(F) = \frac{\langle Df(F), u \rangle_{\mathcal{H}}}{\langle DF, u \rangle_{\mathcal{H}}}$. Hence,

$$\begin{aligned}
\mathbb{E}[f'(F)G] &= \mathbb{E}\left[\frac{\langle Df(F), u \rangle_{\mathcal{H}}}{\langle DF, u \rangle_{\mathcal{H}}} G \right] \\
&= \mathbb{E}\left[\left\langle Df(F), \frac{Gu}{\langle DF, u \rangle_{\mathcal{H}}} \right\rangle_{\mathcal{H}} \right], \quad \frac{Gu}{\langle DF, u \rangle_{\mathcal{H}}} \in Dom(\delta) \\
&= \mathbb{E}\left[f(F) \delta \left(\frac{Gu}{\langle DF, u \rangle_{\mathcal{H}}} \right) \right]. \quad \text{(by Equation 2.10)}
\end{aligned}$$

□

Remark 3.1. *It is easy to see that when $u = DF$, Equation 3.3 becomes*

$$\mathbb{E}[f'(F)G] = \mathbb{E}\left[f(F)\delta\left(\frac{GDF}{||DF||^2_{\mathcal{H}}}\right)\right].$$

Bismut [3] and Elworthy and Li [40] obtained elliptic results of Delta, namely the so-called Bismut-Elworthy-Li (BEL) formula.

Proposition 3.2 (BEL formula)**.** *Assume that b and σ in SDE* (2.16) *are continuously differentiable with bounded partial derivatives and that the matrix σ satisfies the uniform ellipticity condition, i.e., Assumption 2.1 and 2.2 hold. Then, for any $f : \mathbb{R}^d \to \mathbb{R}$ of polynomial growth and $a \in \Lambda_n$, we have*

$$\Delta = \mathbb{E}[\mathrm{e}^{-rT} f(X_{t_1}, \cdots, X_{t_n})\delta(w^{delta})|X_0 = x], \tag{3.4}$$

where $w^{delta} = a(t)(\sigma^{-1}(t, X_t)Y_t)^T$, δ denotes the Skorohod integral, and r is the constant risk-free interest rate.

Proof. This proof is based on the idea shown in [13, page 399] and [29, page 159]. To avoid heavy notations, in this proof we denote w^{delta} by w.

Step 1: Weaker conditions on the payoff function f. Let C_K^∞ represent the set of infinitely differentiable functions with compact support. We first show that if the result holds for any function in set C_K^∞, then it will also be true for any element of L^2.

Assume that the result holds for any function in C_K^∞ and take f that is only in L^2. Then since C_K^∞ is dense in L^2, there exists a sequence $(f_k)_{k\in\mathbb{N}}$ of C_K^∞ elements converging to f in L^2. Let us denote by

$$u_k(x) = \mathbb{E}[\mathrm{e}^{-rT} f_k(X_{t_1}, \cdots, X_{t_n})|X_0 = x] = \mathbb{E}[F_k|X_0 = x]$$

and

$$u(x) = \mathbb{E}[\mathrm{e}^{-rT} f(X_{t_1}, \cdots, X_{t_n})|X_0 = x] = \mathbb{E}[F|X_0 = x]$$

the price associated with F_k and F (discontinuous payoff functions) with the initial asset price x, respectively. Since u_k satisfies the L^2 convergence, it is clear that u_k simply converges to the function u, i.e., $u_k(x) \to u(x)$ as $k \to \infty$, $\forall x \in \mathbb{R}$.

This result holds for any payoff function in C_K^∞, which will lead to the fact that we can write the derivative of u_k as the expectation of the product of F_k and a Malliavin weight $\delta(w)$:

$$\frac{\partial}{\partial x} u_k(x) = \mathbb{E}[F_k \delta(w)|X_0 = x],$$

where $w = a(t)(\sigma^{-1}(t, X_t)Y_t)^T$ is the weight function generator. Define also the function

$$g(x) := \mathbb{E}[F\delta(w)|X_0 = x].$$

Then by the Cauchy-Schwartz inequality we have

$$\left|\frac{\partial}{\partial x} u_k(x) - g(x)\right| = |\mathbb{E}[(F_k - F)\delta(w)|X_0 = x]| \le \epsilon_k(x)h(x), \tag{3.5}$$

where

$$\epsilon_k(x) = (\mathbb{E}[(F_k - F)^2])^{\frac{1}{2}}, \quad h(x) = (\mathbb{E}[(\delta(w))^2)])^{\frac{1}{2}}.$$

Since u_k converges in L^2, then by definition, we have that $\epsilon_k(x) \to 0$ as $k \to \infty$. Hence,

$$\frac{\partial}{\partial x} u_k \to g \quad \text{uniformly on compact subsets of } \mathbb{R}^n.$$

Thus, we can conclude that the function u is continuously differentiable and that

$$\frac{\partial}{\partial x}\mathbb{E}[F|X_0 = x] = \mathbb{E}[F\delta(w)|X_0 = x].$$

Step 2: Interchanging the order of differentiation and expectation. By Step 1, we assume that f is an element of C_K^∞, then it is continuously differentiable with bounded partial derivatives, and we have

$$\frac{\mathrm{e}^{-rT} f(X_{t_1}^{x+h}, \cdots, X_{t_n}^{x+h}) - \mathrm{e}^{-rT} f(X_{t_1}^{x}, \cdots, X_{t_n}^{x})}{||h||} - \frac{\langle \mathrm{e}^{-rT}\sum_{i=1}^n \frac{\partial}{\partial x_i} f(X_{t_1}, \cdots, X_{t_n}) Y_{t_i}, h\rangle}{||h||} \to 0 \text{ a.s.}$$

as $||h|| \to 0$, where $Y_{t_i} = \frac{\partial X_{t_i}}{\partial x}$ and X_t^{x+h} denotes the solution X_t satisfying the initial condition $X_0^{x+h} = x + h$. This can also be written as follows:

$$\frac{F^{x+h} - F^x}{||h||} - \frac{\langle \frac{\partial}{\partial x} F, h\rangle}{||h||} \to 0 \text{ a.s. as } ||h|| \to 0$$

Since the payoff function f is assumed to have bounded partial derivatives, the term $\frac{\langle \frac{\partial}{\partial x} F, h\rangle}{||h||}$ is uniformly integrable in h. Let M denote a uniform bound of the partial derivatives of f, then by the Taylor-Lagrange theorem, we have

$$\frac{||F^{x+h} - F^x||}{||h||} \le M \sum_{i=1}^n \frac{||X_{t_i}^{x+h} - X_{t_i}^x||}{||h||}.$$

Moreover, we can show that $\sum_{i=1}^n \frac{||X_{t_i}^{x+h} - X_{t_i}^x||}{||h||}$ is uniformly integrable (see [41, page 246]), implying the uniform integrability of $\frac{||F^{x+h} - F^x||}{||h||}$. Then by the Dominated Convergence Theorem, it tells us that $\frac{F^{x+h} - F^x}{||h||} - \frac{\langle \frac{\partial}{\partial x} F, h\rangle}{||h||}$ converges to zero in L^1. Therefore, we conclude that

$$\frac{\partial}{\partial x}\mathbb{E}[F|X_0 = x] = \mathbb{E}\left[\frac{\partial}{\partial x} F|X_0 = x\right].$$

Step 3: Malliavin integration by parts. Following Step 2, we now compute Delta as follows:

$$\begin{aligned}
\Delta &= \frac{\partial}{\partial x}\mathbb{E}[\mathrm{e}^{-rT} f(X_{t_1}, \cdots, X_{t_n})|X_0 = x] \\
&= \mathbb{E}\left[\mathrm{e}^{-rT}\sum_{i=1}^n \frac{\partial}{\partial x_i} f(X_{t_1}, \cdots, X_{t_n}) \frac{\partial X_{t_i}}{\partial x} |X_0 = x\right] \\
&= \mathbb{E}\left[\mathrm{e}^{-rT}\sum_{i=1}^n \frac{\partial}{\partial x_i} f(X_{t_1}, \cdots, X_{t_n}) Y_{t_i} |X_0 = x\right],
\end{aligned}$$

where Y_{t_i} is the first variation process associated with X_{t_i}. Assume that $X_{t_i} \in \mathbb{D}^{1,2}$ and let $a(t) \in \Lambda_n$, then by Lemma 3.1 we have

$$Y_{t_i} = \int_0^T D_t X_{t_i} a(t)\sigma^{-1}(t, X_t) Y_t \mathrm{d}t.$$

Therefore, we have

$$\begin{aligned}
\Delta &= \mathbb{E}\left[\int_0^T \mathrm{e}^{-rT} \sum_{i=1}^n \frac{\partial}{\partial x_i} f(X_{t_1}, \cdots, X_{t_n}) D_t X_{t_i} a(t)\sigma^{-1}(t, X_t) Y_t \mathrm{d}t | X_0 = x\right] \\
&= \mathbb{E}\left[\int_0^T D_t(\mathrm{e}^{-rT} f(X_{t_1}, \cdots, X_{t_n})) a(t)\sigma^{-1}(t, X_t) Y_t \mathrm{d}t | X_0 = x\right] \quad \text{(by Proposition 2.1)} \\
&= \mathbb{E}\left[\mathrm{e}^{-rT} f(X_{t_1}, \cdots, X_{t_n}) \int_0^T a(t)(\sigma^{-1}(t, X_t) Y_t)^T \mathrm{d}W_t | X_0 = x\right] \quad \text{(by Proposition 3.1)} \\
&= \mathbb{E}\left[\mathrm{e}^{-rT} f(X_{t_1}, \cdots, X_{t_n}) \delta(w) | X_0 = x\right],
\end{aligned}$$

where $w^{delta} = a(t)(\sigma^{-1}(t, X_t) Y_t)^T$. □

Proposition 3.3. *Let Assumption 2.1 and 2.2 hold and let $f : \mathbb{R}^d \to \mathbb{R}$ be a function of polynomial growth. Then for any $a \in \Lambda_n$ we have*

$$\begin{aligned}
\Gamma &= \mathbb{E}[\mathrm{e}^{-rT} f(X_{t_1}, \cdots, X_{t_n}) \delta(w^{gamma}) | X_0 = x] \\
&= \mathbb{E}\left[\mathrm{e}^{-rT} f(X_{t_1}, \cdots, X_{t_n}) \delta\left(w^{delta}\delta(w^{delta}) + \frac{\partial}{\partial x} w^{delta}\right) | X_0 = x\right],
\end{aligned} \tag{3.6}$$

where $w^{delta} = a(t)(\sigma^{-1}(t, X_t) Y_t)^T$, δ denotes the Skorohod integral, and r is the constant risk-free interest rate.

Proof. We assume that f is continuously twice differentiable with bounded first and second order derivatives. We have

$$\begin{aligned}
\Gamma &= \frac{\partial^2}{\partial x^2} \mathbb{E}[\mathrm{e}^{-rT} f(X_{t_1}, \cdots, X_{t_n}) | X_0 = x] \quad \text{(by the definition of Gamma)} \\
&= \frac{\partial}{\partial x} \mathbb{E}[\mathrm{e}^{-rT} f(X_{t_1}, \cdots, X_{t_n}) \delta(w^{delta}) | X_0 = x] \\
&= \mathbb{E}\left[\mathrm{e}^{-rT} \delta(w^{delta}) \sum_{i=1}^n \frac{\partial}{\partial x_i}(f(X_{t_1}, \cdots, X_{t_n})) \frac{\partial X_{t_i}}{\partial x} | X_0 = x\right] \\
&\quad + \mathbb{E}\left[\mathrm{e}^{-rT} f(X_{t_1}, \cdots, X_{t_n}) \frac{\partial}{\partial x}(\delta(w^{delta})) | X_0 = x\right] \quad \text{(by Proposition 2.1)}
\end{aligned}$$

The first term on the RHS can be calculated very similar to the one in the computation of Delta, which is equal to $\mathbb{E}[\mathrm{e}^{-rT} f(X_{t_1}, \cdots, X_{t_n}) \delta(w^{delta}) \delta(w^{delta}) | X_0 = x]$. Therefore,

$$\begin{aligned}
\Gamma &= \mathbb{E}\left[\mathrm{e}^{-rT} f(X_{t_1}, \cdots, X_{t_n}) \left(\delta(w^{delta})\delta(w^{delta}) + \frac{\partial}{\partial x}(\delta(w^{delta}))\right) | X_0 = x\right] \\
&= \mathbb{E}\left[\mathrm{e}^{-rT} f(X_{t_1}, \cdots, X_{t_n}) \left(\delta(w^{delta})\delta(w^{delta}) + \delta\left(\frac{\partial}{\partial x}(w^{delta})\right)\right) | X_0 = x\right],
\end{aligned}$$

where we used the fact that one could invert the Skorohod integral operator $\delta(\cdot)$ and the differential operator $\frac{\partial}{\partial x}$ by the Dominated Convergence Theorem. Moreover, applying the linearity of the Skorohod integral operator can lead to the following:

$$\Gamma = \mathbb{E}\left[\mathrm{e}^{-rT} f(X_{t_1}, \cdots, X_{t_n})\delta\left(w^{delta}\delta(w^{delta}) + \frac{\partial}{\partial x} w^{delta}\right) |X_0 = x\right]$$

□

Next, we are going to apply the above theories to both the classical Heston model and the fBS model.

3.2 Example 1: Classical Heston model

We now consider the classical Heston model. Here we assume that the stochastic volatility in the model is mean reverting to make sure that the volatility will not settle to become zero or infinite. In addition, we assume that the model follows a square root diffusion.

Definition 3.1 (classical Heston model [42]). Heston's stochastic volatility model is specified as follows:

$$\begin{cases} \mathrm{d}S_t = S_t(r\mathrm{d}t + \sqrt{v_t})\mathrm{d}W_t^1, & (3.7) \\ \mathrm{d}v_t = \kappa(\theta - v_t)\mathrm{d}t + \nu\sqrt{v_t}\mathrm{d}W_t^2, & (3.8) \\ \mathrm{d}\langle W^1, W^2\rangle_t = \rho\mathrm{d}t, \quad \rho \in (-1, 1), & (3.9) \end{cases}$$

where S_t is the asset spot price at time t ($t \in [0, T]$), r is the drift coefficient (const risk-free interest rate). v_t is the variance driven by a square-root process with $\kappa > 0$ the mean reversion rate, $\theta > 0$ the long-run variance and $\nu > 0$ the volatility of variance. Moreover, W^1 and W^2 denote two correlated Wiener processes (standard Brownian motions) with correlation parameter ρ. We assume that the dynamics of the Heston model above are under a risk-neutral measure chosen by the market.

In this subsection, we will consider both European and digital (also called binary) Call options under the Heston model.

Closed-form Greeks First, we state Gatheral's [43, page 16] explicit solution to European-type option price for Heston model, which closely follows the original derivation of the Heston formula in [42] but with some changes in intermediate definitions.

Theorem 3.1. *In the classical Heston model, let S_0 denote the initial stock price, K the strike price, r risk-free interest rate, T maturity, $F = S_0\mathrm{e}^{rT}$ the time T forward price of the spot asset and $x = \log\left(\frac{F}{K}\right)$. Then the closed-form European Call option price at time 0 is*

$$C_0 = K\mathrm{e}^{-rT}(\mathrm{e}^x P_1 - P_0) = S_0 P_1 - K\mathrm{e}^{-rT} P_0, \tag{3.10}$$

where the first term is the present value of the asset upon optimal exercise, and the second term is the current value of the strike-price payment. Moreover, for $j = 0, 1$,

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left\{\frac{\exp(C_j\theta + D_j v_0 + \mathtt{i}kx)}{\mathtt{i}k}\right\}\mathrm{d}k$$

is Heston's characteristic function and

$$\begin{aligned}
C_j &= \kappa\left\{r_-(j)T - \frac{2}{\nu^2}\log\left(\frac{1-g(j)\mathrm{e}^{-d(j)T}}{1-g(j)}\right)\right\},\\
D_j &= r_-(j)\frac{1-\mathrm{e}^{-d(j)T}}{1-g(j)\mathrm{e}^{-d(j)T}},\\
g(j) &= \frac{r_-(j)}{r_+(j)},\\
r_\pm(j) &= \frac{\beta(j) \pm d(j)}{\nu^2},\\
\alpha(j) &= -\frac{k^2}{2} - \frac{\mathtt{i}k}{2} + \mathtt{i}jk,\\
\beta(j) &= \kappa - \rho\nu j - \rho\nu\mathtt{i}k,\\
d(j) &= \sqrt{\beta^2(j) - 4\alpha(j)\gamma},\\
\gamma &= \frac{\nu^2}{2}.
\end{aligned}$$

Theorem 3.2. *The closed-form Greeks for European Call options at time 0 under the Heston model have the form:*

$$\begin{aligned}
\Delta &= \frac{\partial C_0}{\partial S_0} = \frac{\partial C_0}{\partial x}\frac{\partial x}{\partial S_0} = K\mathrm{e}^{-rT}(\mathrm{e}^x P_1 + \mathrm{e}^x P_1' - P_0')\cdot\frac{1}{S_0},\\
\Gamma &= \frac{\partial^2 C_0}{\partial S_0^2} = K\mathrm{e}^{-rT}(\mathrm{e}^x P_1' + \mathrm{e}^x P_1'' + \mathrm{e}^x P_0' - P_0'')\cdot\frac{1}{S_0^2},
\end{aligned}$$

where for $j = 0, 1$*:*

$$\begin{aligned}
P_j' &= \frac{\partial P_j}{\partial x} = \frac{1}{\pi}\int_0^\infty \mathrm{Re}[\exp(C_j\theta + D_j v_0 + \mathtt{i}kx)]\mathrm{d}k,\\
P_j'' &= \frac{\partial^2 P_j}{\partial x^2} = \frac{1}{\pi}\int_0^\infty \mathrm{Re}[\mathtt{i}k\exp(C_j\theta + D_j v_0 + \mathtt{i}kx)]\mathrm{d}k.
\end{aligned}$$

Proof. Notice that both C_j and D_j are independent of x, we can easily drive the above results from Equation 3.10. We omit the details. □

By applying the same technique as in Gatheral's paper [43], we can derive the following result.

Theorem 3.3. *In the classical Heston model, the closed-form price at time 0 for a digital Call option with payoff* $\mathbb{1}_{\{S_T>K\}}$ *is*

$$D_0 = \mathrm{e}^{-rT}P_0 \tag{3.11}$$

with P_0 *and other parameters as in Theorem 3.1.*

Equation 3.11 can also be derived from Equation 3.10 as follows. Note that for sufficiently small $\varepsilon > 0$, we have the approximation

$$\mathbb{1}_{\{S_T > K\}} \approx \frac{(S_T - (K - \varepsilon))^+ - (S_T - K)^+}{-\varepsilon},$$

that is, the digital Call price can be expressed as the negative derivative of European Call price with respect to the strike K:

$$\begin{aligned} D_0 &= -\frac{\mathrm{d}C_0}{\mathrm{d}K} \quad \text{(see Equation 3.10 for } C_0) \\ &= \mathrm{e}^{-rT} P_0 \end{aligned}$$

Based on this, we can easily derive the following Greeks.

Theorem 3.4. *The closed-form Greeks for digital Call options at time 0 under the Heston model are*

$$\begin{aligned} \Delta &= \frac{\partial D_0}{\partial S_0} = \frac{\partial D_0}{\partial P_0} \frac{\partial P_0}{\partial x} \frac{\partial x}{\partial S_0} = \mathrm{e}^{-rT} P_0' \cdot \frac{1}{S_0}, \\ \Gamma &= \frac{\partial^2 D_0}{\partial S_0^2} = \mathrm{e}^{-rT} P_0'' \cdot \frac{1}{S_0^2} \end{aligned}$$

with P_0' and P_0'' as in Theorem 3.2.

Finite-difference Greeks The form of the finite difference approximation is well-known.

Theorem 3.5. *The Greeks for options with price C_0 at time 0 computed by using finite difference method (central difference scheme) have the form*

$$\begin{aligned} \Delta &\approx \frac{C_0(S_0 + h) - C_0(S_0 - h)}{2h}, \\ \Gamma &\approx \frac{C_0(S_0 + h) - 2C_0(S_0) + C_0(S_0 - h)}{h^2}, \\ \mathcal{V} &\approx \frac{C_0(\sigma + h) - C_0(\sigma - h)}{2h} \end{aligned}$$

with sufficiently small h.

Malliavin Greeks Now we drive the Malliavin Greeks for the Heston model. First, we state the Novikov's condition [44].

Theorem 3.6 (Novikov's condition)**.** *Suppose that $(X_t)_{0 \le t \le T}$ is a real-valued adapted process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $(W_t)_{0 \le t \le T}$ is an adapted Brownian motion. Define the following stochastic process:*

$$\varepsilon\left(\int_0^t X_s \mathrm{d}W_s\right) = \exp\left(\int_0^t X_s \mathrm{d}W_s - \frac{1}{2}\int_0^t X_s^2 \mathrm{d}s\right), \quad t \in [0, T],$$

where ε denotes the Doléans-Dade exponential. Then it is a martingale under the probability measure $\mathbb{P}$ and the filtration $\mathcal{F}$ if the condition $\mathbb{E}\left[\mathrm{e}^{\frac{1}{2}\int_0^T |X|_t^2 \mathrm{d}t}\right] < \infty$ is satisfied.

Theorem 3.7 (Malliavin Delta for the classical Heston model)**.** *In the classical Heston model (see Definition 3.1), let the initial stock price* $S_0 = x > 0$*. Suppose that* b, κ, θ, ν *and the initial variance* v_0 *are all strictly positive, and the parameters satisfy* $2\kappa\theta > \nu^2$*, then for any* $f : \mathbb{R}^d \to \mathbb{R}$ *of polynomial growth we have*

$$\Delta = \mathbb{E}\left[e^{-rT} f(S_T) \int_0^T \frac{1}{xT\sqrt{1-\rho^2}\sigma_s} dZ_s \right], \tag{3.12}$$

where Z *is a standard Brownian motion independent of* W^2*.*

Proof. Here we follow the idea stated in [30, Section 3.5]. In order to ensure the existence and uniqueness of the solution to the Heston model, the parameters b, κ, θ, ν and initial variance v_0 all need to be strictly positive. In addition, the condition $2\kappa\theta > \nu^2$ (Novikov's condition in this special case) enables that

$$\sup_{0 \le t \le T} \mathbb{E}[\sigma_t^{-2}] < \infty,$$

which was proved by Cass and Friz in [31, page 17]. This condition guarantees that the variance process is always positive, i.e., $v_t > 0$ for all $t > 0$, and that

$$\int_0^T \frac{1}{\sigma_s} dW_s \in L^1,$$

where W denotes an arbitrary standard Brownian motion. Consider the square root process

$$\sigma_t := \sqrt{v_t}. \tag{3.13}$$

Applying Itô's lemma:

$$\begin{aligned}
d\sigma_t &= \frac{\partial}{\partial t}(\sqrt{v_t})dt + \frac{\partial}{\partial v}(\sqrt{v_t})dv_t + \frac{1}{2}\nu^2 v_t \frac{\partial^2}{\partial v^2}(\sqrt{v_t})dt \\
&= \frac{1}{2\sqrt{v_t}}dv_t - \frac{\nu^2}{8\sqrt{v_t}}dt \\
&= \frac{1}{2\sigma_t}\kappa(\theta - \sigma_t^2)dt + \frac{1}{2\sigma_t}\nu\sigma_t dW_t^2 - \frac{\nu^2}{8\sigma_t}dt \quad \text{(substituting (3.8) and } \sigma_t \text{ in)} \\
&= \left(\frac{\kappa\theta}{2\sigma_t} - \frac{\kappa\sigma_t}{2} - \frac{\nu^2}{8\sigma_t}\right)dt + \frac{\nu}{2}dW_t^2 \\
&= \left\{\left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right)\frac{1}{\sigma_t} - \frac{\kappa\sigma_t}{2}\right\}dt + \frac{\nu}{2}dW_t^2
\end{aligned} \tag{3.14}$$

In particular, we note that the Novikov's condition implies that the factor appearing in the drift term of σ_t satisfies

$$\frac{\kappa\theta}{2} - \frac{\nu^2}{8} \ge 0.$$

According to Yamada-Wanaatabe's lemma (see for example, [45, page 291, Proposition 2.13]), under the Novikov's condition, we have that v_t admits a unique strong solution to the SDE (3.8) for volatility process.

Assume that the volatility σ_t is Malliavin differentiable, i.e., $\sigma_t \in \mathbb{D}^{1,2}$. Let

$$W_t^1 = \rho W_t^2 + \sqrt{1-\rho^2} Z_t, \tag{3.15}$$

where Z is a standard Brownian motion independent of W^2. Then substituting (3.15) into the SDE (3.7) we get

$$\mathrm{d}S_t = S_t(r\mathrm{d}t + \sqrt{v_t}(\rho \mathrm{d}W_t^2 + \sqrt{1-\rho^2}\mathrm{d}Z_t)), \quad S_0 = x.$$

Define the logarithmic price $X_t = \log S_t$ to ensure that the asset price is always positive. By Itô's lemma, we have

$$\begin{aligned}
\mathrm{d}X_t &= \frac{\partial(\log S_t)}{\partial S_t}\mathrm{d}S_t + \frac{\partial(\log S_t)}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2(\log S_t)}{\partial S_t^2}\mathrm{d}S_t\mathrm{d}S_t \\
&= \frac{1}{S_t}\mathrm{d}S_t - \frac{1}{2S_t^2}\mathrm{d}S_t\mathrm{d}S_t \\
&= r\mathrm{d}t + \sqrt{v_t}(\rho\mathrm{d}W_t^2 + \sqrt{1-\rho^2}\mathrm{d}Z_t) - \frac{1}{2}(v_t\rho^2\mathrm{d}t + v_t(1-\rho^2)\mathrm{d}t) \\
&= \left(r - \frac{v_t}{2}\right)\mathrm{d}t + \sqrt{v_t}\rho\mathrm{d}W_t^2 + \sqrt{v_t}\sqrt{1-\rho^2}\mathrm{d}Z_t. \\
&= \left(r - \frac{\sigma_t^2}{2}\right)\mathrm{d}t + \sigma_t\rho\mathrm{d}W_t^2 + \sigma_t\sqrt{1-\rho^2}\mathrm{d}Z_t.
\end{aligned} \tag{3.16}$$

SDEs (3.14) and (3.16) can be written in integral forms as follows:

$$X_t = \log x + \int_0^t \left(r - \frac{\sigma_s^2}{2}\right)\mathrm{d}s + \int_0^t \rho\sigma_s\mathrm{d}W_s^2 + \int_0^t \sqrt{1-\rho^2}\sigma_s\mathrm{d}Z_s \tag{3.17}$$

$$\sigma_t = \sigma_0 + \int_0^t \left(\left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right)\frac{1}{\sigma_s} - \frac{\kappa\sigma_s}{2}\right)\mathrm{d}s + \int_0^t \frac{\nu}{2}\mathrm{d}W_s^2 \tag{3.18}$$

Then, writing (3.17) and (3.18) in matrix form:

$$\begin{pmatrix} X_t \\ \sigma_t \end{pmatrix} = \begin{pmatrix} \log x \\ \sigma_0 \end{pmatrix} + \int_0^t \begin{pmatrix} r - \frac{\sigma_s^2}{2} \\ \left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right)\frac{1}{\sigma_s} - \frac{\kappa\sigma_s}{2} \end{pmatrix}\mathrm{d}s + \int_0^t \underbrace{\begin{pmatrix} \sqrt{1-\rho^2}\sigma_s & \rho\sigma_s \\ 0 & \frac{\nu}{2} \end{pmatrix}}_{A} \begin{pmatrix} \mathrm{d}Z_s \\ \mathrm{d}W_s^2 \end{pmatrix} \tag{3.19}$$

The inverse matrix of A is calculated as follows:

$$\det A = \frac{\nu\sqrt{1-\rho^2}\sigma_s}{2},$$

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} \frac{\nu}{2} & -\rho\sigma_s \\ 0 & \sqrt{1-\rho^2}\sigma_s \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-\rho^2}\sigma_s} & \frac{-2\rho}{\nu\sqrt{1-\rho^2}} \\ 0 & \frac{2}{\nu} \end{pmatrix} := \sigma(s, X_s)^{-1}$$

The first variation process Y_t of $\begin{pmatrix} X_t \\ \sigma_t \end{pmatrix}$ is $Y_t := \frac{\partial}{\partial x}\begin{pmatrix} X_t \\ \sigma_t \end{pmatrix} = \begin{pmatrix} \frac{1}{x} \\ 0 \end{pmatrix}$. Then, by the BEL formula (see Proposition 3.2), for function $f^* : \mathbb{R}^d \to \mathbb{R}$ of polynomial growth, we have

$$\Delta = \mathbb{E}\left[\mathrm{e}^{-rT} f^*(X_T)\int_0^T a(s)(\sigma^{-1}(s, X_s)Y_s)^T\mathrm{d}W_s | X_0 = \log x\right]$$

$$= \mathbb{E}\left[\mathrm{e}^{-rT} f^*(X_T) \int_0^T a(s) \left(\begin{pmatrix} \frac{1}{\sqrt{1-\rho^2}\sigma_s} & \frac{-2\rho}{\nu\sqrt{1-\rho^2}} \\ 0 & \frac{2}{\nu} \end{pmatrix} \begin{pmatrix} \frac{1}{x} \\ 0 \end{pmatrix} \right)^T \begin{pmatrix} \mathrm{d}Z_s \\ \mathrm{d}W_s^2 \end{pmatrix} \right].$$

Choosing $a(s) = \frac{1}{T}$ and applying $(AB)^T = B^T A^T$, we obtain

$$\begin{aligned} \Delta &= \mathbb{E}\left[\mathrm{e}^{-rT} f^*(X_T) \int_0^T \frac{1}{T} \begin{pmatrix} \frac{1}{x} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1-\rho^2}\sigma_s} & 0 \\ \frac{-2\rho}{\nu\sqrt{1-\rho^2}} & \frac{2}{\nu} \end{pmatrix} \begin{pmatrix} \mathrm{d}Z_s \\ \mathrm{d}W_s^2 \end{pmatrix} \right] \\ &= \mathbb{E}\left[\mathrm{e}^{-rT} f^*(X_T) \int_0^T \frac{1}{T} \begin{pmatrix} \frac{1}{x\sqrt{1-\rho^2}\sigma_s} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d}Z_s \\ \mathrm{d}W_s^2 \end{pmatrix} \right] \\ &= \mathbb{E}\left[\mathrm{e}^{-rT} f^*(X_T) \int_0^T \frac{1}{T} \frac{1}{x\sqrt{1-\rho^2}\sigma_s} \mathrm{d}Z_s \right]. \end{aligned}$$

Finally, by applying it to the function $f := f^* \circ \exp$ we get the result

$$\Delta = \mathbb{E}\left[\mathrm{e}^{-rT} f(S_T) \int_0^T \frac{1}{T} \frac{1}{x\sqrt{1-\rho^2}\sigma_s} \mathrm{d}Z_s \right].$$

□

Remark 3.2. *Cass and Friz have carefully proved in [31] that the conclusion in Theorem 3.7 holds for any function $f \in \mathcal{J}(\mathbb{R})$ as an application of the extended BEL formula. This enables us to deal with European-type option payoffs as well as exotic ones, for instance, digital payoffs.*

Theorem 3.8 (Malliavin Gamma for the classical Heston model)**.** *In the classical Heston model (see Definition 3.1), let the initial stock price $S_0 = x > 0$. Suppose that b, κ, θ, ν and the initial variance v_0 are all strictly positive, and the parameters satisfy $2\kappa\theta > \nu^2$, then for any $f : \mathbb{R}^d \to \mathbb{R}$ of polynomial growth we have*

$$\Gamma = \mathbb{E}\left[\frac{\mathrm{e}^{-rT} f(S_T)}{x^2 T\sqrt{1-\rho^2}} \left\{ \frac{1}{T\sqrt{1-\rho^2}} \left(\left(\int_0^T \frac{1}{\sigma_s} \mathrm{d}Z_s \right)^2 - \int_0^T \frac{1}{\sigma_s^2} \mathrm{d}s \right) - \int_0^T \frac{1}{\sigma_s} \mathrm{d}Z_s \right\} \right]. \tag{3.20}$$

Proof. According to Proposition 3.3, we have

$$\Gamma = \mathbb{E}\left[\mathrm{e}^{-rT} f(S_T) \left(\delta(w^{delta})\delta(w^{delta}) + \delta\left(\frac{\partial}{\partial x}(w^{delta}) \right) \right) | S_0 = x \right],$$

where $\delta(w^{delta}) = \int_0^T \frac{1}{xT\sqrt{1-\rho^2}\sigma_s} \mathrm{d}Z_s$. Then

$$\delta\left(\frac{\partial w^{delta}}{\partial x} \right) = \frac{\partial}{\partial x}\delta(w^{delta}) = -\int_0^T \frac{1}{x^2 T\sqrt{1-\rho^2}\sigma_s} \mathrm{d}Z_s,$$

and

$$\begin{aligned} \delta(w^{delta})\delta(w^{delta}) &= \left(\int_{s=0}^T \frac{1}{xT\sqrt{1-\rho^2}\sigma_s} \mathrm{d}Z_s \right) \left(\int_{t=0}^T \frac{1}{xT\sqrt{1-\rho^2}\sigma_t} \mathrm{d}Z_t \right) \\ &= \left(\int_0^T \frac{1}{xT\sqrt{1-\rho^2}\sigma_s} \mathrm{d}Z_s \right)^2 - \left(\int_0^T \mathbb{E}\left[\left(\frac{1}{xT\sqrt{1-\rho^2}\sigma_s} \right)^2 \right] \mathrm{d}s \right) \end{aligned}$$

$$= \frac{1}{x^2T^2(1-\rho^2)} \left\{ \left(\int_0^T \frac{1}{\sigma_s} \mathrm{d}Z_s \right)^2 - \int_0^T \frac{1}{\sigma_s^2} \mathrm{d}s \right\}.$$

Therefore,

$$\Gamma = \mathbb{E}\left[\frac{\mathrm{e}^{-rT} f(S_T)}{x^2 T\sqrt{1-\rho^2}} \left\{ \frac{1}{T\sqrt{1-\rho^2}} \left(\left(\int_0^T \frac{1}{\sigma_s} \mathrm{d}Z_s \right)^2 - \int_0^T \frac{1}{\sigma_s^2} \mathrm{d}s \right) - \int_0^T \frac{1}{\sigma_s} \mathrm{d}Z_s \right\} \right].$$

□

3.3 Example 2: Fractional Black-Scholes model

Many studies have indicated that the so-called long-range dependence property is widespread in economics and finance (see e.g., [46,47]). However, the standard Brownian motion has no memory. Thus, the classical Black-Scholes model could be modified by replacing with a process that has long memory. We will see later that the fBM with an additional parameter $H \in (1/2, 1)$ has such property and is a suitable source of randomness. After such replacement, we obtain the so-called fractional Black-Scholes model (driven by fBM).

3.3.1 Fractional Brownian motion

The fBM was first introduced by Kolmogorov [48] in a study of turbulence [47] and developed by Mandelbrot and van Ness [49], in which the author demonstrated a stochastic integral representation of fBM in terms of a standard Brownian motion. We refer the reader to [35, 50, 52, 53] for further applications of fBM. Now we give its definition.

Definition 3.2 (fBM)**.** The fBM $\{W_t^H, t \geq 0\}$ is a continuous and centered Gaussian process with covariance function

$$\mathbb{E}[W_s^H W_t^H] = R_H(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

for all $s, t \geq 0$, where H is a real number in $(0, 1)$ and is called Hurst index or Hurst parameter associated with the fBM. H models the dependence structure in the stock price.

Due to the fact that fBM is not a semimartingale for $H \neq 1/2$ (see for example, [35, page 275]), we cannot apply the classical stochastic calculus. However, as stated in [35, 54, 55], the stochastic calculus of variations is valid on general Wiener spaces (i.e., valid for an arbitrary Gaussian process). Therefore, since the fBM is indeed a Gaussian process, we can apply the Malliavin calculus (e.g., Section 2) to it. Moreover, the so-called fractional white-noise theory has been constructed by many authors; based on this, the stochastic (Wick-Itô) integration and the Malliavin differentiation for fBM can be introduced. We refer the reader to [56–59] for details of white-noise theory, and to [35, 55, 60–62] for detailed Malliavin calculus with respect to fBM; here we list some basic definitions and properties taken from them.

Definition 3.3 (self-similarity). A stochastic process $X = (X_t)_{t\geq 0}$ such that $X_t \in \mathbb{R}$ is said to be self-similar if for any $a \geq 0$, there exists a function $g : \mathbb{R}^+ \to \mathbb{R}$ such that

$$X_{at} \stackrel{d}{=} g(a)X_t.$$

Property 3.1. *(see for example, [62]) By Definition 3.2, we can obtain that a fBM $(W_t^H)_{t\geq 0}$ with Hurst parameter $H \in (0,1)$ has the following properties:*

(1) $W_0^H = 0$ and $\mathbb{E}[W_t^H] = 0$ for all $t \geq 0$.

(2) W^H is a Gaussian process and $\mathbb{E}[(W_t^H)^2] = t^{2H}$, that is, $W_t^H \sim \mathcal{N}(0, t^{2H})$ for all $t \geq 0$.

(3) W^H is a self-similar process satisfying $W_{at}^H \stackrel{d}{=} a^H W_t^H$ for any $a \geq 0$, where $W_{at}^H \sim \mathcal{N}(0, (at)^{2H})$.

(4) W^H has stationary increments, i.e., $W_{t+s}^H - W_s^H$ has the same law of W_t^H (i.e., $W_{t+s}^H - W_s^H \stackrel{d}{=} W_t^H \sim \mathcal{N}(0, t^{2H})$) for $s, t \geq 0$. This can be shown by the following:

$$\begin{aligned}\mathbb{E}[(W_t^H - W_s^H)^2] &= \mathbb{E}[(W^H)_t^2] + \mathbb{E}[(W^H)_s^2] - 2\mathbb{E}[W_s^H W_t^H] \\ &= t^{2H} + s^{2H} - (t^{2H} + s^{2H} - |t-s|^{2H}) \\ &= |t-s|^{2H},\end{aligned}$$

so taking $t := t + s$, we can obtain $W_{t+s}^H - W_s^H \sim \mathcal{N}(0, t^{2H})$; hence, W^H has stationary increments.

(5) W^H has continuous trajectories.

Definition 3.4 (long-range dependence). A stationary sequence $(X_n)_{n\in\mathbb{N}}$ exhibits long-range dependence if the autocovariance functions $\rho(n) := \mathrm{Cov}(X_k, X_{k+n})$ satisfy

$$\lim_{n\to\infty} \frac{\rho(n)}{cn^{-\alpha}} = 1$$

for some constant c and $\alpha \in (0,1)$. In this case, the dependence between X_k and X_{k+n} decays slowly as n tends to infinity and

$$\sum_{n=1}^{\infty} \rho(n) = \infty.$$

Now we set $X_k := W_k^H - W_{k-1}^H$ and $X_{k+n} := W_{k+n}^H - W_{k+n-1}^H$ for $k \geq 1$. Then $\{X_k, k \geq 1\}$ is a Gaussian stationary sequence, which has unit variance and the covariance function as follows:

$$\begin{aligned}\rho_H(n) &= \mathbb{E}[X_k X_{k+n}] = \mathbb{E}[(W_k^H - W_{k-1}^H)(W_{k+n}^H - W_{k+n-1}^H)] \\ &= \mathbb{E}[W_k^H W_{k+n}^H] - \mathbb{E}[W_k^H W_{k+n-1}^H] - \mathbb{E}[W_{k-1}^H W_{k+n}^H] + \mathbb{E}[W_{k-1}^H W_{k+n-1}^H] \\ &= \frac{1}{2}(k^{2H} + (k+n)^{2H} - n^{2H}) - \frac{1}{2}(k^{2H} + (k+n-1)^{2H} - (n-1)^{2H}) \\ &\quad - \frac{1}{2}((k-1)^{2H} + (k+n)^{2H} - (n+1)^{2H}) + \frac{1}{2}((k-1)^{2H} + (k+n-1)^{2H} - n^{2H}) \\ &= \frac{1}{2}[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}] \\ &\approx H(2H-1)n^{2H-2} \to 0, \quad \text{as} \quad n \to \infty.\end{aligned}$$

The last line above can be obtained easily by binomial expansion or the central difference approximation of the second derivative of $f(n) := n^{2H}$ with $h = 1$.

In particular,

$$\lim_{n\to\infty} = \frac{\rho_H(n)}{H(2H-1)n^{2H-2}} = 1.$$

For $H > 1/2$, $\rho_H(n) > 0$ and

$$\sum_{n=1}^{\infty} \rho_H(n) = \infty,$$

and for $H < 1/2$, $\rho_H(n) < 0$ and

$$\sum_{n=1}^{\infty} |\rho_H(n)| < \infty.$$

Therefore, by Definition 3.4 we can conclude that the sequence $\{X_k, k \geq 1\}$ has the long-range dependence property for $H > 1/2$.

The long-range dependence and the self-similarity properties make the fBM (with $H \in (1/2, 1)$) a suitable model to describe financial quantities. Hence, from now on, we only consider the Hurst parameter $H \in (1/2, 1)$.

Remark 3.3. *If the Hurst parameter $H = 1/2$, it is easy to derive that the covariance $R_{\frac{1}{2}}(s,t) = \min(s,t)$, and the process $W^{\frac{1}{2}}$ is just a standard Brownian motion. Thus, the increments of the process are independent for $H = 1/2$. When $H > 1/2$ ($H < 1/2$), the increments of the process are positively (negatively) correlated.*

It is then clear that the second partial derivative of the covariance function R_H is integrable and has the form

$$\frac{\partial^2 R_H}{\partial t \partial s} = H(2H-1)|t-s|^{2H-2}.$$

We denote the coefficient by $H(2H-1) := \alpha_H$. Then, we can write

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s |r-u|^{2H-2} \mathrm{d}u \mathrm{d}r. \tag{3.21}$$

From Equation 3.21 we can obtain the scalar product in the Hilbert space $\mathcal{H}$:

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T |r-u|^{2H-2} \varphi_r \psi_u \mathrm{d}u \mathrm{d}r$$

for any pair of step functions φ and ψ in $\mathcal{E}$.

We denote by $|\mathcal{H}|$ the linear space of measurable functions $\varphi : [0,T] \to \mathbb{R}$ such that

$$||\varphi||^2_{|\mathcal{H}|} = \alpha_H \int_0^T \int_0^T |r-u|^{2H-2} |\varphi_r||\psi_u| \mathrm{d}u \mathrm{d}r < \infty.$$

From this we can find a linear space of functions contained in $\mathcal{H}$. In addition, $|\mathcal{H}|$ can be shown to be a Banach space with the norm $|| \cdot ||_{|\mathcal{H}|}$.

Remark 3.4. *As in the case of the classical Brownian motion (see Equation 2.12), we can also interpret the divergence operator associated with fBM W^H as a stochastic integral (see [55, 57, 58, 60]).*

3.3.2 Fractional Black-Scholes model

Definition 3.5 (fBS market)**.** [57, page 24] The fractional Black-Scholes market consists of two investment probabilities:

(1) A bank account or a bond with the short rate of interest $r > 0$ (constant):

$$\mathrm{d}B_t = rB_t\mathrm{d}t, \quad t \geq 0.$$

(2) A stock, which has price dynamics modelled by geometric fractional Brownian motion (GfBM):

$$\mathrm{d}S_t = \mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t^H, \quad t \geq 0 \tag{3.22}$$

with $S_0 = x > 0$, where the volatility $\sigma > 0$ and the underlying rate μ is greater than the short rate r, i.e., $\mu > r > 0$.

Remark 3.5. *[57, Example 3.14] The GfBM S_t (see Equation 3.22) has the solution*

$$S_t = x\exp\left(\mu t + \sigma W_t^H - \frac{1}{2}\sigma^2 t^{2H}\right), \quad t \geq 0. \tag{3.23}$$

There are two possibilities defining a stochastic integral with respect to the fBM: path-wise integrals and Wick-type integrals. Using the path-wise integral concept, the existence of arbitrages in the fBS model have been proved by many people, e.g., Roger [63], Shiryaev [47] and Cheridito [64]. On the other hand, Hu and Øksendal in [57], and Elliott and van der Hoek in [59] (see also [51, 58]) have suggested that the fBS model is "free of arbitrage" if we use Wick integral instead, that is, Equation 3.22 becomes

$$\mathrm{d}S_t = \mu S_t\mathrm{d}t + \sigma S_t \diamond \mathrm{d}W_t^H, \quad t \geq 0. \tag{3.24}$$

Applying the fractional white noise calculus with respect to the Wick integral, the following results were derived in [57, 59].

Theorem 3.9 (fBS formula)**.** *The European Call option price at time $t \in [0, T]$ under the fBS model is*

$$C_t = S_t\Phi(d_+) - K\mathrm{e}^{-r(T-t)}\Phi(d_-),$$

where Φ denotes the CDF of the standard normal distribution and

$$d_\pm = \frac{\log(S_t/K) + r(T-t) \pm \frac{\sigma^2}{2}(T^{2H} - t^{2H})}{\sigma\sqrt{T^{2H} - t^{2H}}}.$$

Remark 3.6. *At time 0, the Call price is*

$$C_0 = S_0\Phi(d_+) - K\mathrm{e}^{-rT}\Phi(d_-) \tag{3.25}$$

with

$$d_\pm = \frac{\log(S_0/K) + rT \pm \frac{\sigma^2}{2}T^{2H}}{\sigma T^H}.$$

Proof. Here we prove the fBS formula at time 0. We have the pricing formula:

$$C_0 = \mathrm{e}^{-rT} \int_{\mathbb{R}} \underbrace{\left(S_0 \exp\left[rT + \sigma y - \frac{1}{2}\sigma^2 T^{2H} \right] - K \right)^+}_{=:(S_T - K)^+ =: f(S_T)} \cdot \frac{\mathrm{e}^{-\frac{y^2}{2T^{2H}}}}{T^H \sqrt{2\pi}} \mathrm{d}y. \tag{3.26}$$

To derive the fBS formula, we need to evaluate the integral. First, $(\cdots) > 0$, where

$$S_0 \exp\left[rT + \sigma y - \frac{1}{2}\sigma^2 T^{2H} \right] - K > 0,$$

which leads to

$$y > \frac{\log(K/S_0) - rT + \frac{1}{2}\sigma^2 T^{2H}}{\sigma} := c.$$

Hence,

$$\begin{aligned}
C_0 &= S_0 \int_c^\infty \exp\left(\sigma y - \frac{1}{2}\sigma^2 T^{2H} \right) \frac{\mathrm{e}^{-\frac{y^2}{2T^{2H}}}}{T^H \sqrt{2\pi}} \mathrm{d}y - K\mathrm{e}^{-rT} \int_c^\infty \frac{\mathrm{e}^{-\frac{y^2}{2T^{2H}}}}{T^H \sqrt{2\pi}} \mathrm{d}y \\
&= S_0 \int_c^\infty \exp\left\{ -\frac{1}{2T^{2H}} \left(y^2 - 2\sigma y T^{2H} + \sigma^2 T^{4H} \right) \right\} \frac{1}{T^H \sqrt{2\pi}} \mathrm{d}y - K\mathrm{e}^{-rT} \int_{c/T^H}^\infty \frac{\mathrm{e}^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \mathrm{d}y \\
&= S_0 \int_c^\infty \exp\left\{ -\frac{1}{2T^{2H}} (y - \sigma T^{2H})^2 \right\} \mathrm{d}y - K\mathrm{e}^{-rT} \left(1 - \Phi\left(\frac{c}{T^H} \right) \right) \\
&= S_0 \int_{\frac{c - \sigma T^{2H}}{T^H}}^\infty \frac{\mathrm{e}^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \mathrm{d}y - K\mathrm{e}^{-rT} \Phi\left(-\frac{c}{T^H} \right) \\
&= S_0 \Phi\left(-\frac{c - \sigma T^{2H}}{T^H} \right) - K\mathrm{e}^{-rT} \Phi\left(-\frac{c}{T^H} \right).
\end{aligned}$$

Let

$$d_+ := -\frac{c - \sigma T^{2H}}{T^H} = \frac{\log(S_0/K) + rT + \frac{\sigma^2}{2} T^{2H}}{\sigma T^H},$$

and

$$d_- := -\frac{c}{T^H} = \frac{\log(S_0/K) + rT - \frac{\sigma^2}{2} T^{2H}}{\sigma T^H}.$$

Therefore, we obtain

$$C_0 = S_0 \Phi(d_+) - K\mathrm{e}^{-rT} \Phi(d_-).$$

□

Remark 3.7. *Regarding the contradiction about "arbitrage" issue proposed by two groups of people, Björk and Hult [65, page 198] analysed that this is because "[t]he very definitions of portfolio value and/or self-financing portfolios are completely different from their standard counterparts". They also claimed that the definition of Wick-financing portfolios and hence the arbitrage concept were not economically meaningful. This reveals one of the major drawbacks of the fBS model.*

Since the main purpose of this thesis is to apply Malliavin weighted scheme to compute Greeks, we ignore this aspect here. We then compute the Greeks in the fBS model.

Closed-form Greeks

Theorem 3.10. *The closed-form Greeks for European Call option at time 0 under the fBS model are*

$$\begin{aligned}\Delta &= \Phi(d_+),\\ \Gamma &= \frac{\phi(d_+)}{S_0\sigma T^H},\\ \mathcal{V} &= S_0 T^H \phi(d_+).\end{aligned}$$

Proof. The results can be derived easily from Equation 3.25. We omit the details. □

Finite-difference Greeks See Theorem 3.5.

Malliavin Greeks The Malliavin Greeks in the fBS model can be computed very similar to that in the classical Black-Scholes's case by applying Proposition 2.10 and Proposition 3.1.

Theorem 3.11. *The Greeks for European Call option at time 0 computed by using Malliavin weighted scheme under the fBS model are as follows:*

$$\begin{aligned}\Delta &= \mathrm{e}^{-rT}\mathbb{E}\left[f(S_T)\frac{W_T^H}{S_0\sigma T^{2H}}\right]\\ \Gamma &= \mathrm{e}^{-rT}\mathbb{E}\left[f(S_T)\left(\frac{(W_T^H)^2}{\sigma T^{2H}} - W_T^H - \frac{1}{\sigma}\right)\frac{1}{S_0^2\sigma T^{2H}}\right]\\ \mathcal{V} &= \mathrm{e}^{-rT}\mathbb{E}\left[f(S_T)\left(\frac{(W_T^H)^2}{\sigma T^{2H}} - W_T^H - \frac{1}{\sigma}\right)\right]\end{aligned}$$

Proof. Here we give a simpler proof by using the classical integration-by-parts formula (but with the same ideology as in the Malliavin weighted scheme). We prove the expression for Delta only. Gamma and Vega can be derived similarly. Let $S_0 = x > 0$ and use Equation 3.26, we have

$$f(S_T) = (S_T - K)^+, \quad S_T = x\exp\left(rT + \sigma y - \frac{1}{2}\sigma^2 T^{2H}\right).$$

Then

$$\begin{aligned}\frac{\partial f(S_T)}{\partial y} &= \frac{\partial f}{\partial S_T}\cdot\frac{\partial S_T}{\partial y} = \frac{\partial S_T}{\partial y} = \sigma S_T \Longrightarrow S_T = \frac{1}{\sigma}\frac{\partial f}{\partial y},\\ \frac{\partial f(S_T)}{\partial x} &= \frac{\partial S_T}{\partial x} = \frac{S_T}{x} = \frac{1}{x\sigma}\frac{\partial f}{\partial y}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta = \frac{\partial C_0}{\partial x} &= \frac{\partial}{\partial x}\int_{\mathbb{R}} \mathrm{e}^{-rT} f(S_T)\cdot\frac{\mathrm{e}^{-\frac{y^2}{2T^{2H}}}}{T^H\sqrt{2\pi}}\mathrm{d}y\\ &= \mathrm{e}^{-rT}\int_{\mathbb{R}}\frac{\partial f(S_T)}{\partial x}\cdot\frac{\mathrm{e}^{-\frac{y^2}{2T^{2H}}}}{T^H\sqrt{2\pi}}\mathrm{d}y\\ &= \mathrm{e}^{-rT}\int_{\mathbb{R}}\frac{1}{x\sigma}\frac{\partial f}{\partial y}\cdot\frac{\mathrm{e}^{-\frac{y^2}{2T^{2H}}}}{T^H\sqrt{2\pi}}\mathrm{d}y\end{aligned}$$

$$= \mathrm{e}^{-rT} \int_{\mathbb{R}} f(S_T) \frac{y}{x\sigma T^{2H}} \cdot \frac{\mathrm{e}^{-\frac{y^2}{2T^{2H}}}}{T^H\sqrt{2\pi}} \mathrm{d}y \quad \text{(classical integration by parts)}$$
$$= \mathrm{e}^{-rT} \mathbb{E}\left[f(S_T) \frac{W_T^H}{S_0 \sigma T^{2H}} \right], \quad y = W_T^H \sim \mathcal{N}(0, T^{2H}). \qquad \square$$

4 Numerical experiments and discussions

In the previous sections, we have derived the closed-form, finite-difference and Malliavin Greeks in the Heston model and the fBS model, respectively. In this section, we conduct the numerical experiments in these two models to show the efficiency of the Malliavin weighted scheme by comparing it with the finite difference method. We first compute the closed-form Greeks by using Mathematica (see Appendix A for the code). Then, we apply Monte Carlo methods to simulate the (Malliavin and finite-difference) Greeks by using C++. Finally, the results as well as future research are discussed.

4.1 Classical Heston model

Here, we consider the classical Heston model. We first compute the closed-form Greeks. The parameters used and the outputs are summarised in Table 2 and Table 3, respectively.

S_0	K	r	T	v_0	θ	κ	ν	ρ
100	100	0	1	0.1	0.08	4.0	0.6	−0.7

Table 2 The parameters used in Heston model.

	Call Price	Delta	Gamma
European Call	11.0659	0.6072	0.0142
Digital Call	0.4966	0.0142	−0.0001

Table 3 Exact values of both European and digital Call option prices and the associated Greeks at time 0 under the Heston model (accurate up to four decimal places).

It is notable that in our case, the exact Delta for a digital Call option with payoff $\mathbb{1}_{\{S_T>K\}}$ is the same as that for a European Call option with payoff $(S_T - K)^+$. As stated in [66, Section 17.4.1], the payoff of the digital Call option is identical (but not equal) to the Delta of a vanilla Call option. As a consequence, according to [67, Section 8.3.2], Marroni and Perdomo pointed out that “qualitatively, the Greeks of the digital option can be thought of higher orders of the Greeks of a vanilla call option”; that is to say, for example, the Delta for a digital Call has the same shape as the Gamma for a vanilla Call. In our case, we take the risk-free interest rate to be zero, i.e., $r = 0$. All these support our findings. In particular, we give an easy example under the Black-Scholes (BS) model.

	Call Price	Delta	Gamma
European Call	$S_0\Phi(d_1) - K\mathrm{e}^{-rT}\Phi(d_2)$	$\Phi(d_1)$	$\frac{\partial \Phi(d_1)}{\partial S_0}$
Digital Call	$\mathrm{e}^{-rT}\Phi(d_2) = \mathrm{e}^{-rT}\Phi(d_1 - \sigma\sqrt{T})$	$\mathrm{e}^{-rT}\frac{\partial \Phi(d_1 - \sigma\sqrt{T})}{\partial S_0}$	< 0

Table 4 Closed-form expressions of Call price and the associated Greeks for European and digital Call options (payoff $\mathbb{1}_{\{S_T > K\}}$) at time 0 in the BS model.

From Table 4, it is clear in the sense that the Call price of digital Call has the same shape of the Delta for a European Call, and so on.

Now we introduce the key simulation procedures for both a European Call and a digital Call. We first implement Equation 3.17 and 3.18, where X is the logarithmic price. From this, we can obtain $S_T = \exp(X_T)$ at time T. Then, the payoff $f(S_T) = (S_T - K)^+$ for European Call and $f(S_T) = \mathbb{1}_{\{S_T > K\}}$ for digital Call can be calculated. Finally, based on Theorem 3.5 and Theorem 3.7, we can get the simulated Greeks by applying Monte Carlo methods. Note that all the finite integrals involved here are approximated by using left Riemann sum method; each integral is divided into $n = 10000$ small partitions and a numerical inversion is done for each interval. In order to ensure that the volatility σ never hits zero, we tried two different methods: the one is to keep generating a new Brownian motion W^2 until $\sigma > 0$ (since in Equation 3.18, W^2 is the only random number), the other is to set a small positive lower bound for σ, for example, we can set $\sigma = T/n$ when it is non-positive. As a result, they both perform well. Last but not least, the number of simulations N from 1 to 80000 verses the value of Delta is plotted again by using Mathematica. Notice that the step size is set to be $s = 80$, which means that we independently

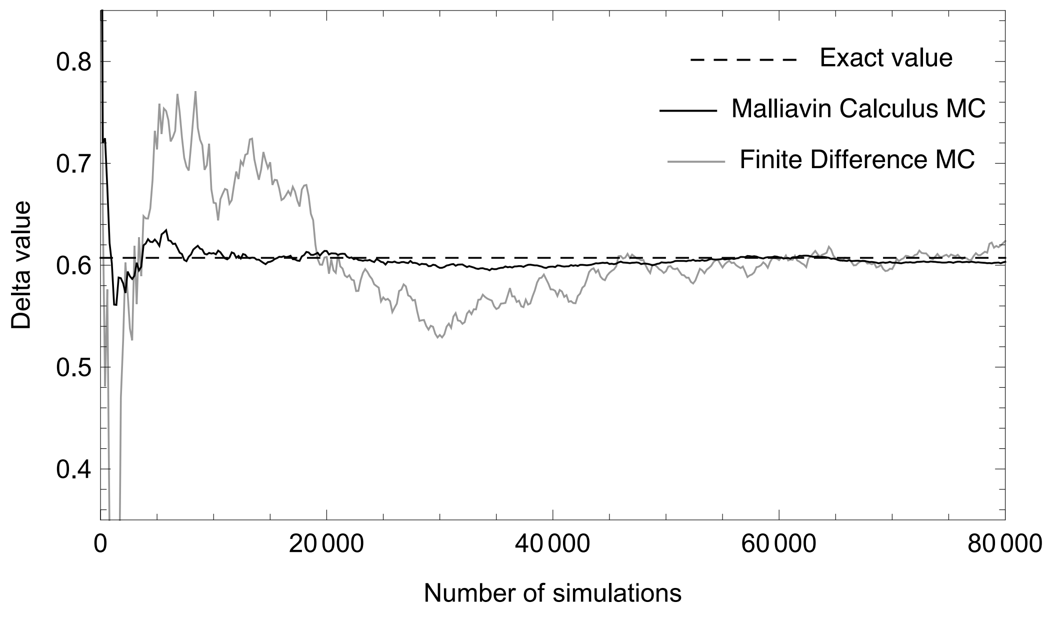

Figure 1 Delta for a European Call option with payoff function $f(x) = (x - K)^+$ and parameters as in Table 2 under the Heston model.

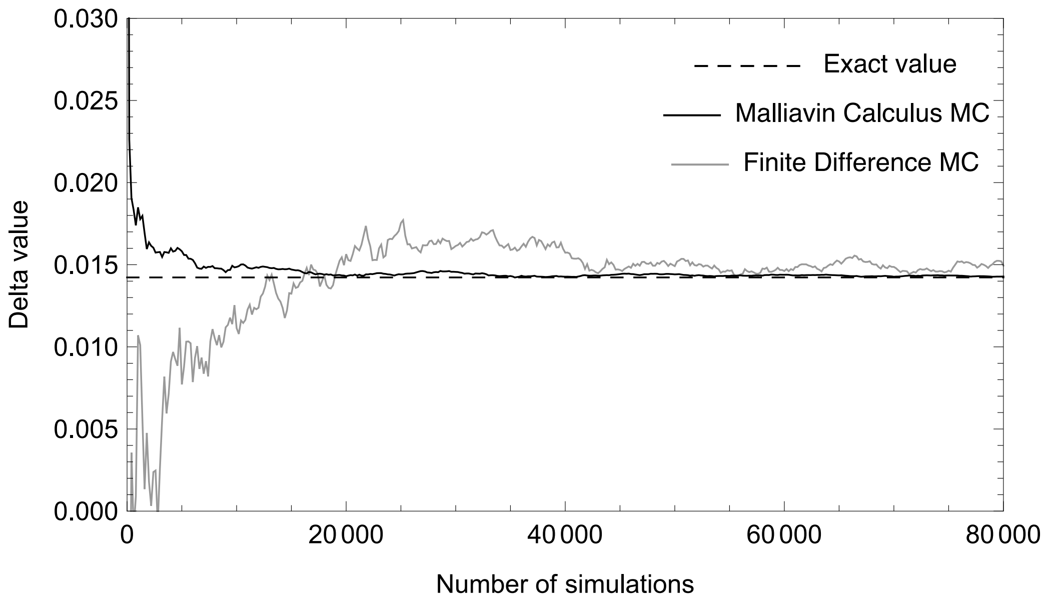

Figure 2 Delta for a binary Call option with payoff function $f(x) = \mathbb{1}_{\{x>K\}}$ and parameters as in Table 2 under the Heston model.

conduct Monte Carlo N/s times in total. The purpose of this is to make the graph clear and save some time. Furthermore, in the case of finite difference method, we choose the central difference $h = 1$ for European Call and $h = 0.7$ for digital Call.

In the case of a European Call option, we tried several different small values of the central difference h, e.g., $h = 0.001, 0.02, 0.05, 0.07$. However, the error of the difference between $C_0(S_0+h)$ and $C_0(S_0 - h)$ is quite big, and the output Delta even becomes a large negative number. Due to this, we then tried some bigger hs, e.g., $h = 1, 1.2, 1.5$. The results shown that when $h = 1$, Delta converges the fastest and the graph is plotted below.

From Figure 1 we can see that the Malliavin weighted scheme significantly outperforms the finite difference method. This is surprising since in theory, when the payoff function is smooth enough (e.g., for vanilla options), the finite difference method would perform better than the Malliavin calculus method. After discussing with Dr Thomas Cass, we tried a different strike price $K = 80$ and kept other parameters the same (i.e., starting from in-the-money). However, the Malliavin calculus method still gives a better convergence. This is puzzling, and it is worth investigating the reasons behind this output.

In the case of a digital Call (with discontinuous payoff), we again tried many hs when using the finite difference method, and we found that $h = 0.7$ gives the best convergence. As can be seen from Figure 2, the Malliavin weighted scheme is much more efficient than the finite difference method, as expected. The reason is that the former has lower simulation variance and converges faster (see for example, [29, Section 2.2.2]).

4.2 Fractional Black-Scholes model

In this subsection, we implement the Greeks in the fBS model. First, we compute the closed-form Greeks. The parameters used and the outputs are summarised in Table 5 and Table 6, respectively.

S_0	K	r	σ	T	H	h
100	100	0.05	0.15	1	0.7	0.5

Table 5 The parameters used in fBS model.

Call Price	Delta	Gamma	Vega
5.6560	0.6585	0.0245	36.7032

Table 6 Exact values of European Call option price and the associated Greeks at time 0 under the fBS model (accurate up to four decimal places).

We then simulate the Delta, Vega and Gamma for a European Call option, respectively. We first apply the stock price at time T, $S_T = S_0 \exp\left[rT + \sigma y - \frac{1}{2}\sigma^2 T^{2H}\right]$, where $y = W^H \sim \mathcal{N}(0, T^{2H})$. Then based on Theorem 3.5 and 3.11, we apply Monte Carlo methods. We finally plot the number of simulations N verses the value of Greeks in Figure 3, 4 and 5. Here N is from 1 to 20000 with step size $s = 80$; beyond $N = 20000$, the convergence is not notably improved, so we only plot this range of N. We find that for European Call options, Malliavin weighted scheme underperforms the finite difference method, as expected. Moreover, the Malliavin calculus method is more efficient for second-order Greeks (for example, Gamma) than first-order ones (for example, Delta).

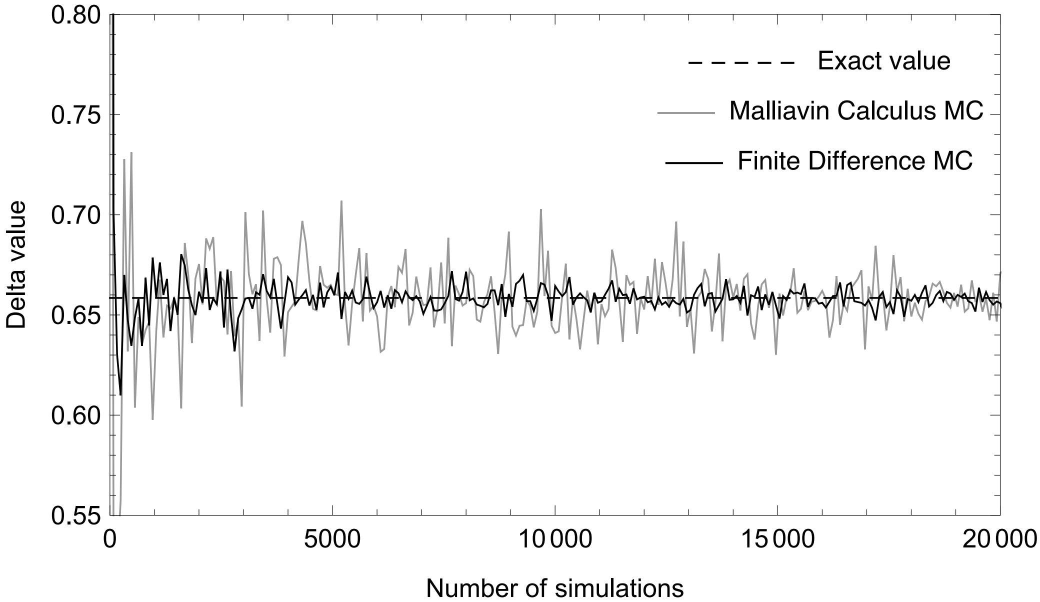

Figure 3 Delta for a European Call option with parameters as in Table 5 under the fBS model.

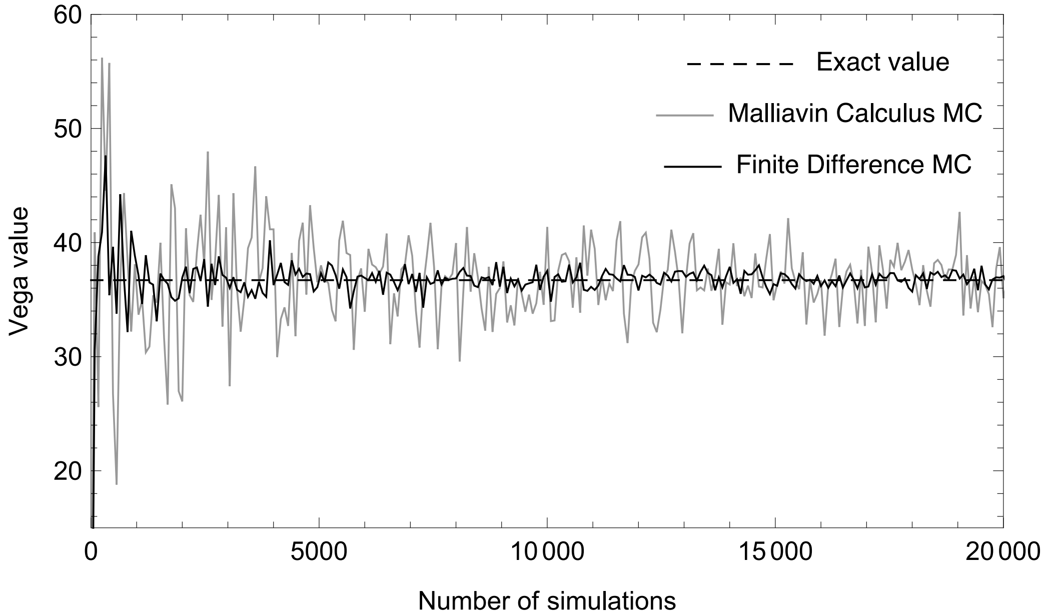

Figure 4 Vega for a European Call option with parameters as in Table 5 under the fBS model (in this case, $h = 0.00025$).

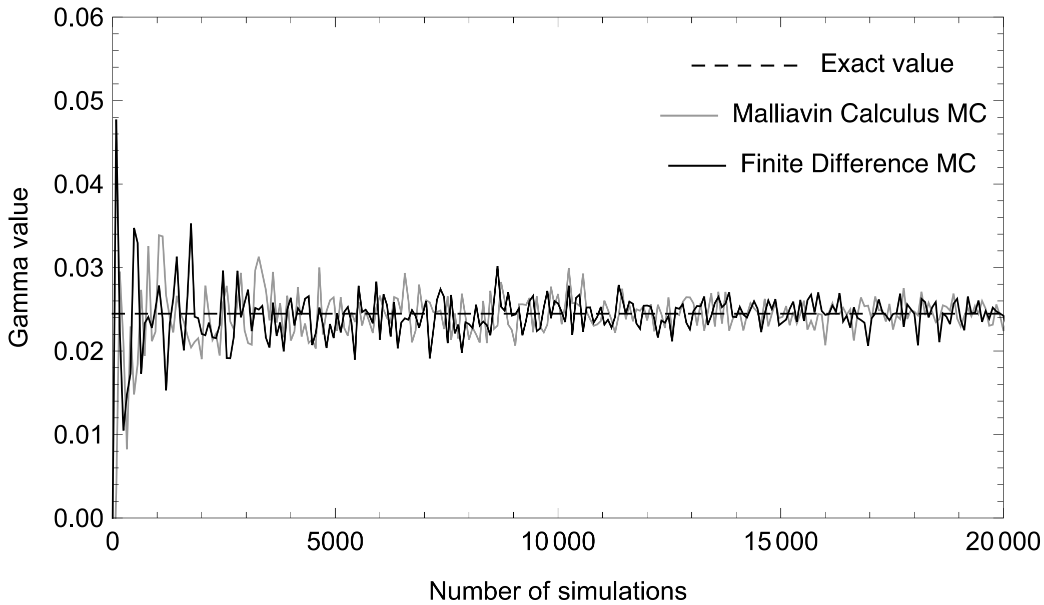

Figure 5 Gamma for a European Call option with parameters as in Table 5 under the fBS model.

4.3 Wider implications and future research

So far, we have investigated the applications of Malliavin weighted scheme in both the classical Heston model and the fBS model. As shown in [53], log-volatility can be well modelled using the fBM with Hurst parameter of order 0.1. Now we consider the fractional versions of Heston model as defined in [68], called the rough Heston model.

Definition 4.1 (rough Heston model)**.** The rough Heston model has dynamics:

$$\begin{cases} \mathrm{d}S_t = S_t\sqrt{v_t}\mathrm{d}W_t^1, \quad S_0 = x > 0 & (4.1) \\ v_t = v_0 + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}\kappa(\theta - v_s)\mathrm{d}s + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}\nu\sqrt{v_s}\mathrm{d}W_s^2 & (4.2) \\ \mathrm{d}\langle W^1, W^2\rangle_t = \rho\mathrm{d}t, \quad \rho \in (-1,1) & (4.3) \end{cases}$$

The parameters κ, θ, v_0 and ν in (4.2) are all strictly positive and have the same meaning as in the classical Heston model (see Definition 3.1). $\alpha \in (1/2, 1)$ denotes the smoothness of the volatility sample paths. When $\alpha = 1$, this becomes the classical Heston model. Here we set the risk-free interest rate $r = 0$.

Following the similar procedures as in the proof of Theorem 3.7, we can obtain

$$X_t = \log x - \int_0^t \frac{v_s}{2}\mathrm{d}s + \int_0^t \rho\sqrt{v_s}\mathrm{d}W_s^2 + \int_0^t \sqrt{1-\rho^2}\sqrt{v_s}\mathrm{d}Z_s, \tag{4.4}$$

where $X_t = \log S_t$ is the logarithmic price, W^2 and Z are two independent standard Brownian motions. Applying again the left Riemann sum method to the integrals contained in Equation (4.2) and (4.4), we can get the estimated value of X_T, and hence the stock price at time T, $S_T = \exp(X_T)$. Thus, the Call price can be calculated by simply implementing option's payoff function. From this, the finite difference approximation can be conducted straightforward.

Now we try to derive the Malliavin Greeks in this model. Consider the square root process $\sigma_t := \sqrt{v_t}$ and apply Itô's lemma to it, and define $g(s) := \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}$ $(s \in [0,t))$, we can obtain

$$\sigma_t = \sigma_0 + \int_0^t \left(\left(\frac{\kappa\theta}{2}\cdot g(s) - \frac{\nu^2}{8}\right)\frac{1}{\sigma_s} - \frac{\kappa\sigma_s}{2}\cdot g(s)\right)\mathrm{d}t + \int_0^s \frac{\nu}{2}\cdot g(s)\mathrm{d}W_s^2. \tag{4.5}$$

In this case, Novikov's condition particularly implies that

$$\frac{\kappa\theta}{2}\cdot g(s) - \frac{\nu^2}{8} \geq 0 \tag{4.6}$$

for all $s \in [0,t)$. It is easy to see that function $g(s)$ is strictly increasing for $\alpha \in (1/2, 1)$.

Writing (4.4) and (4.5) in matrix form:

$$\begin{pmatrix} X_t \\ \sigma_t \end{pmatrix} = \begin{pmatrix} \log x \\ \sigma_0 \end{pmatrix} + \int_0^t \begin{pmatrix} -\frac{\sigma_s^2}{2} \\ \left(\frac{\kappa\theta}{2}g(s) - \frac{\nu^2}{8}\right)\frac{1}{\sigma_s} - \frac{\kappa\sigma_s}{2}g(s) \end{pmatrix}\mathrm{d}s + \int_0^t \underbrace{\begin{pmatrix} \sqrt{1-\rho^2}\sigma_s & \rho\sigma_s \\ 0 & \frac{\nu}{2}\cdot g(s) \end{pmatrix}}_{\sigma(s,X_s)} \begin{pmatrix} \mathrm{d}Z_s \\ \mathrm{d}W_s^2 \end{pmatrix}$$

Then, we can derive the following:

$$\sigma(s, X_s)^{-1} = \begin{pmatrix} \frac{1}{\sqrt{1-\rho^2}\sigma_s} & \frac{-2\rho}{\nu g(s)\sqrt{1-\rho^2}} \\ 0 & \frac{2}{\nu g(s)} \end{pmatrix} \tag{4.7}$$

and the first variation process

$$Y_t = \begin{pmatrix} \frac{1}{x} \\ 0 \end{pmatrix}. \tag{4.8}$$

Choosing $a(s) = \frac{1}{T}$ and applying the BEL formula,

$$\begin{aligned}
\Delta &= \mathbb{E}\left[\mathrm{e}^{-rT} f(S_T) \int_0^T a(s)(\sigma^{-1}(s, X_s)Y_s)^T \mathrm{d}W_s | X_0 = \log x\right] \\
&= \mathbb{E}\left[\mathrm{e}^{-rT} f(S_T) \int_0^T \frac{1}{T} \left(\begin{pmatrix} \frac{1}{\sqrt{1-\rho^2}\sigma_s} & \frac{-2\rho}{\nu g(s)\sqrt{1-\rho^2}} \\ 0 & \frac{2}{\nu g(s)} \end{pmatrix} \begin{pmatrix} \frac{1}{x} \\ 0 \end{pmatrix}\right)^T \begin{pmatrix} \mathrm{d}Z_s \\ \mathrm{d}W_s^2 \end{pmatrix}\right] \\
&= \mathbb{E}\left[\mathrm{e}^{-rT} f(S_T) \int_0^T \frac{1}{T} \begin{pmatrix} \frac{1}{x} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1-\rho^2}\sigma_s} & 0 \\ \frac{-2\rho}{\nu g(s)\sqrt{1-\rho^2}} & \frac{2}{\nu g(s)} \end{pmatrix} \begin{pmatrix} \mathrm{d}Z_s \\ \mathrm{d}W_s^2 \end{pmatrix}\right] \\
&= \mathbb{E}\left[\mathrm{e}^{-rT} f(S_T) \int_0^T \frac{1}{T} \frac{1}{x\sqrt{1-\rho^2}\sigma_s} \mathrm{d}Z_s\right].
\end{aligned}$$

We finally get the Malliavin Delta in rough Heston model, which is the same as in the classical Heston model (see Theorem 3.7).

Remark 4.1. *In this rough Heston model, we also need to find the condition that ensures*

$$\mathbb{P}(\{v_t > 0, \forall t > 0\}) = 1$$

and

$$\sup_{0 \le t \le T} \mathbb{E}[\sigma_t^{-2}] < \infty.$$

This is what we may explore in the future.

At this state, we can only test some values of parameters to see how the graph may look like. Choosing the value of parameters as reported in Table 7 (the same as in Table 2 under the classical Heston model with an extra parameter $\alpha = 0.7$; they satisfy (4.6)), we again plot the number of simulations N verses the value of Delta for both a European Call and a digital Call (see Figure 6 and Figure 7).

S_0	K	T	v_0	θ	κ	ν	ρ	α
100	100	1	0.1	0.08	4.0	0.6	−0.7	0.7

Table 7 The parameters used in the rough Heston model.

We can see from Figure 6 that the performance of Malliavin calculus method and finite difference method is debatable, while in the case of a digital Call option, the Malliavin weighted scheme clearly gives better convergence, as expected.

After figuring out the Malliavin weights and the conditions parameters need to satisfy in the rough Heston model, future research could be conducted in the general case of stochastic volatility models driven by the fBM, and various option styles could be tested, e.g., vanilla, Asian, barrier, binary, and exotic options.

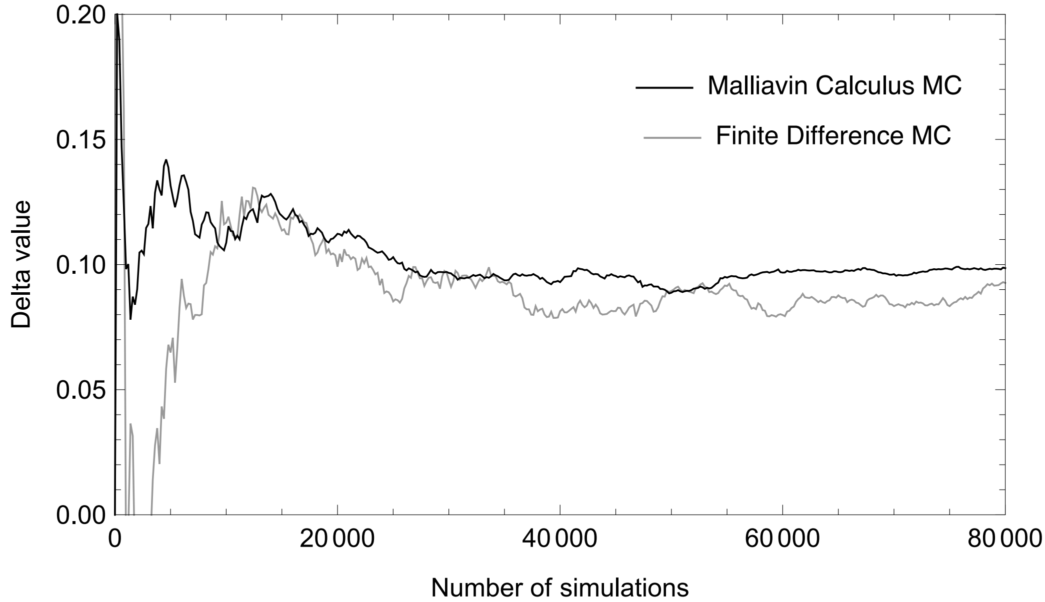

Figure 6 Delta for a European Call option with payoff function $f(x) = (x - K)^+$ and parameters as in Table 7 under the rough Heston model.

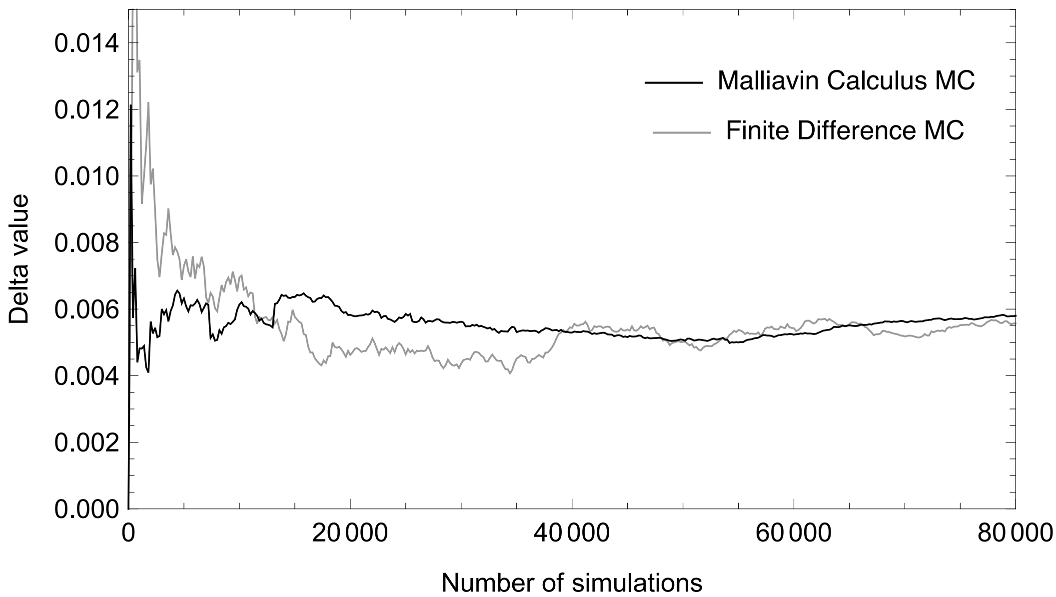

Figure 7 Delta for a digital Call option with payoff function $f(x) = \mathbb{1}_{\{x>K\}}$ and parameters as in Table 7 under the rough Heston model.

5 Conclusion

In this thesis, we have studied a new technique of computing Greeks based on the Malliavin integration-by-parts formula, called the Malliavin weighted scheme. This approach was then compared to the finite difference method (central scheme). The numerical experiments were conducted in the classical Heston, the fBS, and the rough Heston models. We conclude that the Malliavin weighted scheme remarkably outperforms the finite difference method for discontinuous payoff options. In particular, Malliavin calculus method is more efficient for the simulation of Gamma than first-order Greeks.

A Mathematica code for closed-form results

A.1 Classical Heston model: European Call and digital Call

A.1.1 Parameters and functions

```
In[1]:= ClearAll["Global`*"];
        S0 = 100;
        K = 100;
        r = 0.0;
        v0 = 0.1;
        theta = 0.08;
        kappa = 4.0;
        nu = 0.6;
        rho = -0.7;
        T = 1;
        F = S0 * Exp[r * T];
        x = Log[F / K];
        gamma = nu ^ 2 / 2;
```

```
In[14]:= alpha[j_] := -k ^ 2 / 2 - I * k / 2 + I * j * k;
         beta[j_] := kappa - rho * nu * j - rho * nu * I * k;
         d[j_] := Sqrt[beta[j] ^ 2 - 4 * alpha[j] * gamma];
         r1[j_] := (beta[j] + d[j]) / nu ^ 2;
         r2[j_] := (beta[j] - d[j]) / nu ^ 2;
         g[j_] := r2[j] / r1[j];
         CC[j_] :=
           kappa * (r2[j] * T - (2 / nu ^ 2) * Log[(1 - g[j] * Exp[-d[j] * T]) / (1 - g[j])]);
         DD[j_] := r2[j] * (1 - Exp[-d[j] * T]) / (1 - g[j] * Exp[-d[j] * T]);
         P[j_] := 1 / 2 + (1 / Pi) * NIntegrate[
              Re[Exp[CC[j] * theta + DD[j] * v0 + I * k * x] / (I * k)], {k, 0, Infinity}];
         DP1[j_] := (1 / Pi) * NIntegrate[Re[Exp[CC[j] * theta + DD[j] * v0 + I * k * x]],
             {k, 0, Infinity}];
         DP2[j_] := (1 / Pi) * NIntegrate[Re[I * k * Exp[CC[j] * theta + DD[j] * v0 + I * k * x]],
             {k, 0, Infinity}];
```

A.1.2 Closed-form Call price and the associated Greeks at time 0

```
In[25]:= HEuroCallPrice = K * Exp[-r * T] * (Exp[x] * P[1] - P[0]) // N
Out[25]= 11.0659
```

```
In[26]:= HEuroDelta = (K * Exp[-r * T] / S0) * (Exp[x] * P[1] + Exp[x] * DP1[1] - DP1[0]) // N
Out[26]= 0.607228
```

```
In[27]:= HEuroGamma = (K * Exp[-r * T] / S0 ^ 2) *
           (Exp[x] * DP1[x] + Exp[x] * DP2[1] + Exp[x] * DP1[0] - DP2[0]) // N

Out[27]= 0.0142267

In[28]:= HDigitalCallPrice = Exp[-r * T] * P[0] // N

Out[28]= 0.496569

In[29]:= HDigitalDelta = Exp[-r * T] * DP1[0] / S0 // N

Out[29]= 0.0142267

In[30]:= HDigitalGamma = Exp[-r * T] * DP2[0] / (S0 ^ 2) // N

Out[30]= -0.000140434
```

A.2 Fractional Black-Scholes model: European Call

A.2.1 Parameters

```
In[1]:= Clear["Global`*"]
        S0 = 100;
        K = 100;
        r = 0.05;
        sigma = 0.15;
        T = 1;
        H = 0.7;
```

A.2.2 Closed-form Call price and the associated Greeks at time 0

```
In[8]:= d1 = (Log[S0 / K] + r * T + (sigma ^ 2) * T ^ (2 * H) / 2) / (sigma * T ^ H);
        d2 = (Log[S0 / K] + r * T - (sigma ^ 2) * T ^ (2 * H) / 2) / (sigma * T ^ H);

In[10]:= C0 = S0 * CDF[NormalDistribution[0, 1], d1] - K * CDF[NormalDistribution[0, 1], d2]

Out[10]= 5.65603

In[11]:= fBSDelta = CDF[NormalDistribution[0, 1], d1]

Out[11]= 0.658486

In[12]:= fBSGamma = PDF[NormalDistribution[0, 1], d1] / (S0 * sigma * T ^ H)

Out[12]= 0.0244688

In[13]:= fBSVega = S0 * T ^ H * PDF[NormalDistribution[0, 1], d1]

Out[13]= 36.7032
```

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