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A Risk Methodology for Cross-Asset Volatility Trading Strategies

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Declaration

The work contained in this thesis is my own work unless otherwise stated.

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Abstract

Volatility as an asset class has greatly expanded over the past years. Inside the volatility trading spectrum, risk premia strategies provide hedge funds with multiple opportunities to develop strategies that provide risk-adjusted returns that have interesting characteristics. The most important issue with those is that they yield large drawdowns in some environments. Hence, risk modeling plays an important role in the definition and execution of a trading strategy. The goal of this paper is to exhibit a risk methodology for cross-asset volatility trading strategies.

To do so, we will carry out a theoretical study on the error on risk, which results from a linear approximation of market returns. In practice, we will study the co-movements of the traded volatility products through a beta sensitivity model derived from Greek and PnL time series. Then, we will use this beta model to propagate extreme local market moves to the whole market through the definition of relevant sparse-risk scenarios. Finally, we will integrate these scenarios to a simple volatility trading strategy and will assess their impact through various performance metrics.

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Introduction

The notion of risk is one of the major components of modern portfolio management theory. Markowitz [14] pioneered this theory by introducing the notion of mean-variance portfolio optimization, that aims to maximize the expected returns of a portfolio under risk constraints, modeled by the variance. Later, Sharpe [17] and Lintner [13] introduced the Capital Asset Pricing Model (CAPM), a statistical approach to risk in finance. They demonstrate that the expected returns of an asset must be linearly related to its *beta*, a measure of the correlation between the asset's returns and the market portfolio's returns: for asset k,

$$\mathbf{E}[r_k] = \alpha_0 + \alpha_1 \beta_k.$$

Lintner [13] suggests a two-pass approach. The beta estimates $\hat{\beta}_k$ are first computed through linear regression, before computing estimates for α_0 and α_1 through cross-sectional regression

$$\mathbf{E}[r_k] \simeq \overline{r_k} = \alpha_0 + \alpha_1 \hat{\beta}_k + \eta_k.$$

Fama and MacBeth [7] refine this model by computing monthly rolling betas. Gibbons [8], on the other hand, uses a maximum likelihood estimation approach to the problem and argues this solves the problem of error-in-variables, introduced by the two-pass approach [7], that arise when the number of assets N in the portfolio increases. Shanken [16] discusses the relationship between these two approaches and suggests solutions to the error-in-variables issue in Fama and MacBeth's approach [7] when N increases.

Meanwhile, the securities traded on markets started to become more and more complex, and more purely mathematical approaches to risk modeling have been developed. Option pricing in particular, pioneered by Bachelier [2] and followed by the Black-Scholes model [6], introduced new tools to measure risk. In particular, the sensitivity of option prices to the variables characterizing them, or *Greeks*, still play a fundamental role in hedging portfolios. Hull and White [10] discussed the notions of Delta-Gamma and Delta-Vega hedging in foreign exchange markets in a stochastic volatility framework. Later, Jarrow and Turnbull [11] describe a delta-hedging framework for interest rate portfolios. And Greeks are still greatly discussed in textbooks [9, Chapter 14, pp.299-329].

In this paper, we reconcile the statistical and mathematical approaches to risk modeling by suggesting a simple risk methodology for specific volatility trading strategies. In Chapter 1, we start from the fundamental assumptions of the CAPM and linear regression [13][7] and show how controlling the error in regression by constructing correlated portfolios leads to a control of the errors on the Greeks, hence on the portfolio risks. We also discuss several methods to construct such portfolios and present the *beta model*, which we use in practice to estimate the correlations between asset returns. In Chapter 2, we discuss the construction of tail-risk scenarios on asset returns. We present a statistical methodology to define base-case scenarios for selected benchmarks and use the beta model to propagate them to the whole universe of assets. Finally, in Chapter 3, we apply this methodology to a volatility trading strategy, the Gamma scalping strategy, and discuss the results of such approach.

Chapter 1

Aggregation via the beta model

In this Chapter, we introduce the notion of Greeks in the fundamental option pricing models (Black-Scholes [6] and Bachelier [2]) and discuss their relation to risk. Then, we derive the theoretical formulas for spot-related Greeks for option portfolios with multiple underlyings and show how controlling the approximation errors resulting from the linear regression of asset returns affects the error on the Greeks. Finally, we present the beta model, a practical implementation of the linear regression theory.

1.1 The notion of Greeks

1.1.1 Greeks and their relationship to risk

As discussed in [9, Chapter 14, pp.299-300], when an investor trades options in the over-the-counter markets, an important component of their trading strategy is the risk management. If similar options are traded on exchange markets as well, it is easy for investors to hedge their exposure as they simply can buy or sell these. However, it is more difficult to quantify risk for options that are either listed exclusively on a single market or ones that are tailored for specific clients. A different approach to this problem is to consider options in a framework, such as the Black-Scholes model [6], that yields a formula for option pricing. In that case, option prices become functions of a set of parameters, and we can define risk as the sensitivity of the option prices to these parameters. Formally, we define risk for trading options as the partial derivatives of the option prices with respect to the parameters that characterize them. These partial derivatives are refered to as the options Greeks.

Consider an option with fixed strike and maturity. In the most common option pricing models, the main parameters that characterize this option's price P are the underlier's spot price S, the volatility σ, the time t and the risk-free interest rate r. The main Greeks associated to these parameters are

- Delta $\Delta := \partial_S P$, which is the sensitivity of the option's price to that of the underlier. Basically, if the underlier's price increases by one unit of currency, the option's price increases by Δ units (it decreases if $\Delta < 0$);

- Gamma $\Gamma := \partial_{SS} P$, which is the sensitivity of the delta to the underlier's price. It is a measure of convexity of the option's price;

- Vega $v := \partial_\sigma P$, which is the sensitivity of the option's price to the implied volatility. In other words, the option's price increases by v units of currency when the implied volatility increases by one point. In practice, we can also define the Vega as the sensitivity of the option's price to an increase of 1% in volatility;

- Theta $\Theta := \partial_t P$, which measures the variation of the option's price with time;

- Rho $\rho := \partial_r P$, which measures the variation of the option's price with the risk-free interest rate.

We can define more greeks as second-order, third-order and cross-variable partial derivatives, but we focus on these ones as they are the main drivers of risk. In practice, we can use Greeks to hedge our positions. The most simple case of Greek hedging is the Delta hedging. In a derivatives portfolio where all the products share the same underlier, it consists in buying $-\Delta$ shares of the underlier, so that the total Delta of the portfolio become zero. Other cases of hedging through Greeks have been studied, such as Delta-Gamma and Delta-Vega hedging. In the following sections, we will only consider the risk related to the underlier's spot and volatility, namely the Delta, Gamma and Vega. However, we will only give theoretical results for the Delta and Gamma, since the results for the Vega are roughly the same as those of the Delta (we simply replace S by σ and the returns by the change in volatility).

1.1.2 Greeks for single-asset vanilla options

From now on, we will consider a Call option[1] in a market with no risk-free interest rate (i.e. $r = 0$). We will look at two major option pricing frameworks, namely the Black-Scholes and the Bachelier models. In these frameworks, since we have a formula for the Call option price, we can easily compute Greeks by differentiation.

[1]Once we have results for Call options, getting the same results for Put options is straightforward. To do so, we can use the Call-Put parity, which is a relationship between the Put and Call prices, assuming a no-arbitrage market.

The Black-Scholes model

In the Black-Scholes framework [6], we assume that the Call option price C is solution to the partial derivative equation given by

$$\partial_t C + \frac{1}{2}\sigma^2 \partial_{SS} C = 0 \tag{1.1.1}$$

where

- (S_t) is the price process of the underlying asset;
- σ is the annualised standard deviation or implied volatility of the stock return

with boundary conditions

- $C(0,t) = 0$;
- $C(S,t) \to S$ when $S \to +\infty$;
- $C(S,T) = (S-K)^+$, with K and T respectively the strike and maturity of the option.

Note that, to simplify the notations, we write $C(S,t)$ for $C(S,K,T,t,\sigma)$. Upon resolution[2], we get

$$C(S,t) = S\Phi(d_1) - K\Phi(d_2) \tag{1.1.2}$$

with

- $d_{1,2} = \frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{S}{K} \pm \frac{1}{2}\sigma^2(T-t)\right)$;
- Φ the cumulative distribution function of a standard Gaussian random variable.

Since we now have a formula for the option's price, we can compute its Delta and Gamma by differentiation. We get

Proposition 1.1.1.

$$\Delta = \Phi(d_1)\ ,\ \Gamma = \frac{1}{S\sigma\sqrt{T-t}\phi(d_1)},$$

where ϕ is the first-order derivative of Φ.

Proof. First, note that

$$\begin{aligned}
\frac{\phi(d_2)}{\phi(d_1)} = \exp\left(-\frac{1}{2}(d_2^2 - d_1^2)\right) &= \exp\left(-\frac{1}{2}(d_2-d_1)(d_2+d_1)\right) \\
&= \exp\left(\frac{1}{2}\sigma\sqrt{T-t}\cdot 2\frac{\ln\frac{S}{K}}{\sigma\sqrt{T-t}}\right) \\
&= \frac{S}{K}
\end{aligned}$$

[2]The reader can find a detailed proof in [6, pp.640-645].

and that $\partial_S d_1 = \partial_S d_2$. Then, we have

$$\Delta = \Phi(d_1) + S\partial_S d_1 \phi(d_1) - K\partial_S d_2 \phi(d_2) = \Phi(d_1)$$

and

$$\Gamma = \partial_S \Delta = \partial_S d_1 \phi(d_1) = \frac{1}{S\sigma\sqrt{T-t}}\phi(d_1).$$

□

The Bachelier model

The Bachelier model [2] is mainly used to price interest rate derivatives. In this framework, since we assume there is no risk-free interest rate, the price of a Call option follows the stochastic differential equation given by

$$\mathrm{d}S_t = \sigma \mathrm{d}W_t,$$

where W is a Brownian motion. Using the same argument as in Black-Scholes [6] and Merton [15], we can derive a formula for the Call price, given by

$$C(S,t) = (S-K)\Phi(z) + \sigma\sqrt{T-t}\phi(z), \tag{1.1.3}$$

where $z := \frac{S-K}{\sigma\sqrt{T-t}}$. Again, we can compute the Delta and Gamma by differentiation, and we have

Proposition 1.1.2.

$$\Delta = \Phi(z),\ \Gamma = \frac{\phi(z)}{\sigma\sqrt{T-t}}.$$

Proof. First, note that $\dot{\phi}(z) = -z\phi(z)$. Then

$$\Delta = \frac{S-K}{\sigma\sqrt{T-t}}\phi(z) + \Phi(z) + \dot{\phi}(z) = z\phi(z) + \Phi(z) - z\phi(z) = \Phi(z)$$

and

$$\Gamma = \frac{1}{\sigma\sqrt{T-t}}\phi(z).$$

□

1.2 Computing portfolio Greeks

In a portfolio of derivatives that all share the same underlying, it is straightforward to compute the portfolio Greeks, as the differentiation operator is linear. Hence, the risk of any linear combination of such assets is simply the linear combination of the individual risks. However, when trading portfolios with multiple underlyings, we cannot simply sum risks. To understand that, we can look at the delta of a portfolio with two options, with prices P_1 and P_2, and underlying prices S_1 and S_2. Let $S := S_1 + S_2$ and $P := P_1 + P_2$. In general, there is no reason to have

$$\partial_S P = \partial_{S_1} P_1 + \partial_{S_2} P_2.$$

Using the same notations, we consider a portfolio of n options, with a price P_k and an underlying price S_k for option k. Note that we can consider the n assets to have different underlyings, since we can aggregate Greeks for options with the same underlying as discussed above. Let $P := \sum_{k=1}^{n} P_k$ and $S := \sum_{k=1}^{n} S_k$.

Throughout the rest of this study, we will use the notation $\bar{f} := \partial_{S_1} f$, given a function f that is differentiable in S_1. We will also assume the S_k are differentiable functions of S_1.

1.2.1 First-order derivatives, the delta

Proposition 1.2.1. *The delta of the portfolio P is given by*

$$\Delta = \beta \sum_{k=1}^{n} \beta_k \Delta_k, \tag{1.2.1}$$

where

- $\beta := \partial_S S_1$;
- $\beta_k := \partial_{S_1} S_k$;
- $\Delta_k := \partial_{S_k} P_k$ *is the Delta of option k.*

Proof. Using the chain rule, we have

$$\Delta = \partial_S P = \sum_{k=1}^{n} \partial_S P_k = \sum_{k=1}^{n} \partial_S S_1 \partial_{S_1} S_k \partial_{S_k} P_k = \beta \sum_{k=1}^{n} \beta_k \Delta_k.$$

□

In this formula, the Delta of the portfolio is written as a weighted sum of individual option Deltas. However, altough the formula looks simple, it is difficult to use it to compute the portfolio Delta, as the β and β_k are not known.

1.2.2 Second-order derivatives, the gamma

Proposition 1.2.2. *The Gamma of the portfolio P is given by*

$$\Gamma = \sum_{k=1}^{n} \left(\bar{\beta}\beta_k + \beta^2 \bar{\beta}_k\right) \Delta_k + \beta^2 \sum_{k=1}^{n} \beta_k^2 \Gamma_k, \tag{1.2.2}$$

where

- β, β_k *are defined in Proposition 1.2.1;*
- $\Gamma_k := \partial_{S_k S_k} P_k = \partial_{S_k} \Delta_k$ *is the Gamma of option k.*

Proof. Again, using the chain rule, we have

$$\begin{aligned}
\Gamma = \partial_S \Delta &= \sum_{k=1}^{n} \left(\partial_S \beta \beta_k + \beta \partial_S \beta_k\right) \Delta_k + \beta \sum_{k=1}^{n} \beta_k \partial_S \Delta_k \\
&= \sum_{k=1}^{n} \left(\bar{\beta}\beta_k + \beta \partial_S S_1 \partial_{S_1} \beta_k\right) \Delta_k + \beta \sum_{k=1}^{n} \beta_k \partial_S S_1 \partial_{S_1} S_k \partial_{S_k} \Delta_k \\
&= \sum_{k=1}^{n} \left(\bar{\beta}\beta_k + \beta^2 \bar{\beta}_k\right) \Delta_k + \beta \sum_{k=1}^{n} \beta_k \beta \beta_k \Gamma_k \\
&= \sum_{k=1}^{n} \left(\bar{\beta}\beta_k + \beta^2 \bar{\beta}_k\right) \Delta_k + \beta^2 \sum_{k=1}^{n} \beta_k^2 \Gamma_k.
\end{aligned}$$

□

The formula for the portfolio Gamma is even more complex than that of the portfolio Delta as it requires computing β, β_k, $\bar{\beta}$ and $\bar{\beta}_k$. In practice, it is clearly not usable.

1.2.3 The linear case

A simple case of this problem is to assume that the returns $r_k := \partial_t \ln S_k = \frac{\partial_t S_k}{S_k}$ are linearly dependent. In that case,

Proposition 1.2.3. *Assume that, for all $1 \leq k \leq n$, $r_k = c_k r_1$, where $c_k \in \mathbf{R}^*$, and that $r_1 \neq 0$ a.s. Let $B := \sum_{k=1}^{n} \beta_k / c_k$. Then*

- $\beta_k = c_k \frac{S_k}{S_1}$;

- $\bar{\beta}_k = c_k(c_k - 1)\frac{S_k}{S_1^2}$;
- $\beta = \frac{B}{B^2 + S\bar{B}}$;
- $\bar{\beta} = -\beta^3 \left[2\bar{B} + S\frac{\bar{\bar{B}}}{B}\right]$.

Proof. The proof is slightly technical and can be found in Appendix A.1. □

1.3 Simplification through linear regression

The problem of computing the Delta and Gamma in a derivatives portfolio with multiple underlyings whose returns are linearly dependent is fairly easy to solve. Therefore, we can intuitively imagine that, if we find a high enough correlation[3] between the returns r_1 and r_2 of two stocks, we can use an approximation $r_2 = c_2 r_1$, where $c_2 \in \mathbf{R}^*$, with an error that is small enough so that we can neglect it while looking at data. In practice, to do so we use linear regression.

1.3.1 Linear regression

Consider N random variables $r_1, \cdots, r_N$. The linear regression equation of r_k on r_1 is given by

$$r_k = a_k + c_k r_1 + \varepsilon_k, \tag{1.3.1}$$

where $a_k, c_k \in \mathbf{R}$ and ε_k is a random variable. Consider a sample $(r_{1,i}, r_{2,i})$, $1 \leq i \leq M$ of outcomes of $(r_1, \cdots, r_N)$. Under a simplified version of the framework presented in [16], we use the following assumptions:

- The ε_k and r_1 are independent random variables;
- The ε_k are centered Gaussian random variables with variance $\sigma_{\varepsilon_k}^2$.

In the case of stock returns, we can assume that $a_k = 0$[4]. Then, for all k, an estimator of c_k, the least square estimator, is given by

$$\hat{c_k} := \arg\min_{c_k} \|r_k - c_k r_1\|^2 = \rho_k \frac{\sigma_k}{\sigma_1},$$

where ρ_k is the Pearson correlation between r_k and r_2, and σ_k (resp. σ_r) the standard deviation of r_1 (resp. r_k).

[3] in absolute value.
[4] We can justify this assumption by invoking the no-arbitrage principle.

1.3.2 Controlling the regression error

From now on, we will consider $c_k = \hat{c}_k$ and will write c_k for simplicity. Since the ε_k and r_1 are independent random variables, we can compute the variance in (1.3.1) as following

$$\sigma_k^2 = c_k^2\sigma_1^2 + \sigma_{\varepsilon_k}^2. \tag{1.3.2}$$

The idea is then to set a threshold $\bar{\rho}$ on the correlations so that, for all k such that $|\rho_k| \geq \bar{\rho}$, the variance $\sigma_{\varepsilon_k}^2$ be bounded by a constant of our choice[5]. The following proposition yields such result.

Proposition 1.3.1. *Let $r_1, \cdots r_N$ N random variables and $\bar{\rho} \in [0,1]$. Even if it means reordering the r_k, suppose that[6] $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_N$. Let $n \geq N$ such that $\rho_k \geq \bar{\rho}$ for all $1 \leq k \leq N$. Then, under the assumptions of 1.3,*

$$0 \leq \max_{k\leq n} \sigma_{\varepsilon_k}^2 \leq \sigma^2(1-\bar{\rho}^2) =: C, \tag{1.3.3}$$

where $\sigma^2 := \max_{k\leq N} \sigma_k^2$.

Proof. From Equation (1.3.2), we have, by definition of c_k,

$$\sigma_{\varepsilon_k}^2 = \sigma_k^2 - \rho_k^2\frac{\sigma_k^2}{\sigma_1^2}\sigma_1^2 = \sigma_k^2(1-\rho_k^2).$$

Then, we have

$$\begin{aligned}
\max_{k\leq n} \sigma_{\varepsilon_k}^2 &= \max_{k\leq n}(\sigma_k^2(1-\rho_k^2)) \\
&\leq \max_{k\leq n} \sigma_k^2(1-\min_{k\leq n}\rho_k^2) \\
&\leq \max_{k\leq N} \sigma_k^2(1-\bar{\rho}^2) \\
&= \sigma^2(1-\bar{\rho}^2).
\end{aligned}$$

□

1.3.3 Controlling the error on Greeks

Suppose we have a portfolio of N assets with returns $r_1, \cdots, r_N$ and suppose we choose a threshold $\bar{\rho} \in [0,1]$ and $n \leq N$, as in Proposition 1.3.1, that yields a constant C, defined in Equation (1.3.3).

[5] This constant can depend on the r_k though.

[6] Note that $\rho_1 = 1$, so the variable r_1 against which we do the regressions does not change its position.

Throughout this section, we will denote by $\sigma_X^2 = (\sigma_X^2 | r_1, t)$ the variance of a random variable X conditional to t and r_1, and, unless stated, all the inequalities between will be almost sure inequalities. This is to simplify notations[7].

Error on the β_k

Let $\hat{\beta}_k := c_k \frac{S_k}{S_1}$, as seen in Proposition 1.2.3. Then

Proposition 1.3.2. *Let $\delta\beta_k := \beta_k - \hat{\beta}_k$. We have*

$$\sigma_{\delta\beta_k}^2 = \frac{S_k^2}{r_1^2 S_1^2} \sigma_{\varepsilon_k}^2 \leq \frac{S_k^2}{r_1^2 S_1^2} C. \tag{1.3.4}$$

Proof. By definition, we have

$$r_k = \frac{\partial_t S_1 \partial_{S_1} S_k}{S_k} = \beta_k \frac{r_1 S_1}{S_k} = \beta_k \frac{r_k - \varepsilon_k}{c_k} \frac{S_1}{S_k},$$

hence

$$\begin{aligned}
\beta_k &= \frac{r_k}{r_k - \varepsilon_k} c_k \frac{S_k}{S_1} \\
&= c_k \frac{S_k}{S_1} \left(1 + \frac{\varepsilon_k}{r_k - \varepsilon_k}\right) \\
&= \hat{\beta}_k + \frac{\varepsilon_k}{r_1} \frac{S_k}{S_1},
\end{aligned}$$

which yields $\delta\beta_k = \frac{\varepsilon_k}{r_1} \frac{S_k}{S_1}$. Finally, by taking the conditional variance on r_1 and t,

$$\sigma_{\delta\beta_k}^2 = \frac{S_k^2}{r_1^2 S_1^2} \sigma_{\varepsilon_k}^2 \leq \frac{S_k^2}{r_1^2 S_1^2} C.$$

□

Error on B

Let $\hat{B} := \sum_{k=1}^n \hat{\beta}_k / c_k$ and $B := \sum_{k=1}^n \beta_k / c_k$. Then

Proposition 1.3.3. *Let $\delta B := B - \hat{B}$ and assume the ε_k are independent. We have*

$$\sigma_{\delta B}^2 = \frac{1}{r_1^2 S_1^2} \sum_{k=1}^n \frac{S_k^2}{c_k^2} \sigma_{\varepsilon_k}^2 \leq \frac{\sigma_1^2}{\bar{\rho}^2 r_1^2 S_1^2} \sum_{k=1}^n \frac{S_k^2}{\sigma_k^2} C. \tag{1.3.5}$$

[7] This makes sense in practice, as we want to define scenarios for market moves, starting at time t, and assuming certain base-case scenarios, i.e. r_1, as we will see in Chapter 2

Proof. Assuming the ε_k are independent, we can sum the variances in δB, and we have

$$\begin{aligned}\sigma_{\delta B}^2 = \sum_{k=1}^{n} \frac{\sigma_{\delta\beta_k}^2}{c_k^2} &= \frac{1}{r_1^2 S_1^2} \sum_{k=1}^{n} \frac{S_k^2}{c_k^2} \sigma_{\varepsilon_k}^2 \\ &= \frac{\sigma_1^2}{\rho^2 r_1^2 S_1^2} \sum_{k=1}^{n} \frac{S_k^2}{\sigma_k^2} \sigma_{\varepsilon_k}^2 \\ &\leq \frac{\sigma_1^2}{\rho^2 r_1^2 S_1^2} \sum_{k=1}^{n} \frac{S_k^2}{\sigma_k^2} C.\end{aligned}$$

□

The following propositions are proven using the same methods. However, as they are more technical, their proof is given in Appendix A.2 and **??**.

Error on $\overline{\beta}_k$

Let $\overline{\beta}_k := \partial_{S_1}\beta_k$ and $\hat{\overline{\beta}}_k := \partial_{S_1}\overline{\beta}_k$. Then

Proposition 1.3.4. *Let $\delta\overline{\beta}_k := \overline{\beta}_k - \hat{\overline{\beta}}_k$ and suppose that S_1 is a C^2 function of t. We have*

$$\sigma_{\delta\overline{\beta}_k}^2 = \frac{1}{r_1^2 S_1^2} \left(\frac{S_k^2}{r_1^2 S_1^2} \sigma_{\varepsilon_k}^2 + \hat{\beta}_k^2 \right) \sigma_{\varepsilon_k}^2$$

and

$$\sigma_{\delta\overline{\beta}_k}^2 \leq \frac{1}{r_1^2 S_1^2} \left(\frac{S_k^2}{r_1^2 S_1^2} C + \frac{\sigma_k^2}{\sigma_1^2} \right) C.$$

Proof. The proof is slightly technical and can be found in Appendix A.2. □

Error on $\overline{B}$

Let $\hat{\overline{B}} := \sum_{k=1}^{n} \hat{\overline{\beta}}_k / c_k$ and $\overline{B} := \sum_{k=1}^{n} \overline{\beta}_k / c_k$. Then

Proposition 1.3.5. *Let $\delta\overline{B} := \overline{B} - \hat{\overline{B}}$ and assume the ε_k are independent. We have*

$$\sigma_{\delta\overline{B}}^2 = \sum_{k=1}^{n} \frac{1}{c_k^2 r_1^2 S_1^2} \left(\frac{S_k^2}{r_1^2 S_1^2} \sigma_{\varepsilon_k}^2 + \hat{\beta}_k^2 \right) \sigma_{\varepsilon_k}^2 \tag{1.3.6}$$

and

$$\sigma_{\delta\overline{B}}^2 \leq \frac{\sigma_1^2}{\sigma_k^2 r_1^2 S_1^2} \left(\frac{S_k^2}{r_1^2 S_1^2} C + \frac{\sigma_k^2}{\sigma_1^2} \right) \frac{C}{\rho^2}.$$

Proof. First, note that

$$\overline{B} = \sum_{k=1}^{n} \frac{\overline{\beta}_k - \hat{\overline{\beta}}_k}{c_k} = \sum_{k=1}^{n} \frac{\delta\overline{\beta}_k}{c_k}.$$

Then, we have

$$\sigma^2_{\delta\overline{B}} = \sum_{k=1}^{n} \frac{\sigma^2_{\delta\overline{\beta}}}{c_k^2} = \sum_{k=1}^{n} \frac{1}{c_k^2 r_1^2 S_1^2} \left(\frac{S_k^2}{r_1^2 S_1^2} \sigma^2_{\varepsilon_k} + \hat{\beta}_k^2 \right) \sigma^2_{\varepsilon_k}$$
$$\leq \frac{1}{c_k^2 r_1^2 S_1^2} \left(\frac{S_k^2}{r_1^2 S_1^2} C + \frac{\sigma_k^2}{\sigma_1^2} \right) C$$
$$\leq \frac{\sigma_1^2}{\sigma_k^2 r_1^2 S_1^2} \left(\frac{S_k^2}{r_1^2 S_1^2} C + \frac{\sigma_k^2}{\sigma_1^2} \right) \frac{C}{\overline{\rho}^2}.$$

□

Note that, by definition, $\frac{C}{\overline{\rho}} = \sigma^2 \left(\frac{1}{\overline{\rho}^2} - 1 \right)$ decreases with $\overline{\rho}$, hence controls the error on $\sigma^2_{\delta\overline{B}}$.

Control on β

Using the same techniques as previously, we can compute a boundary for β, which we can then use to compute a boundary on the Delta and Gamma. However, the calculations for that become too complex and far exceed the purpose of this section. However, we can show that, if we consider a portfolio with enough weight on the benchmark asset, we can use the approximation $\beta \simeq 1$. More formally, we have the following result.

Proposition 1.3.6.

$$\lim_{S_1 \to +\infty} \beta = 1.$$

Proof. Since $S = \sum_{k=1}^{n} S_k$, we have

$$\beta = \partial_S S_1 = \partial_S \left(S - \sum_{k=2}^{n} S_k \right) = 1 - \sum_{k=2}^{n} \partial_S S_k$$
$$= 1 - \sum_{k=2}^{n} \partial_S S_1 \partial_{S_1} S_k$$
$$= 1 - \beta \sum_{k=2}^{n} \beta_k$$
$$= 1 - \beta \sum_{k=2}^{n} \left(c_k + \frac{\varepsilon_k}{r_1} \right) \frac{S_k}{S_1}$$
$$= 1 - \beta \sum_{k=2}^{n} \left(\rho \frac{\sigma_k}{\sigma_1} + \frac{\varepsilon_k}{r_1} \right) \frac{S_k}{S_1}.$$

Consequently, when dividing by β ($\beta \neq 0$ since the portfolio should contain the benchmark asset),

$$|1/\beta - 1| \leq \sum_{k=2}^{n} (\rho \frac{\sigma_k}{\sigma_1} + \frac{\varepsilon_k}{r_1}) \frac{S_k}{S_1} \to 0 \text{ as } S_1 \to +\infty,$$

since $\sigma_1 \to +\infty$ and r_1 is independent of the scale of S_1. Finally, by continuity of $x \mapsto \frac{1}{x}$ in 1, we have

$$\lim_{S_1 \to +\infty} \beta = 1.$$

□

1.4 The linear approximation model

1.4.1 Overview and methodology

The goal of the linear approximation model (or beta model) is to simplify risk scenarios. Assume, for instance, that we manage a portfolio of derivatives with 30 underlying securities, and that we want to create tail risk scenarios on these. To keep it simple, assume that we only consider scenarios on spot moves (i.e. the spot of asset A goes up or down). In such case, we have 2^{30} –over a billion– possible scenarios.

However, as we showed previously, the linear approximation of returns allows reducing the amount of scenarios while controlling the error on returns and Greeks. This hypothesis greatly simplifies the model: suppose we cluster the 30 assets in 4 groups and that all the assets' returns are well correlated within each group[8]. Then, if we select one asset per cluster to define scenarios on returns, then we can propagate these scenarios on the other assets via the linear approximation. This reduces the number of possible scenarios to $2^4 = 16$. Another advantage of such model is that it yields historically accurate scenarios. Since the $\hat{\beta}_k$ are computed through data analysis, the market scenarios resulting from such model will reflect the reality of markets, and we can avoid unrealistic scenarios that can introduce noise in risk analysis.

Using Kapoor's framework [12], we suggest the following methodology:

- Group the assets in clusters and select a benchmark per group, which is an asset that is representative of the cluster. There are multiple approaches to that problem, as we discuss in Section 1.4.2;
- Compute the $\hat{\beta}_k$ by computing the c_k through linear regression. In reality, many discussions arise, as we see in Section 1.4.3.

1.4.2 Benchmark selection and asset clustering

In this model, we want to group assets in clusters and select a representative benchmark per cluster. Then, the idea would be to propagate scenarios defined on these benchmarks

[8] in the sense that they exceed the threshold $\overline{\rho}$ that defines the control on the errors.

to their respective clusters through the beta model. To do so, we have three different approaches.

Clustering with fixed $\overline{\rho}$

The first and maybe most straightforward approach relies on fixing a threshold $\overline{\rho}$, hence a control constant C that bounds the error resulting from the linear approximation. Assuming we fix $\overline{\rho}$, we use the following clustering algorithm.

Algorithm 1: Clustering and benchmark selection with fixed $\overline{\rho}$

Consider a list of assets $L = [A_1, \cdots, A_N]$ and a threshold $\overline{\rho} \in [0, 1]$;
Compute the correlation matrix R.;
while *L is not empty* **do**
  Select the row i of R that maximizes the number of $|R_{i,j}| > \overline{\rho}$, with $A_j \in L$;
  Form a cluster with the A_j, and choose A_i as a benchmark;
  Remove the A_j from L;
end
return *clusters*

This algorithm yields the minimal number of clusters such that, for each cluster, the absolute values of the correlations between the returns of assets and those of the benchmark exceed $\overline{\rho}$.

Clustering with fixed number of clusters n_C

Another approach to asset clustering is to fix the number of clusters n_C and to try to maximize correlations inside each cluster. This approach is done through the following algorithm.

Algorithm 2: Clustering and benchmark selection with fixed number of clusters n_C

Consider a list of assets $L = [A_1, \cdots, A_N]$ and a number of clusters n_C;
Compute the correlation matrix R;
Define a graph $\mathcal{G}$ with N vertices $V_1, \cdots, V_N$ and no edges;
$n \leftarrow N$;
while $n > n_C$ **do**
  Select the pair (i, j) such that $|R_{i,j}|$ be maximal;
  Add an edge between A_i and A_j to $\mathcal{G}$;
  $n \leftarrow n - 1$;
end
return *clusters*

This algorithm yields n_C clusters such that, for each cluster, the absolute values of the correlations between the returns of assets are as big as possible. Then, we have

$$\overline{\rho} = \min_{c \text{ cluster}} \min_{A_j \in c} \{\rho_{i,j} \ : \ A_i \in c \text{ is the benchmark of cluster } c\},$$

Clustering by optimization over $\overline{\rho}$ and n_c

The correlation threshold $\overline{\rho}$ and the number of clusters n_c play opposite roles in the beta model. Indeed, if we select a high threshold, the clusters become smaller as less and less asset returns are correlated. This results in the creation of multiple clusters. On the other hand, if we reduce the number of clusters, they become bigger and start containing more assets with less correlated returns[9]. Hence, n_C increases as $\overline{\rho}$ increases.

To solve this problem, we can maximize the following metric over $\overline{\rho}$:

$$\phi(\overline{\rho}, n_C) := \overline{\rho} - \frac{n_C}{N},$$

where N is the number of assets in the portfolio. Note that we re-scaled n_C so that it fall between 0 and 1 in order to avoid any scale effect in the maximization problem. Since n_C decreases as $\overline{\rho}$ increases, we have a simple algorithm to solve this problem.

Algorithm 3: Clustering and benchmark selection by optimization over $\overline{\rho}$ and n_c

Consider a list of assets $L = [A_1, \cdots, A_N]$;
Set $n_C = N$ and $\phi = 0$;
Note that in that case $\overline{\rho} = 1$ *as each element is alone in its cluster, hence* $\phi(\overline{\rho}, n_C) = 1 - \frac{N}{N} = 0$.;
while $n_C > 1$ **do**
  Use Algorithm 2. with L and n_C and compute the corresponding $\overline{\rho}$;
  if $\phi(\overline{\rho}, n_C) > \phi$ **then**
    $\phi \leftarrow \phi(\overline{\rho}, n_C)$;
    $n_C \leftarrow n_C - 1$
  end
  else
    break
  end
end
return *clusters*

This algorithm makes a compromise between n_C and $\overline{\rho}$. Note that we can change the metric ϕ without altering the algorithm.

[9] We can think of the extreme cases $\overline{\rho} = 1$ and $\overline{\rho} = 0$. The first case is equivalent to not using the beta model, hence yields as many clusters as there are assets. In the second case, we cluster all assets together regardless of their correlations.

1.4.3 Computing betas

As we discussed in Section 1.3, the $\hat{\beta}_k$ are computed through linear regressions of the returns. Recall that, for asset k, the $\hat{\beta}_k$ is defined as

$$\hat{\beta}_k := c_k \frac{S_k}{S_1},$$

where $c_k := \rho_k \frac{\sigma_k}{\sigma_1}$ is the least-square estimator computed through the linear regression of the returns r_k of asset k against the returns r_1 of the benchmark. Hence, there are two main components to the beta model:

- The correlations of the returns ρ_k;
- The variances of the returns σ_k^2.

In practice, when computing variances and correlations, some discussions arise.

The time span of the regression

The goal of the beta model is to devise a risk methodology. Therefore, it should take extreme-case scenarios into account. However, since these are scarce and take place in a limited amount of time, computing variances over an entire dataset would smooth the effects of such events. On the other hand, computing variances over a short amount of time would result in noisy betas, which might introduce bias in the perception of risk[10]. Following the general idea of Fama and MacBeth [7], Kapoor [12] suggests computing rolling betas over three time spans:

- Short-time betas, where we consider data over the previous month for computation. These betas capture the effects of quick yet violent market moves on asset prices;
- Mid-term betas, where we consider data over the previous three months for computation. These betas reflect the general trend of the market at the time they are computed (eg. during a crisis);
- Long-term betas, where we consider data over the previous two years for computation. These betas reflect the overall trend of the market.

Then, we can use these betas separately to compute multiple risk scenarios for the risk model.

[10] Consider a situation where markets have just recovered from a crisis, as in mid-2020. A model that is calibrated only on a few months of data would overestimate risks.

The problem of market asynchronicity

When looking at correlations between multiple financial time series, we must take into account that markets do not open and close at the same hours. For instance, computing correlations between the daily returns at the local closing time $(r_1(t))$ of indices that trade in the United States against the returns $(r_2(t))$ of indices that trade in Asia is not very relevant[11]. Bergomi [4] suggests that if we consider two asynchronous assets with time processes of returns $(r_1(t))$ and $(r_2(t))$, and if we denote by $(r_2'(t))$ the shifted time series defined by $r_2'(t) := r_2(t-1)$ for all t, we simply compute the correlation between the two assets' returns as

$$\rho = \rho_{r_1,r_2} + \rho_{r_1,r_2'}.$$

Table 1.4.3 gives the results of a numerical application of this methodology. We can see how the correlations are higher in reality, which allows better asset clustering and more accuracy in scenario definition. The indices we consider are Asian indices that trade in Hong-Kong (HSI, HSCEI) and Japan (NKY). The correlations have been computed with data from 01/01/20 to 30/06/20.

Index	ρ_{r_1,r_2}	ρ
HSI	-0.47	-0.73
HSCEI	-0.45	-0.69
NKY	0.43	0.78

Table 1.1: Correlations between the returns of equity indices and SPX with and without taking market asynchronicity into account.

The volatility betas

Computing volatility betas is done fundamentally the same way as for spots, except that we look at implied volatility changes $\delta\Sigma$ instead returns r. However, when the spot of an option depend only on its underlying product, it is not the case for volatility. In stochastic volatility models, the volatility of an option mainly depends on two parameters: the time to maturity T of the option and its moneyness, defined as the ratio of the price of the underlying and the strike of the option $\frac{S}{K}$. The implied volatilities $\left(\Sigma(T, \frac{S}{K})\right)_{T,S/K}$ are referred to as the volatility surface.

In [12], the idea is to compute a beta surface by interpolation of the implied volatili-

[11] Since the market closing times in that case are interspersed, we can imagine that part of the correlation on day t is achieved by comparing $r_1(t)$ to $r_2(t)$ and part of it is achieved by comparing $r_1(t)$ to $r_2(t+1)$.

ties of market quotes on a grid $\left(T_i, \left(\frac{S}{K}\right)_j\right)_{i,j}$. To do so, we apply the following algorithm

Algorithm 4: The Benchmark Equivalent Risk model for volatility betas.

for i, j **do**
  for *time* t **do**
    Select the option with maturity $T(t)$ and moneyness $\frac{S(t)}{K}$ that minimizes

$$\frac{|T(t) - T_i|}{\max_k T_k} + \frac{|\frac{S(t)}{K} - \left(\frac{S}{K}\right)_j|}{\max_k \left(\frac{S}{K}\right)_k}$$

    For its strike $K(t)$, compute the volatility change over this fixed strike, i.e. $\delta\Sigma(t) = \Sigma(T-1, \frac{S(t+1)}{K}) - \Sigma(T, \frac{S(t)}{K})$;
  end
  Compute the beta for pair (i, j) on the time series $\delta\Sigma$.
end

For the rest of the study and to simplify notations, we will consider a fixed maturity and a fixed moneyness. This will allow to have one volatility beta, β'_k per underlying k in the portfolio.

Chapter 2

Construction and calibration of risk scenarios

In this Chapter, we use the beta model defined in Chapter 1 to define risk scenarios. We start by defining the notion of risk scenarios and show how the beta model contributes to simplify and improve the quality of scenarios. Then, we devise a statistical approach that allow computing base-case risk scenarios through various metrics. Finally, we define the notion of scenario equivalence and present a methodology to calibrate scenarios equivalent to the base-case risk scenario, by resorting to constrained optimization algorithms.

2.1 The notion of risk scenarios

The main goal of our risk methodology is to study the behavior of markets in extreme cases in order to estimate and control the potential losses of the trading strategy in place. Therefore, a natural idea is to apply stress tests to the markets in order to check their behavior. To do so, we introduce the notion of risk scenarios.

Consider a portfolio of N assets and assume we defined a benchmark, as we discussed in Chapter 1. A risk scenario on the portfolio $\mathcal{S} := (\delta S_1, \delta\Sigma_1, \{\beta_k\}, \{\beta'_k\})$ is defined by

- The spot move of the benchmark δS_1;
- The volatility move of the benchmark $\delta\Sigma_1$;
- Spot betas for the underlyings in the portfolio β_k;
- Volatility betas for the underlyings in the portfolio β'_k.

Then, we can propagate the spot and volatility moves of the assets in the portfolio by using the linear regression approximation, and we set

$$\delta S_k := \beta_k \delta S_1 \ , \ \delta \Sigma_k := \beta'_k \delta \Sigma_1 .$$

The idea of the risk methodology is to

- Find base case risk scenarios. In other words, build a scenario $\mathcal{S}$ where the β_k and β'_k are computed as seen in Chapter 1 and where δS_1 and $\delta\sigma_1$ are set. The discussion about methods to set δS_1 and $\delta\Sigma_1$ takes place in Section 2.2;

- Build multiple risk scenarios that are equivalent to scenario $\mathcal{S}$. In other words, build scenarios $\mathcal{S}^i$ where the β_k^i and $\beta_k'^i$ are computed as seen in Chapter 1, and where δS_1^i and $\delta\Sigma_1^i$ are calibrated in order for scenarios $\mathcal{S}$ and $\mathcal{S}^i$ to be equivalent in some sense. The notions of scenario equivalence and calibration are discussed in Section 2.3.

2.2 Creation of base-case scenarios

A natural idea to define a base-case risk scenario is to look at the biggest historical returns and volatility moves for the benchmark. However, returns and volatility are not independent, hence we must look at their joint distribution to determine extreme case scenarios. Figure 2.1 displays such distribution for SPX and shows that there are almost no scenarios where both returns and volatility changes are positive.

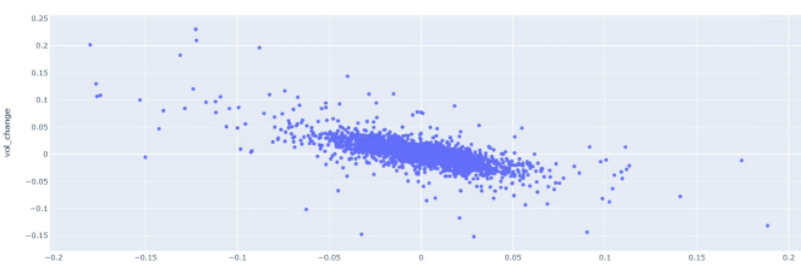

Figure 2.1: Joint distribution of daily returns r_1 and volatility changes $\delta\Sigma_1$ for SPX between 01/01/12 and 30/06/20.

In order to define δS_1 and $\delta\sigma_1$ for the base-case scenario $\mathcal{S}$, we define a score

$$s : (r, \delta\Sigma) \mapsto s(r, \delta\Sigma).$$

The score is simply a function that aggregates the returns and volatility changes. Then, we define δS_1 and $\delta\sigma_1$ as

$$\delta S_1 := S_1 \mathbf{E}[r_1 | s(r_1, \delta\Sigma_1) \geq q]$$

and

$$\delta\sigma_1 := \mathbf{E}[\delta\Sigma_1 | s(r_1, \delta\Sigma_1) \geq q],$$

where $q \geq 0$. In practice, we define q as the 95%-quantile of $s(r, \delta\Sigma)$.

The question remaining is that of the definition of the score s in practice. In Figure 2.3, we consider

$$s(r, \delta\Sigma) := \mathbf{1}_{\{r \geq 0, \delta\Sigma < 0\}},$$

whereas in Figure 2.2, we consider

$$s(r, \delta\Sigma) := \|(r, \delta\Sigma)\|_1 = |r| + |\delta\Sigma|$$

and

$$s(r, \delta\Sigma) := \|(r, \delta\Sigma)\|_2^2 = r^2 + \delta\Sigma^2.$$

Figure 2.2: Data for SPX between 01/01/12 and 30/06/20 that is above the 95%-quantile for the score $s(r, \delta\Sigma) := \mathbf{1}_{\{r \geq 0, \delta\Sigma < 0\}}$.

The three scores have different interpretations and yield different results:

- The first score is biased, in the sense that it assumes a risk scenario to be a significant increase of the benchmark's underlying price with a decrease in its implied volatility. This scenario makes sense when looking at Figure 2.1, as the returns and changes in implied volatility for SPX seem negatively correlated. However, it is only a risk scenario when holding certain short positions on options (eg. short Call). Hence, it should only be used as a risk scenario when holding such positions in the portfolio;

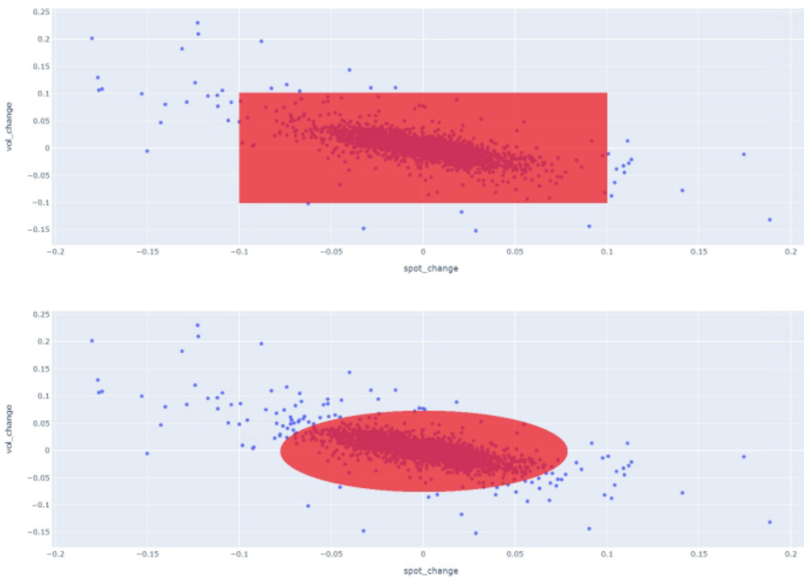

Figure 2.3: Data for SPX between 01/01/12 and 30/06/20 that is above the 95%-quantile for the scores $s(r, \delta\Sigma) := \|(r, \delta\Sigma)\|_1$ (top) and $s(r, \delta\Sigma) := \|(r, \delta\Sigma)\|_2^2$ (bottom).

- The second score in unbiased, contrary to the first one. However, the risk scenario might be attenuated as we average returns and changes in implied volatility to compute δS_1 and $\delta\Sigma_1$ (the negative and positive returns, for instance, will compensate each other). An idea to improve that would be to compute

$$\delta S_1 := \gamma S_1 \mathbf{E}[|r_1||s(r_1, \delta\Sigma_1) \geq q]$$

and

$$\delta\sigma_1 := -\gamma \mathbf{E}[|\delta\Sigma_1||s(r_1, \delta\Sigma_1) \geq q],$$

where $\gamma = \pm 1\}$ is defined according to the positions we hold in the portfolio.

- The third score works and can be improved exactly the same way as the second one. The only difference between them is that the third score is smoother, hence more risk-averse[1].

[1]In the third score, more pairs $(r, \delta\Sigma)$ are considered withing the risky zone than in the second one. Hence, the third score is more risk-averse than the second one.

2.3 Generation and calibration of equivalent risk scenarios

A base-case risk scenario is not enough to fully measure and understand the risk in a portfolio strategy. In order to cover more cases, we might want to change the parameters of the scenario (the benchmark, for instance), or create custom scenarios, such as having multiple benchmarks or modifying the direction of the benchmark's returns and implied volatility changes. Therefore, we want to create custom risk scenarios by hand. However, such scenarios can introduce bias in risk assessment, as they can be unrealistic or they can over/underestimate risk, compared to the base-case scenarios. Therefore, we need a way to assert a custom scenario is equivalent to the base-case scenario, in the sense that it is as realistic and that it yields risk that is of the same magnitude.

2.3.1 Notion of risk equivalence

For a scenario $\mathcal{S}$, consider the time process $\delta P(\mathcal{S}, S_1, \Sigma_1, \cdot)$, defined by

$$\delta P(\mathcal{S}, S_1, \Sigma_1, t) := P(S_1 + \delta S_1, K, \Sigma_1 + \delta\Sigma_1, T-1, t+1) - P(S_1, K, \Sigma_1, T, t). \quad (2.3.1)$$

The process δP is the time series of the daily price change of the option on the benchmark, if it is subject to the risk scenario $\mathcal{S}$. Then, define the expected shortfall of scenario $\mathcal{S}$ as

$$\mathrm{ES}_{q_{\mathcal{S}}} := \mathbf{E}[\delta P(\mathcal{S}, S_1, \Sigma_1, \cdot) \mid \delta(\mathcal{S}, S_1, \Sigma_1, \cdot) \leq q_{\mathcal{S}}],$$

where $q_{\mathcal{S}}$ is the 1%-quantile of $\delta P(\mathcal{S}, S_1, \Sigma_1, \cdot)$. It is the average of the loss under scenario $\mathcal{S}$ over the 1% worst days.

Consider a base-case scenario $\mathcal{S} := (\delta S_1, \delta\Sigma_1, \{\beta_k\}, \{\beta'_k\})$ and a custom scenario $\mathcal{S}^i := (\delta S_1^i, \delta\Sigma_1^i, \{\beta_k^i\}, \{\beta_k'^i\})$. Consider an option of fixed strike K and maturity T on the benchmark asset, and denote by $P(S_1, K, \Sigma_1, T, t)$ its price at time t.

Definition 2.3.1. The scenario $\mathcal{S}'$ is equivalent to the scenario $\mathcal{S}$ if

$$\mathrm{ES}_{q_{\mathcal{S}}} = \mathrm{ES}_{q'_{\mathcal{S}}}.$$

In other words, a custom scenario is equivalent to the base-case scenario when, if we hold a position on the benchmark of the base-case scenario, we lose the same amount in extreme cases in both scenarios. Note that this definition is not reciprocal[2] if the benchmark for the custom scenario is different from that of the base-case scenario.

[2] in the sense that $\mathcal{S}^i$ being equivalent to $\mathcal{S}$ does not mean that $\mathcal{S}$ is equivalent to $\mathcal{S}^i$.

2.3.2 Calibration of the scenario parameters

Consider a base-case scenario $\mathcal{S} := (\delta S_1, \delta\Sigma_1, \{\beta_k\}, \{\beta'_k\})$. Then, consider a custom scenario $\mathcal{S}^i := (\delta S_1^i, \delta\Sigma_1^i, \{\beta_k^i\}, \{\beta_k'^i\})$ that we want to calibrate so that it be equivalent to the base-case risk scenario. When defining a custom scenario, we often set constraints on the parameters $(\delta S_1^i, \delta\Sigma_1^i) \in \mathcal{C}$. For instance, if we want a scenario where the volatility moves up by at least 5% while the returns go down, we would set

$$\mathcal{C} := \mathbf{R}^- \times [0.05, +\infty).$$

In practice, it is virtually impossible to achieve the equivalence relationship, and it is generally impossible to find an explicit formula that yields δS_1^i and $\delta\Sigma_1^i$. Besides, there is no guarantee that perfect equality can even be achieved. Therefore, the problem of scenario calibration can be expressed as a problem of optimization under constraints. The values δS_1^{i*} and $\delta\Sigma_1^{i*}$ of δS_1^i and $\delta\Sigma_1^i$ we take satisfy the following problem:

$$(\delta S_1^{i*}, \delta\Sigma_1^{i*}) = \arg\min_{(\delta S_1^{i*}, \delta\Sigma_1^{i*}) \in \mathcal{C}} (\mathrm{ES}_{q_S} - \mathrm{ES}_{q'_S})^2. \tag{2.3.2}$$

2.3.3 Solving the calibration problem

In order to solve the problem in Equation (2.3.2), we use an optimized version of the BFGS algorithm [18]. Consider a differentiable function $f : x \in \mathbf{R}^n \to \mathbf{R}$ that we wish to minimize. Let H an initial guess of the Hessian matrix of f and $x \in \mathbf{R}^n$ an initial guess of the minimizer of f. The BFGS algorithm works as following.

Algorithm 5: The BFGS algorithm.

```
H, x, ε > 0;
while ‖∇f(x)‖ > ε do
    p ← solution to Hp = −∇f(x)^a;
    α ← optimal step found using a line search algorithm;
    x̄ ← x + αp =: x + s;
    y ← ∇f(x̄) − ∇f(x);
    H ← (yy^⊤)/(y^⊤s) − (Hss^⊤H)/(s^⊤Hs);
end
```

[a] This is a linear system of equations and can easily be solved with matrix inversion, for example by using Gauss-Jordan elimination. The reader can refer to [1] for more details.

This algorithm uses a line search algorithm, which works as following.

Algorithm 6: The line search algorithm.

$p, x, \varepsilon' > 0$;
while $\|\nabla f(x)\| > \varepsilon'$ **do**
  $\alpha \leftarrow$ minimum of $\alpha \mapsto f(x + \alpha p)$;
  $x \leftarrow x + \alpha p$;
end

Note that the problem of minimization of $\alpha \mapsto f(x+\alpha p)$ is fairly easy to solve since $\alpha \in \mathbf{R}$. Table 2.3.3 gives some examples of calibrated scenarios with this method.

Scenario	δS_1	$\delta \Sigma_1$
Base case	+322.47	-0.08
Spot up, volatility up	+203.91	+0.04
Spot up, volatility static	+467.23	+0.00

Table 2.1: Results of the calibration of custom scenarios using the BFGS algorithm. In all cases, the benchmark is SPX and the position in the portfolio is a Short 3 month at-the-money Straddle (see Section 3.2 for more details).

Chapter 3

Application to a gamma scalping strategy

In this Chapter, we apply the methodology outlined in Chapter 2 to a simple volatility trading strategy. We start by defining the Gamma scalping strategy, which will give an overall idea of the strategy we test our model on, and will show how such strategy can make profit in volatility trading. Then, we will define the at-the-money straddles, the main structures traded in this strategy, and discuss how to calibrate risk scenarios on them. Finally, we will define some performance metrics to show the impact of the risk methodology on the strategy.

3.1 The Gamma scalping strategy

3.1.1 Gamma scalping

In [3, Chapter 2, pp.40-41], Bennett refers to constant Delta-hedging as Gamma scalping. When continuously Delta-hedging a portfolio, an investor is no longer exposed to moves of the underlying's price. Instead, such strategy makes profit based on the rate at which the Delta changes, making it an interesting volatility strategy: the higher the volatility, the higher the profits.

Consider a volatility portfolio with both long and short positions (i.e. a portfolio where positions make PnL based on volatility). Profit is generated on long positions when the volatility is high, while short positions make profit when volatility is low. In other words, risk on long positions is that the underlying's price stays still, whereas risk from short positions comes from significant market moves. Gamma scalping is a strategy that reduces such risk.

In short, the Gamma scalping stategy can be summarized as following.

- Buy a volatility product (for instance, a Straddle, see Section 3.2 for more details);
- Delta-hedge the portfolio;
- If the underlying rises, add short stock in order to keep a Delta-neutral portfolio;
- If the underlying falls, add long stock in order to keep a Delta-neutral portfolio.

We will see through a simple example how this strategy can make profit from volatility.

3.1.2 A practical example

Consider a three-period model ($t \in \{0, 1, 2\}$).

$t = 0$

Suppose that the underlying is at $S_0 = 100$ and that we buy a long Straddle of strike $K = S_0 = 100$. Suppose that the portfolio is Delta-hedged[1]. Table 3.1.2 summarizes the situation of the portfolio at $t = 0$.

S_0	100
Stocks	0
Cash	0

Table 3.1: Gamma scalping portfolio at time $t = 0$.

$t = 1$

Consider the following cases

- The underlying increases in price, $S_1 = 110$. Then, the Delta of the portfolio Δ_1 is positive, and we short-sell Δ_1 units of stock. Table 3.1.2 summarizes the situation of the portfolio at $t = 1$.

S_1	110
Stocks	$-\|\Delta_1\|$
Cash	$+110\|\Delta_1\|$

Table 3.2: Gamma scalping portfolio at time $t = 1$ when $S_1 > S_0$.

- The underlying decreases in price, $S_1 = 90$. Then, the Delta of the portfolio Δ_1 is negative, and we long $|\Delta_1|$ units of stock. Table 3.1.2 summarizes the situation of the portfolio at $t = 1$.

[1] In theory, an at-the-money Straddle has a Delta of 0, hence is already Delta-hedged on its own.

S_1	90
Stocks	$+\|\Delta_1\|$
Cash	$-90\|\Delta_1\|$

Table 3.3: Gamma scalping portfolio at time $t = 1$ when $S_1 < S_0$.

$t = 2$

Now, suppose the underlying is back at its original price $S_2 = S_0 = 100$. Then, the Delta on the Straddle becomes 0 again while the Delta on the stocks remains the same, hence is equal to Δ_1 in both cases. Consequently, the total Delta of the portfolio is Δ_1, no matter the case.

- In the first case, the price decreases from $S_1 = 110$ to $S_2 = 100$, hence we long $|\Delta_1|$ units of stock. Table 3.1.2 summarizes the situation of the portfolio at $t = 2$.

S_2	100
Stocks	0
Cash	$+110\|\Delta_1\| - 100\|\Delta_1\| = 10\|\Delta_1\|$

Table 3.4: Gamma scalping portfolio at time $t = 2$ when $S_1 > S_0$.

- In the second case, the price increases from $S_1 = 90$ to $S_2 = 100$, hence we short-sell $|\Delta_1|$ units of stock. Table **??** summarizes the situation of the portfolio at $t = 2$.

S_2	100
Stocks	0
Cash	$-90\|\Delta_1\| + 100\|\Delta_1\| = 10\|\Delta_1\|$

Table 3.5: Gamma scalping portfolio at time $t = 2$ when $S_1 < S_0$.

In all cases, the Gamma scalping strategy makes a profit of $10|\Delta_1|$. This simple example illustrates how such continuous Delta-hedging volatility structures can make profit.

3.2 At-the-money Straddles

3.2.1 Definition

A Straddle option is a derivative that involves buying a Call and Put with same strike and maturity. When a Straddle is bought at-the-money (i.e. when the strike is close to the underlying price) profit from the Straddle is independent from the direction in which the underlying price moves.

At expiration, if the underlying price is close to the strike price, the Straddle leads to a loss. However, the Straddle makes significant profit if there is a sufficiently large move of the underlying price in either direction. In other words, a Long Straddle position makes more profit as the implied volatility of the Call and Put options that constitute it increase. Therefore, a Straddle is a commonly used instrument in volatility trading strategies.

In our risk framework, we look at Short Straddle positions, as these can theoretically yield unlimited losses. The payoff function of a Short Straddle is given in Figure 3.1. The profit of such position is limited to the premium received from the sale of the Put and Call, and a maximum profit at maturity is achieved if the underlying security trades exactly at the strike price of the Straddle.

Figure 3.1: Payoff of a short straddle of strike \$4 where the total price (call + put) is \$3.

3.2.2 Pricing at-the-money straddles

In order to calibrate the scenarios as seen in Chapter 2, we need to compute the process $\delta P(\mathcal{S}, S_1, \Sigma_1, \cdot)$ defined in Equation (2.3.1). In the Black-Scholes framework [6], we have the following result.

Proposition 3.2.1.

$$\delta P(\mathcal{S}, S_1, \Sigma_1, t) = \delta S_1(4\Phi(d) - 1),$$

where $d := \sigma\sqrt{T - t}$.

Proof. The proof is given in Appendix A.3. □

3.3 Scaling the strategy through risk scenario analysis

3.3.1 The notion of Lambdas

Once scenarios have been defined and calibrated, we should use them to scale the trading strategy: if a majority of scenarios tell us a certain position is risky, we should scale it down so we meet certain risk requirements. Formally, suppose we have I scenarios $\mathcal{S}^i$ ($\mathcal{S}^1$ being the base-case scenario) and that they are all calibrated. Suppose also that we define a maximum loss target L on a certain position, which is the maximum amount we are willing to lose on that position. The goal is to find functions

$$\lambda : (\mathcal{S}^1, \cdots, \mathcal{S}^I, S_1, \Sigma_1, t) \mapsto \lambda(\mathcal{S}^1, \cdots, \mathcal{S}^I, S_1, \Sigma_1, t)$$

that yield a risk PnL on the positions in the considered portfolio and that aggregates the scenarios. Bergomi [5] suggests defining the benchmark's Lambda in the form

$$\lambda_1(\mathcal{S}^1, \cdots, \mathcal{S}^I, S_1, \Sigma_1, t) := -\left(\sum_{i=1}^{I} -\delta P(\mathcal{S}^i, S_1, \Sigma_1, t)^a \mathbf{1}_{(\delta P(\mathcal{S}^i, S_1, \Sigma_1, t)<0)}\right)^{\frac{1}{a}},$$

where $a > 0$. We can extend the definition of Lambdas to any asset k in the portfolio by defining

$$\lambda_k(\mathcal{S}^1, \cdots, \mathcal{S}^I, S_1, \Sigma_1, t) := -\left(\sum_{i=1}^{I} -\delta P(\mathcal{S}^i, \beta_k^i S_1, \beta_k'^i \Sigma_1, t)^a \mathbf{1}_{(\delta P(\mathcal{S}^i, \beta_k^i S_1, \beta_k'^i \Sigma_1, t)<0)}\right)^{\frac{1}{a}}.$$

Figure 3.2 shows the time series of the Lambda for SPX. They are given in absolute value and computed for a short position on a 3 month at-the-money Straddle. Note how the Lambdas drop during major crises such as the subprimes in late 2008 and COVID-19 in 2020, suggesting that a Short Straddle strategy would yield more PnL during crises.

3.3.2 Computing scaling factors

These Lambdas aggregate scenarios and yield a representative risk PnL for the considered position. For instance, if we consider holding a position on a certain option on asset k, the operation would potentially yield a loss of $\lambda_k(\mathcal{S}^1, \cdots, \mathcal{S}^I, t)$. Therefore, if we wish to meet a loss target L, we should hold a position of

$$\frac{L}{\lambda_k(\mathcal{S}^1, \cdots, \mathcal{S}^I, S_k, \Sigma_k, t)}$$

units of the considered option. By doing so, we re-scale the positions on all underlyings in the portfolio so that they yield the same potential loss L: all assets in the portfolio

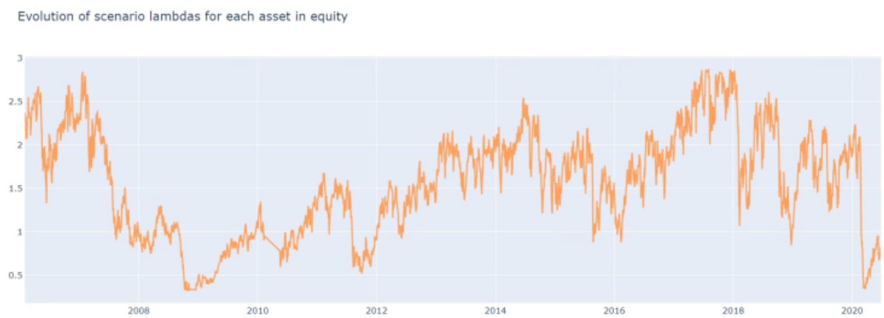

Figure 3.2: Lambda time series for SPX between 2006 and 2020, given in absolute value.

are equally weighted with respect to risk.

3.4 Application to a US equity index based strategy

In this section, we consider a Gamma scalping strategy[2] on US equity indices (SPX, NDX, RUT), using 3 month at-the-money straddles. We will denote by $\Pi(t)$ the value of the portfolio at time t.

3.4.1 Definition of success metrics

In order to evaluate the impact of the scenario-based risk methodology on this strategy, we will compare several metrics on the strategy with and without using it. We will use the following metrics.

- The maximum drawdown, $\mathrm{MDD} := \min_t(\Pi_t - \max_{t'<t} \Pi_{t'})$, which is the maximum observed loss from a peak to a through of the portfolio;
- The yearly Sharpe Ratio, $\mathrm{SR} := \frac{1}{Y}\sum_{y=1}^{Y} \frac{\Pi(y)-\Pi(y-1)}{\sigma_\Pi(y)}$, where $\Pi(y)$ is computed at the end of the calendar year y, and $\sigma_\Pi(y)$ is the sample variance of the portfolio, computed over year y. Note that we still assume the risk-free rate to be zero;
- The yearly Delta-hedged PnL,

$$\mathrm{PnL} := \frac{1}{Y}\sum_{y=1}^{Y}\sum_{t=365(y-1)}^{365y} \frac{\Pi(t)-\Pi(t-1)-\Delta(t-1)(S(t)-S(t-1))}{365}.$$

[2]In reality, the strategy is slightly more complicated than that, but I cannot delve in the details. This is why, for instance, the Sharpe ratios can seem abnormally high for such simple strategy.

3.4.2 Summary of the results

Table 3.4.2 presents the results of the backtests of the strategy between 2006 and mid-2020, with and without using the risk methodology.

Metric	Without the risk methodology	With the risk methodology
MDD (kUSD)	-77.2	-64.6
SR	2.01	2.12
PnL (kUSD)	+60.02	+68.23

Table 3.6: Summary of the evolution of the success metrics for the risk methodology between 2006 and mid-2020.

We can see that applying the risk methodology improves the strategy. Not only does it help increase the Sharpe ratio, which illustrates the performance of the strategy, but it also increases the maximum drawdown, which is a pure risk metric. In other words, the methodology helps the model loose less in case of loss.

Conclusion

Overall, we have successfully built a risk methodology for volatility trading strategies. After introducing the notion of Greeks and their relation to risk,we carried out a theoretical study to show that, when asset returns are highly correlated, the approximation errors resulting from the linear regression yield a good control of the error on the Greeks. This study justifies the use of the beta model, as it still yields relevant results while greatly simplifying the risk methodology.

Then, we used the beta model to define risk scenarios and saw how clustering assets through various algorithms and propagating simple market moves on all clusters reduces the dimension of the problem of scenario definition while keeping the scenarios relevant. We devised a practical methodology to define base-case scenarios and to calibrate custom scenarios to be equivalent in terms of risk profile.

Finally, we applied this risk methodology to Gamma scalping, a fairly simple volatility trading strategy. We displayed its impact through various performance metrics.

Though this model seems to yield interesting results, we can discuss further improvements. Among the many interesting potential improvements to the study and methodology, we can cite the following.

- Refine the assumptions in Equation 1.3.1 and compute a better boundary on the error on Greeks, which will eventually allow better clustering of assets;
- Extend the methodology to other asset classes and study its behavior. It is particularly interesting to see how the methodology works on interest rate options, as these are very different from equity options;
- Extend the methodology to multi-class strategies, and see how the methodology behaves when multiple asset classes are traded in a portfolio For instance, in practice, will the methodology cluster assets by class?
- Improve the study of the methodology's performance metrics. Though the risk methodology seems to improve the performance of the Gamma scalping strategy

overall, performance might be due to effects of both alpha and scaling through risk, and not purely from risk. Hence, we might find better metrics to assess the impact of the risk alone, independently from the way the strategy behaves.

Appendix A

Technical Proofs

In this section, we present detailed proofs of some propositions mentioned in this paper. These proofs are gathered here to make the paper more readable.

A.1 Proof of Proposition 1.2.3

Proof. First, note that, by the chain rule and by linearity of the differentiation operator,

$$\partial_S B = \partial_S S_1 \partial_{S_1} B = \beta \bar{B}$$

and

$$\partial_S \bar{B} = \partial_S S_1 \partial_{S_1} \bar{B} = \beta \bar{\bar{B}}.$$

Then,

- By using the chain rule, we have

$$r_k = \frac{\partial_t S_1 \partial_{S_1} S_k}{S_k} = \beta_k \frac{r_1 S_1}{S_k} = \frac{\beta_k}{c_k} \frac{S_1}{S_k} r_k$$

  hence, since $r_k \neq 0$ a.s.,

$$\beta_k = c_k \frac{S_k}{S_1}.$$

- By differentiation of β_k,

$$\bar{\beta}_k = c_k \frac{S_1 \partial_{S_1} S_k - S_k}{S_1^2} = c_k \frac{\beta_k S_1 - S_k}{S_1^2} = c_k \frac{c_k S_k - S_k}{S_1^2} = c_k (c_k - 1) \frac{S_k}{S_1^2};$$

- From the first point, we have

$$S = \sum_{k=1}^{n} S_k = \sum_{k=1}^{n} \frac{\beta_k}{c_k} S_1 = B S_1,$$

hence[1]

$$\beta = \partial_S S_1 = \frac{1}{B} - S\frac{\partial_S B}{B^2}$$
$$= \frac{1}{B} - S\beta\frac{\bar{B}}{B^2}.$$

This yields

$$\beta\left(1 + S\frac{\bar{B}}{B^2}\right) = \frac{1}{B},$$

hence

$$\beta = \frac{1}{B + S\frac{\bar{B}}{B}} = \frac{B}{B^2 + S\bar{B}}.$$

- Note that $\beta(B^2 + S\bar{B}) = B$. Then, we have

$$\begin{aligned}
\bar{\beta} = \partial_S \beta &= \frac{\partial_S B(B^2 + S\bar{B}) - B(2B\partial_S B + \bar{B} + S\partial_S \bar{B})}{(B^2 + S\bar{B})^2} \\
&= \frac{\beta\bar{B}(B^2 + S\bar{B}) - B(2\beta B\bar{B} + \bar{B} + S\beta\bar{\bar{B}})}{(B^2 + S\bar{B})^2} \\
&= \frac{\bar{B}B - B(2\beta B\bar{B} + \bar{B} + S\beta\bar{\bar{B}})}{(B^2 + S\bar{B})^2} \\
&= -\beta B\frac{2B\bar{B} + S\bar{\bar{B}}}{(B^2 + S\bar{B})^2} \\
&= -\beta^3\frac{2B\bar{B} + S\bar{\bar{B}}}{B} \\
&= -\beta^3\left[2\bar{B} + S\frac{\bar{\bar{B}}}{B}\right].
\end{aligned}$$

□

[1] Here, the division by $\sum_{k=1}^{n} \beta_k / c_k$ is justified by the fact that for a non-empty portfolio, $S > 0, S_1 > 0$ a.s., hence $S/S_1 > 0$ a.s.

A.2 Proof of Proposition 1.3.4

Proof. We have

$$\overline{\beta}_k = \partial_{S_1}\hat{\beta}_k + \partial_{S_1}\delta\beta_k = \overline{\hat{\beta}}_k + \partial_{S_1}\delta\beta_k,$$

hence

$$\begin{aligned}
\delta\overline{\beta}_k = \partial_{S_1}\delta\beta_k &= \partial_{S_1}\left(\frac{S_k}{r_1 S_1}\varepsilon_k\right) \\
&= \partial_{S_1}\frac{1}{\partial_t S_1}S_k\varepsilon_k + \partial_{S_1}S_k\frac{\varepsilon_k}{\partial_t S_1} + \frac{S_k}{\partial_t S_1}\partial_{S_1}\varepsilon_k \\
&= -\frac{\partial_{S_1,t}S_1}{(\partial_t S_1)^2}S_k\varepsilon_k + \beta_k\frac{\varepsilon_k}{\partial_t S_1} \\
&= -\frac{\partial_{t,S_1}S_1}{r_1^2 S_1^2}S_k\varepsilon_k + \beta_k\frac{\varepsilon_k}{r_1 S_1} \\
&= \beta_k\frac{\varepsilon_k}{r_1 S_1} \\
&= \left(\hat{\beta}_k + \frac{S_k}{r_1 S_1}\varepsilon_k\right)\frac{\varepsilon_k}{r_1 S_1},
\end{aligned}$$

by independence of S_1 and ε_k, and by Schwartz's theorem. Besides we have the following result.

Lemma A.2.1. *Let* $X \sim \mathcal{N}(0, \sigma_X^2)$ *and* $a, b \in \mathbf{R}$. *Then*

$$\mathrm{Var}(X(aX+b)) = \sigma_X\sigma_Y + \sigma_X\mathbf{E}[Y]^2 + \sigma_Y\mathbf{E}[X]^2.$$

Proof. Let $Y := aX + b$. We have

$$\begin{aligned}
\mathrm{Var}(XY) &= \mathbf{E}[X^2Y^2] - \mathbf{E}[XY]^2 \\
&= \mathrm{Cov}(X^2, Y^2) + \mathbf{E}[X^2]\mathbf{E}[Y^2] - (\mathrm{Cov}(X,Y) + \mathbf{E}[X]\mathbf{E}[Y])^2.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\mathrm{Cov}(X^2, Y^2) = \mathrm{Cov}(X^2, a^2X^2 + 2abX + b^2) &= a^2\mathrm{Var}(X^2) + 2ab\mathrm{Cov}(X^2, X) \\
&= 2a^2\sigma_X^4,
\end{aligned}$$

since $(X/\sigma_X^2)^2 \sim \chi_1^2$ (hence $\mathrm{Var}(X^2) = 2\sigma_X^4$) and, by symmetry,

$$\mathrm{Cov}(X^2, X) = \mathrm{Cov}((-X)^2, -X) = -\mathrm{Cov}(X^2, X).$$

Then,

$$\mathbf{E}[X^2]\mathbf{E}[Y^2] = \sigma_X^2(a^2\sigma_X^2 + 2ab\mathbf{E}[X] + b^2) = \sigma_X^2(a^2\sigma_X^2 + b^2).$$

Finally,

$$(\mathrm{Cov}(X,Y) + \mathbf{E}[X]\mathbf{E}[Y])^2 = \mathrm{Cov}(X,Y)^2 = a^2\sigma_X^4.$$

Following that, we have

$$\mathrm{Var}(XY) = a^2\sigma_X^4 + b^2\sigma_X^2$$

□

Consequently, using this lemma with $X = \varepsilon_k$ $a = \frac{S_k}{r_1 S_1}$ and $b = \hat{\beta}_k$, we have[2]

$$\sigma^2_{\delta\overline{\beta}_k} = \frac{1}{r_1^2 S_1^2}\left(\frac{S_k^2}{r_1^2 S_1^2}\sigma^2_{\varepsilon_k} + \hat{\beta}_k^2\right)\sigma^2_{\varepsilon_k} \leq \frac{1}{r_1^2 S_1^2}\left(\frac{S_k^2}{r_1^2 S_1^2}C + \frac{\sigma_k^2}{\sigma_1^2}\right)C$$

by definition of $\hat{\beta}_k$. □

[2]Recall we are computing variances conditional to t and r_1, hence we can "get the S_k and r_1 outside the variances".

A.3 Proof of Proposition 3.2.1

Proof. First, note that, when $S_1 = K$, we have $d_1 = -d_2 = \sigma\sqrt{T-t} =: d$. Besides, note that, by symmetry of the Gaussian distribution, $\Phi(x) + \Phi(-x) = 1$. Then, we have the price of the at-the-money Call at time t

$$\begin{aligned} C(S_1, t) = S\Phi(d_1) - K\Phi(d_2) &= S_1(\Phi(d) - \Phi(-d)) \\ &= S_1(2\Phi(d) - 1), \end{aligned}$$

and by Call-Put parity, we have the price of the Put at time t

$$P(S_1, t) = C(S_1, t),$$

hence the price of the Straddle at time t

$$(C + P)(S, t) = 2S_1(2\Phi(d) - 1).$$

At time $t+1$, if we shock the benchmark's spot and the Straddle's implied volatility with the scenario $\mathcal{S}$, we have the Call and Put prices at time $t+1$

$$C(S_1 + \delta S_1, t+1) = (S_1 + \delta S_1)(2\Phi(d) - 1)$$

and

$$P(S_1 + \delta S_1, t+1) = C(S_1 + \delta S_1, t+1) - \delta S_1,$$

again by the Call-Put parity. Note that, in these formulas, the definition of d_1 does not change, as $T - t$ is constant in the definition of $\delta P(\mathcal{S}, S_1, \Sigma_1, \cdot)$. Hence, we have the price of the Straddle at time $t+1$

$$(C + P)(S_1 + \delta S_1, t+1) = 2(S_1 + \delta S_1)(2\Phi(d) - 1) - \delta S_1,$$

and

$$\begin{aligned} \delta P(\mathcal{S}, S_1, \Sigma_1, t) &= (C + P)(S_1 + \delta S_1, t+1) - C + P)(S_1, t) \\ &= 2\delta S_1(2\Phi(d) - 1) + \delta S_1 \\ &= \delta S_1(4\Phi(d) - 1). \end{aligned}$$

□

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