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OPTIMAL TRADING STRATEGY AND TRADING BEHAVIOUR

by

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Declaration

The work contained in this thesis is my own work unless otherwise stated.

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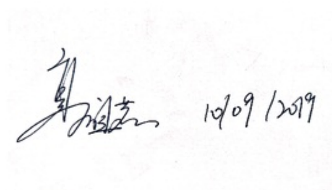

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Abstract

Motivated by industry practice of stocks/assets trading, we study the optimal strategies for an investor to purchase and subsequently liquidate a position over an infinite time horizon, subject to the market entry cost or namely, the cost of the stocks/assets and transaction costs.

To do so requires the construction of dynamic strategies and optimal static. Modelling the price by geometric Brownian motion, inspired by Henderson's liquidation model, we apply a probabilistic methodology and give a rigorous derivation of an investor's objective function, as well as the optimal price thresholds for both entering and exiting the market. Both analytical and numerical results are provided to illustrate the dependence of optimal strategies on model parameters such as market entry cost and the coefficient of risk aversion.

Contents

1 Introduction

1.1 Brief literature review of prospect theory

It has been widely observed that, in reality, investors are more likely to hold on losing assets and sell gaining assets. This phenomenon could arise due to the fact that some investors have prospect theory preference, suggested by Shefrin and Statman (1985) [18] and Odean (1998) [14]. Prospect theory, an analysis of the decision making under uncertainty proposed by Kahneman and Tversky (1979) [10]. It is an expressive and alternative model to reassess the expected utility theory in the view of decisions made under risky assets display several common effects that are discrepant from the basic principles of expected utility theory. To be more intuitive, we take a gambling game as an example. Suppose a stock price x takes possible value x_i (discrete) with probability p_i, and define $u(x_i)$ as its valuation for each possible outcome x_i, under the expected utility theory, we have

$$\mathbb{E}\big[u(x)\big] = p_1 \cdot u(x_1) + p_2 \cdot u(x_2) + p_3 \cdot u(x_3) + ...$$

which is assumed to be maximised.

However, in the presence of uncertainty, people may not choose the stock with a higher expected value. For example, if we have the following gambling situation:

$$\begin{cases} \text{Choice 1: a player wins £800 with probability 1.} \\ \text{Choice 2: a player wins £2000 with probability 0.5,} \\ \qquad\qquad\qquad\qquad \text{or nothing with probability 0.5.} \end{cases}$$

According to the expected utility theory, the expected value of choice 1 is £800 and that of choice 2 is £1000, so a player should choose the second choice with a higher expected value. While this may be true, people with enough risk aversion may prefer choice 1 over choice 2 - people choose the one with much more certainty. In order to investigate more into the invalidity of expected utility theory in such case, an alternative theory aroused - prospect theory, in which utility is derived from the realised gains and losses rather than the final wealth, as well as the decision weights are in place of the probabilities. Moreover, the utility is defined in a way such that it compares the difference between the asset price and a reference level, this concept is proposed by Markowitz (1952) [13]. The utility function exhibits concavity for gains and convexity for losses, such feature is known as loss aversion.

1.2 Behavioural finance leads to the discussion of the disposition effect

Disposition effect by Shefrin and Statman (1985) [18], is a phenomenon found in many experimental studies that investors are unwilling to sell an asset at a price below the level it was purchased. This effect has been observed in many settings such as individual investors (see Odean 1998 [14]), institutional investors (see Grinblatt and Keloharju 2001 [8]), the real estate market (see Genesove and Mayer 2001 [7]), and traded options markets (Posteshman and Serbin 2003 [16]). A strong disposition effect is where investors never sell an asset at a loss willingly but a gain.

In particular, Shefrin and Statnman (1985) [18] demonstrate the disposition effect arises via a numerical example [1] if an investor's utility is derived from realised gains and losses, and the utility function captures loss aversion as described at the end of Section 1.1.

1.3 When to purchase and liquidate an asset?

While observing the prevailing market prices, a speculative investor may consider to enter/exit the market immediately or to seek a future opportunity. This motivates our interest in studying the optimal strategy of trades.

One natural question to ask is: When is the best time for investors to purchase and liquidate their assets? This is an important step in understanding the financial behaviour of individual and institutional investors. Given the dynamics of some risky asset prices, our investigation is conducted on the side of an optimal double stopping problem. In particular, our formulation leads to an optimal double stopping problem that provides both the optimal decisions for entering and exiting the market. We develop both analytical solutions and numerical results to this market entry & exit problem, especially, we incorporate a market entry transaction cost to see how this affects an investor's decisions.

The model formulation we study contains piecewise power S-shaped functions and asset prices following geometric Brownian motion. This is a tractable approach to the problem we are interested in as the model applies to any diffusion (price) process that is time-homogeneous, and to conventional utility functions, which enables the model to be applied more widely in this area.

1.4 Aims and structure of the thesis

The main purpose of this thesis is to study the optimal opportunities for investors to open and close a position with respect to the market entry cost. Our main results provide an analytical

[1] Suppose an investor purchased a stock for £50 three months ago. The stock is now being traded at £40. The investor is deciding whether to sell the stock now at a realised loss of £10 or to wait for a future opportunity. Suppose the stock price will either increase by £10 or decrease by £10 for the next period, with equal probability. Equal chances for another £10 loss or break-even if the investor chooses to wait. Shefrin and Statnman (1985) [18] suggest that because the choice is associated with the convex part of the S-shaped function, an investor with prospect theory preference would wait and gamble on a possible break-even. However, if breaking even is sufficiently unlikely, the investor may sell the stock and realise a loss at the current stage instead.

expression for the value function and stopping thresholds of the stopping problem subject to the model parameters. The rest of the thesis is structured as follows. In Section 2, we briefly review the key results that are crucial to our solutions, followed by an introduction and an application of Henderson's liquidation model [9]. We then formulate the general trading problem subject to the market entry cost in Section 3, analytical solutions and numerical results will be given for determining the entry & exit strategies. Finally, we close with a conclusion.

2 Utility, Value Function and the Optimal Exit Strategy

2.1 Realised utility related to the prospect theory preference

The utility function $U(\cdot)$ will be used to denote the prospect theory preferences, as suggested by Tversky and Kahneman (1992) [20], we use power functions to construct an S-shaped utility function:

$$U(x) = \begin{cases} x^{\alpha_1} & x > 0 \\ -k(-x)^{\alpha_2} & x \le 0 \end{cases} \tag{2.1}$$

where $\alpha_1, \alpha_2 \in (0, 1)$.

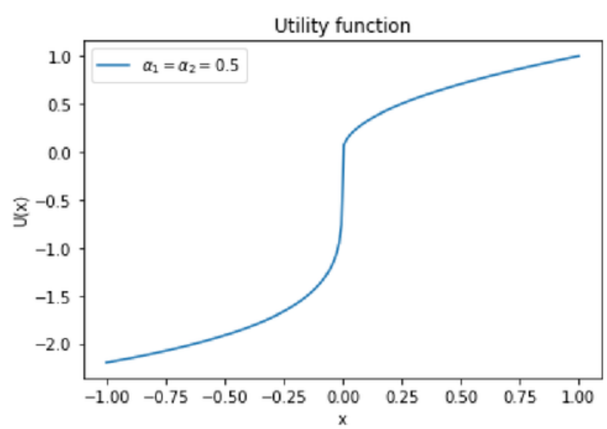


Figure 1: Example of a utility function $U(x)$ with $\alpha_1 = \alpha_2 = 0.5$.

The function $U(\cdot)$ being concave (i.e., $U'' \le 0$) for $x \ge 0$ reflects risk aversion, the function is more concave when α_1 is smaller, the parameter $1 - \alpha_1$ measures the level of risk aversion. Likewise, the function exhibits more convexity as α_2 decreases. and the parameter $1 - \alpha_2$ measures the level of risk-seeking. The parameter $k > 1$ gives asymmetrical graph which introduces the feature of loss aversion. (experimental values of the parameters: $\alpha_1 = \alpha_2 = 0.88, k = 2.25$, see Tversky and Kahneman 1992 [20]).

Another crucial property of this function is that its first derivative at zero from the left and the right are both equal to infinity, i.e., $U'(0+) = U'(0-) = \infty$, indicating infinite marginal utility at the origin.

2.2 The value function and its characterisation

In this thesis, we will be considering a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with historical probability measure $\mathbb{P}$ and a standard Brownian motion $W = \{W_t; t \geq 0\}$ under $\mathbb{P}$. The asset price Y_t follows a time-homogeneous diffusion process with the state space $\eth \subseteq \mathbb{R}$ driven by the stochastic differential equation:

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t, \qquad Y_0 = y \tag{2.2}$$

for some Borel functions $\mu : \eth \to \mathbb{R}$, and $\sigma : \eth \to (0, \infty)$. We assume that the state space $\eth$ is an interval with boundaries $-\infty \leq a < b \leq \infty$, and that for any y_0, $y_1 \in (a, b)$, if starting at y_0, Y reaches y_1 with positive probability.

Our solutions in the next section rely heavily on the key theorems and corollaries on the side of an optimal stopping problem. For the best understanding of our approaches and solutions, we first draw our attention to the following setups and results presented in Dayanik and Karatzas 2003 [3].

Suppose we have a diffusion process Y of type (2.2), and Y is stopped instantaneously when it reaches either the endpoints ϕ or ψ. Define $V(y)$ as the following ([3] eq (3.1)) which is usually referred as the value function

$$V(y) := \sup_{\tau \in \mathcal{F}} \mathbb{E}_y \left[h(Y_\tau)\right], \quad y \in [\phi, \psi] \tag{2.3}$$

where $h(\cdot)$ is a bounded Borel measurable function such that $\mathbb{E}_y\big[h(Y_\tau)\big]$ is well-defined for every $\mathcal{F}_t$ - stopping time, $h(\cdot)$ is commonly interpreted as the *reward function.*

In order to find the optimal stopping time or, equivalently, for which τ that $\mathbb{E}_y\big[h(Y_\tau)\big]$ is maximised within the constraints in (2.3). Our main task is to firstly identify the value function $V(\cdot)$. In such a way, despite the fact that sometimes an explicit expression for $V(\cdot)$ may be less available, we will still be able to recognise the optimal stopping time based on the shape of the function

$V(\cdot)$. We assume that $h(\cdot)$ is a positive function since if $h(\cdot) \leq 0$, then trivially we get $V = 0$, and $\tau = \infty$ is an optimal stopping time.

N.B. The following theorem is based on the setup ([3] eq (1.2)):

$$V(y) := \sup_{\tau \in \mathcal{F}} \mathbb{E}_y \left[e^{-\beta\tau} h(Y_\tau) \right] \tag{2.4}$$

where discounted optimal stopping is considered if $\beta > 0$. However, we are primarily interested in the optimal stopping in the absence of discounting, i.e., $\beta = 0$. Our purpose to include this notation here is to provide further insight into the following key results, meanwhile, to smoothly develop our solution.

Theorem 2.1. *(Dynkin,1963 [4]) Given that $h(\cdot)$ is lower semi-continuous function. The value function $V(\cdot)$ of (2.4) is the smallest β-excessive majorant of $h(\cdot)$ on $\eth$ with respect to the process Y.*

where $\beta-$*excessive* function (for the process Y) is defined [19] to be the nonnegative functions $\delta(\cdot)$ such that

$$\delta(y) \geq \mathbb{E}_y \big[e^{-\beta\tau} \delta(Y_\tau) \big], \quad \forall y \in \eth$$

Though we may not be able to derive the value function explicitly from Theorem 2.1, it is often employed as a method of identification. For example, to suppose a value function and then to examine it with Theorem 2.1, this is also commonly seen in extant literature. Theorem 2.2 and Corollary 2.3 will be primarily applied to obtain our solutions to the optimal exit & entry strategies.

Theorem 2.2. *(Dynkin and Yushkevich, 1969) Every 0-excessive (or simply, excessive) function for one-dimensional Brownian motion Y is concave, and vice-versa.*

Corollary 2.3. *The value function $V(\cdot)$ of (2.4) is the smallest non-negative concave majorant of $h(\cdot)$ under the following assumptions:*

Assume we are in the situation of (2.3) and there exists a standard Brownian motion Y starting in $[\phi, \psi]$ which is a closed and bounded interval, with $\mu(y) = 0$, $\forall y \in [\phi, \psi]$, $\sigma(\phi) = \sigma(\psi) = 0$, and $\sigma(y) = 1$, $\forall y \in (\phi, \psi)$, and the state space $\eth = [\phi, \psi]$ for some $-\infty < \phi < \psi < \infty$. ([3], pp.175)

2.3 The idea of involving the scale function

We now formulate a general solution to the optimal stopping problem. In order to solve for an investor's optimal strategies, we introduce the idea of *scale function* $s(\cdot)$ of the process Y_t following

(2.2). As suggested by Revuz and Yor 1999 [17], we use the *scale function* $s(\cdot)$ to transform the process Y_t such that $s(Y_{t\wedge\tau}), t \geq 0$ is a (local) martingale, the benefits of such manipulation will be discussed later in this section. The properties of the solutions to one-dimensional SDEs can be more apparent under such a transformation. Consider the solution to the one-dimensional SDE (2.2). The technique is to transform Y into a local martingale by defining $X_t = s(Y_t)$ and assuming that $s \in C^2(\mathbb{R})$, applying Itô's formula to give

$$dY_t = s'(Y_t)\sigma(Y_t)dW_t + \left[s'(Y_t)\mu(Y_t) + \frac{1}{2}s''(Y_t)\sigma^2(Y_t)\right]dt$$

next we would like to find a function $s(\cdot)$ such that

$$s'(y)\mu(y) + \frac{1}{2}s''(y)\sigma^2(y) = 0$$

such function $s(\cdot)$ is defined to be the *scale function* of diffusion Y_t [Cass]. We shall lead into a more formal definition of the scale function later in this section (see Proposition 2.5).

The following propositions pave the way for our study of the investor's optimal decisions. The proofs of Proposition 2.4 and Proposition 2.5 can be found in Revuz and Yor (1999, pp.300-312) [17].

Proposition 2.4. *(Revuz and Yor 1999 or see [3] prop 2.2). There exists a continuous and strictly increasing function $s(\cdot)$ on $\eth$ such that for any $l, u, x \in \eth$, with $\phi \leq l < x < u \leq \psi$, we have*

$$\mathbb{P}_x(\tau_u < \tau_l) = \frac{s(x) - s(l)}{s(u) - s(l)}, \quad \text{and} \quad \mathbb{P}_x(\tau_u > \tau_l) = \frac{s(u) - s(x)}{s(u) - s(l)} \tag{2.5}$$

If there exists another function $\bar{s}$ with these properties, then $\bar{s}$ is an Affine transformation of s, i.e., $\bar{s} = cs + d$ for some $c > 0$ and $d \in \mathbb{R}$. Thus the function s is unique up to Affine transformations, and is called the "scale function" of Y.

Proposition 2.5. *(Revuz and Yor 1999 or see [3] prop 2.3). A Borel function f that is locally bounded, is a scale function, if and only if the process $Y_t := f(X_{t\wedge\xi\wedge\tau_a\wedge\tau_b})$, $t \geq 0$, is a local martingale. Moreover, if X is driven by the stochastic differential equation (2.2), then for any arbitrary $c \in \eth$ (fixed), we have*

$$s(x) = \int_c^x \exp\left\{\int_c^y -\frac{2\mu(z)}{\sigma^2(z)}dz\right\}dy, \quad x \in \eth$$

When trying to obtain an explicit expression for the value function, the following result by Karatzas and Sudderth (1999) [11] can be therefore convenient.

Proposition 2.6. *(Karatzas and Sudderth 1999 [11] or see [3] prop 3.3). On the interval $[s(\phi), s(\psi)]$, let $g(\cdot)$ be the smallest nonnegative concave majorant of the function $h(s^{-1}(y))$. Then $V(y) = g(s(y))$, $\forall y \in [\phi, \psi]$.*

The following proposition characterises the existence of the optimal stopping time τ^*.

Proposition 2.7. *(Dayanik and Karatzas 2003 [3] prop 3.4) If $h(\cdot)$ is continuous on $[\phi, \psi]$, subsequently, so is $V(\cdot)$, and the stopping time τ^* is optimal, where we define τ^* to be*

$$\Lambda := \{y \in [\phi, \psi] : V(y) = h(y)\} \quad \textit{and} \quad \tau^* := \inf\{t \geq 0 : Y_t \in \Lambda\}$$

The proof of this result is similar to that in Dynkin and Yushkevich (1969, pp.112-119 [6]).

In this thesis we are mainly concerned with the situation where μ and $\sigma > 0$ are constants. For simplicity, we denote the constant parameter $\beta := 1 - \frac{2\mu}{\sigma^2}$, the term $\frac{\mu}{\sigma^2}$ can be interpreted as the reflection of the asset's expected performance (per unit variance). We know that the general solution to (2.2) is $Y = Y_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$ (with μ, $\sigma > 0$ constants), therefore we have the following results:

1. If $\beta < 0$, then $Y_t \to \infty$ as t $\to \infty$.
   The scale function is $s(y) = -(y)^\beta$.
2. If $\beta > 0$, then $Y_t \to 0$ almost surely as $\to \infty$.
   The scale function is $s(y) = y^\beta$.
3. If $\beta = 0$, the scale function is given by $s(y) = ln(y)$.

Provided that the asset price Y is a time-homogeneous diffusion process, the solution will be taking the form of an interval, for instance, (y_1, y_2), and the process Y is to be stopped when it exits this interval. The problem is more tractable and more transparent to solve when we work with martingales as suggested at the beginning of this section. Let $X_t = s(Y_t), X_0 = s(y_0)$.

Remark 2.8. Let $\tau^Y_{y_1,y_2}$ denote the optimal stopping time of the process Y in the interval (y_1, y_2), then let $\tau^Y_{y_1,y_2} := \inf\{t > 0 : Y_t \notin (y_1, y_2)\}$. Likewise, let $\tau^X_{a,b} := \inf\{t > 0 : X_t \notin (s(y_1), s(y_2))\}$, where we have defined the transformed interval by $a = s(y_1), b = s(y_2)$.

By Remark 2.8 we can obtain the exit time(s) for the investor from the transformed exit price(s) of the asset price Y. Define the function $g(x) := U(f(s^{-1}(x)) - f_R)$ and observe that $g(\cdot)$ is necessarily an increasing function. Then by Proposition 2.4, for any fixed interval $(y_1, y_2) \in \eth$ such that $(s(y_1), s(y_2))$ is bounded, we have

$$\begin{aligned} \mathbb{E}\left[U(Y_\tau - R) \mid Y_0 = y\right] &= \mathbb{E}\left[U(s^{-1}(X_\tau) - R) \mid X_0 = x\right] \\ &= \mathbb{E}\left[g(X_\tau) \mid X_0 = x\right] \\ &= g(a)\frac{b-x}{b-a} + g(b)\frac{x-a}{b-a} \end{aligned} \tag{2.6}$$

where we have applied the probabilities of the (bounded) martingale $(X_t)_{t \leq \tau}$ hitting each end of the interval.

Now the ultimate step is to find the optimal interval (a,b), i.e., to solve

$$\sup_{a<x<b}\left\{g(a)\frac{b-x}{b-a}+g(b)\frac{x-a}{b-a}\right\} \tag{2.7}$$

to which the solution is given by the least concave majorant of $g(\cdot)$, by Proposition 2.6. Define the least concave majorant of $g(\cdot)$ as $\bar{g}(\cdot)$.

Figure 2 demonstrates a stylised example plot of the function $g(\cdot)$. With the help of the graph, we can give a more intuitive explanation to illustrate that the smallest concave majorant is indeed the solution to the interval-typed strategies (2.7). We would like to choose the optimised interval to maximise the quantity in the curly brackets of (2.7). For instance, if the investor decides to open a position at the point x_2, or in fact, any other starting point $x \in (A,B)$, the quantity in (2.7) is maximised if we take $a=A$ and $b=B$. i.e., the stopping points A,B are the best choices. Nevertheless, if the investor purchases an asset at the point x_1 then (2.7) is maximised by taking $a=b=x_1$ because all other values of exiting points give relatively lower values. This corresponds to an immediate liquidation since we stop if we are outside the interval (A,B). Therefore, for any transformed price $x \in (A,B)$ the strategy is to stop when X_t reaches either endpoints of this interval. Any points outside the interval the strategy is to stop immediately.

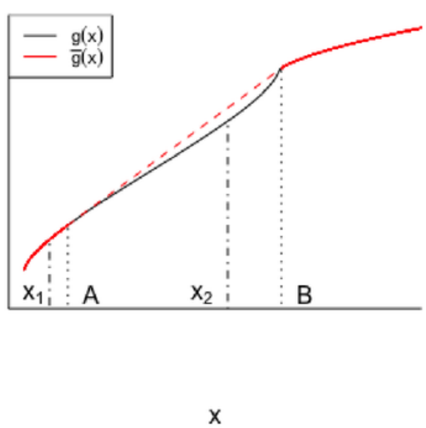


Figure 2: Stylised example graph of the function $g(x)$ as a function of transformed price x, where $x = s(y)$.

N.B. the function $g(x)$ represents the value of the game when the investor closes a position immediately. The smallest concave majorant $\bar{g}(x)$ of $g(x)$ is given by the straight dashed line for $x \in (A, B)$ and the function $g(x)$ itself for $x \geq B$ and $x \leq A$.

The following proposition underlies this intuition.

Proposition 2.9 ([9] (Proposition 1)). *On the interval $(s(\phi_\eth), s(\psi_\eth))$, where $(\phi_\eth, \psi_\eth)$ is any fixed interval in $\eth$, let $\bar{g}(x)$ be the smallest concave majorant of $g(x) := U(s^{-1}(x) - R)$.*

1. *Suppose $s(\phi_\eth) = -\infty$, then $V_1(y) = U(f(\psi_\eth) - f_R)$, $y \in (\phi_\eth, \psi_\eth)$.*
2. *Suppose $s(\phi_\eth) > -\infty$, then $V_1(y) = \bar{g}(s(y))$, $y \in (\phi_\eth, \psi_\eth)$.*

This result follows from optimal stopping theory (see Oksendal 2005 [15], and specifically Dynkin 1965 [5], Dynkin and Yushkevich 1969 [6], and more recently, Dayanik and Karatzas 2003).

Proposition 2.9 paves a way to determine the investor's behaviour under different circumstances analytically (see Proposition 2.10 [9] and the proof).

2.4 The exit problem

In this thesis we would like to consider the case where an investor can only sell or buy an entire position, partial sales are excluded, but are included in our discussion for further research in **??**. We also make the assumption that one can liquidate his/her position at any time of his/her choice, this leads to our model to be over an infinite time horizon. For each unit of an asset Y, we denote the payoff as $f(Y)$, where $f(\cdot)$ is a non-decreasing function. As the common practice in the literature, whether the investor is "wining" or "losing" when selling the asset is compared to a reference level, here we denote it by f_R, $f_R \geq 0$. An appropriate interpretation of the reference level f_R is the price the investor paid for the asset, or equivalently, the break-even level. The investor's objective can be derived from his/her utility with the help of the value function.

At the liquidation time, the investor's objective can be written as

$$V_1(y) = \sup_{\tau \in \mathcal{F}} \mathbb{E}\left[U(f(Y_\tau) - f_R) \mid Y_0 = y\right] \tag{2.8}$$

where $U(\cdot)$ is an increasing function (which is consistent with most theoretical models) and the supreme is taken over the set of all $\mathcal{F}_t$ - stopping time. At the time of sale, utility is derived from the realised gains or losses and the investor evaluates utility by taking the difference between the payoff and his/her reference level, hence we have the objective function $V_1(y)$ (2.8).

This leads to the analysis of an optimal stopping problem, which is exactly to find the value function, as well as an optimal stopping time τ^* for which the supreme is attained if such a time exists.

Provided that a price process Y_t follows a geometric Brownian motion, we are primarily interested in the optimal trading decisions for an investor with respect to the market entry cost parameter. We take $f(y) = y$ and $f_R = R$, where $R \geq 0$ is assumed to be the break-even level (or price paid when purchasing the asset). At the time of sale τ, the investor receives a payoff Y_τ and then compare this with his/her reference level R. This would be appropriate for modelling the trades of a stock or a real estate.

2.4.1 The prospect theory liquidation model

section2.4.1 Now we state and prove the main results that are applied to solve the exit problem (2.8) with respect to the utility $U(\cdot)$ given by (2.1) and the price dynamics we are interested in studying.

Proposition 2.10 (Henderson 2012 [9] prop 2). *Consider an investor facing the exit problem*

$$\sup_\tau \mathbb{E}\left[U(Y_\tau - R) \mid Y_0 = y\right]$$

where Y follows geometric Brownian motion: $dY = \mu Y dt + \sigma Y dW$, and we denote $\beta = 1 - \frac{2\mu}{\sigma^2}$, and $U(\cdot)$ is given by Tversky and Kahneman (1992) [20] as in (2.1). *The solution is given in three different cases, subjecting to the relative model parameters.*

i) If $\beta \leq 0$, or $0 < \beta < \alpha_1 < 1$, it is optimal for the investor to wait indefinitely at all price levels and to never exit the market (see Figure 3a and Figure 3b).

ii) If $0 < \alpha_1 < \beta \leq 1$ or $\alpha_1 = \beta < 1$, the investor would stop at a point y_u, where $y_u > R$. In other words, the investor would wait for the price reaches beyond the break-even level, and therefore liquidates at a gain. (see Figure 3c and Figure 3d).

iii) If $\beta > 1$, the investor's optimal strategy is to liquidate at either endpoints y_l and y_u (as points A, B in Figure 2). Since these two points are on either side of the break-even point, i.e., $y_l < R < y_u$, therefore the investor may either sell the asset at a gain or at a loss (see Figure 3e and Figure 3f).

Let us prove Proposition 2.10 case by case. Define $\bar{g}(x)$ to be the least concave majorant of the fuction $g(x)$.

Proof. If $\beta < 0$, by Section 2.4 i) we have $s^{-1}(x) = (-x)^{\frac{1}{\beta}}$ where x is the transformed price of price y, $x \in (-\infty, 0)$ and $s(R) = -(R)^\beta$. Recall that we use the utility defined in (2.1) and

$g(x) \coloneqq U(s^{-1}(x) - R)$ which can be reexpressed as

$$g(x) = \begin{cases} ((-x)^{\frac{1}{\beta}} - R)^{\alpha_1} & x > -R^{\beta} \\ -k(R - (-x)^{\frac{1}{\beta}})^{\alpha_2} & x \leq -R^{\beta} \end{cases} \tag{2.9}$$

Perform differentiations with respect to x we get

$$g'(x) = \begin{cases} -\frac{\alpha_1}{\beta}(-x)^{\frac{1}{\beta}-1}((-x)^{\frac{1}{\beta}} - R)^{\alpha_1 - 1} & x > -R^{\beta} \\ -\frac{\alpha_2}{\beta}k(-x)^{\frac{1}{\beta}-1}(-(-x)^{\frac{1}{\beta}} + R)^{\alpha_2 - 1} & x \leq -R^{\beta} \end{cases} \tag{2.10}$$

Note that the gradient of $g(x)$ at the left and the right limit of the investor's reference level (after transformation) both tend to infinity, i.e., $g'(-(R)^{\beta}_{-}) = g'(-(R)^{\beta}_{+}) = \infty$. It can also be shown that $g'(-\infty) = 0$ and $g'(0) = \infty$. This is to be expected, as a negative β corresponds to a high positive $\frac{2\mu}{\sigma^2}$, in other words, a high excess return per unit variance. The investor would always wait for a higher price level regardless of his/her extent of risk aversion, as displayed in Figure 3a.

$$g''(x) = \begin{cases} \frac{\alpha_1}{\beta}(-x)^{\frac{1}{\beta}-2}((-x)^{\frac{1}{\beta}} - R)^{\alpha_1 - 2}\left[\frac{(\alpha_1-\beta)}{\beta}(-x)^{\frac{1}{\beta}} - R(\frac{1}{\beta} - 1)\right] & x > -R^{\beta} \\ \frac{\alpha_2}{\beta}k(-x)^{\frac{1}{\beta}-2}(-(-x)^{\frac{1}{\beta}} + R)^{\alpha_2 - 2}\left[\frac{(\beta-\alpha_2)}{\beta}(-x)^{\frac{1}{\beta}} + R(\frac{1}{\beta} - 1)\right] & x \leq -R^{\beta} \end{cases} \tag{2.11}$$

We find that $g''(x) \geq 0$, $\forall x \leq -R^{\beta}$. For the first expression in (2.11), $g''(x)$ is nonnegative if $(-x)^{\frac{1}{\beta}} < \frac{R(1-\beta)}{\alpha_1 - \beta}$, since $R < \frac{R(1-\beta)}{\alpha_1-\beta}$ we have that $g''(x)$ is convex $\forall x \in (-\infty, 0)$. If we were to draw the least concave majorant of $g(x)$, it will be a horizontal line of value equal to the maximum value of $g(x)$ which is less applicable since $g'(0) = \infty$.

If $\beta > 0$, we have that $s^{-1}(x) = x^{\frac{1}{\beta}}$, $x \in (0, \infty)$ and $s(R) = R^{\beta}$. Hence

$$g(x) = \begin{cases} (x^{\frac{1}{\beta}} - R)^{\alpha_1} & x > R^{\beta} \\ -k(R - x^{\frac{1}{\beta}})^{\alpha_2} & x \leq R^{\beta} \end{cases} \tag{2.12}$$

Note that $g(0) = -kR^{\alpha_2}$, differentiate $g(x)$ to give

$$g'(x) = \begin{cases} \frac{\alpha_1}{\beta}x^{\frac{1}{\beta}-1}(x^{\frac{1}{\beta}} - R)^{\alpha_1 - 1} & x > R^{\beta} \\ \frac{\alpha_2}{\beta}kx^{\frac{1}{\beta}-1}(-x^{\frac{1}{\beta}} + R)^{\alpha_2 - 1} & x \leq R^{\beta} \end{cases} \tag{2.13}$$

If $0 < \beta < 1$, calculations give $g'(0) = 0$ and $g'(0) = \infty$ if $\beta > 1$. Moreover, if $\beta > \alpha_1$ we have that $g'(\infty) = 0$ and $g'(\infty) = \infty$ if $\beta < \alpha_1$.

$$g''(x) = \begin{cases} \frac{\alpha_1}{\beta}x^{\frac{1}{\beta}-2}(x^{\frac{1}{\beta}} - R)^{\alpha_1 - 2}\left[\frac{(\alpha_1-\beta)}{\beta}x^{\frac{1}{\beta}} - R(\frac{1}{\beta} - 1)\right] & x > R^{\beta} \\ \frac{\alpha_2}{\beta}kx^{\frac{1}{\beta}-2}(-x^{\frac{1}{\beta}} + R)^{\alpha_2 - 2}\left[\frac{(\beta-\alpha_2)}{\beta}x^{\frac{1}{\beta}} + R(\frac{1}{\beta} - 1)\right] & x \leq R^{\beta} \end{cases} \tag{2.14}$$

Recall that $1 - \alpha_1$ represents the coefficient of risk aversion. By also examining the concavity/convexity of $g(x)$, these lead to three possible shapes of $g(x)$.

① If $0 < \beta < 1$ and $\beta < \alpha_1$, $g(x)$ is increasing convex, it is also optimal for the investor to wait indefinitely due to the moderate value of $\frac{2\mu}{\beta}$ and relatively low risk aversion level, as shown in Figure 3b.

② If $0 < \beta < 1$ and $\beta > \alpha_1$, then $g(x)$ is convex for $x < R^\beta$ and concave for $x > R^\beta$ as shown in Figure 3c and Figure 3d. In which case the least concave majorant $\bar{g}(x)$ is to be drawn by a straight line from the point $(0, g(0))$ to the point where this line segment touches $g(x)$ (that is to the right of the break-even level) and finished with duplicating the function $g(x)$ beyond this point. In this case, the investor's risk aversion level is higher so that the investor would sell at a gain at some point above the break-even level.

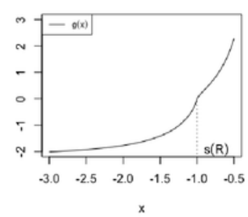
③ If $\beta > 1$, $g(x)$ is convex for $\frac{\beta-1}{\beta-\alpha_2}R < x^{\frac{1}{\beta}} < \frac{1-\beta}{\alpha_1-\beta}R$, since $\frac{\beta-1}{\beta-\alpha_2}R < R$ and $R < \frac{1-\beta}{\alpha_1-\beta}R$, we deduce that $g(x)$ switches from concave to convex and then to concave as in Figure 3e ad Figure 3f. Then the smallest concave majorant $\bar{g}$ is given by taking the line segment joining the two points $(g(x_l),\ g(x_u))$ on $g(x)$, where $x_l \in (0, R^\beta)$, $x_u \in (R^\beta, \infty)$ and the function $g(x)$ itself for other values of x. In this case the excess return per unit variance is very low, or in fact, a negative expected excess return. Therefore the investor will choose to sell the asset at a point above the break-even level with a relatively small gain, moreover, it is also optimal for the investor to liquidate at a loss under such circumstance.

The special cases where $\beta = 0, 1$ and $\beta = \alpha_1$ can be treated similarly.

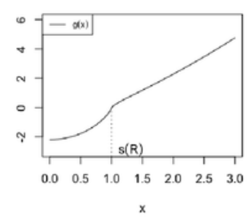
□

Figure 3: Optimal sale of an asset under the model of Proposition 2.10

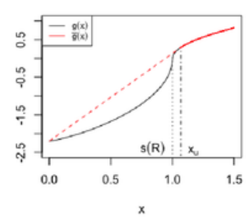
(a) *i)* $\beta = -0.5$, $\alpha_1 = \alpha_2 = 0.75$, $s(R) = -1$. The investor never liquidates for any values of x, equivalently, waits for all prices y.



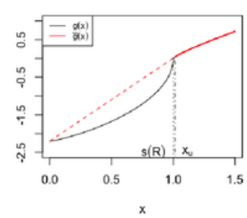
(b) *i)* $\beta = 0.5, \alpha_1 = \alpha_2 = 0.75$, $s(R) = 1$. The investor never liquidate for any values of x, equivalently, waits for all prices y.



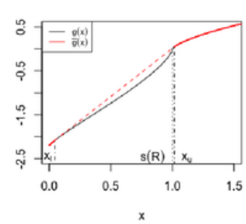
(c) *ii)* $\beta = 0.8$, $\alpha_1 = \alpha_2 = 0.5$, $s(R) = 1$. The investor liquidates at a gain, i.e., for $x \geq x_u = 1.0676$ or equivalently, for $y \geq 1.085$.



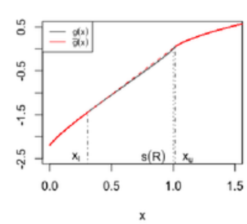
(d) $\beta = 0.8$, $\alpha_1 = 0.8$, $\alpha_2 = 0.5$, $s(R) = 1$. The investor liquidates at a gain, i.e., for $x \geq 1.0155$ or equivalently, $y \geq 1.019$.



(e) *iii)* $\beta = 1.2$, $\alpha_1 = \alpha_2 = 0.7$, $s(R) = 1$. The investor waits for $x \in (x_l = 0.0461, x_u = 1.0144)$ and liquidates otherwise. Equivalently, waits for $y \in (y_l = 0.078, y_u = 1.012)$.



(f) *iii)* $\beta = 1.2$, $\alpha_1 = 0.7$ $\alpha_2 = 0.9$, $s(R) = 1$. The investor waits for $x \in (x_l = 0.307, x_u = 1.0165)$ and liquidates otherwise. Equivalently, waits for $y \in (y_l = 0.055, y_u = 1.014)$.



Each graph illustrates the function $g(x) := U(s^{-1}(x) - R)$. The transformed reference level is denoted by $s(R)$ in each panel plot and indicated with a dotted line. Moreover, the dash-dot lines are drawn for labelling the exit price levels in plots (c),(d),(e) and (f), which are resulted from the smallest concave majorant of $g(x)$.

Due to the fact that the smallest concave majorant function is determined by the shape of the original function $g(\cdot)$, the solution to our problem will be purely depending on the characterisations of such function $g(\cdot)$, which in turn is determined only by the parameters (i.e., the price dynamics) and the form of utility function involved. Albeit we only focus on one particular utility function (2.1) here other possible utility functions can be as well easily addressed.

3 Optimal strategies to enter & exit the market

3.1 The entry & exit problem

Our interest is not only in investigating investors' behaviour while they have already owned some assets but also in the combined problem, i.e., when is the best time for investors to buy and sell their assets, subjecting to underlying parameters such as transaction costs. From now on we consider a zero risk free rate and assume that $\alpha_1 = \alpha_2 = \alpha$, where α_1 and α_2 are as defined in (2.1). We will be primarily consider the asset price Y_t follows a geometric Brownian motion, i.e., a one-dimensional regular diffusion of the type (2.2) with constants μ and $\sigma > 0$, namely

$$dY_t = \mu Y_t dt + \sigma Y_t dW_t \qquad Y_0 = y \tag{3.1}$$

The general problem we are considering here is the investor's objective function:

$$\sup_{\substack{\tau_1,\tau_2 \\ \tau_1 \le \tau_2}} \mathbb{E}\left[U(Y_{\tau_2} - \lambda Y_{\tau_1} - R) \mid Y_0 = y\right] \tag{3.2}$$

where the supreme is taken over the set of all $\mathcal{F}_t$ - stopping time, τ_1 and τ_2 denote the time of entry and exit respectively hence the condition $\tau_1 \le \tau_2$. $R \ge 0$ indicates the investor's initial reference level. $\lambda > 1$ represents the parameter of entry cost - the cost of purchasing the asset itself and the transaction costs. People may also be charged an additional transaction cost for selling a position, we will discuss this situation later in Section 3.5. We can reexpress (3.2) as

$$\sup_{\tau_1} \mathbb{E}\big[\sup_{\tau_2} \mathbb{E}\big[U(Y_{\tau_2} - \lambda Y_{\tau_1} - R) \mid \mathcal{F}_{\tau_1}\big] \mid Y_0 = y\big] \tag{3.3}$$

given the Markov property of the diffusion processes the inner supreme can be written as

$$\begin{aligned}&\sup_{\tau_2\geq\tau_1}\mathbb{E}\left[U(Y_{\tau_2}-\lambda Y_{\tau_1}-R)\mid Y_{\tau_1}=y\right]\\ =&\sup_{\tau_2\geq 0}\mathbb{E}\left[U(Y_{\tau_2}-\lambda y-R)\mid Y_0=y\right]\\ =&V_1(y;\lambda y+R)\end{aligned} \tag{3.4}$$

where we define $V_1(y;H) := \sup_{\tau\geq 0}\mathbb{E}\big[U(Y_\tau - H)\mid Y_0=y\big]$. Ultimately, we will obtain a function of Y_{τ_1} and then maximise it over τ_1.

Given the expression (3.3) and the asset price Y_t follows (3.1), we are interested in studying the sequential optimal double stopping time for trades, or equivalently, the optimal strategies to buy and sell an asset Y. We will divide our investigation into two steps: first, solve the optimal exit (sale) problem (see Section 2.4) subject to some fixed time τ_1, after which we apply a similar method to solve the optimal entry (purchase) problem.

Our approach is to treat the problem as two "exit problem", in other words, suppose that τ_1 (the entry time) is fixed at some time T which leads (3.2) to become

$$\sup_{\tau_2}\mathbb{E}\left[U(Y_{\tau_2}-(\lambda Y_T+R))\mid Y_0=y\right] \tag{3.5}$$

this is our first "exit problem" and its value is given by the general solution to the exit problem (see Section 2.4) with R replaced by $\lambda Y_T + R$. During which step, an expression for the value function $V_1(y;H)$ defined in (3.4) will be obtained. To determine the optimal entry time, the problem can be treated as an "exit problem" again because we will still be dealing with an optimal stopping problem. More precisely, our goal is to determine the shape of the resulted value function $V_1(y;\lambda y+R) := g_2(y) = g_2(s^{-1}(x))$, which would enable us to discuss its least concave majorant function and this will be the solution to our optimal entry problem.

3.2 Optimal entry strategy under the case $\mu = 0$

For the following subsections in Section 3, we consider our problem subject to three different cases of μ and try to express the solution of the strategy thresholds explicitly and therefore to determine investors' behaviour with respect to the underlying parameters. We start with the special case where $\mu = 0$ ($\beta = 1$), to better understand the problem which will also enable us to obtain a closed-form solution more easily. Hence we have the price dynamic

$$dY_t = \sigma Y_t dW_t \qquad Y_0 = y \tag{3.6}$$

then there is no necessity of the scale function here as the process Y_t is naturally a martingale (i.e., Y_t is in natural scale, $s(y) = y$ up to Affine transformation), and the utility function yields

$$U(y) = \begin{cases} y^\alpha & y > 0 \\ -k(-y)^\alpha & y \leq 0 \end{cases} \tag{3.7}$$

We formulate the exit problem as the investor's value function:

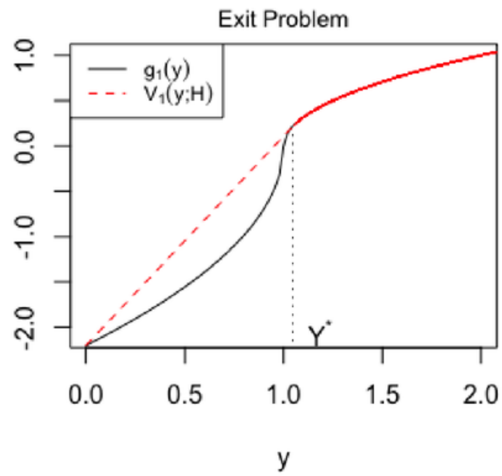
$$\begin{aligned} V_1(y; H) &:= \sup_\tau \mathbb{E}\left[U(Y_\tau - H) \mid Y_0 = y\right] \\ &= \sup_\tau \mathbb{E}\left[g_1(Y_\tau) \mid Y_0 = y\right] \end{aligned} \tag{3.8}$$

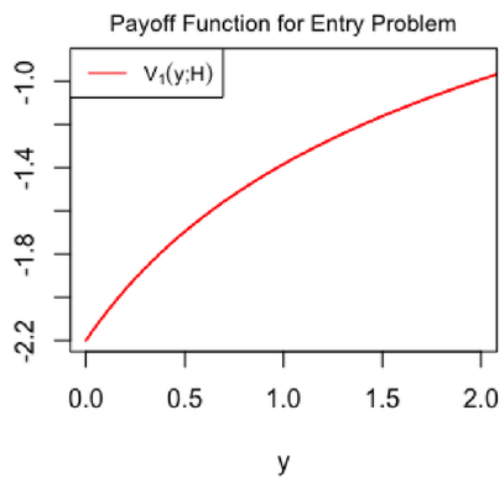
where $g_1(y) := U(y - H)$, and H denotes the investor's reference level. Hence we have the expression for $g_1(y)$:

$$g_1(y) = \begin{cases} (y - H)^\alpha & y > H \\ -k(H - y)^\alpha & y \leq H \end{cases} \tag{3.9}$$

The value function $V_1(y; H)$ is simply the least concave majorant of the function $g_1(y)$.

Figure 4: Stylised graphs of function $g_1(y)$ and the corresponding value function $V_1(y;H)$ with $\alpha = 0.5$, $k = 2.2$, $H = 1$, $\lambda = 1.01$.





Now we solve for the value function $V_1(y;H)$ analytically to obtain an explicit expression and solve the problem by determining its least concave majorant function. Denote Y^* to be the horizontal axis coordinate of the touching point of $g_1(y)$ and $V_1(y;H)$, where Y^* satisfies

$$
\begin{aligned}
g_1'(Y^*) &= \frac{g_1(Y^*) - g_1(0)}{Y^*} \\
\alpha(Y^* - H) &= \frac{(Y^* - H)^\alpha + kH^\alpha}{Y^*} \\
\alpha\frac{Y^*}{H}(\frac{Y^*}{H} - 1)^{\alpha-1} &= (\frac{Y^*}{H} - 1)^\alpha + k
\end{aligned}
\tag{3.10}
$$

which suggests Y^* is taking the form such that $Y^* = cH$, where $c > 1$ is a constant given by the solution of the equation (substitute $Y^* = cH$ into (3.6)):

$$
\alpha c(c-1)^{\alpha-1} = (c-1)^\alpha + k \tag{3.11}
$$

Then we have an explicit expression for the value function $V_1(y; H)$:

$$
V_1(y; H) = \begin{cases} \alpha H^{\alpha-1}(c-1)^{\alpha-1}y - kH^\alpha & y \leq cH \\ (y - H)^\alpha & y > cH \end{cases} \tag{3.12}
$$

Now the joint problem of entry and exit (3.2) can be represented as

$$
\sup_{\tau_1} \mathbb{E}\left[V_1(Y_{\tau_1}; \lambda Y_{\tau_1} + R) \mid Y_0 = y\right] \tag{3.13}
$$

This motivates us to find the form, or geometrically speaking, the shape of

$$
g_2(y) := V_1(y; \lambda y + R) \tag{3.14}
$$

with parameters $\lambda \geq 1, R \geq 0$.

Since we know that $y \leq c(\lambda y + R)$, the first case where $y \leq cH$ always applies. Hence we can further simplify the expression of $V_1(y; H)$ to

$$
V_1(y; \lambda y + R) = \alpha H^{\alpha-1}(c-1)^{\alpha-1}y - kH^\alpha \tag{3.15}
$$

where an explicit function of y and its shape depends on the parameters involved.

This is the value function while selling the asset. In order to study the optimal time for purchasing the asset, we would like to study the shape of the function $g_2(y)$ defined in (3.14), after which we will be able to find the least concave majorant function of $g_2(y)$ to solve the entry problem. We consider the second derivative of $g_2(y)$ to examine the concavity/convexity of the function.

$$
g_2''(y) = \underbrace{\lambda\alpha(1-\alpha)(\lambda y + R)^{\alpha-3}}_{:=\gamma(y)}\left[\underbrace{\lambda(\lambda k - \alpha(c-1)^{\alpha-1})}_{:=a} y + R\underbrace{(\lambda k - 2(c-1)^{\alpha-1}}_{:=b}\right] \tag{3.16}
$$

As a result, we have $g_2''(y)$ is in the form of $g_2''(y) = \gamma(y)(ay + b)$, where $\gamma(y) > 0$, a, b are constants. So the concavity/convexity of $g_2(y)$ depends on the signs of a and b which depend on the parameters and the result follows:

Proposition 3.1. *Consider an investor facing the optimal purchase problem*

$$\sup_{\tau_1} \mathbb{E}\left[V_1(Y_{\tau_1}; \lambda Y_{\tau_1} + R) \mid Y_0 = y\right]$$

where Y follows a driftless geometric Brownian motion $dY = \sigma Y dW$ and $V_1(\cdot;\cdot)$ is given by (3.12). The value function of the problem is $\bar{g}_2$ which is defined as the least concave majorant of g_2. The solution consists of three cases, depending on relative parameter values.

i) If $\lambda \leq \frac{\alpha(c-1)^{\alpha-1}}{k}$, then $g_2(y)$ is increasing concave $\forall y \geq 0$. Therefore the least concave majorant function $\bar{g}_2(y)$ is $g_2(y)$ itself. Hence, $\bar{g}_2(y) = g_2(y)$.

ii) If $\frac{\alpha(c-1)^{\alpha-1}}{k} < \lambda \leq \frac{2(c-1)^{\alpha-1}}{k}$, then

$$\bar{g}_2(y) = \begin{cases} g_2(y) & y \leq y^* \\ g_2(y^*) & y > y^* \end{cases}, \qquad y \geq 0$$

where $y^* = \frac{R((c-1)^{\alpha-1} - \lambda k)}{\lambda(\lambda k - \alpha(c-1)^{\alpha-1})}$ given by $g_2'(y^*) = 0$.

i.e., $\bar{g}_2(y)$ coincides with g_2 up to the maximum point y^* of $g_2(y)$ and followed by a horizontal line valued at this maximum.

iii) If $\lambda \geq \frac{2(c-1)^{\alpha-1}}{k}$, then $g_2(y)$ is decreasing convex $\forall y \geq 0$. Thus, the least concave majorant $\bar{g}_2$ is a horizontal line valued at $g_2(0)$. Hence, $\bar{g}_2(y) = g(0) = -kR^\alpha$.

Proof. We begin our proof with some investigation of the monotonicity of $g_2(y)$

$$g_2'(y) = \alpha(\lambda y + R)^{\alpha-2}\left[\lambda(\alpha(c-1)^{\alpha-1} - \lambda k)y + R((c-1)^{\alpha-1} - \lambda k)\right] \tag{3.17}$$

Recall from (3.16) that g_2 can be written as $g_2''(y) = \gamma(y)(ay + b)$.

i) If $a < 0$ and $b < 0$, i.e., $\lambda < \frac{\alpha(c-1)^{\alpha-1}}{k}$ and $\lambda < \frac{2(c-1)^{\alpha-1}}{k}$, subsequently if $\lambda < \frac{\alpha(c-1)^{\alpha-1}}{k}$, by (3.17), g_2 is monotonically increasing $\forall y \geq 0$, then $g_2(y)$ is increasing concave $\forall y \geq 0$. Thus $\bar{g}_2$ is the function g_2 itself.

ii) If $a > 0$ and $b < 0$, i.e., $\frac{\alpha(c-1)^{\alpha-1}}{k} < \lambda < \frac{2(c-1)^{\alpha-1}}{k}$, then $g_2(y)$ is concave for $0 \leq y < \frac{R(2(c-1)^{\alpha-1} - \lambda k)}{\lambda(\lambda k - \alpha(c-1)^{\alpha-1})}$ and convex for $y > \frac{R(2(c-1)^{\alpha-1} - \lambda k)}{\lambda(\lambda k - \alpha(c-1)^{\alpha-1})}$. In addition, from (3.17) $g_2(y)$ is increasing for $0 \leq y < \frac{R(2(c-1)^{\alpha-1} - \lambda k)}{\lambda(\lambda k - \alpha(c-1)^{\alpha-1})}$ and decreasing for $y > \frac{R(2(c-1)^{\alpha-1} - \lambda k)}{\lambda(\lambda k - \alpha(c-1)^{\alpha-1})}$. Graphically speaking, the function $g_2(y)$ is a concave inverted U-shape to start with, followed by decreasing convexity. Under such a situation, $\bar{g}_2$ is consists of the function g_2 up to the point $y^* = \frac{R((c-1)^{\alpha-1} - \lambda k)}{\lambda(\lambda k - \alpha(c-1)^{\alpha-1})}$ where $g_2(y)$ is maximised and a flat line valued at this maximum.

iii) If $a \geq 0$ and $b \geq 0$, i.e., $\lambda \geq \frac{\alpha(c-1)^{\alpha-1}}{k}$ and $\lambda \geq \frac{2(c-1)^{\alpha-1}}{k}$, subsequently if $\lambda \geq \frac{2(c-1)^{\alpha-1}}{k}$, by (3.17), $g_2(y)$ is monotonically decreasing $\forall y \geq 0$, then $g_2(y)$ is decreasing convex $\forall y \geq 0$. Hence the least concave majorant $\bar{g}_2$ is a horizontal line valued at $g_2(0)$.

$a < 0$ and $b > 0$ would lead to a contradiction as $\alpha < 1$.

□

Proposition 3.2. *Phrased economically, with the corresponding enumeration in Proposition 3.1.*

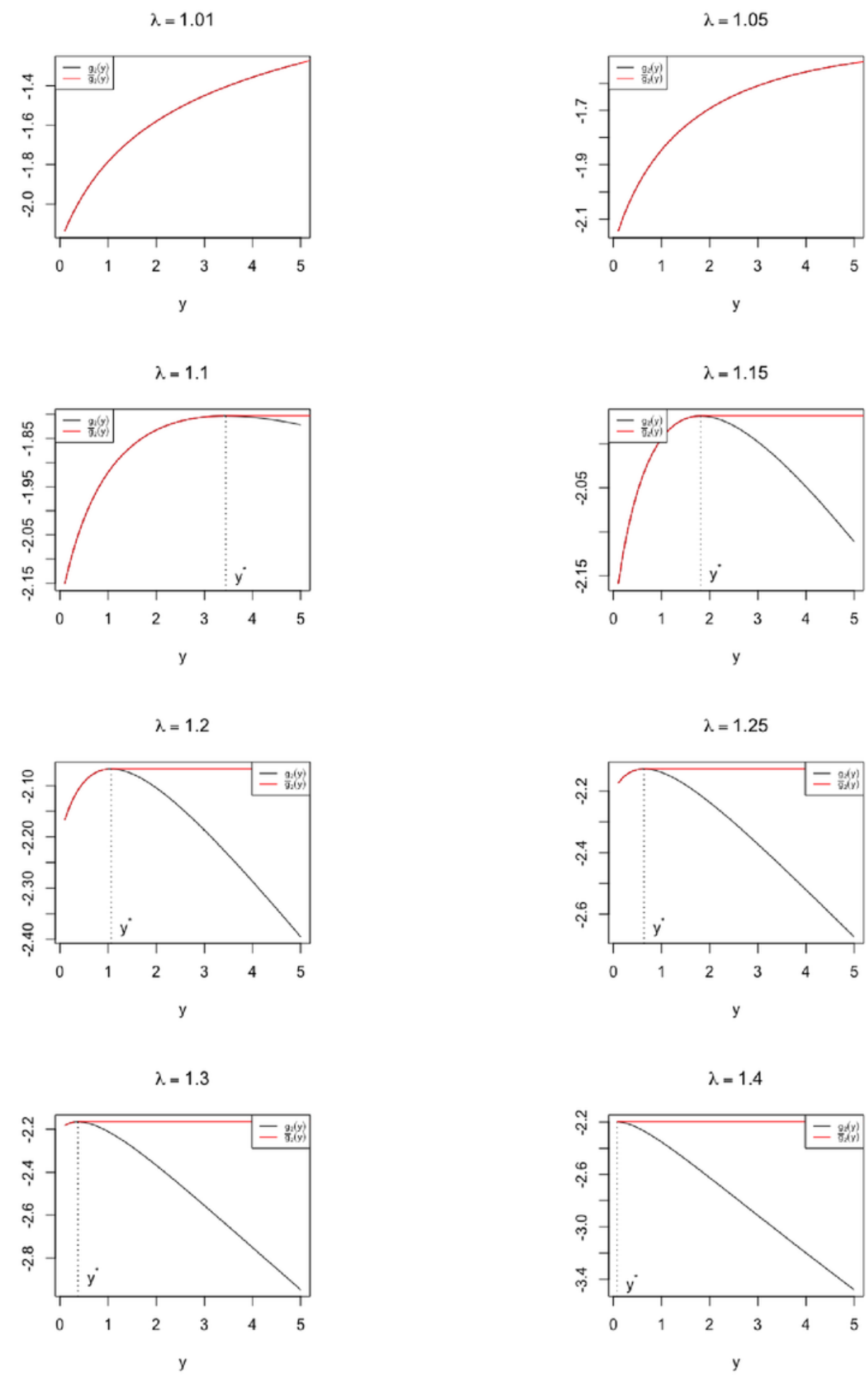
i) When λ is relatively small, it is optimal for the investor to open a position immediately at all price levels.

ii) When λ is of intermediate value (namely case ii) in Proposition 3.1), the investor will choose to buy the asset at or below the price level $y^* = \frac{R((c-1)^{\alpha-1}-\lambda k)}{\lambda(\lambda k-\alpha(c-1)^{\alpha-1})}$.

iii) When λ is relatively large, the investor never purchases the asset (or to stop at $y = 0$ which would not happen as a geometric Brownian motion will not hit zero, this corresponds to the optimal stopping time $\tau_1^* = \infty$ which means that the investor never buys the asset, subsequently, $\tau_2^* = \infty$, i.e., the investor never sells. Such a strategy leads to a 0 payoff from trading and a realised utility $U(-R) = -kR^\alpha$).

This provides us with a clearer picture of the properties of $g_2(y)$.

Figure 5: Graphs of the value function g_2 and its smallest concave majorant function $\bar{g}_2$ with different values of λ for $\mu = 0$, denote y^* to be the maximum point of $g_2(y)$.



Each graph in Figure 5 is plotted with $\alpha = 0.7$, $R = 1$, $c = 1.0213$ where c is calculated from (3.11) and $k = 2.2$. More specifically, the two thresholds in Proposition 3.1 for λ are 1.0093 and 2.8836, corresponds to the other parameter values we have substituted in. When λ is relatively small (i.e., the first two cases in Figure 5) then th least concave majorant function $\bar{g}_2$ coincides with g_2. As for larger value of λ, e.g., $\lambda \geq 1.1$, we observe that the curve of g_2 starts to decrease and switches from concave to convex as y increases, as a consequence, the least concave majorant of g_2 will become a flat line valued at where g_2 is maximised, as illustrated in Figure 5. For sufficiently large value of λ, the least concave majorant of g_2 will be a horizontal line valued at $g_2(0)$.

In addition, if we are in the situation where $\frac{\alpha(c-1)^{\alpha-1}}{k} < \lambda \leq \frac{2(c-1)^{\alpha-1}}{k}$, we can nevertheless investigate how the optimal purchase boundaries change with the model parameters. Recall that g_2 switches from concave to convex at the point $\tilde{y} = \frac{R(2(c-1)^{\alpha-1}-\lambda k)}{\lambda(\lambda k-\alpha(c-1)^{\alpha-1})}$, thanks to the explicit expression for $\tilde{y}$, we deduce the following, ceteris paribus:

- if λ increases (within the range), as a result, $\tilde{y}$ decreases. This would correspond to the case where the investor chooses to buy the asset when the price level is relatively low, which is reasonable as the entry cost increases. Vice versa for decreased λ.
- If k increases, i.e., a higher extent of loss aversion, will also resulting in a decreased value of $\tilde{y}$. This is intuitive because investors with higher loss aversion are more reluctant to purchase assets with a comparatively high initial price. Vice versa for investors with less aversion towards losses.

3.3 Optimal entry strategy under the case $\mu > 0$ and $\alpha \leq \beta < 1$

Now we explore the case where $\mu > 0, \alpha \leq \beta < 1$ and $\alpha_1 = \alpha_2 = \alpha$. The diffusion process Y follows (3.1), with the help of the scale function, we denote the transformed price process $X_t = s(Y_t)$, and $s(H) = H^\beta$. Recall that $g_1(x) \coloneqq U(s^{-1}(x) - H)$, then the function $g_1(x)$ yields

$$g_1(x) = \begin{cases} -k(H - x^{\frac{1}{\beta}})^\alpha & x < H^\beta \\ (x^{\frac{1}{\beta}} - H)^\alpha & x \geq H^\beta \end{cases} \tag{3.18}$$

Recall that $g_1(\cdot)$ is an increasing function regardless of the value of λ (for fixed H). We approach the problem in a similar fashion as in the previous Section, i.e., to find X^* satisfies

$$\begin{aligned} g_1'(X^*) &= \frac{g_1(X^*) - g_1(0)}{X^*} \\ \frac{\alpha}{\beta}(X^* - H)^{\alpha-1} X^{*\frac{1}{\beta}-1} &= \frac{(X^{*\frac{1}{\beta}-1} - H)^\alpha + kH^\alpha}{X^*} \end{aligned} \tag{3.19}$$

which suggests that X^* takes the form $X^* = c^\beta H^\beta$, where $c > 1$ is a constant and solves the equation

$$\frac{\alpha}{\beta}(c-1)^{\alpha-1}c = (c-1)^{\alpha} + k \tag{3.20}$$

from which $c = 1.0737$ is given by taking $\alpha = 0.5$, $\beta = 0.8$, $k = 2.2$, and this value of c will be later used in the graphs.

Thus the least concave majorant $V_1(x; H)$ of the function $g_1(x)$ reads

$$V_1(x;H) = \begin{cases} g_1'(X^*)x - kH^{\alpha} & x < c^{\beta}H^{\beta} \\ g_1(x) = (-H + x^{\frac{1}{\beta}})^{\alpha} & x \geq c^{\beta}H^{\beta} \end{cases} \tag{3.21}$$

$$V_1(x;H) = \begin{cases} \frac{\alpha}{\beta}(c-1)^{\alpha-1}c^{1-\beta}H^{\alpha-\beta}x - kH^{\alpha} & x < c^{\beta}H^{\beta} \\ (-H + x^{\frac{1}{\beta}})^{\alpha} & x \geq c^{\beta}H^{\beta} \end{cases} \tag{3.22}$$

Provided that $c > 1, \lambda \geq 1, R \geq 0$, and H depends on x: $H(x) = \lambda x^{\frac{1}{\beta}} + R$, we have $x < c^{\beta}(\lambda x^{\frac{1}{\beta}} + R)^{\beta} = c^{\beta}H(x)^{\beta}$, hence the first situation in (3.22) always applies. For simplicity of notation, define $g_2(x) \coloneqq V_1(x; \lambda x^{\frac{1}{\beta}} + R)$.

Likewise, we explore properties of the function g_2 by investigating its first and the second order differentiation

$$g_2(x) = H(x)^{\alpha-\beta}\Big(\frac{\alpha}{\beta}(c-1)^{\alpha-1}c^{1-\beta}x - kH(x)^{\beta}\Big)$$

We first consider the first derivative

$$\begin{aligned} g_2'(x) &= \frac{\alpha}{\beta}H(x)^{\alpha-\beta-1}\big[(c-1)^{\alpha-1}c^{1-\beta}(R + \frac{\alpha\lambda}{\beta}x^{\frac{1}{\beta}}) - k\lambda x^{\frac{1}{\beta}-1}H(x)^{\beta}\big] \\ &= \frac{\alpha}{\beta}H(x)^{\alpha-\beta-1}x^{\frac{1}{\beta}}\big[(c-1)^{\alpha-1}c^{1-\beta}(Rx^{-\frac{1}{\beta}} + \frac{\alpha\lambda}{\beta}) - k\lambda(\lambda + Rx^{-\frac{1}{\beta}})^{\beta}\big] \\ &= \frac{\alpha}{\beta}x^{\frac{1}{\beta}}H(x)^{\alpha-\beta-1}h_1(x^{-\frac{1}{\beta}}) \\ &= \frac{\alpha}{\beta}x^{\frac{1}{\beta}}(\lambda x^{\frac{1}{\beta}} + R)^{\alpha-\beta-1}h_1(x^{-\frac{1}{\beta}}) \end{aligned}$$

where $h_1(z) \coloneqq (c-1)^{\alpha-1}c^{1-\beta}(Rz + \frac{\alpha\lambda}{\beta}) - k\lambda(\lambda + Rz)^{\beta}$.

Check that $g_2'(0) = \frac{\alpha}{\beta}(c-1)^{\alpha-1}c^{1-\beta}R^{\alpha-\beta} > 0$, also note that the sign of $g_2'(\infty)$ is governed by the sign of $h_1(0)$.

$$h_1(0) = \lambda\Big(\frac{\alpha}{\beta}(c-1)^{\alpha-1}c^{1-\beta} - k\lambda^{\beta}\Big) \quad \begin{cases} > 0 & if \quad \lambda < \big(\frac{\alpha}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\big)^{\frac{1}{\beta}} \\ < 0 & if \quad \lambda > \big(\frac{\alpha}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\big)^{\frac{1}{\beta}} \end{cases}$$

Now we check the monotonicity of $h_1(\cdot)$.

$$\begin{aligned} h_1'(z) &= (c-1)^{\alpha-1}c^{1-\beta}R - \frac{k\lambda\beta R}{(Rz+\lambda)^{1-\beta}} \\ &\geq (c-1)^{\alpha-1}c^{1-\beta}R - \frac{k\lambda\beta R}{\lambda^{1-\beta}} \\ &= \Big(\frac{(c-1)^{\alpha-1}c^{1-\beta}}{\beta} - k\lambda^{\beta}\Big)R\beta \end{aligned} \tag{3.23}$$

from the first line of (3.23) we observe that h_1 is an increasing function in z, i.e., $h_1'(z) \geq 0$. This observation assists us to deduce that

- If $\lambda < \big(\frac{1}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\big)^{\frac{1}{\beta}}$, then h_1 is increasing.
- If $\lambda > \big(\frac{1}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\big)^{\frac{1}{\beta}}$, then h_1 is first decreasing and then increasing.

Note that $h_1(\infty) > 0$ which means $g_2(x)$ is always increasing for small x regardless the value of λ. One can also check that $h_1''(z) = k\beta(1-\beta)R^2(Rz+\lambda)^{\beta-2} > 0$ which indicates that $h_1(\cdot)$ is a convex function.

Thus:

- If $\lambda < \big(\frac{\alpha}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\big)^{\frac{1}{\beta}}$, then $h_1(0) > 0$ and since h_1 is increasing, $h_1(z) > 0, \forall z \geq 0$.
- If $\big(\frac{\alpha}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\big)^{\frac{1}{\beta}} < \lambda < \big(\frac{1}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\big)^{\frac{1}{\beta}}$, then $h_1(0) < 0$ and h_1 is increasing.
- If $\lambda > \big(\frac{1}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\big)^{\frac{1}{\beta}}$, then $h_1(0) < 0$ and h_1 is initially decreasing and then increasing as z increases.

we therefore deduce that h_1 goes through the x-axis ($h_1(z) = 0$) at most once. Especially, for $\lambda > \big(\frac{\alpha}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\big)^{\frac{1}{\beta}}$, h_1 crosses 0 exactly once and it goes from negative to positive, this results in g_2 initially increases and then decreases.
We treat $g_2''(x)$ in a similar fashion. Write

$$
\begin{aligned}
g_2''(x) &= H(x)^{\beta-\alpha}x^{\frac{1}{\beta}-2}\Big\{(c-1)^{\alpha-1}c^{1-\beta}x\Big[\frac{\lambda\alpha}{\beta^2}H(x)-\frac{\lambda}{\beta}(\beta-\alpha+1)(R+\frac{\alpha\lambda}{\beta}x^{\frac{1}{\beta}})\Big] \\
&\quad - k\lambda H(x)^{\beta}\Big[-\lambda(1-\frac{\alpha}{\beta})x^{\frac{1}{\beta}}+R(\frac{1}{\beta}-1)\Big]\Big\} \\
&= \frac{\lambda\alpha}{\beta}H(x)^{\alpha-\beta-2}x^{\frac{1}{\beta}-2}\Big\{(c-1)^{\alpha-1}c^{1-\beta}x\Big[-\frac{\lambda^2\alpha}{\beta^2}(\beta-\alpha)x^{\frac{1}{\beta}}+\frac{R}{\beta}(\frac{\alpha}{\beta}-\beta+\alpha-1)\Big] \\
&\quad - kH(x)^{\beta}\Big[-\lambda(1-\frac{\alpha}{\beta})x^{\frac{1}{\beta}}+R(\frac{1}{\beta}-1)\Big]\Big\} \\
&= \frac{\lambda\alpha}{\beta}H(x)^{\alpha-\beta-2}x^{\frac{1}{\beta}-2}x^{\frac{1}{\beta}}\Big\{(c-1)^{\alpha-1}c^{1-\beta}x\Big[-\frac{\lambda^2\alpha}{\beta^2}(\beta-\alpha)+\frac{R}{\beta}(\frac{\alpha}{\beta}-\beta+\alpha-1)x^{-\frac{1}{\beta}}\Big] \\
&\quad - kx(\lambda+Rx^{-\frac{1}{\beta}})^{\beta}\Big[-\lambda(1-\frac{\alpha}{\beta})+R(\frac{1}{\beta}-1)x^{-\frac{1}{\beta}}\Big]\Big\} \\
&= \frac{\lambda\alpha}{\beta}H(x)^{\alpha-\beta-2}x^{\frac{2}{\beta}-1}h_2(x^{-\frac{1}{\beta}}) \\
&= \frac{\lambda\alpha}{\beta}(\lambda x^{\frac{1}{\beta}}+R)^{\alpha-\beta-2}x^{\frac{2}{\beta}-1}h_2(x^{-\frac{1}{\beta}})
\end{aligned}
$$

where $h_2(z) \coloneqq (c-1)^{\alpha-1}c^{1-\beta}\big[-\frac{\lambda\alpha}{\beta^2}(\beta-\alpha)+\frac{R}{\beta}(\frac{\alpha}{\beta}-\beta+\alpha-1)z\big]-k(\lambda+Rz)^{\beta}\big[-\lambda(1-\frac{\alpha}{\beta})+R(\frac{1}{\beta}-1)z\big]$.

$$
h_2(0) \quad \begin{cases} >0 & if \quad \lambda > \big(\frac{\alpha}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\big)^{\frac{1}{\beta}} \\ <0 & if \quad \lambda < \big(\frac{\alpha}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\big)^{\frac{1}{\beta}} \end{cases}
$$

Note that $h_2'(0) < 0$ provided that $\lambda < \big(\frac{\alpha}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\big)^{\frac{1}{\beta}}$. Direct differentiation yields that $h_2''(z) < 0$, therefore $h_2(z)$ exhibits concavity.
Hence

- If $\lambda < \big(\frac{\alpha}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\big)^{\frac{1}{\beta}}$, then $h_2(0) < 0$, $h_2'(0) < 0$ and $h_2(z) < 0$, $\forall z \geq 0$.
- If $\lambda > \big(\frac{\alpha}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\big)^{\frac{1}{\beta}}$, then $h_2(0) > 0$ and $h_2(z)$ initially increases then decreases as z grows.

therefore $h_2(z)$ crosses the x-axis exactly once (since $h_2(z) \xrightarrow{\infty} -\infty$).

Calculations show $g_2(0) = -kR^{\alpha} \leq 0$, and keep in mind that $z = x^{-\frac{1}{\beta}}$. Our result is as follows:

Proposition 3.3. *Consider an investor facing the optimal purchase problem*

$$
\sup_{\tau_1} \mathbb{E}\left[V_1(Y_{\tau_1}; \lambda Y_{\tau_1} + R) \mid Y_0 = y\right]
$$

where Y follows geometric Brownian motion $dY = Y(\mu dt + \sigma dW)$ with $\mu > 0$, we write $\beta = 1 - \frac{2\mu}{\sigma^2} < 1$ and $V_1(\cdot;\cdot)$ is given by (3.22). Define the transformed price $X_t = s(Y_t)$, and the scale function yields $s(y) = y^{\beta}$. The value function of the problem is $\bar{g}_2(y^{\beta})$ where $\bar{g}_2(x)$ is defined as *the least concave majorant of $g_2(x)$. The solution consists of three cases, depending on relative parameter values.*

i) If $\lambda < \left(\frac{\alpha}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\right)^{\frac{1}{\beta}}$, then $\bar{g}_2(x)$ is the function $g_2(x)$ itself. Hence, $\bar{g}_2(x) = g_2(x)$.

ii) If $\lambda > \left(\frac{\alpha}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\right)^{\frac{1}{\beta}}$,

$$\bar{g}_2(x) = \begin{cases} g_2(x) & x \leq x^* \\ g_2(x^*) & x > x^* \end{cases}, \qquad x > 0$$

where x^* satisfies

$$\begin{aligned} g_2'(x^*) &= 0 \\ \Leftrightarrow h_1(x^{*-\frac{1}{\beta}}) &= 0 \\ \Leftrightarrow (c-1)^{\alpha-1}c^{1-\beta}(Rz^* + \frac{\alpha\lambda}{\beta}) &= k\lambda(\lambda + Rz^*)^{\beta} \end{aligned}$$

where $z^* = x^{*-\frac{1}{\beta}}$.

Hence, $\bar{g}_2(x)$ coincides with $g_2(x)$ up to the maximum point of $g_2(x)$ and becomes a flat line valued at this maximum.

Proof. Recall that $z = x^{-\frac{1}{\beta}}$.

i) When $\lambda < \left(\frac{\alpha}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\right)^{\frac{1}{\beta}}$, $g_2'(x) > 0$ and $g_2''(x) < 0$ (since $h_2 < 0$), then $g_2(x)$ is an increasing concave function, so the least concave majorant $\bar{g}_2$ is the function g_2.

ii) Since h_1 passes through the x-axis at most once, we can deduce that $g_2(x)$ has at most one stationary point. When $\lambda > \left(\frac{\alpha}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\right)^{\frac{1}{\beta}}$, $h_1(0) < 0 \Rightarrow g_2'(x) < 0$ as $x \to \infty$, $h_1(z)$ changes sign from negative to positive as z increases, equivalently, $g_2'(x)$ goes from positive to negative as x increases. Additionally, $h_2(z) \xrightarrow{\infty} -\infty \Rightarrow g_2''(x) < 0$ and $h_2(0) > 0 \Rightarrow g_2''(x) > 0$ as $x \to \infty$. Thus $g_2(x)$ is concave for small x and convex for large x. Also notice that h_2 passes through the x-axis exactly once gives that $g_2(x)$ changes concavity/convexity exactly once. Combing all of which gives us the least concave majorant $\bar{g}_2(x)$ of $g_2(x)$ is an initially increasing concave function until the maximum point x^* of $g_2(x)$, followed by a horizontal line with the y-axis value of $g_2(x^*)$. The ideas can be more intuitively and further demonstrated by Figure 6.

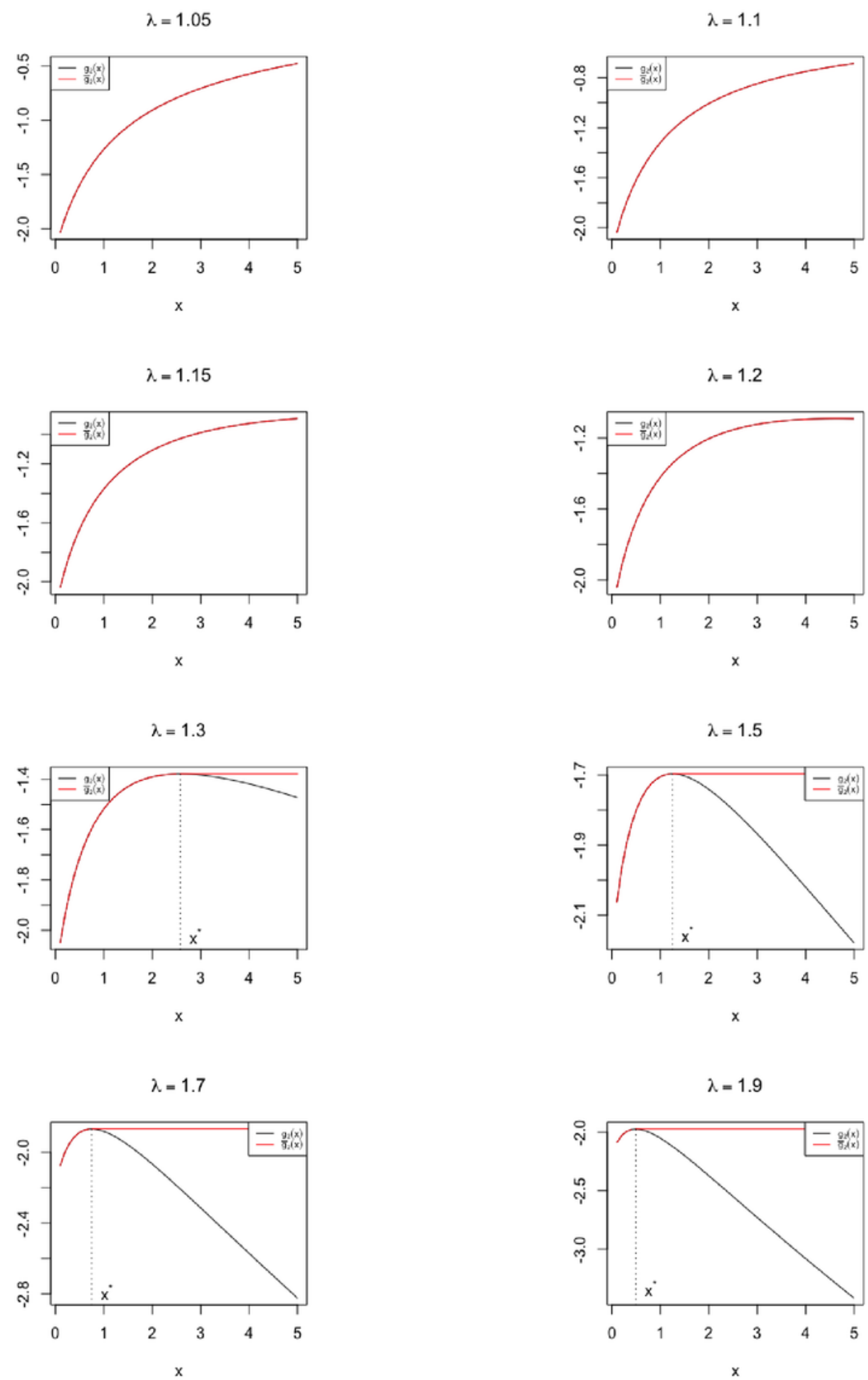
□

Proposition 3.4. *Followed from Proposition 3.3, in the context of financial behaviour, the optimal purchase opportunity for an investor under different circumstances.*

i) If the entry cost parameter λ is comparatively small, the investor will purchase the asset immediately at all price levels (e.g., see the first four graphs in Figure 6).

ii) If λ is comparatively large, the investor's optimal purchase opportunity will be constrained up to a certain price level (namely, x^* in Proposition 3.3), in other words, the investor will choose not to purchase the asset when the price is beyond this level (e.g., see the last four graphs in Figure 6).

Figure 6: Graphs of the value function g_2 and its smallest concave majorant function $\bar{g}_2$ with respect to different values of λ for $\mu > 0$, denote x^* to be the maximum point of $g_2(x)$.



Each graph in Figure 6 is plotted with $\alpha = 0.5$, $\beta = 0.8$, $H = 1$, $c = 1.0737$ and $k = 2.2$. The two λ thresholds are $\left(\frac{\alpha}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\right)^{\frac{1}{\beta}}$ and $\left(\frac{1}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\right)^{\frac{1}{\beta}}$ or, more specifically 1.0772 and 2.562 in our particular case. As one can observe, for $\lambda = 1.3$ and onwards, the curve of $g_2(x)$ begins to decrease and becomes convex as x increases. Thus the least concave majorant $\bar{g}_2(x)$ for larger value of λ is a horizontal line with value $g_2(x^*)$ after the maximum point x^* of $g_2(x)$.

Similar to our discussion in the $\mu = 0$ case, we investigate the problem further by studying the optimal purchase level with respect to the model parameters. We have showed that $g_2'(x) = \frac{\alpha}{\beta}x^{\frac{1}{\beta}}H^{\alpha-\beta-1}h_1(x^{-\frac{1}{\beta}})$, write $z = x^{-\frac{1}{\beta}}$ and we have an expression for $h_1(\cdot)$, our goal is to find the point where g_2 obtains its maximum, i.e., for which point z such that $h_1(z) = 0$. We have

$$(c-1)^{\alpha-1}c^{1-\beta}(Rz + \frac{\alpha\lambda}{\beta}) = k\lambda(\lambda + Rz)^{\beta} \tag{3.24}$$

We know that h_1 crosses 0 for $\lambda > \left(\frac{\alpha}{\beta k}(c-1)^{\alpha-1}c^{1-\beta}\right)^{\frac{1}{\beta}}$. Ceteris paribus,

- When λ increases, since the highest order of the term λ on the LHS is λ and is $\lambda^{\beta+1}$ on the RHS of (3.24), in order to balance the magnitude on both sides of the equation, z will increase (i.e., x will decrease). In other words, the investor would purchase the asset at a lower price if the entry cost is relatively high.

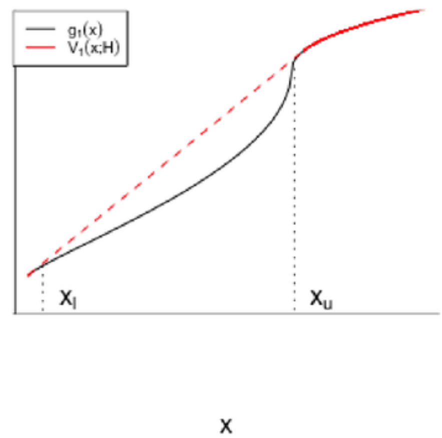
- When k increases, similarly, comparing the magnitude on both sides of the equation will result in an increased value of z (i.e., a decreased value of x). Thus investors with a higher level of loss aversion are more likely to purchase the asset at a lower price level.

3.4 Optimal entry strategy under the case $\mu < 0$

In this section we study the case where $\mu < 0$ ($\beta > 1$), as discussed in Section 2.4, we have $s(H) = H^{\beta}$ and the function $g_1(x)$ takes the form

$$g_1(x) = \begin{cases} -k(H - x^{\frac{1}{\beta}})^{\alpha} & x < H^{\beta} \\ (x^{\frac{1}{\beta}} - H)^{\alpha} & x \geq H^{\beta} \end{cases} \tag{3.25}$$

Figure 7: Stylised representation of the function $g_1(x)$ and its least concave majornt $V_1(x;H)$ for $\mu < 0$ (here we used parameter values $\alpha_1 = \alpha_2 = 0.5$, $\beta = 1.2$, $k = 2.2$).



The existence of the thresholds described follows from the proof of Proposition 2.7. We adapt the same methods as previously to obtain an expression for the value function $V_1(x;H)$. We solve a pair of critical points $x_l < x_u$ at which the gradient of $g_1(x)$ coincides the gradient of line segment joining them, i.e., use the relationship

$$g_1'(x_l) = g_1'(x_u) = \frac{g_1(x_u) - g_1(x_l)}{x_u - x_l} \tag{3.26}$$

and calculations give

$$\begin{cases} -\frac{\alpha k}{\beta}(H - x_l^{\frac{1}{\beta}})^{\alpha-1} x_l^{\frac{1}{\beta}-1} & = \frac{(H-x_u^{\frac{1}{\beta}})^{\alpha} + k(H-x_l^{\frac{1}{\beta}})^{\alpha}}{x_u - x_l} \\ \frac{\alpha}{\beta}(H - x_u^{\frac{1}{\beta}})^{\alpha-1} x_u^{\frac{1}{\beta}-1} & = \frac{(H-x_u^{\frac{1}{\beta}})^{\alpha} + k(H-x_l^{\frac{1}{\beta}})^{\alpha}}{x_u - x_l} \end{cases} \tag{3.27}$$

suggesting that $x_u = c_u^{\beta} H^{\beta}, x_l = c_l^{\beta} H^{\beta}$, where $c_u > 1$ and $c_l < 1$ are constants solve the pair of equations given by the relationship in (3.26).

$$\begin{aligned} \frac{\alpha_1}{\beta}(c_u - 1)^{\alpha_1 - 1} c_u^{1-\beta} &= \frac{(c_u - 1)^{\alpha}_{1} + k(1 - c_l)^{\alpha}_{1}}{c_u^{\beta} - c_l^{\beta}} \\ \frac{k\alpha_1}{\beta}(1 - c_l)^{\alpha_1 - 1} c_l^{1-\beta} &= \frac{(c_u - 1)^{\alpha}_{1} + k(1 - c_l)^{\alpha}_{1}}{c_u^{\beta} - c_l^{\beta}} \end{aligned} \tag{3.28}$$

[We will use parameter values $c_l = 0.3096$ and $c_u = 1.0070$ calculated from (3.28) later where other parameter values $\beta = 1.5$, $\alpha = 0.7$ and $k = 2.2$ were substituted.]

Hence the value function $V_1(x;H)$ yields

$$V_1(x;H) = \begin{cases} -k(H - x^{\frac{1}{\beta}})^{\alpha} & x \le c_l^{\beta} H^{\beta} \\ \delta H^{\alpha-\beta} x - k(1-c_l)^{\alpha} H^{\alpha} - \delta c_l^{\beta} H^{\alpha} & c_l^{\beta} H^{\beta} < x < c_u^{\beta} H^{\beta} \\ (x^{\frac{1}{\beta}} - H)^{\alpha} & x \ge c_u^{\beta} H^{\beta} \end{cases} \tag{3.29}$$

where $\delta := \frac{(c_u-1)^{\alpha}+k(1-c_l)^{\alpha}}{c_u^{\beta}-c_l^{\beta}}$, and H depends on x, $H(x)^{\beta} = (\lambda x^{\frac{1}{\beta}} + R)^{\beta}$ and $\lambda > 1, R \ge 0$, we deduce that only the first two cases in (3.29) apply. Define $g_2(x) := V_1(x; \lambda x^{\frac{1}{\beta}} + R)$.

$$g_2'(x) := V_1'(x; \lambda x^{\frac{1}{\beta}} + R) = \begin{cases} \frac{\alpha k}{\beta}(1-\lambda)\big(H(x) - x^{\frac{1}{\beta}}\big)^{\alpha-1} x^{\frac{1}{\beta}-1} & x \le c_l^{\beta} H(x)^{\beta} \\ (\alpha-\beta)\delta H(x)^{\alpha-\beta-1} H'(x) x + \delta H(x)^{\alpha-\beta} & \\ \quad -\alpha H(x)^{\alpha-1} H'(x)\big(k(1-c_l)^{\alpha} + \delta c_l^{\beta}\big) & c_l^{\beta} H(x)^{\beta} < x < c_u^{\beta} H(x)^{\beta} \end{cases} \tag{3.30}$$

Because the case where $\mu < 0$ is far more complicated to be analysed in generality, thus analytical results in terms of λ dependence are less available, nonetheless, numerical results will be provided here.

Figures 8, 9 and 10 demonstrate some stylised representations of the function $g_2(x)$ along with its least concave majorant, subject to different values of λ and shared parameter values $\alpha = 0.7$, $\beta = 1.5$, $R = 1$, $k = 2.2$, $c_l = 0.3096$ and $c_u = 1.007$, define $\bar{g}_2(x)$ to be the least concave majorant of $g_2(x)$. The optimal entry price thresholds are given to the stylised graphs correspondingly.

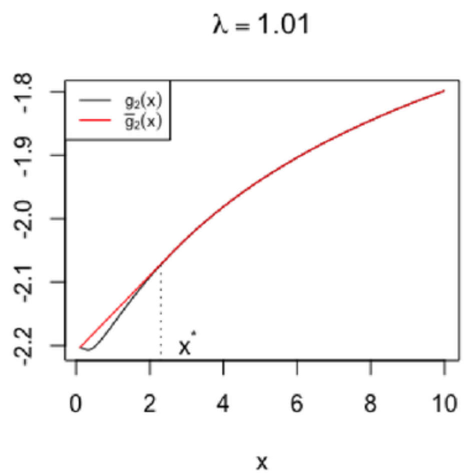


Figure 8: Graph of the functions $g_2(x)$ and $\bar{g}_2(x)$ for $\mu < 0$ with $\alpha = 0.7$, $\beta = 1.5$, $k = 2.2$ and $\lambda = 1.01$.

Note that in (3.30), if $x \le c_l^\beta H(x)^\beta$, $g_2(x)$ is decreasing since $\lambda > 1 \Rightarrow g_2'(x) < 0$. Therefore in our stylised Figure 8, x^* is in the region $[c_l^\beta H(x^*)^\beta, c_u^\beta H(x^*)^\beta]$. This corresponds to the situation where λ is relatively small, the investor will choose to enter the market when the price is sufficiently large, i.e., for $x \ge x^*$, where x^* satisfies

$$g_2'(x^*) = \frac{g_2(x^*) - g_2(0)}{x^*}, \quad c_l^\beta H(x^*)^\beta < x^* < c_u^\beta H(x^*)^\beta \tag{3.31}$$

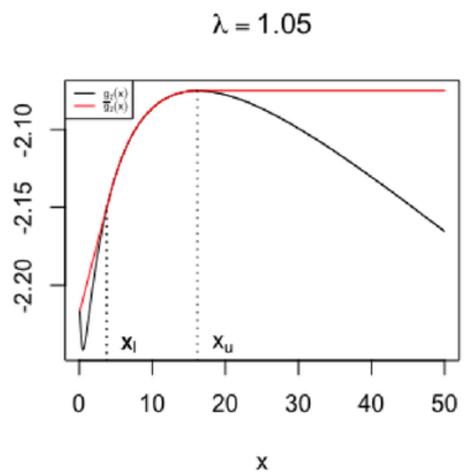


Figure 9: Graph of the functions $g_2(x)$ and $\bar{g}_2(x)$ for $\mu < 0$ with $\alpha = 0.7$, $\beta = 1.5$, $k = 2.2$ and $\lambda = 1.05$.

In Figure 9, λ is of certain intermediate value, the function $g_2(x)$ starts to exhibit convexity for large values of x. This results in the least concave majorant $\bar{g}_2(x)$ of the function $g_2(x)$ formed by a chord from $(0, -kH^\alpha)$ to $(x_l, g_2(x_l))$ where x_l satisfies (3.31) for the same reason, and the function $g_2(x)$ itself until the price level reaches x_u, finally, followed by a horizontal line valued at $g_2(x_u)$, where x_u is the maximum point of $g_2(x)$ satisfies

$$(\alpha - \beta)\delta H'(x_u)x_u + \delta H(x_u) = \alpha H(x_u)^\beta H'(x_u)\big(k(1-c_l)^\alpha + \delta c_l^\beta\big), \quad c_l^\beta H(x_u)^\beta < x_u < c_u^\beta H(x_u)^\beta$$

Under this circumstance, the investor will choose to buy the asset if the price level $x \in (x_l, x_u)$ and wait for all other values of x.

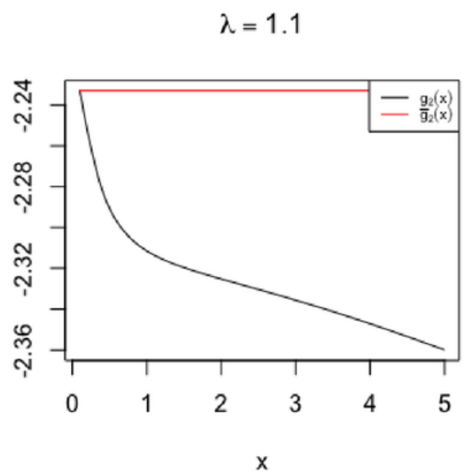


Figure 10: Graph of the functions $g_2(x)$ and $\bar{g}_2(x)$ for $\mu < 0$ with $\alpha = 0.7$, $\beta = 1.5$, $k = 2.2$ and $\lambda = 1.1$.

If λ is relatively large, $g_2(x)$ will become a decreasing convex function, $\bar{g}_2(x)$ is therefore taking the structure of a horizontal line at $g_2(0)$ starting from $x = 0$, i.e., $\bar{g}_2(x) = g_2(0)$. Which can be interpreted as the situation where the investor waits indefinitely.

When considering the case where $\mu < 0$ or , equivalently $\beta > 1$, one approach is to convert this back in terms of μ and σ. This choice of β does not have a very sensible magnitude, however, this is a value that some existing literature have been using. One way of arguing a large β value is that we can think of the drift of the asset also capturing some kind of discounting effect. For example, imagine that we replace the price diffusion process Y_t by $e^{-\delta t}Y_t$, the δ here can be referred as some subjective discount rate, then the notation β we used throughout this thesis will become $1 - \frac{2(\mu-\delta)}{\sigma^2}$, where the existence of the discount rate δ will result in an increased value of β.

3.5 Problem extended with exit costs

We have discussed how the investor's behaviour depends on the entry cost parameter λ, in general, we find that it is optimal not to purchase assets with sufficiently large entry costs, which is very reasonable and pronounced. However, it is also common in practice that investors have to pay extra fees when closing a position, i.e., investors are charged some proportion of their sale as a transaction cost. Then how would investors behave correspondingly? If we denote the exit price to be γY_{τ_2} where $\gamma \leq 1$ is a constant (before we only had Y_{τ_2}). Then (2.8) become

$$\sup_{\substack{\tau_1,\tau_2\\ \tau_1<\tau_2}} \mathbb{E}\left[U(\gamma Y_{\tau_2} - \lambda Y_{\tau_1} - R) \mid Y_0 = y\right] \tag{3.32}$$

with respect to the same filtration and state space.

This extended problem can be handled in the same fashion. We still consider the utility function (2.1) and due to the fact that $U(Cx) = C^{\alpha}U(x)$ for some constant $C > 0$, we have:

1. If $\alpha_1 = \alpha_2 = \alpha$,

$$\begin{aligned}&\sup_{\substack{\tau_1,\tau_2\\ \tau_1<\tau_2}} \mathbb{E}\left[U(\gamma Y_{\tau_2} - \lambda Y_{\tau_1} - R) \mid Y_0 = y\right]\\ =&\sup_{\substack{\tau_1,\tau_2\\ \tau_1<\tau_2}} \mathbb{E}\left[\gamma^{\alpha}U(Y_{\tau_2} - \frac{\lambda}{\gamma}Y_{\tau_1} - \frac{R}{\gamma}) \mid Y_0 = y\right]\\ =&\gamma^{\alpha}\sup_{\substack{\tau_1,\tau_2\\ \tau_1<\tau_2}} \mathbb{E}\left[U(Y_{\tau_2} - \frac{\lambda}{\gamma}Y_{\tau_1} - \frac{R}{\gamma}) \mid Y_0 = y\right]\end{aligned}$$

2. If $\alpha_1 \neq \alpha_2$, the utility function (2.1) with argument Cx yields

$$U(Cx) = \begin{cases} C^{\alpha_1}x^{\alpha_1} & Cx > 0 \\ -kC^{\alpha_2}(-x)^{\alpha_2} & Cx \leq 0 \end{cases}$$

We can, in turn, rewrite the second line as

$$-C^{\alpha_1} \underbrace{k\frac{C^{\alpha_2}}{C^{\alpha_1}}}_{:=\tilde{k}}(-x)^{\alpha_2}$$

then continue with the analysis conducted in Section 3 with the parameter k replaced by $\tilde{k}$.

4 Further Research and Discussion

4.1 Piecewise exponential utility

Another S-shaped utility function often discussed in the literature is the piecewise exponential utility, where the utility function is concave exponential above the investor's reference point and convex exponential below the reference point. As studied in Kyle et al. (2006) [12],

$$U(x) = \begin{cases} k_1(1 - e^{-\gamma_1 x}) & x \geq 0 \\ k_2(e^{\gamma_2 x} - 1) & x < 0 \end{cases}, \qquad \text{where } \gamma_1,\, \gamma_2,\, k_1,\, k_2 > 0.$$

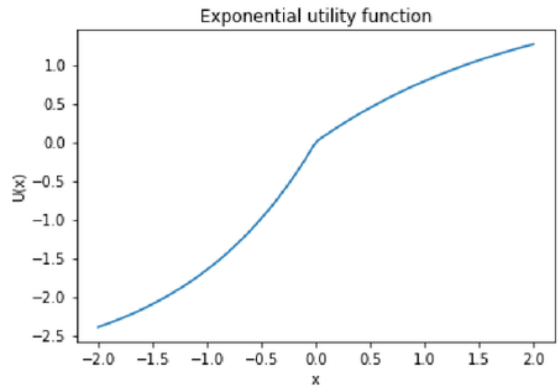


Figure 11: Stylised representation of an exponential utility function

Observe that different from our utility in (2.1), the limits $U'(0_-) > U'(0_+)$ are finite, thus the investor is more sensitive to losses around the origin. If taking $Y = 0$ to represent the investor's reference point, γ_1 represents the local absolute risk aversion above this reference point, whereas γ_2 represents the local absolute risk-seeking below this reference point. In their model they also assume $k_1\gamma_1 < k_2\gamma_2$ to ensure loss aversion.

Their results also consist of dependence on the excess expected return (per unit variance) and the coefficient of loss aversion, which can be found in Kyle et al. (2006) [12] (pp.280-283). However, in their model, there is not a case where an investor would wait and liquidate an asset at a loss, i.e., our scenario *iii*) in Proposition 2.10. We may nonetheless apply our methodology in Section 2 and Section 3 to determine an investor's optimal strategies.

4.2 Piecewise linear utility

Up to this point, we have been considering the S-shaped realised utility. Barberis and Xiong (2012) [1] suggest a dynamic stopping model whereby it is possible for an investor to reinvest in a previously sold asset. In which model they consider a piecewise linear function for utility,

$$U(x) = \begin{cases} x & x > 0 \\ kx & x \leq 0 \end{cases}$$

where $k > 1$ determines an investor's relative sensitivity to realised losses as against realised gains. Furthermore, an investor's decisions in their model are found to be consistent with many situations that have been observed in the markets. However, they concede that their model predicts a strong disposition effect because, in their model, an investor will only liquidate at a gain, never at a loss, unless forced to exit the market by a liquidity stock. A natural direction of research could be testing the implications of this realisation utility.

4.3 Partial liquidations

Another interesting extension of this thesis would be to consider partial liquidations. For example, if the investor has $N \geq 1$ units of claim on the asset Y, each unit i with payoff $f_i(Y)$, $1 \leq i \leq N$, and the investor can choose to sell any units at different times, we denote τ_i for unit i of the asset Y sold, $1 \leq i \leq N$. If the investor realises utility over gains and losses on each partial liquidation, indeed, we can regard the sale of each claim as a separate exit problem. Then the optimal trading strategy follows immediately from Proposition 2.10 for all units of the asset Y. Another possible approach could be supposing that the investor derives utility from the aggregated realised gains and losses over all partial sales. In which case the investor's objective can be expressed as (Henderson 2012 [9] eq (9))

$$V_n(y,\theta) = \sup_{\tau_n \leq \ldots \leq \tau_1} \mathbb{E}\big[U(\sum_{i=1}^{n} f(Y_{\tau_i}) - nf_R + \theta) \mid Y_0 = y\big] \tag{4.1}$$

where $n \leq N$ denotes the number of units remaining and θ represents the wealth at the current stage. This formulation allows us to work backwards from the solution given by N=1. Note that if $N = 1$ and $\theta = 0$, we will recover the case in (2.8). This function represents the value when an investor holding $n \leq N$ units of claim, with initial asset price y and each unit with the identical reference level. Thus the solutions have the same structure regardless of whether the investor realises utility over each individual partial liquidation or over accumulated partial liquidations.

Conclusion

At the beginning of this thesis, we introduce some results in the extant literature that are crucial to our approach to determine the investor's best trading strategies. Motivated by Henderson's Liquidation Model [9], we extend it to a double stopping problem by including the best opportunity to enter the market/purchase an asset. Upon which, we have derived explicit expressions for the value function under certain parameter values, together with optimal price thresholds for entering and exiting the trading market subject to the market entry cost. Now we finish with a reminder of several important contributions.

Firstly, the model and the methodology provide an approach to determine an investor's best behaviour subject to the market entry cost, as well as a characterisation for the value function for both optimal entry and exit. The model itself integrates the preference specification of Tversky and Kahneman (1992) [20] and the common practice that asset prices following geometric Brownian motion.

Secondly, we find that if $\mu \geq 0$ (i.e., nonnegative instantaneous expected excess return), and the market entry cost parameter λ is relatively small, an investor will purchase the corresponding assets immediately. On the contrary, if λ is relatively large, an investor will purchase the asset within a certain price interval and wait otherwise. Although explicit solutions are less available for the case where $\mu < 0$, we are still able to proceed with numerical analysis and study how the optimal strategies may be chosen.

Finally, our extension to include other possible utility functions, namely, piecewise linear utility suggested by Barberis and Xiong (2012) [1] and piecewise exponential utility by Kyle et al. (2006) [12], to provide us with other aspects to study investor's prevalent behaviour observed in the market. Furthermore, partially liquidating enables an investor to have a greater combination of selling strategies, which may potentially improve his/her realised utility.

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