Declaration

The work contained in this thesis is my own work unless otherwise stated.

Signature and date:

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Hazard rate surface model and its application to Southern European Sovereign bonds

by

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Abstract

The purpose of this study is to create and caliber a model for the short-term hazard rate. The introduction to hazard rates surface, generated as stochastic processes (CIR or Exponential Vasicek). Also, using numerical method schemes allow a new vision for short-term hazard rate. This innovation allows the short-term hazard rate to fit the CDS curves, and the bonds market pricing, under the risk neutral probability.
Moreover, I will discuss the necessity of adding a jump process to model the hazard rate as close as possible to market reality; this thesis contains a discussion of the representation of this jump process, as it could occurs during a crisis, and the consequences of using diverse scheme to determine the short-term hazard rate.
The simulation part is focused on Southern European Sovereign bonds, as in August 2018, the Greek government has been allowed to come back on debt market, bringing some risk on economy with similar structure.

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1 Introduction

As in August 2018, we saw the reintroduction of Greece in the debt market, I decided to focus my thesis on how the default probability of Southern European countries could be modelled.
The probability of default have been a major problem for euro-zone bonds, as the last crisis in the euro-zone was a sovereign debt crisis with its paroxysm the Greek debt. As countries with similar economic environment have also suffered a surge in yield, denoting a mis-pricing and an arising fear on the market.

The following thesis will explain my work about that particular topic, in which I will discuss the pricing of risky bonds and their probability of default, and especially the pricing of sovereign European bonds. As in the euro-zone the German Sovereign bonds are considered as the less risky, I will consider them as risk-free bonds. Of course, you could use my work for any other risky bonds, with another free-risky bond, for another currency.The advantage of my method is you get a realistic hazard rate for any period, without any shifting to fit the market, of course such a method needs more work for calibration.
The distribution of the default event time is also really important as the payment of the interest are conditioned to the non-default of the issuer.
I will use stochastic models for those hazard rates and succeed to fit the CDS market and price risky bonds while describing short-term hazard rate as uniquely stochastic. I will study the credit event probability I will call default probability (a clear definition is given in the definition 1.1). As you will notice my models will be less tractable than many other models in past studies, but as you will see it succeed to describe hazard rates as purely stochastic thanks to the introduction of hazard rates surface. This introduction of this surface comes from my perception of the default risk priced by the market. The hazard rate implied by such a model will be a multiple factors model, comparable to the models described in [16], which is a two parameters model.
The purpose is to fit the CDS market, as well as the past probability default priced by the market, or at least try to understand how the market would price a CDS. In a non-arbitrage market, as it could expect for the liquid European Sovereign CDS market, the probability of default should be rated by the market with a higher yield (definition given in the definition 1.7).
A way to get the default probability valued by the market is studying the difference between, the yield of the bonds, and the risk-free rate. With daily yield data, you can extract for everyday the hazard rate implied by the bond market. As the model should reflect the pricing of the market, this method seems to have legitimacy. In my opinion, there is no advantage to look at a shorter interval, as the variation will be tiny, or captured on the next day.
The first observation anyone can make on CDS spread behaviour is that jumps occur because of unexpected events, they occur for both single names (Companies) or for sovereign debt. Jump

models have been many times discussed as in [2] for hazard rates, however adding a jump process will generate much less tractability, as well as difficulties to estimate the parameters of the models. Th reason of the jump are multiple, for instances: when there is a crisis or when an unexpected election result comes up. Those movements have been explained in [9], [22], [23]. As the default probability jumps, the bonds' yield do the same, and the bond price varies as in [11]. In my study you will follow an analysis of the probability of default under the risk neutral world, which is not the real world, the one used to price financial assets, as the financial asset in a non-arbitrage market we use the risk-neutral probability.

As a CDS spread tends to explode during stress periods named crises, as the risk of default estimated by the market surges during some period designated as crisis. It is then logical to model hazard rate, the parameter that will drive the default probability, will be variable, and will vary with low volatility under a calm market, and a jump process to model the jumps that could occur. As there exists economic differences between different geographic area I will study Southern European Sovereign bonds, as southern Europe gives riskier sovereign bonds. I propose to study sovereign risky bonds, inside the euro-zone to allow a homogeneity in the parameters, with high correlation between the risk of default. Any news that can have a direct impact are political elections and decision, or the economical context. The ECB monetary policy will lead to a high correlation between those countries as they have the similar profile, and they will all have the same monetary policy. Such a work has been done by M. Piazzesi in 2005 for US Treasuries in [13], except that work was done uniquely for US Treasuries.

The main innovation of the paper is the idea of surface of equivalent hazard rate under the risk neutral probability which is equivalent to the measure of the bonds, yield. I will create this surface by linear combinations between the calibration points. To simulate such surface the equivalent hazard rate will follow a CIR or Exponential Vasicek, with the same speed reversion, as two different parameters would give an unwanted spread due to different speed of relaxation. In addition, the key feature for those hazard rates is the mean reversion behaviour, meaning that the hazard rate will come back to the mean reversion level if a shock moves their value.

Many papers have treated about stochastic hazard rates, with mean reversion model with one or two parameters. The CIR model introduced in [10] allows such behaviour, as well as the Exponential Vasicek model.

I will also be base my study on those models as they both have a non-explosive variance. As calibration is thought, what should be a good parameter of volatility if the volatility explode as the time goes to infinity. Moreover the hazard rates by its definition can't be negative, as a negative hazard rate will imply that the probability of default is negative, or that the risky bond is more likely to be paid back than the risk-free bond. Both CIR and Exponential Vasicek models do not allow negative values.

In the previous paper, Brigo and El-Bachir [2] have modelled the short-term hazard rate (this is not the equivalent hazard rate cited above) by a sum of a stochastic process, a jump process, and a deterministic function to fit accurately the CDS curve implied by the market. The hazard rate fit well the market, but it is not purely stochastic.
I will also try to caliber my model to fit the CDS curves, to do so the parameters, as mean reversion level, mean reversion speed, variance, and then amplitude of the jump, will be estimated with the past yield value for the bonds with maturity 1Y,2Y,3Y,4Y,5Y,10Y, when the data is available and liquid. As with such models the hazard rate λ will have a lognormal or non-central -square distribution, thanks to the estimation of their mean and variance, I could get an estimation of each parameter.
A major problem to caliber the model is also to determine how long should be the period of observation, with the mean reversion level and variance to be determined. If the period is too long I might consider data that is no longer relevant, the economic environment might have changed, so the parameter μ will be badly caliber by considering data that should be excluded today. This parameter μ is the mean reversion level, and this parameter is defined as: (definition of this 1.3). $\lim_{t \to +\infty} \mathbb{E}(\lambda_n(t)) = \mu_n$ for the CIR model for example. Same if we had a lot of moves due to a former economic problem, if the situation is now better, I am likely to overvalue μ_n, and as the model is mean reverting, the long-term expectation of hazard rate should converge to this value.
Of course, the same remark stands for the parameter υ_n. For such reason the calibration of this model is hard, and some CDS spread data might be use-full. For example with 5Y bonds I can get the different parameters, conditioning on good fitting for CDS with maturity 5Y, as a high volatility will imply a high skew in the distribution.
The problem of the estimation of the volatility parameter could be solved by looking at the implied volatility in the default swaptions market and using the Merton, Black and Scholes model [1] to get the volatility of the CDS spread given the price of the default swaptions. The CDS spread is proportional to the hazard rate, an estimation of the volatility of the hazard rate will be determined by this method. Another issue raised by this paper is that the implied volatility depends on the default swaptions. The volatility of the hazard rate seen today of the hazard rate on the period [0;1Y] and [0;10Y] is not the same, as well as the parameter μ_n. That is why the one parameter stochastic model with no deterministic function does not work or does not fit properly the market. Also, the primary purpose is to introduce hazard rate, $\lambda_n(t)$ for different n, creating a surface to find purely stochastic hazard rates. By allowing variable μ_n and υ_n the model will be less tractable, but there is no need to add a deterministic function, that will maybe be wrong if the configuration change, for instance, if the CDS spread curve is increasing with maturity, and then rapidly change to decreasing as the hazard rate surges. It is difficult to know what kind of behaviour this deterministic function should adopt.

My thesis will solve some of the above questions, to do so I need to cover major mile stones, first I will defined hazard rate models and the crucial assets to price it, then look at the simple stochastic model without jumps. I will also look at the hazard rate surface to realize a close fitting to the market. Finally, I will add jumps in this model to represent as close as possible the market.
During this thesis, the data used were at the beginning Italian, Greek, French, Portuguese, and Spanish bonds, as Italy had an election that leads to a yield jump on June 2018; this jump has propagated to the rest of Euro bonds.
My study will as well be an empirical one, as I will try to fit the CDS market from the bonds historical data, previous empirical studies carried out on interest rates for example in [24] and [25].

1.1 Definition of the market model that will be used

As a beginning, we should define and agree on the environment of the study, thus I will give some hypothesis about the market to use it in the following development.
The market is a non-arbitrage market, where the bonds are well priced, meaning that the price of the bonds reflect correctly the default probability of the issuer, and when it should happen. On the first hand the default probability implied by the market is not the real probability of default, but it is the subjective opinion of the market of the company or the government to pay back the debt it has contracted and will contract during the period. On the other hand, you could estimate the cashflows of a risky bond under the risk-neutral probability, with numeraire the risk-free rate bond.
The hypothesis of the bonds priced under realistic hazard rate can is most of the time true as I will deal with Euro sovereign bonds, as Italian bonds, for non-exotic maturities.
As a consequence, the fair price of every asset in this paper will be the expectation under the risk-neutral probability, and its numeraire, the risk-free bond which is for the euro-zone the yield of German bonds.
Under the risk-neutral probability the price and so the yield of the bonds will directly reflect the hazard rate, as it will be the parameter to describe the probability of default, as well as describe it with respect to the time/maturity.
Using the argument of the risk-neutral pricing, the expected cashflow of the risk-free bonds with coupon equals to the yield (for the Euro bonds the German Yield will be the risk-free yield) should be the same as the risky bonds with coupon equals the market yield. Of course the expectation of payment of the interest (yearly interest rate equals to the yield) and the nominal will be dependent on the default or not.

1.2 The Default Probability

Definition 1.1. The hazard rate λ is a parameter that estimates the probability of default event.

Definition 1.2. A default event could be a default, a restructuring, or any other events on the debt issued. The hazard rate takes into consideration more events than the bankruptcy.

Moreover as stated before the default probability model used makes the probability of default dependent on the hazard rate λ. I will use the notation τ as the time when the default occurs. The probability of default of the risky bonds before time t can be expressed, under the risk-neutral probability, with the following formula:

$$\mathbb{E}_Q[\mathbb{1}_{[\tau \leq t]}|\tau > 0] = \mathbb{Q}(\tau \leq t) \tag{1.1}$$

The exponential hazard rate model is the model that determines this probability:

$$Q(\tau \leq t|\tau > 0) = 1 - e^{\int_0^t -\lambda^{\mathbb{Q}}(u)}\mathrm{d}u$$
$$Q(\tau > t|\tau > 0) = e^{\int_0^t -\lambda^{\mathbb{Q}}(u)}\mathrm{d}u$$

where $\lambda^{\mathbb{Q}}$(u) is the hazard rate at time u as described in [27].

Remark 1.3. This hazard rate under the risk-neutral probability is not the same as the probability in the real world, the probability $\mathbb{P}$ also, $\mathbb{Q}$ are not the same.
For the rest of the thesis, I will assume that both $\lambda_n^{\mathbb{P}}(t)$ and $\lambda_n^{\mathbb{Q}}(t)$ follows the same model, close in value but with different parameters as mentioned in [32] with the curve made by Lehman Brothers.

The default probability under the real probability $\mathbb{P}$ will be:

$$P(\tau \leq t|\tau > 0) = 1 - e^{\int_0^t -\lambda^{\mathbb{P}}(u)}\mathrm{d}u$$
$$P(\tau > t|\tau > 0) = e^{\int_0^t -\lambda^{\mathbb{P}}(u)}\mathrm{d}u$$

To go further, I need to add a small part on bond pricing that will be used get the value of the constant $\lambda_{implied\ for\ n\ Years}$ implied by the market.
The $\lambda_{implied}$ from the bond market is of course not constant over the maturity of the bonds, but with such information, no one could say the term structure of the hazard rate. This $\lambda_{implied}$ could be considered as constant; the integral takes the simple form:

$$\int_0^n \lambda(u)\mathrm{d}u = n\lambda_{implied\ for\ n\ years}(0)$$

The bonds market could serve to determine the mean of the hazard rate until maturity, priced by the market.

Remark 1.4. There is no necessity to beginning the interval of default observation at 0, then: To simulate a credit event, given a hazard rate, and given that the credit event didn't occur before the time of observation.
the following formula gives the probability under the risk-neutral probability of the default event between t_a and t_b:

$$\mathcal{Q}(t_a \leq \tau \leq t_b | \ t_a < \tau) = 1 - e^{-\int_{t_a}^{t_b} \lambda^{\mathbb{Q}}(u) \mathrm{d}u}$$

As I use daily hazard rate, then the probability of default on one day, considering the hazard rate is constant over the day:

$$\mathcal{Q}(t_i \leq \tau \leq t_{i+1} | \ t_i < \tau) = 1 - e^{-\frac{1}{252}\lambda^{\mathbb{Q}}(t_i)}$$

Remark 1.5. For the rest of the thesis, I will divide the year in 100 intervals where the $\lambda(t)$ will be constant. The $\lambda(t)$ can take 100 different values during a year.

1.3 Hazard rate definitions and notations

In the next pages, it is important to be clear with the notations, as I will use other hazard rates notations than the one used above ($\lambda_n^{\mathbb{Q}}$):
•$\lambda_i(t)$ is the average hazard rate for the period (t;t+i) seen at time t, so:

$$\mathbb{Q}(t < \tau < t + i | \tau > t) = 1 - e^{-i\lambda_i^{\mathbb{Q}}(t)}$$

This is the probability of default of the risky bond from time t to time t+i.
•$\lambda_{i-j}^{\mathbb{Q}}(t)$ is the hazard rate seen at time t, for the period [i;j], then the default probability is:

$$\mathbb{Q}(t + i < \tau < t + j | \tau > t + i) = 1 - e^{-(j-i)\lambda_{i-j}^{\mathbb{Q}}(t)}$$

• One can notice that there is a direct relation between the two hazard rates above. For example if t=0, the relation is quite easy:

$$\mathbb{Q}(\tau < j | \tau > 0) = 1 - e^{-j\lambda_j^{\mathbb{Q}}(0)} = 1 - e^{-j\lambda_{0-j}^{\mathbb{Q}}(0)}$$

So $\lambda_j^{\mathbb{Q}}(0) = \lambda_{0-j}^{\mathbb{Q}}(0)$
Then $1 - e^{-j\lambda_{0-j}^{\mathbb{Q}}(0)} = 1 - e^{-\sum_{i=1}^{j} \lambda_{i-1-i}^{\mathbb{Q}}(0)}$
For j=2, the equation is now: $\lambda_2^{\mathbb{Q}}(0) = \frac{1}{2}(\lambda_{0-1}^{\mathbb{Q}}(0) + \lambda_{1-2}^{\mathbb{Q}}(0))$

Remark 1.6. for any $\leq t_1 < t_2$:

$$\mathbb{Q}(t_1 \leq \tau < t_2) = \mathbb{Q}(\tau \leq t_2) - \mathbb{Q}(\tau < t_1)$$

$$\mathbb{Q}(t_1 \leq \tau < t_2) = 1 - e^{-\int_0^{t_2} \lambda^{\mathbb{Q}}(u)\mathrm{d}u} - 1 + e^{-\int_0^{t_1} \lambda^{\mathbb{Q}}(u)\mathrm{d}u}$$

$$\mathbb{Q}(t_1 \leq \tau < t_2) = -e^{-\int_0^{t_2} \lambda^{\mathbb{Q}}(u)\mathrm{d}u} + e^{-\int_0^{t_1} \lambda^{\mathbb{Q}}(u)\mathrm{d}u}$$

• The last notation is the surface hazard rate by abuse of language $\lambda^{\mathbb{Q}}(t)$ will be the $\lambda^{\mathbb{Q}}$ at time t as we are now at 0, this notation referrers to the short-term daily hazard rate. If I write $\lambda^{\mathbb{Q}}(t)$, it means the short-term hazard rate for the day number t, or for the day at time t.

1.4 Introduction to bond pricing

In this part, every hazard rate will be the hazard rate under the risk-neutral probability $\mathbb{Q}$, if not it is specified. First of all, let's give the definition of the yield of a bond, the definition is from [21]:

Definition 1.7. The yield is the fixed coupon that would be needed to make the price of a bond equals to its nominal, considering that the bond can't default.
Of course the higher the yield, the lower the bond price, and the higher the risk of default before maturity is. There exists many studies on European yield as they have jumped on the last Euro bonds crisis [29], in this paper the author has his analysis on Irish bonds, which are now quite safe with low default probability implied by the market.

Definition 1.8. The recovery rate $\mathcal{R}$ is the part of the nominal that will be paid by the issuer of the bonds when a credit event occurs. Typically it is the part of the nominal of the bond that will be paid back, by selling the assets of the issuer.

Most of the time the bonds are quoted thanks to their yield, here I need to compute the discounted cash-flows given the yield. The topic of this thesis is not computing interest rates, but find a representation of default probability, to do so you could take the German Sovereign bond's yield as a reference to discount the cash-flows, as it is mainly used so in the industry, to add some weight to my choice I would like to cite the work done in [28]. Of course, my model could be adapted with some other risk-free bonds, for another currency.
If the maturity is n years, interests i, risk-free rates r (in this simple case the interest rate is constant, and the yearly equivalent rate). If the coupon is paid every year, a price for such bond could be as in [21]:

$$P(0) = \left(\sum_{j=1}^{n} \frac{100 * i}{(1+r)^j}\right) + \frac{100}{(1+r)^n}$$

with a nominal of 100.
Of course, if the free risk rates is not constant you can adapt the formula:

$$P(0) = \left(\sum_{j=1}^{n} \frac{100 * i}{(r[j])}\right) + \frac{100}{(r[n])}$$

$$r[h] = \mathrm{e}^{\int_0^h r(u)\mathrm{d}u}$$

Those formulas are for risk-free bonds, as there is no default. To add some risk, I need to add the possibility of a default event. Let's say the default time is τ.

$$P(0) = \mathbb{E}_{\mathbb{Q}}[Discounted\ Cashflows|\mathcal{F}_0]$$

with:

$$Cashflows = \left(\sum_{j=1}^{n} \frac{100 * i}{r[j]} \nVdash(\tau > t_i)\right) + \frac{100 * i}{r[t_k, \tau]} \nVdash(t_{k+1} \geq \tau > t_k) + \frac{100}{r[n]} + \frac{\mathcal{R} * 100}{r[\tau]} \nVdash(\tau < t_n)$$

As I add the hypothesis of Independence between default event, and so the default intensity, and the interest rates used for discounting. By changing the measure and using the risk-neutral probability $\mathbb{Q}$, the numeraire is now the German bond with the adequate maturity.
The rate r is now the yield of German bonds.

As I add the hypothesis of independence between the default event, and so the default intensity as the function exponential is a bijection, and the interest rates used for discounting.

$$P(0) = \mathbb{E}_{\mathcal{Q}}\left(\left(\sum_{j=1}^{n} \frac{(t_j - t_{j-1}) * 100 * i}{r[j]} \nVdash_{\{\tau > t_j\}} + \frac{(\tau - t_k) * 100 * i}{r[\tau]} \nVdash_{\{t_{k+1} \geq \tau > t_k\}}\right) + \frac{100}{r[n]} \nVdash_{\{\tau > t_n\}} + \frac{\mathcal{R}^{\mathbb{Q}} * 100}{r[\tau]} \nVdash_{\{\tau < t_n\}}\right)$$

$$P(0) = \sum_{j=1}^{n} \left(\frac{(t_j - t_{j-1}) * 100 * i}{r[j]} \mathbb{Q}(\tau > t_j)\right) + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{(u - t_{j-1}) * 100 * i}{r[u]} \mathrm{d}\mathbb{Q}(t_{j-1} < \tau \leq u)$$
$$+ \frac{100}{r[n]} \mathbb{Q}(\tau > t_n) + \frac{\mathcal{R}^{\mathbb{Q}} * 100}{r[\tau]} \mathbb{Q}(\tau < t_n)$$

$$\mathrm{d}Q(u) = \lambda^{\mathbb{Q}}(u) e^{-\lambda^{\mathbb{Q}}(u)} \mathrm{d}u$$

$$P(0) = \sum_{j=1}^{n} \left(\frac{(t_j - t_{j-1}) * 100 * i}{r[j]} \mathbb{Q}(\tau > t_j)\right) + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{(u - t_{j-1}) * 100 * i}{r[u]} \lambda(u)\big(1 - \mathbb{Q}(\tau \leq \approxeq)\big)\mathrm{d}u$$
$$+ \frac{100}{r[n]} \mathbb{Q}(\tau > t_n) + \frac{\mathcal{R}^{\mathbb{Q}} * 100}{r[\tau]} \mathbb{Q}(\tau < t_n)$$

For the discounting factor r[t], in the next pages, I will use the discounting rate $r[t] = \frac{1}{(1+r)^t}$ the constant r in this fraction will also be y_n,in this section, the German bonds' yield, seen as discounting rate.
The rates are now computed discretely, and not continuously, as the coupon is paid at a fixed date. I can also argue that the market gives a price for coupon paid quarterly, semi annually, or

annually, so there is no need to use the continuous discounted rates, as it will only add difficulties, without any advantage.
In addition, the hazard rate λ_n constant on the nY period could be determined with the CDS market, as we are looking at the price of the bond at a fixed time 0.
Of course in this particular case, the German bond yield is used as r(u) this rate is constant over the time, however, varies from maturity to another.
Is the independence between risk-free rates and implied hazard rate exact? The following table show the result of the Italian bonds.
The independence hypothesis is questionable, as shown by the correlation results, done with 2500 days:

Maturity (n)	Correlation between λ_n and the yield of German bonds with maturity n
1	0.1670
2	0.1756
3	0.1585
4	0.1195
5	0.0620
10	0.0002

To get the market implied λ the way I have used is to price the bonds at their yield and with the risk-free yield, as the price of a bond which pays a coupon equals to its yield should be 100. The hazard rate implied by the bonds market could be estimated as below:

$$P(0) = \sum_{j=1}^{n} \left(\frac{100i}{(1+y_n)^j} e^{-j\lambda_n^{\mathbb{Q}}(0)} \right) + \frac{100e^{-n\lambda_n^{\mathbb{Q}}(0)}}{(1+y_n)^n} + 50\lambda_n^{\mathbb{Q}}(0) \left(1 - \frac{e^{-n\lambda_n^{\mathbb{Q}}(0)}}{(1+y_n)^n} \right) \frac{1}{\lambda_n^{\mathbb{Q}}(0) \ln(1+y_n)}$$

Using a parameter $\mathcal{R}^{\mathbb{Q}}$=0.5, definition 1.8. For the proof you could refer to the appendix A.
This analysis gives the equivalent constant hazard rate at time 0 (time of issuing of the bond) till the maturity of the bond.
As this is the equivalent to the mean of the hazard rate on the period of study, the formula can now be adapted for the probability of default under the risk-neutral probability.

$$\mathbb{Q}[The\ counterparty\ default\ before\ nYears](t) = \mathbb{Q}[t < \tau < nYear{+}t](seen\ at\ time\ t) = 1 - e^{-n\lambda_n^{\mathbb{Q}}(t)}$$

$$\mathbb{Q}[The\ counterparty\ default\ before\ nYears](t) = 1 - e^{-n\lambda_n^{\mathbb{Q}}(t)}$$

This equivalent hazard rate $\lambda_n^{\mathbb{Q}}$(t) will be considered as stochastic in the following part.
For each past days, you can get the equivalent yearly hazard rate, and so get the historical mean and the historical volatility.
There is no closed form for $\lambda_n^{\mathbb{Q}}$ the way, a dichotomy method will be applied to determine the past

$\lambda_n^{\mathbb{Q}}$, as this method convergences quickly from my results for European bonds, even with a large set of data, I will take a small error parameter.

1.5 Debt introduction

At the beginning of my thesis, I have explained how you could fine an approximation of the hazard rate through the bonds market. But some entities could have CDS on their debt, but very illiquid bonds market. Using the hypothesis of non-arbitrage market , the yield of the new bonds, and so their price could be determined by this method. This is useful in case were only few bonds are on the market, or if the maturity of the new bonds is not quoted.

Remark 1.9. The hazard rate is unchanged if the new debt issued is minimal. Such a method could be inapplicable for large debt issued, and a more developed model with the impact of debt issued could be used, looking at the previous impact on similar companies.

Remark 1.10. In the situation the entity is issuing a large amount of debt, the default event has a higher probability to happen at the maturity of the bonds. The fresh cash raised with the debt will be added to the treasury, to invest the short-term hazard rate should decrease, but the hazard rate at the maturity should increase. The CDS curve is likely to steep.

Remark 1.11. Even if the curve is likely to be steeper, it is impossible to be sure that the short-term hazard rate will go down, as it depends for what the debt is raised, if the company is in difficulties, or investing, the investors could see a higher risk for the debt, the hazard rate even in the short rate could raise if the money is spent immediately, but the remark on the steepness of the market is still valid.

As seen in chapter 5 the price of the bonds could be express as:

$$\frac{P}{100} = \left(1 + \frac{y}{m}\right)^{-mT} + \frac{c}{y}\left(1 + \frac{y}{m}\right)^{-mT}$$

with P the clean price of the bond, m the number of payment of interest per year, T the maturity, y is the yield.
Using a Newton-Rapson scheme to find the fair yield, then for the $(n+1)^{th}$ iteration the yield is:

$$y_{n+1} = y_n - \frac{P(y_n) - 100}{P'(y_n)}$$

You can continue the iterations till $|y_{n+1} - y_n|$ is small enough for you.
With P(y_n) obtained by Monte Carlo method, using the $\lambda^{\mathbb{Q}}(t)$ simulated, and estimated the expectation under the risk-neutral probability.

Remark 1.12. To get $P'(y_n)$, you could use this formula: $P'(y_n) = \frac{\partial P}{\partial y}(y_n) = \lim_{h \to 0} \frac{P(y_n+h) - P(y_n-h)}{2h}$

2 Hazard rate surface and its use

2.1 $\lambda_n^{\mathbb{Q}}(t)$ implied from the market

The main point to design such an algorithm is to understand that the excess yield compared to the German bonds' yield is due to the risk of default, as the liquidity is high, there is no problem of illiquidity effect, or to be accurate I will not into account those factors, however such a problem, if it exists, could appear in crisis, when the hazard rate bounces, and liquidity drops.

Here is how I did make my dichotomy algorithm to find the equivalent yearly past hazard rate:

Given a hazard rate $\lambda_n^{\mathbb{Q}}(t)$ you could get the price of the bond with maturity nY at the time t:

• If the hazard rate is adapted the expectation of the discounted cashflows under the risk-neutral measure is 100. A bond pays interest while there is no default. If there is a default it pays $\mathcal{R}$nominal.

• By setting the limits of $\lambda_n^{\mathbb{Q}}(t)$ to 0 and 1, the looping algorithm is looking at the price of the bond with a coupon equal to the yield at time t, and with a probability of default is at each time T equals $\mathbb{Q}(\tau > T)$, the fair value of the payment of interest at time 1Y is for example given that the issuer didn't default yet, $\frac{100 * y_{bonds\ I\ am\ looking\ at} * \mathbb{Q}[\tau>1Y]}{1+y_{German\ bonds}}$
with $\mathbb{Q}[\tau > 1Y] = 1 - e^{-\lambda_n^{\mathbb{Q}}}$

• If my price is too high (P>100) then I should increase the value of $\lambda_n^{\mathbb{Q}}$ and if the price is to low (100>P) then I should decrease the value of $\lambda_n^{\mathbb{Q}}$. if the price is too high, then I should increase the value of $\lambda_n^{\mathbb{Q}}$.

• As the algorithm was fast, a few seconds for about 5000 different $\lambda_n^{\mathbb{Q}}$, I decided to take a minimal value for the error, meaning the interval where the really $\lambda_n^{\mathbb{Q}}$ belongs is small, for my study I took 0.00001, while a classical value of hazard rate belongs to a few bips (0.0001), to some hundreds/thousand bips (0.01-0.1).

• At the i^{th} iteration lambda belongs to [x;y] if the price of the bond with the hazard rate $\frac{x+y}{2}$ gives a bond price too high, the next interval will be [$\frac{x+y}{2}$;y], on the contrary, the next interval will be [x;$\frac{x+y}{2}$] until the interval is too small.

2.2 Pertinence of stochastic processes to represent hazard rate

In the previous definition of hazard rate, the equivalent yearly hazard rate $\lambda_n^{\mathbb{Q}}$ implied by the bonds market is highly correlated to the yield of the bonds, as yield is often modelled as stochastic processes, it could be appropriate to try find a model for the hazard rates.

For those wondering why the correlation is not perfect between yield for nY maturity bonds and $\lambda_n^{\mathbb{Q}}$, I would say that the yield can be constant, but if the risk-free rates is moving then the spread between the risk-free rates, and the yield of the bond changes, and so the value of the hazard rate

$\lambda_n^{\mathbb{Q}}$ changes, as the spread between risk-free rate and yield of a risky bond is a direct consequence of its default probability.

To determine a feasible model, I have to look at the distribution of the past hazard rates obtained by the bond yield on the market, as in [30].

Here is the distribution of the equivalent hazard rate for the next 3Y for the Italian bonds, for the past 10 years, meaning this is the hazard rates implied by the yield of the 3Y maturities Italian bonds:

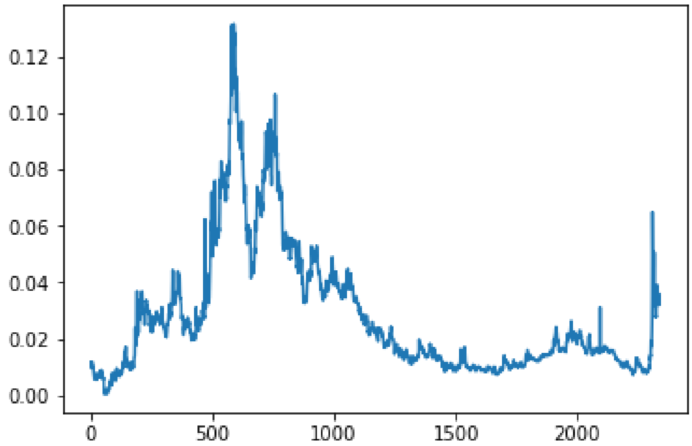


Figure 1: Historical implied λ_{3Y} for Italy

And its distribution with Chi-square(blue) and lognormal(red) estimate distributions:

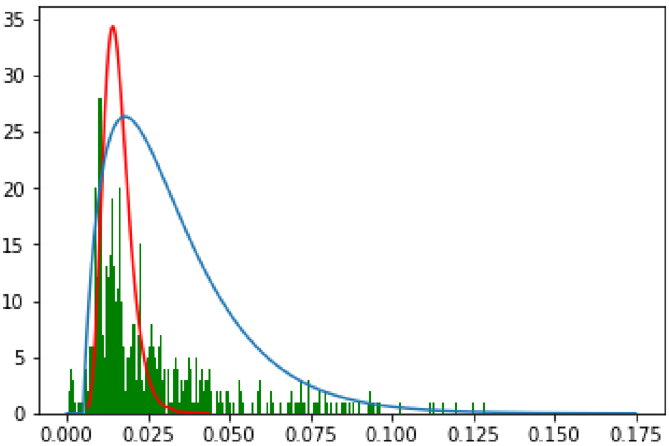


Figure 2: Historical distribution of implied λ_{3Y} for Italy

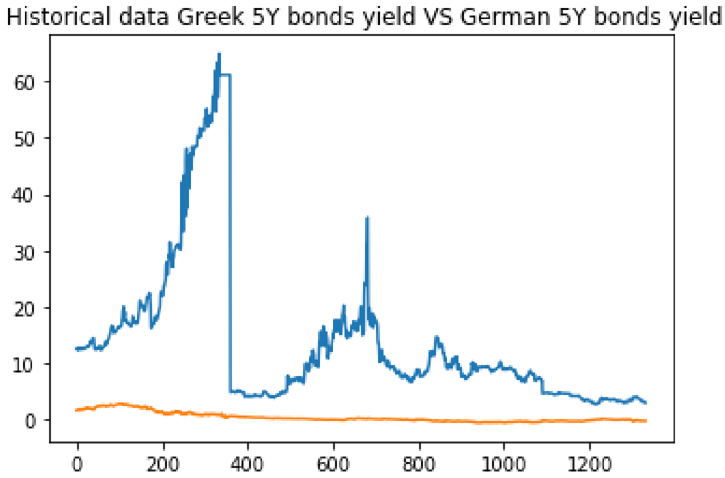
One could notice, that such a distribution could be simulated with a stochastic process, more-

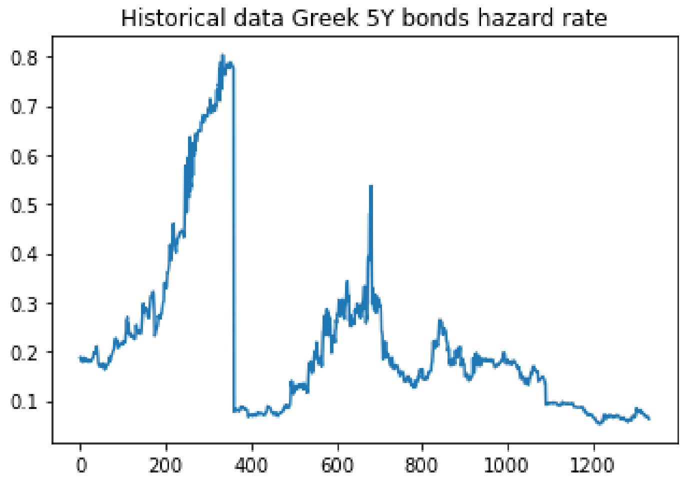
over using this data I can conclude that it is possible to use a CIR model, or an Exponential Vasicek to models the constant hazard rates. The equivalent hazard rate could be considered as a yield or an interest rate, as they are often model as stochastic processes by the industry with Vasicek, CIR or exponential Vasicek. On the contrary to euro-zone interest rates, the hazard rates can't be negative; there is no point to use a Vasicek model, even if this model is highly tractable. The distribution of the latter model is Normal, so that the rates can be negative. In addition the distribution has a positive skewness, you will find in the appendix the same property about the two models remaining.
Another essential feature of the graph appears, sometimes a jump happens. The jumps process can be justified because of a special event, for example, an election, with unexpected results, as it has occurred in Italy in June 2018. The jump is evident on the graph(around the abscissas point 2400). I will logically add a jump process to describe as close as possible the market behaviour. Of course, adding a jump process causes difficulties as the models become less or not tractable. I will also look at the influence of the jump process and at its necessity in the models.
The ideal model would be an Exponential Vasicek, or a CIR with a pure jumps process. The main problem is to find the frequency of those jumps, and the amplitude of those jumps.
The last observation I want to mention is that the distribution of hazard rates is fat-tailed, compared to the fitted data χ^2-distribution, or lognormal distribution. A possible modelling would be adding the now famous jump process, and by mean-reversion, the hazard rate will go back near the mean, whereas the volatility part of the diffusion process causes the variations around the mean.

Remark 2.1. This explanation for fat-tailed distribution is highly questionable, as the fat tail is caused, for this particular example, by multiple small jumps, and steady growth from the point 500 to the point 600, after those high values the value of $\lambda_{3Y}^{\mathbb{Q}}$ came back down with a slow relaxation to the mean reversion level. Of course, multiple jumps can occur in a short time, however, it looks like the volatility on $\lambda_{3Y}^{\mathbb{Q}}$ for Italian bonds is clustered, with long calm periods(point 1000 to point 2300).

The jumps can also decrease the hazard rate $\lambda_n^{\mathbb{Q}}$, as in the following graph of the Greek bonds crisis in 2010 when euro-zone countries/bondholders agreed to reduce and restructure the debt of Greece. The jump was big, expected by the market, as Greece default on some bonds, there were no arbitrage, but the remaining bonds were more likely to be paid back as the debt was much lower and sustainable, the hazard rate for the next few years was so much lower.
Here is the yield of German bonds and Greek bonds, as the hazard rates and the difference of yield between German bonds (risk-free rate) and the Greek bonds are directly correlated.

Here is the Yield implied by those past data:





The highest the yield the highest the hazard rate is, as the Greek bonds are riskier than Italian bonds. Also the volatility of the Greeks hazard rates $\lambda_n^{\mathbb{Q}}$ is higher than for the Italian one. The jump of the Greek bonds was the Euro-zone crisis that happened in 2011, it ended with the payment by a part of the debt by the others countries of the zone as well as restructuring. Then the debt was more sustainable according to the investors that lead to a lower yield. Of course the German bonds have a lower yield than the Greek bonds as no bond could have a lower yield than the risk-free yield in a non-arbitrage market.
Finally the idea of a purely stochastic model to fit correctly the market could be implemented by modelling stochastic equivalent hazard rate over the time, then finding the short-term hazard rate, that will be strictly stochastic.

Remark 2.2. This idea comes from the fact that shifted the stochastic hazard rate model and the shifting was hard to define, I decided to try to do a purely stochastic process, without any deterministic term. The paper from Brigo and El-Bachir [2] gives explanation with details.

2.3 Market implied $\lambda^{\mathbb{Q}}_{i,i+1}(t)$

The main idea is first to get the yields of the bonds on the market, and to price the risk-free bond with a coupon equals to its yield, and the discounting rates will be the yield of the German bonds as I am dealing with euro-zone bonds.
I use the yield data, as in a non-arbitrage market the excess of yield is the result of the default risk, so the difference between the German yield (risk-free bond, as if Germany default, your euros are worthless). By pricing others bonds at their yield, I get an estimation of the value that the issuer of the bond will not paid back.
Then using the past values of $\lambda^{\mathbb{Q}}_n$, for each limit of interval, I can generate the stochastic $\lambda^{\mathbb{Q}}_n$ and so get the hazard rate $\lambda^{\mathbb{Q}}_{n,(n+1)}$. Depending on the scheme and model that I choose for the short-term hazard rate.
As I already mentioned before it is hard to know what should be the mean reversion level, depending on the data I take, is the 10-year-old data relevant, or does the market as change, and at the same time if I take only a few days as data, there is no way I could capture the mean reversion phenomenon. It is the same for the volatility of the $\lambda^{\mathbb{Q}}_n$.
Of course, the interval should be determined with a liquid market, that is why I will decompose the timeline as follows:
$[0Y;1Y];[1Y;2Y];[2Y;3Y];[3Y;4Y];[4Y;5Y];[5Y;10Y]$
It could be important to mention that for t belonging to [0Y;1Y] is:

$$\lambda^{\mathbb{Q}}(t) = \lambda^{\mathbb{Q}}_{0,1Y}(t)$$

2.4 The different models for $\lambda^{\mathbb{Q}}_n(t)$ and their impacts on $\lambda^{\mathbb{Q}}_{n,n+1}(t)$

As you could have figured it out by the statements above the short-term hazard rate $\lambda^{\mathbb{Q}}(t)$ will be describe a bit differently that in many other articles. In those papers the hazard rate is stochastic, with a shift or with multiple parameters, but no article has created a model where the equivalent hazard rate $\lambda^{\mathbb{Q}}_n(t)$ defined as above will be stochastic. For the short-term hazard rates many simpler stochastic models exist, however, this approach needs calibration with a deterministic function to force the model to take the market value, so here I ask for a pure stochastic model, with no deterministic function to caliber the model. From the different $\lambda^{\mathbb{Q}}_n$ I will define schemes to get the $\lambda^{\mathbb{Q}}(t)$.
As the deterministic function might not work when the market conditions change, and it could lead to a different shift each day. A excellent study would be to get the evolution depending on the value a short-term hazard rates of the deterministic function, and if there is patterns or at least some repetition behaviours.
Before to find the short-term hazard rate $\lambda^{\mathbb{Q}}_{i,i+1}$, I will first look at the hazard rate on a year

period. The idea is to determine the hazard rate $\lambda^{\mathbb{Q}}_{i,i+1}$ for the next year, and the forward hazard rate for the 1 year. For instance, the hazard rate priced by the market for one year in 2 years, so the hazard rate for the period (2Y-3Y). By simulating the equivalent hazard rate under the chosen model, then by using the following theorems.

2.5 CIR model, and its impact on $\lambda^{\mathbb{Q}}_{i,(i+1)}$(t)

Let's say the hazard rates $\lambda^{\mathbb{Q}}_{n}$ follow a Cox Ingold R PDE, for the two first years here is what I found:

$$2k\mu_1 > \upsilon_1^2 \textit{ and } 2k\mu_2 > \upsilon_2^2$$
$$\mathrm{d}\lambda^{\mathbb{Q}}_1(t) = k(\mu_1 - \lambda^{\mathbb{Q}}_1(t)) + \upsilon_1\sqrt{\lambda^{\mathbb{Q}}_1(t)}\mathrm{d}W_1(t)$$
$$\mathrm{d}\lambda^{\mathbb{Q}}_2(t) = k(\mu_2 - \lambda^{\mathbb{Q}}_2(t)) + \upsilon_2\sqrt{\lambda^{\mathbb{Q}}_2(t)}\mathrm{d}W_2(t)$$

Then by analyzing the moves of bonds' yield, for the bonds with maturities 1Y and 2Y once can get the correlation between the hazard rates, the PDE for the second hazard rate is:

$$\mathrm{d}\lambda^{\mathbb{Q}}_2(t) = k(\mu_2 - \lambda^{\mathbb{Q}}_2(t)) + \upsilon_2\sqrt{\lambda^{\mathbb{Q}}_2(t)}\left(\rho_{1-2}\mathrm{d}W_1(t) + \sqrt{1-\rho^2_{1-2}}\mathrm{d}W_1^{'}(t)\right)$$

Here I estimate the parameter ρ_{1-2} with the past values of the bonds, and the risk-free rates. The standard Brownian motion $W_1^{'}$ is independent from the standard Brownian motion W_1.
Then the resultant hazard rate for the time 1Y-2Y is $\lambda^{\mathbb{Q}}_{1-2}(t)$ for this period under this model verifies the following PDE:

$$\mathrm{d}\lambda^{\mathbb{Q}}_{1-2}(t) = 2\left(k(\mu_2-\lambda^{\mathbb{Q}}_2(t))+\upsilon_2\sqrt{\lambda^{\mathbb{Q}}_2(t)}\left(\rho_{1-2}\mathrm{d}W_1(t)+\sqrt{1-\rho^2_{1-2}}\mathrm{d}W_1^{'}(t)\right)\right)-\left(k(\mu_1-\lambda^{\mathbb{Q}}_1(t))+\upsilon_1\sqrt{\lambda^{\mathbb{Q}}_1(t)}\mathrm{d}W_1(t)\right)$$
$$-2\rho_{1-2}\upsilon_2\upsilon_1\sqrt{\lambda^{\mathbb{Q}}_1(t)\lambda^{\mathbb{Q}}_2(t)}\mathrm{d}t$$

as:

$$\lambda^{\mathbb{Q}}_{1-2}(t) = 2\lambda^{\mathbb{Q}}_2(t) - \lambda^{\mathbb{Q}}_1(t)$$

I can now deduct the following theorem:

Theorem 2.3. *If equivalent hazard rate over the maturities iY, and (i+1)Y follow CIR PDE the equivalent hazard rate $\lambda^{\mathbb{Q}}_i(t)$ and $\lambda^{\mathbb{Q}}_{i+1}(t)$ for the the period [iY;(i+1)Y] follow the following PDE:*

$$\mathrm{d}\lambda^{\mathbb{Q}}_{i,(i+1)}(t) = (i+1)\mathrm{d}\lambda^{\mathbb{Q}}_{i+1}(t) - i\mathrm{d}\lambda^{\mathbb{Q}}_i(t)$$

Theorem 2.4. *If equivalent hazard rate over the maturities iY, and (i+1)Y follow CIR PDE the equivalent hazard rate $\lambda^{\mathbb{Q}}_i(t)$ and $\lambda^{\mathbb{Q}}_{i+1}(t)$ for the the period [iY;(i+1)Y] follows the following PDE:*

$$\mathrm{d}\lambda^{\mathbb{Q}}_{i,(i+1)}(t) = (i+1)\left(k(\mu_{i+1}-\lambda^{\mathbb{Q}}_{i+1}(t))\mathrm{d}t+\upsilon_{i+1}\sqrt{\lambda^{\mathbb{Q}}_{i+1}(t)}\left(\rho_{i,(i+1)}\mathrm{d}W_i(t)+\sqrt{1-\rho^2_{i,(i+1)}}\mathrm{d}W_i^{'}(t)\right)\right)$$

$$-i\bigg(k(\mu_i - \lambda_i^{\mathbb{Q}}(t)) + \upsilon_i\sqrt{\lambda_i^{\mathbb{Q}}(t)}\mathrm{d}W_i(t)\bigg)$$

Theorem 2.5. *If equivalent hazard rates $\lambda_i^{\mathbb{Q}}(t)$ and $\lambda_{i+1}^{\mathbb{Q}}(t)$ over the maturities iY, and (i+1)Y follow CIR PDE the equivalent hazard rate for the the period [iY;(i+1)Y] follows the following PDE:*

$$\mathrm{d}\lambda^{\mathbb{Q}}_{i,(i+1)}(t) = A(i,t)\mathrm{d}t + B(i,t)\mathrm{d}W(t) + C(i,t)\mathrm{d}W'(t)$$

with W and W' two independent Brownian motions under the risk-neutral probability.
with $A(i,t) = (i+1)k(\mu_{i+1} - \lambda_{i+1}^{\mathbb{Q}}(t)) - ik(\mu_i - \lambda_i^{\mathbb{Q}}(t))$
$B(i,t) = (i+1)(\upsilon_{i+1}\rho_{i,(i+1)}\sqrt{\lambda_{i+1}^{\mathbb{Q}}(t)}) - i(\upsilon_i\sqrt{\lambda_i^{\mathbb{Q}}(t)})$
$C(i,t) = (i+1)\upsilon_{i+1}\sqrt{(1-\rho^2_{i,(i+1)})\lambda^{\mathbb{Q}}_{i+1}}$

2.6 Exponential Vasicek model, and its impact on $\lambda^{\mathbb{Q}}_{i,(i+1)}$(t)

Let's assume the yearly equivalent hazard rates follow Exponential Vasicek models, then the PDEs which describe their behaviour are:

$$\lambda_i^{\mathbb{Q}}(t) = e^{y_i(t)}$$

$$\lambda_{i+1}^{\mathbb{Q}}(t) = e^{y_{i+1}(t)}$$

with initial conditions: $y_i(0) = \log(\lambda_i^{\mathbb{Q}}(0))$; $y_{i+1}(0) = \log(\lambda_{i+1}^{\mathbb{Q}}(0))$, $\lambda_{i+1}^{\mathbb{Q}}(0)$ and $\lambda_i^{\mathbb{Q}}(0)$ are implied by the spot yield of the bonds.
Moreover:

$$\mathrm{d}y_i(t) = k_i(\mu_i - y_i(t))\mathrm{d}t + \upsilon_i\mathrm{d}W_i(t)$$

$$\mathrm{d}y_{i+1}(t) = k_{i+1}(\mu_{i+1} - y_{i+1}(t))\mathrm{d}t + \upsilon_{i+1}\mathrm{d}W_{i+1}(t)$$

$$\mathrm{d}y_{i+1}(t) = k_{i+1}(\mu_{i+1} - y_{i+1}(t))\mathrm{d}t + \upsilon_{i+1}\bigg(\rho_{i,(i+1)}\mathrm{d}W_i(t) + \sqrt{1-\rho^2_{i,(i+1)}}\mathrm{d}W'_{i+1}(t)\bigg)$$

where W'_{i+1} and W_i are independent standard Brownian motions.

If equivalent hazard rate $\lambda^{\mathbb{Q}}_{i,(i+1)}$ over the maturities iY, and (i+1)Y follows Exponential Vasicek PDE the equivalent hazard rate for the period [iY;(i+1)Y] follows the following PDE:

$$\lambda^{\mathbb{Q}}_{i,(i+1)}(t) = (i+1)\lambda^{\mathbb{Q}}_{i+1}(t) - i\lambda^{\mathbb{Q}}_i(t) = (i+1)e^{y_{i+1}(t)} - ie^{y_i(t)}$$

$$\mathrm{d}\lambda^{\mathbb{Q}}_{i,(i+1)} = \frac{\partial\lambda^{\mathbb{Q}}_{i,(i+1)}}{\partial y_i}\mathrm{d}y_i + \frac{\partial\lambda^{\mathbb{Q}}_{i,(i+1)}}{\partial y_{i+1}}\mathrm{d}y_{i+1} + \frac{\partial^2\lambda^{\mathbb{Q}}_{i,(i+1)}}{\partial y_i\partial y_{i+1}}\mathrm{d} < y_i, y_{i+1} > + \frac{1}{2}\frac{\partial^2\lambda^{\mathbb{Q}}_{i,(i+1)}}{\partial^2 y_i}\mathrm{d} < y_i, y_i >$$
$$+\frac{1}{2}\frac{\partial^2\lambda^{\mathbb{Q}}_{i,(i+1)}}{\partial^2 y_{i+1}}\mathrm{d} < y_{i+1}, y_{i+1} >$$

$$\mathrm{d}\lambda^{\mathbb{Q}}_{i,(i+1)} = -i\lambda_i^{\mathbb{Q}}\mathrm{d}y_i + (i+1)\lambda^{\mathbb{Q}}_{i+1}\mathrm{d}y_{i+1} + \bigg(-\frac{i}{2}\lambda_i^{\mathbb{Q}}\upsilon_i^2 + \frac{i+1}{2}\lambda^{\mathbb{Q}}_{i+1}\upsilon^2_{i+1}\bigg)\mathrm{d}t$$

Concatenating all the developments, I can get the following theorem.

Theorem 2.6. *If $\lambda^{\mathbb{Q}}_{i+1}(t)$ and $\lambda^{\mathbb{Q}}_{i}(t)$ are modeled under the Exponential Vasicek model, then $\lambda^{\mathbb{Q}}_{i,(i+1)}(t)$ is defined by the following PDE:*

$$\mathrm{d}\lambda^{\mathbb{Q}}_{i,(i+1)} = -i\lambda^{\mathbb{Q}}_{i}\Big(k_i(\mu_i - y_i(t))\mathrm{d}t + \upsilon_i \mathrm{d}W_i(t)\Big)$$
$$+(i+1)\lambda^{\mathbb{Q}}_{i+1}\Big(k_{i+1}(\mu_{i+1} - y_{i+1}(t))\mathrm{d}t + \upsilon_{i+1}\Big(\rho_{i,(i+1)}\mathrm{d}W_i(t) + \sqrt{1-\rho^2_{i,(i+1)}}\mathrm{d}W'_{i+1}(t)\Big)\Big)$$
$$+\Big(-\frac{i}{2}\lambda^{\mathbb{Q}}_{i}\upsilon_i^2 + \frac{i+1}{2}\lambda^{\mathbb{Q}}_{i+1}\upsilon^2_{i+1}\Big)\mathrm{d}t$$

$$\mathrm{d}\lambda^{\mathbb{Q}}_{i,(i+1)} = -i\lambda^{\mathbb{Q}}_{i}\upsilon_i\mathrm{d}W_i(t) + (i+1)\lambda^{\mathbb{Q}}_{i+1}\upsilon_{i+1}\Big(\rho_{i,(i+1)}\mathrm{d}W_i(t) + \sqrt{1-\rho^2_{i,(i+1)}}\mathrm{d}W'_{i+1}(t)\Big)$$
$$+\Big(-\frac{i}{2}\lambda^{\mathbb{Q}}_{i}\upsilon_i^2 + \frac{i+1}{2}\lambda^{\mathbb{Q}}_{i+1}\upsilon^2_{i+1} + (i+1)\lambda^{\mathbb{Q}}_{i+1}(t)k_{i+1}(\mu_{i+1} - y_{i+1}(t)) - i\lambda^{\mathbb{Q}}_{i}(t)k_i(\mu_i - y_i(t))\Big)\mathrm{d}t$$

2.7 $\lambda^{\mathbb{Q}}_{i,(i+1)}(t)$'s volatility

With the hypothesis of stochastic $\lambda^{\mathbb{Q}}_{n}$(t), the hazard rate $\lambda^{\mathbb{Q}}_{i,(i+1)}$(t) is stochastic. But the parameters are different between two different intervals [i;i+1] and [j;j+1]. By this method I get a $\lambda^{\mathbb{Q}}_{i,(i+1)}$ as an equivalent two parameters model, as the difference between two Exponential Vasicek processes. In fact you could transform the PDE in a more readable one:

$$\mathrm{d}\lambda^{\mathbb{Q}}_{i,(i+1)}(t) = \mathrm{d}(x(t) + y(t))$$

With x(t) and y(t) are stochastic with the same kind of PDE than $\lambda^{\mathbb{Q}}_{n}$, as x(t)=$(i+1)\lambda^{\mathbb{Q}}_{i+1}(t)$ and y(t)=-i$\lambda^{\mathbb{Q}}_{i}(t)$. First let's try for the Exponential Vasicek model:

$$\mathrm{d}\lambda^{\mathbb{Q}}_{i,(i+1)} = -i\lambda^{\mathbb{Q}}_{i}\upsilon_i\mathrm{d}W_i(t) + (i+1)\lambda^{\mathbb{Q}}_{i+1}\upsilon_{i+1}\Big(\rho_{i,(i+1)}\mathrm{d}W_i(t) + \sqrt{1-\rho^2_{i,(i+1)}}\mathrm{d}W'_{i+1}(t)\Big)$$
$$+\Big(-\frac{i}{2}\lambda^{\mathbb{Q}}_{i}\upsilon_i^2 + \frac{i+1}{2}\lambda^{\mathbb{Q}}_{i+1}\upsilon^2_{i+1} + (i+1)\lambda^{\mathbb{Q}}_{i+1}(t)k_{i+1}(\mu_{i+1} - y_{i+1}(t)) - i\lambda^{\mathbb{Q}}_{i}(t)k_i(\mu_i - y_i(t))\Big)\mathrm{d}t$$
$$\mathrm{d}\lambda^{\mathbb{Q}}_{i,(i+1)} = \Big(-i\lambda^{\mathbb{Q}}_{i}\upsilon_i + (i+1)\lambda^{\mathbb{Q}}_{i+1}\upsilon_{i+1}\rho_{i,(i+1)}\Big)\mathrm{d}W_i(t) + (i+1)\lambda^{\mathbb{Q}}_{i+1}\upsilon_{i+1}\sqrt{1-\rho^2_{i,(i+1)}}\mathrm{d}W'_{i+1}(t)$$
$$+\Big(-\frac{i}{2}\lambda^{\mathbb{Q}}_{i}\upsilon_i^2 + \frac{i+1}{2}\lambda^{\mathbb{Q}}_{i+1}\upsilon^2_{i+1} + (i+1)\lambda^{\mathbb{Q}}_{i+1}(t)k_{i+1}(\mu_{i+1} - y_{i+1}(t)) - i\lambda^{\mathbb{Q}}_{i}(t)k_i(\mu_i - y_i(t))\Big)\mathrm{d}t$$

- The same can be done for the CIR model.

With such behaviour I can adapt the volatility, and the mean of the process $\lambda^{\mathbb{Q}}_{i,i+1}(t)$ as you can see on the section 3.7.

2.8 Default Probability under $\lambda^{\mathbb{Q}}_{i,i+1}(t)$ representation

I have defined the hazard rate for a 1 year period, but you could also have longer or shorter intervals. I can get this hazard rate $\lambda^{\mathbb{Q}}_{i,i+1}(t)$ for 1Y period, and then the market will price at the time t=0 the probability of default as constant over this year time. The default probability could now be found with this theorem.

Theorem 2.7. *For any fixed time of observation t smaller than the maximal time of the study, I calculate $\lambda^{\mathbb{Q}}_{i,(i+1)}(t)$ with the $\lambda^{\mathbb{Q}}_{i}(t)$ and $\lambda^{\mathbb{Q}}_{i+1}(t)$ and will be the hazard rate for the period [i;i+1].* $Q(\tau < T) = 1 - e^{\int_0^T -\lambda^{\mathbb{Q}}(u)}\mathrm{d}u$ *As $\lambda^{\mathbb{Q}}$ is constant for yearly period the probability $\mathcal{Q}$ is now:*

$$Q[\tau < T|\tau > 0](t) = e^{-\sum_{i=1}^{j} \lambda^{\mathbb{Q}}_{i-1,i}(t) - \alpha\lambda^{\mathbb{Q}}_{j,j+1}(t)} = e^{-T\lambda^{\mathbb{Q}}_{T}(t)}$$

where $j < T < j+1$, and $\alpha = T - j$

In the following parts, I will explain that I can consider the short-term hazard rate as stochastic with the same increment as $\lambda^{\mathbb{Q}}_{i,(i+1)}(t)$ for t belonging to [iY;(i+1)Y]. In this particular case the probability of default is now:

$$Q[\tau < T|\tau > 0] = e^{-\sum_{i=1}^{j} \int_{i-1}^{i} \lambda^{\mathbb{Q}}_{i-1,i}(t)\mathrm{d}t - \int_{j}^{j+\alpha} \lambda^{\mathbb{Q}}_{j,j+1}(t)\mathrm{d}t}$$

If the short-term hazard rate $\lambda^{\mathbb{Q}}(t)$ is generated as above with the same PDE as $\lambda_{i,(i+1)}(\mathrm{t})$, the short-term hazard rate will be mean reverted, but its volatility will change depending on the time. Such a model could fit more appropriately the CDS spread market, with no need for deterministic shifting function. An explanation could be that expectation of economical conditions drives the long-term hazard rate. Moreover the equivalent hazard rate is much less volatile for larger maturity, as attenuation for the extreme events will dilute large hazard rates over the period (the number of default is maximum 1).

2.9 Adding a jumps process J(t)

As it could happen with the rates, I will add a jump process to the $\lambda^{\mathbb{Q}}_{n}(t)$ the amplitude will depend on n the maturity, but on the contrary to the work of G. Chacko and S. Das [15] the jump can be up or down with the same variance no matter the direction of the hike.

2.10 Short-term hazard rates schemes

In this section, I will explain the different way to describe the short-term hazard rate $\lambda^{\mathbb{Q}}(t)$, meaning the daily hazard rate for the day at time t, seen with the information at time 0.

As seen in the previous sections the hazard rate will profoundly influence the pricing of bonds. There are different ways to express the hazard rates:
$\Lambda^{\mathbb{Q}}(\mathrm{t})$ the estimation of $\frac{1}{t}\int_0^t \lambda^{\mathbb{Q}}(u)\mathrm{d}u$ so the average of yearly hazard rate on the period, for example for one year maturity: $Q[\tau < 1Y|\tau > 0](t) = 1 - e^{-\int_0^1 \lambda^{\mathbb{Q}}(u)}\mathrm{d}u = 1 - e^{-\lambda_1^{\mathbb{Q}}} = 1 - e^{-\lambda_{0,1}^{\mathbb{Q}}}$. So for every model the the short-term hazard rate for t belonging to [0;1Y] will be defined as the hazard rate $\lambda_1^{\mathbb{Q}}(t)$=$\lambda_{0,1Y}^{\mathbb{Q}}(t)$. But then there could be many different model to defined it. In the followings paragraphs, I will explain the different schemes I use to define the hazard rate $\lambda^{\mathbb{Q}}(t)$.

2.10.1 First scheme for he short-term hazard rate $\lambda^{\mathbb{Q}}(t)$

Once I have the equivalent hazard rate $\lambda_n^{\mathbb{Q}}(\mathrm{t})$ I could then get the hazard rate at each point by linear extension between the points, creating a kind of surface. For example, $\lambda_j^{\mathbb{Q}}(t)$ is purely stochastic, as well as $\lambda_{j-1}^{\mathbb{Q}}(t)$, then the hazard rate between those point for t belongs to $[t_j; t_{j+1}]$ are obtained like:

$$\lambda^{\mathbb{Q}}(t) = \lambda_{j,(j+1)}^{\mathbb{Q}}(t) = \frac{1}{t_{j+1} - t_j}\left(t_{j+1}\lambda_{j+1}^{\mathbb{Q}}(t) - t_j\lambda_j^{\mathbb{Q}}(t)\right)$$

$$\mathbb{Q}[iY < \tau < jY|\mathcal{F}_\rangle] = 1 - e^{-\int_i^j \lambda^{\mathbb{Q}}(u)\mathrm{d}u}$$

$$\mathbb{Q}[iY < \tau < jY|\mathcal{F}_\rangle] = 1 - e^{-\sum_{k=i}^{j-1}\int_k^{k+1} \lambda_{k,k+1}^{\mathbb{Q}}(u)\mathrm{d}u}$$

The interval could also be such that i and j are non-integer, let's define a as the integer just greater than i, and b the integer just greater than j, such that $|i - a| < 1$ and $|j - b| < 1$, the probability of default under the risk-neutral probability is then:

$$\mathbb{Q}[iY < \tau < jY|\mathcal{F}_\rangle] = 1 - e^{-\left[\sum_{k=a}^{b-1}\int_k^{k+1} \lambda_{k,k+1}^{\mathbb{Q}}(u)\mathrm{d}u + \int_i^a \lambda_{a-1,a}^{\mathbb{Q}}(u)\mathrm{d}u + \int_b^j \lambda_{b,b+1}^{\mathbb{Q}}(u)\mathrm{d}u\right]}$$

$$\mathbb{Q}[\tau < jY|\mathcal{F}_\imath] = 1 - e^{-i\lambda_i(0) - \int_i^j \lambda(u)\mathrm{d}u}$$

For each step (i.e, 100 per year), the volatility of $\lambda^{\mathbb{Q}}(t)$ is different as $\lambda_j^{\mathbb{Q}}(t)$ and $\lambda_{j+1}(t)$ have different parameters.

2.10.2 Second scheme for the short-term hazard rate $\lambda^{\mathbb{Q}}(t)$

For the details of the proof please refer to the appendix C. In this paper I consider 100 points per year, so there will be 100 different values of $\lambda^{\mathbb{Q}}(t)$ per year.
The main idea for this scheme is to express the probability $\mathcal{Q}(\tau < t_{i+\frac{1}{100}})$:

$$\mathcal{Q}\left[\tau < t_{i+\frac{1}{100}}|\tau > 0\right] = 1 - e^{-t_i\lambda_i^{\mathbb{Q}}(t) - \int_{t_i}^{t_{i+\frac{1}{100}}} \lambda^{\mathbb{Q}}(u)\mathrm{d}u} = 1 - e^{-t_{i+\frac{1}{100}}\lambda_{i+\frac{1}{100}}^{\mathbb{Q}}}$$

with:

$$\lambda^{\mathbb{Q}}_{i+\frac{k}{100}} = \frac{100-k}{100}\lambda^{\mathbb{Q}}_{i}(t) + \frac{k}{100}\lambda^{\mathbb{Q}}_{i+1}(t)$$

Theorem 2.8. *Under this model, the probability of default under the risk-neutral probability $\mathcal{Q}$ is:*

$$\lambda^{\mathbb{Q}}\left(i+\frac{j}{100}\right) = \left(ij+\frac{j^2}{100}\right)\lambda^{\mathbb{Q}}_{i+1}\left(i+\frac{j}{100}\right)+\left(\left(i+\frac{j}{100}\right)(100-j)-100i\right)\lambda^{\mathbb{Q}}_{i}\left(i+\frac{j}{100}\right)-s\left(i+\frac{j}{100}\right)$$

wih:

$$s\left(i+\frac{j}{100}\right) = 100\left(i+\frac{j-i}{100}\right)\left(\frac{j-1}{100}\lambda^{\mathbb{Q}}_{i+1}\left(i+\frac{j}{100}\right)\right)+\frac{100-j+1}{100}\lambda^{\mathbb{Q}}_{i}\left(i+\frac{j}{100}\right)-100i\lambda^{\mathbb{Q}}_{i}\left(i+\frac{j-1}{100}\right)$$

The proof of this theorem is in appendix C.

Remark 2.9. The main problem of this scheme is the time of computation as it takes much time to get s at each point, as this method need to calculate all the past points again.

2.10.3 Third and last scheme for the short-term hazard rate $\lambda^{\mathbb{Q}}(t)$

This scheme is highly inspired by the first one, but here the discretisation will go further(proves are given in appendix C).

$$\mathcal{Q}\left[\tau < i+\frac{j+1}{100}\right](0) = 1-e^{-(i+\frac{j+1}{100})\lambda^{\mathbb{Q}}_{i+\frac{j+1}{100}}(i+\frac{j+1}{100})}$$

$$\mathcal{Q}\left[\tau < i+\frac{j+1}{100}\right](0) = 1-e^{-(i+\frac{j}{100})\lambda^{\mathbb{Q}}_{i+\frac{j}{100}}(i+\frac{j+1}{100})-\frac{1}{100}\lambda^{\mathbb{Q}}_{(i+\frac{j}{100}),(i+\frac{j+1}{100})}(i+\frac{j+1}{100})}$$

Theorem 2.10. *for t belonging to $[i+\frac{j}{100}; i+\frac{j+1}{100}]$, then $\lambda^{\mathbb{Q}}(t) = \lambda^{\mathbb{Q}}_{(i+\frac{j}{100}),(i+\frac{j+1}{100})}(t)$*
Then I obtain:

$$\lambda^{\mathbb{Q}}(t) = \left(\frac{i}{100}+\frac{2j+1}{100^2}\right)\lambda^{\mathbb{Q}}_{i+1}(t) - \left(\frac{i}{100}+\frac{2j+1-100}{100^2}\right)\lambda^{\mathbb{Q}}_{i}(t)$$

with $t = i+\frac{j}{100}$ and $0 < j < 100$.

Remark 2.11. The theorem 7.3 gives the hazard rate if I have the yearly equivalent hazard rates $\lambda^{\mathbb{Q}}_n$ and $\lambda^{\mathbb{Q}}_{n+1}$, but in case the data is unavailable and the only available yearly equivalent hazard rate are separate by multiple year.

Theorem 2.12. *for t belonging to $[i+\frac{j}{100}; i+\frac{j+1}{100}]$, and to [i;i+k] then $\lambda^{\mathbb{Q}}(t) = \lambda^{\mathbb{Q}}_{(i+\frac{j}{100}),(i+\frac{j+1}{100})}(t)$*
Then I obtain:

$$\lambda^{\mathbb{Q}}(t) = \left(\frac{i}{100k}+\frac{2j+1}{100^2k}\right)\lambda^{\mathbb{Q}}_{i+k}(t) - \left(\frac{i}{100k}+\frac{2j+1-100}{100^2k}\right)\lambda^{\mathbb{Q}}_{i}(t)$$

with $t = i+\frac{j}{100}$ and $0 < j < 100k$.

Remark 2.13. As the time to compute the second scheme could be large, I will not study it.

2.11 Creation of the surface of $\lambda_n^{\mathbb{Q}}(t)$

I have explained how and why I consider $\lambda_n^{\mathbb{Q}}(t)$ as a stochastic process, but if for n not market data is available, the $\lambda_n^{\mathbb{Q}}(t)$ will be obtained with a surface of $\lambda_i^{\mathbb{Q}}$(t) for each i, integer or not, and for each t as I can simulate as long as I need the stochastic processes, with numerical methods.
The idea is to get a surface of equivalent hazard rate $\lambda_n^{\mathbb{Q}}$ over the period till the maturity, implied by the market value of the bonds' yield.
Here is the past hazard rate surface of $\lambda_n^{\mathbb{Q}}$ that we have for Italian bonds:

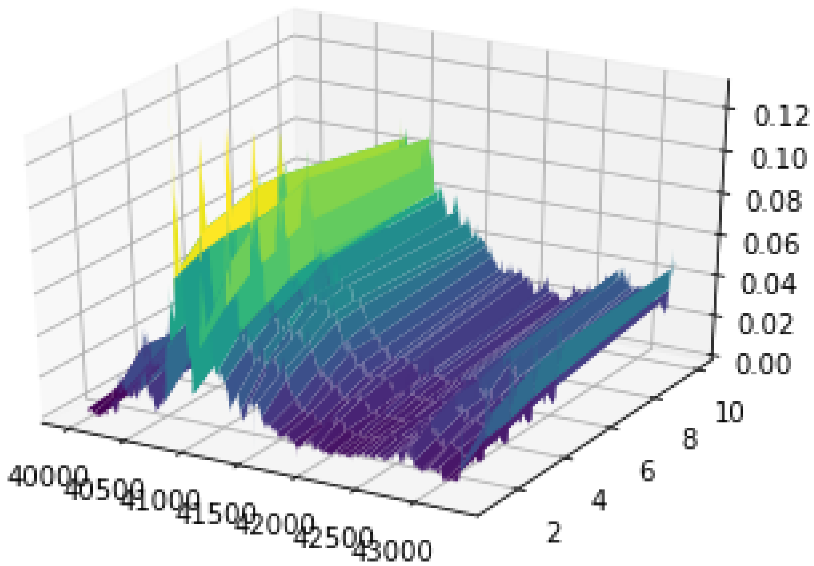


Figure 3: Historical implied λ_{nY} surface for Italy

2.12 CDS pricing

2.12.1 Portfolio Replication

To get a free risk bond, we already have the formula for such bond In a non-arbitrage market, so I could make a portfolio that could replicate the same payoff. The risk-free bond can is replicated by buying a risky bond and a protection on that particular bod, by paying the protection leg of credit default swap (then called CDS, definied) with the same maturity as the bond.
From that assumption, I should be able to price the CDS knowing $\lambda^{\mathbb{Q}}(t)$ in the previous subsection, and so I will be able to find the spread of the CDS. I could then compare the results with the price on the market.

2.12.2 CDS pricing under rsik neutral probability

Many papers have treated about CDS valuation; I would cite [19] and [30].

Definition 2.14. A CDS is a contract in which two counterparties agree to exchange protection on credit event of an obligor against fixed cashflows. One counterparty is paying a fixed amount at fixed dates while there is no credit event from the obligor (the entity on which the contract is) and the protection leg pays (1-$\mathcal{R}$)*Nominal when the default happens. $\mathcal{R}$ is the recovery rate. Of course the payment of (1-$\mathcal{R}$)*Nominal occurs if the default time is before the end of the CDS contract. The one buying protection will get what it should have lost because of the default while holding a bond with the same maturity, on this particular issuer.

The following formulas come from [19], [30], [21]. To replicate a risk-free bond, you can buy a CDS and a risky bond with the same maturity. For more tractability and as I have the data for the old style CDS, I will look at the CDS spread without any upfront.
$B^o(T)$ will be the zero coupon bond, so the discount factor, so $\frac{1}{(1+y)^T}$ The expected payoff of the coupon leg under the $\mathcal{Q}$ probability, with $B^o(T)$ as numeraire and so the fair value of the legs seen now:
s is the CDS spread then,

$$\Pi^*(T) = \sum_j \left(\alpha(T_j - T_{j-1}) B^o(T_j)\mathcal{Q}(T_j) + \int_{t=T_{j-1}}^{T_j} \alpha(t - T_{j-1}) B^o(T_j)[-\mathrm{d}\mathcal{Q}(t)] \right)$$

The risky PV01 is sΠ^* The default leg pays $(1 - \mathcal{R})$*Nominal is there is a credit event the fair value under the probability $\mathcal{Q}$:

$$\Xi(T) = -(1 - \mathcal{R}^{\mathbb{Q}}) \int_{t=0}^{T} B^o(t)\mathrm{d}\mathcal{Q}(t)$$

In the model I am using in the following part:

$$\mathcal{Q}(T) = 1 - e^{-\int_0^T \lambda^{\mathbb{Q}}(u)\mathrm{d}u}$$

$$\mathrm{d}\mathcal{Q}(t) = \lambda(t) e^{-\int_0^t \lambda^{\mathbb{Q}}(u)\mathrm{d}u}$$

$$\Xi(T) = \lambda(t)(1 - \mathcal{R}^{\mathbb{Q}}) \int_0^T B^o(t) e^{-\int_0^t \lambda^{\mathbb{Q}}(u)\mathrm{d}u}\mathrm{d}t$$

$$\Pi^*(T) = \int_0^T B^o(t) e^{-\int_0^t \lambda^{\mathbb{Q}}(u)\mathrm{d}u}\mathrm{d}t$$

As in the way I calculated the $\lambda^{\mathbb{Q}}$ that will be calculated are constant for the maturity of the CDS. Of course two CDS with different maturities will have different $\lambda^{\mathbb{Q}}_n$.

so Ξ and Π^* could be written:

$$\Xi(T) = \lambda_T^{\mathbb{Q}}(1 - \mathcal{R}^{\mathbb{Q}}) \int_0^T B^o(t) e^{-t\lambda_T^{\mathbb{Q}}} \mathrm{d}t$$

$$\Pi^*(T) = \int_0^T B^o(t) e^{-t\lambda_T^{\mathbb{Q}}} \mathrm{d}t$$

The fair spread will be $s = \lambda_n^{\mathbb{Q}}(1 - \mathcal{R}^{\mathbb{Q}})$ with nY the maturity of the CDS.
In a non-arbitrage market, buying protection thanks to a CDS with CDS spread s and risky bonds with yield y, should be the same as buying/holding a risk-free bond. The resultant yield y-s is the coupon paid till the credit event. The risky free bond pays a risk-free rate. The price, so the expectation of the risk-free bond under the probability $\mathcal{Q}$ 1, as the numeraire is this particular bond.
Then the expectation of payoff of this replication portfolio under the probability $\mathcal{Q}$ is at time 0:

$$100 = P(0) + CDS(0)$$

CDS(0) is the value of the expectation under the risk-neutral probability of the protection leg minus the expectation the coupon leg.
If the coupon of the risky bond is equal to the bond coupon, the price of this portfolio is 1, as the bond price is 1, and the CDS price is 0 if the spread is the fair spread. No matter if the credit event happening before the maturity my discounted payoff will be 1, for both free risk bond and portfolio. The above portfolio is a replication portfolio; this strategy will not make any money, as I could have bought the risk-free bond, that will for sure not default before maturity.
There is one point that I will not look at; it is the restructuring event that could happen. For example, the Greek debt has been restructured after the sovereign debt crisis in the euro-zone, but the CDS protection will pay the due part during the restructuring. The hazard rate is not a default parameter, but a parameter that encloses a more significant problem, as it takes to account the default, and all the other credit event. By abusive language, I called it default parameter.
Many papers about hazard rates refer to the distinction to the different event, but the dissociation between the different parameters is a tough task.
$\lambda^{\mathbb{Q}}(t)$ is in fact the sum of different specific parameter of credit events. The probability of bankruptcy and restructuring are not the same for two different entities, as well as there part on the global $\lambda^{\mathbb{Q}}(t)$ could differ. bankruptcy and a restructuring will not have the same impact on the amount of debt paid back, so it might be great to make some difference to estimate $\mathcal{R}$ with precision. In my thesis, I don't make any difference, as $\lambda(t)$ and $\mathcal{R}^{\mathbb{Q}}$ will capture those details, as global parameters.

2.12.3 Using a hazard rate surface

As the hazard rate is not constant, I cannot use the same simplification as before:

$$\Xi(t) = \lambda^{\mathbb{Q}}(t)(1-\mathcal{R}^{\mathbb{Q}})\int_0^T B^o(t)e^{-\int_0^t \lambda^{\mathbb{Q}}(u)\mathrm{d}u}\mathrm{d}t$$

$$\Pi^*(t) = \int_0^T B^o(t)e^{-\int_0^t \lambda^{\mathbb{Q}}(u)\mathrm{d}u}\mathrm{d}t$$

Taking the expectation under the measure $\mathbb{Q}$ and its numeraire $B^o(T)$:

$$\mathbb{E}[\Xi(T)] = \mathbb{E}\left[\frac{1}{B^o(T)}\lambda^{\mathbb{Q}}(t)(1-\mathcal{R}^{\mathbb{Q}})\int_0^T B^o(t)e^{-\int_0^t \lambda^{\mathbb{Q}}(u)\mathrm{d}u}\mathrm{d}t\right]$$

$$\mathbb{E}[\Pi^*(T)] = \frac{1}{B^o(T)}\mathbb{E}\left[\int_0^T B^o(t)e^{-\int_0^t \lambda^{\mathbb{Q}}(u)\mathrm{d}u}\mathrm{d}t\right]$$

If the hazard rate is independent from the risk-free rate:

$$Fair\ Default\ Swap\ Spread = \frac{\mathbb{E}[\Xi(t)]}{\mathbb{E}[\Pi^*(t)]}$$

For any time t, the hazard rate $\lambda_n^{\mathbb{Q}}(t)$ follows this relationship:

$$\lambda_n^{\mathbb{Q}}(t) = \frac{CDS\ Spread\ for\ nY\ CDS}{1-\mathcal{R}^{\mathbb{Q}}}$$

2.13 Using a Monte Carlo method and the previous schemes

As a computation experimentation, I will use 1000 paths of $\lambda(t)$ and estimate the fair CDS spread:

$$n\lambda_n^{\mathbb{Q}}(0) = \int_0^T \lambda^{\mathbb{Q}}(t)dt = \sum_{i=0}^{n-1}\int_{\frac{i}{r}}^{\frac{i+1}{r}} \lambda^{\mathbb{Q}}(u)\mathrm{d}u$$

The hazard rate can takes 100 values per year in my hypothesis, I could set n=100T, then the integral is now the integral of a constant over this period.

$$\int_0^T \lambda^{\mathbb{Q}}(t)\mathrm{d}t = \frac{1}{100}\sum_{i=0}^{n-1}\lambda^{\mathbb{Q}}(\frac{i}{100})$$

I will then set $X = \frac{1}{100}\sum_{i=0}^{100T-1}\lambda^{\mathbb{Q}}(\frac{i}{100})$. Then using the strong law of large number:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}X_i = \mathbb{E}[X]$$

where the X_i are generating independently. For the rest of the thesis I will use 1000 simulations.

Proposition 2.15. *With X defined above, and under a exponential hazard rate model, the following property holds:*

$$E[X] = T\ \frac{Fair\ CDS\ Spread}{(1-\mathcal{R}^{\mathbb{Q}})}$$

$$\lim_{N\to\infty} \frac{1}{T}\frac{1}{N}\sum_{i=1} NX_i = \frac{Fair\ CDS\ Spread}{(1-\mathcal{R}^{\mathbb{Q}})}$$

3 Theory and Calibration with market data

3.1 Calibration of the volatility parameter under both models (CIR and Exponential Vasicek)

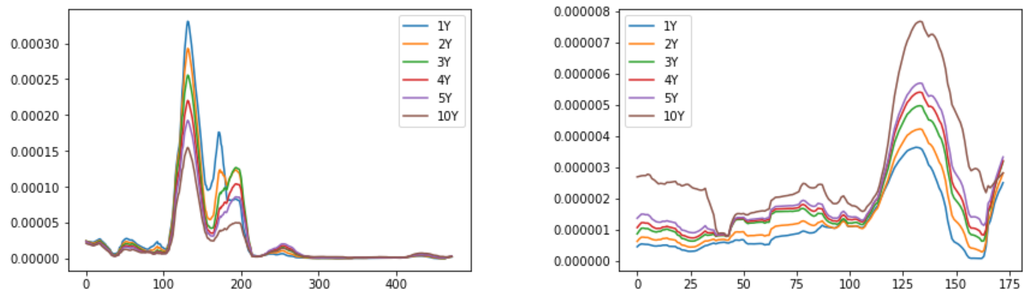
As I have already written it in the introduction 1 the estimation of the parameters is a sensitive question. As the mean reversion level and volatility parameters of the different $\lambda_i^{\mathbb{Q}}$ are not the mean over a past period. In the paper [2], the volatility of the CDS spread and so of the $\lambda_i^{\mathbb{Q}}(t)$ could be determined using a simplistic Black and Scholes model. I will slightly change the volatility parameters to fit the market at time 0 (now).
I will use the market data from the default Swaptions market, to get the implied volatility of the CDS spreads, using a Black and Scholes model.
The paper [26] made by Y. At-Sahalia, Y. Wang, F. Yared. are explaining why such default market implied volatility could be used as it reflect the view of the market on the CDS spread volatility, also, in my thesis, I specifically focus my work on fitting the market.

Remark 3.1. The Black and Scholes model is simple and it seems questionable as on this particular thesis, I use a CIR or Exponential Vasicek model for $\lambda_n^{\mathbb{Q}}$(t), knowing that $\lambda_n^{\mathbb{Q}}(t)(1-\mathcal{R}^{\mathbb{Q}}) = \hat{s}(t)$ with $\hat{s}(t)$ the fair CDS spread at time t for a nY maturities CDS.

Proposition 3.2. *The study of default swaption market, you can determine the volatility of hazard rate, as once you have the implied volatility of the cds spread, then the volatility of the hazard rate $\lambda_n^{\mathbb{Q}}(t)$ where the maturity of the cds you are looking at is nY, with the following formula:*
Var$[\lambda_n^{\mathbb{Q}}(t)] = \frac{1}{(1-\mathcal{R}^{\mathbb{Q}})^2} Var[CDS\ spread]$

3.2 Market Variance to estimate the volatility parameter

If there is one parameter that changes a lot with the time It is volatility, it is known from [**?**], that the volatility is clustered and highly persistent. I have observed such a behaviour for CDS spread as well, making it hard to evaluate the volatility parameters. As seen above the default swaption could provide a fair estimation, but does the past volatility could give a good value to generate the hazard rate $\lambda_n^{\mathbb{Q}}$ to create the surface I have written about?
Let's look at the past volatility for the Italian CDS spread:



((a)) Italian CDS Spread Volatility on a rolling period of 50 points

((b)) Italian CDS Spread Volatility on a rolling period of 50 points, closest data

Figure 4: Italian CDS Spread rolling Volatility

From those figures, it is clear that excluding large Volatility values, due to jumps on the CDS values, as the rolling windows are 50 points long, and the volatility is reminiscent for 50 points. The fair value for the CDS should be lower than the mean over the period if you add a jump process that will capture sizeable rolling volatility surge. Without any jump process, the volatility of the mean reversion process should be the average of the volatility, but that will lead to a poor representation of the hazard rate ($\lambda_n^{\mathbb{Q}}$) process.

Remark 3.3. The higher the maturity, the lower the volatility is during the surge of volatility, during the jumps. During the low volatility period, the volatility for the 10Y CDS spread seems higher, but the normalized volatility is in fact smaller than the other maturity (the normalized volatility is the value of the volatility divided by the mean value of the CDS over the period the volatility is estimated).

Remark 3.4. In the first sentence of the previous remark explains why the jump process should not be reverberated the same way on all the maturities.
As a jump occurs on unexpected events, which will affect the short-term solvency, whereas the long-term CDS spread are also affected, but with a smaller unexpected event effect.

3.3 Black and Scholes for default swaption

Definition 3.5. A default swaption is an option on CDS spread. This asset gives the right, but not the obligation to enter a CDS to seek protection or to protect the counterparty against a credit event of the obligor.

In Black and Scholes paper [1] the evolution of the underlying of an option is driven by an Ito drift-diffusion process and given by:

$$\mathrm{d}X(t) = \mu X(t)\mathrm{d}t + \upsilon X(t)\mathrm{d}Z(t)$$

where W(t) is a standard Brownian motion under the real world probability $\mathcal{P}$.
The easiest way to estimate υ is to set the drift at 0. Let's consider a call on cds spread:

$$\mathrm{d}X(t) = X(t)\upsilon \mathrm{d}W(t)$$

s is log-normal which will be great for the CIR model, but here there is no mean reversion.
As European swaption prices are often approximated by a Black-Scholes formula, which arises from approximating the stochastic forward swap rate volatility by a specific deterministic volatility function. The fair value of the option on CDS spread as in [31] is:

$$V^{Default\ Swaption}(t) = \nVdash_{\{\tau > t\}} V^a(t)[\hat{s}(t)\mathcal{N}(d1) - s^*\mathcal{N}(d2)]$$

where

$$V^a(t) = \mathbb{E}^{\mathbb{Q}}\Big[\sum_{n=1}^{N} \frac{1}{b(T_n)}\delta_n \nVdash_{\{\mathbb{T}_{\ltimes} \leq \tau\}} + \frac{\nVdash}{(\tau)}\delta^* \nVdash_{\{\mathbb{T}_{\nvdash} \leq \mathbb{N}\}} |\mathcal{F}_t\Big]$$

$$d_1 = \frac{\log\left(\frac{\hat{s}(t)}{s^*}\right)}{\upsilon_n^2} + \frac{\upsilon_n^2}{2}$$

$$d_2 = \frac{\log\left(\frac{\hat{s}(t)}{s^*}\right)}{\upsilon_n^2} - \frac{\upsilon_n^2}{2}$$

3.4 $\lambda_n^{\mathbb{Q}}$'s parameters

There is two schemes that I will use in this part, 2.10.1 and 2.10.3, to get the hazard rate for each time t, as there are 100 points per year. I use a Euler scheme as a numerical method to generate the $\lambda_n^{\mathbb{Q}}(t)$ for each n and each t (for the explanation about this scheme go to appendix C). For the first model the CIR, the hazard rate is stochastic such that:
with $2k\mu_n > \upsilon_n^2$

$$\mathrm{d}\lambda_n^{\mathbb{Q}}(t) = k(\mu_n - \lambda_n^{\mathbb{Q}}(t))\mathrm{d}t + \upsilon_n\sqrt{\lambda_n^{\mathbb{Q}}(t)}\mathrm{d}W_t$$

Here the Euler scheme will be:

$$\lambda_n^{\mathbb{Q}}(t+h) = \lambda_n^{\mathbb{Q}}(t) + k(\mu_n - \lambda_n^{\mathbb{Q}}(t))h + \upsilon_n\sqrt{h\lambda_n^{\mathbb{Q}}(t)}Z_{l+1}$$

with $\lambda_n^{\mathbb{Q}}(t+h)$ the $(l+1)^{th}$ $\lambda_n^{\mathbb{Q}}$
here h is the interval of time between t and t+h. Here the Miller scheme will be, from [34]:

$$\lambda_n^{\mathbb{Q}}(t+h) = k(\mu_n - \lambda_n^{\mathbb{Q}}(t))h + \upsilon_n\sqrt{h\lambda_n^{\mathbb{Q}}(t)}Z_{l+1} + \frac{1}{2}\upsilon_n^2\sqrt{\frac{h}{2\lambda_n^{\mathbb{Q}}(t)}}\sqrt{h\lambda_n^{\mathbb{Q}}(t)}[Z_{l+1}^2 - 1]$$

You can't apply this Miler scheme as $\lambda_n^{\mathbb{Q}}(t)$ can potentially go to 0.

Remark 3.6. As we get the data everyday for a long period, the bonds' yield on so the implied λ_n on a long historical period; Moreover We get the hazard rate $\Lambda^{\mathbb{Q}}$ (as defined above) for multiple maturities. The volatility of hazard rates can be done through the default swaptions.
Here is the past hazard rate surface that we have for Italian bonds:

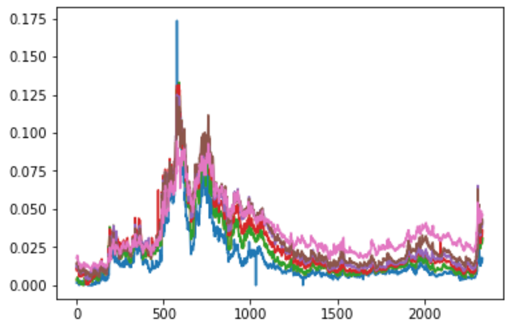


Figure 5: Italy past hazard rates $\lambda_i^{\mathbb{Q}}$ implied by the bond market

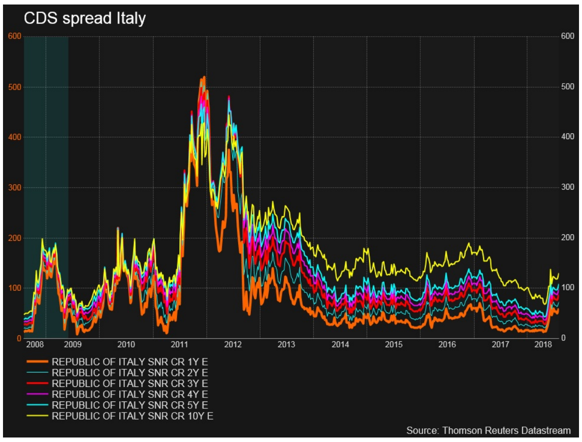


Figure 6: Italy past CDS spread

From this past values, anyone could determine the value of the parameters μ_n and the value of λ_n at time 0.

3.5 Term Structure

With the extracted data, I decided to caliber the correlation between the hazard rate for each maturity, by merely looking at the correlation between the historical hazard rates $\lambda_n^{\mathbb{Q}}$ obtained.

From the market past values of CDS spread (knowing the risk-free yield for the same points), you can also estimate the correlation between the CDS spread and the hazard rate $\lambda_n(t)$, assuming the recovery rate $\mathcal{R}$ constant.

Remark 3.7. If the correlation matrices are positive definite, I can factorise this matrix $\upsilon = LL^T$, with L triangular inferior with positive coefficients.
In many cases you can do so, but for some entities, especially for single names, the correlation matrix is not positive definite.

Remark 3.8. If the correlation matrix is not positive definite, you can't do such factorization, however, you can get an estimate of the matrix L, and continue the following algorithm.

The easiest way to compute the matrix L is given in the appendix. For the Italian bonds, the matrix L is:

1	0	0	0	0	0
0.975	0.2222	0	0	0	0
0.96	0.2520	0.1220	0	0	0
0.934	0.2972	0.1569	0.1211	0	0
0.9195	0.2965	0.1865	0.1312	0.1209	0
0.8608	0.3048	0.2363	0.2027	0.2041	0.1660

You can find the data needed to find it in the appendix G.1.

3.6 Simulation algorithm for $\lambda_n^{\mathbb{Q}}(t)$

The purpose of this section is to explain how to simulate the different $\lambda_n^{\mathbb{Q}}(t)$ where the parameters as the mean reversion level, mean reversion speed, and the volatility parameters would already have been chosen. The key of the algorithm is retaining the correlation between the $\lambda_n^{\mathbb{Q}}(t)$ while keeping all the other properties.
You need to keep in mind that as the $\lambda_n^{\mathbb{Q}}(t)$ have been generated, you have now famous surface of $\lambda_n^{\mathbb{Q}}(t)$, for any n and t, by linear interpolation.
To formalize the algorithm, you could use a matrix representation as the correlation is naturally under matrix representation.

3.6.1 Under CIR model

$$\Lambda(t) = \begin{bmatrix} \lambda_1^{\mathbb{Q}}(t) & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2^{\mathbb{Q}}(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3^{\mathbb{Q}}(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4^{\mathbb{Q}}(t) & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5^{\mathbb{Q}}(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_{10}^{\mathbb{Q}}(t) \end{bmatrix}$$

$$M_\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_{10})^T$$

L is the lower triangular matrix defined as above, as the matrix which multiplied by its transpose gives the correlation matrix.
By generating at each step of the scheme a vector a independent random normal variables, which will represent the random shock:

$$Z = (Z_1, Z_2, Z_3, Z_4, Z_5, Z_6)^T$$

Proposition 3.9. *Under a CIR model, and the simulation under an Euler scheme is:*

$$\Lambda_{t+h} = \Lambda_t + kh(M_\mu - \Lambda_t) + \sqrt{h}\Gamma\Lambda_t' LZ$$

For my example with the $\lambda_n^{\mathbb{Q}}(t)$ calibrated for the Italian example:

$$Z_1^{'}(t) = Z_1(t)$$
$$Z_2^{'}(t) = 0.975Z_1(t) + 0.2222Z_2(t)$$
$$Z_3^{'}(t) = 0.96Z_1(t) + 0.252Z_2(t) + 0.122Z_3(t)$$
$$Z_4^{'}(t) = 0.934Z_1(t) + 0.2972Z_2(t) + 0.1569Z_3(t) + 0.1211Z_4(t)$$
$$Z_5^{'}(t) = 0.9195Z_1(t) + 0.2965Z_2(t) + 0.18648Z_3(t) + 0.1312Z_4(t) + 0.12089Z_5(t)$$
$$Z_6^{'}(t) = 0.8608Z_1(t) + 0.3048Z_2(t) + 0.2363Z_3(t) + 0.2027Z_4(t) + 0.2041Z_5(t) + 0.166Z_6(t)$$

All those Z_i' are random variables with standard normal distribution, but the correlation between between each of them is given by the matrix L.
It follows that:

$$\lambda_i^{\mathbb{Q}}(t+1) = \lambda_i^{\mathbb{Q}}(t) + k(\mu_i - \lambda_i^{\mathbb{Q}}(t))\mathrm{d}h + \upsilon_i\sqrt{h\lambda_i^{\mathbb{Q}}(t)}Z_i^{'}(t)$$

Then by a simple transformation the hazard rate for the period [iY;(i+1)Y]:

$$\lambda_{i,i+1}^{\mathbb{Q}}(t+1) = (i+1)\lambda_{i+1}^{\mathbb{Q}}(t) - \sum_{j=0}^{i-1}\lambda_{j,j+1}^{\mathbb{Q}}(t)$$

Remark 3.10. The PDE of $\lambda^{\mathbb{Q}}_{i,(i+1)}(t)$ is not a straightforward sum difference between the $\lambda^{\mathbb{Q}}_{i+1}(t)$ and the PDE of $\lambda^{\mathbb{Q}}_{i}(t)$ as they are correlated, and the quadratic variation of $Z'_{i+1}(t)$ and $Z'_i(t)$ is not zero. I will prove it later in this thesis.

3.6.2 Under Exponential Vasicek model

Under the Exponential Vasicek model, the derivation of the PDE with a Euler scheme is a bit harder as you will need the correlation of the $y_i(t)$, where $\lambda^{\mathbb{Q}}_i(t) = e^{-y_i(t)}$ and that $y_i(t)$ is modeled by as Vasicek process:
Introducing the matrix Y and using L' as the lower triangular which multiplied by its transpose will give the correlation matrix between the $y_i(t)$, Γ is the volatility parameter matrix for the $y_i(t)$ the equation for the Euler scheme is now:

$$Y_{t+h} = Y_t + kh(M_\mu - Y_t) + \sqrt{h}\Gamma L' Z$$

The matrix L' is given in the appendix G.

3.7 The Variance of the hazard rate $\lambda^{\mathbb{Q}}(t)$ for this first model

With this model I can get any short-term hazard rate $\lambda^{\mathbb{Q}}(t)$ with t belonging to the interval [0;i_{max}], for this thesis i_{max}=10Y.
Then for any t $\in [i + \frac{j}{100}; i + \frac{j+1}{100}]$:

$$Var[\lambda^{\mathbb{Q}}_{i+1}(t)] = \lambda^{\mathbb{Q}}_{i+1}(0)\frac{\upsilon^2_{i+1}}{k_{i+1}}\left(e^{-k_{i+1}t} - e^{-k_{i+1}t}\right) + \frac{\mu_{i+1}}{2k_{i+1}}\upsilon^2_{i+1}\left(1 - e^{-k_{i+1}t}\right)^2$$

$$Var[\lambda^{\mathbb{Q}}_{i}(t)] = \lambda^{\mathbb{Q}}_{i}(0)\frac{\upsilon^2_{i}}{k_{i}}\left(e^{-k_{i}t} - e^{-k_{i}t}\right) + \frac{\mu_{i}}{2k_{i}}\upsilon^2_{i}\left(1 - e^{-k_{i}t}\right)^2$$

as in the thesis I take $k_i = \frac{1}{2}$ for every i.

$$Var[\lambda^{\mathbb{Q}}_{i,(i+1)}(t)] = (i+1)^2\left(\lambda^{\mathbb{Q}}_{i+1}(0)2\upsilon^2_{i+1}\left(e^{-\frac{t}{2}} - e^{-t}\right) + \mu_{i+1}\upsilon^2_{i+1}\left(1 - e^{-\frac{t}{2}}\right)^2\right)$$

$$-i^2\left(\lambda^{\mathbb{Q}}_{i}(0)2\upsilon^2_{i+1}\left(e^{-\frac{t}{2}} - e^{-t}\right) + \mu_i \upsilon i^2\left(1 - e^{-\frac{t}{2}}\right)^2\right)$$

$$-2\rho_{i,(i+1)}i(i+1)\sqrt{\left(e^{-\frac{t}{2}} - e^{-t}\right)2(\lambda^{\mathbb{Q}}_{i}(0)\upsilon^2_i + \lambda^{\mathbb{Q}}_{i+1}(0)\upsilon^2_{i+1}) + \left(1 - e^{-\frac{t}{2}}\right)^2(\mu_i\upsilon^2_i + \mu_{i+1}\upsilon^2_{i+1})}$$

Theorem 3.11. *For the $\lambda_i^{\mathbb{Q}}(t)$ modelled as CIR processes and under the following scheme:*

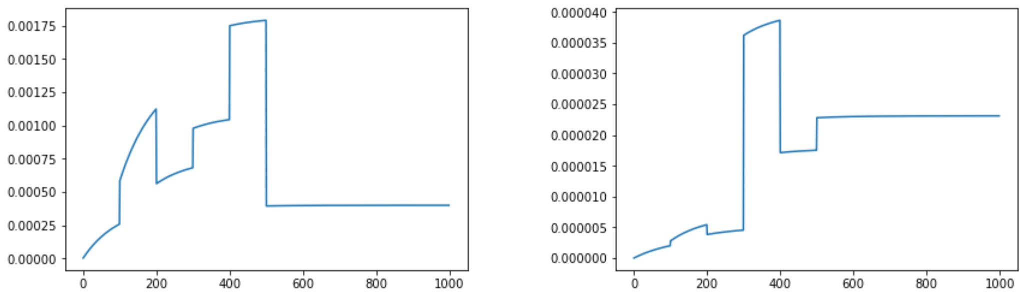
$$\lambda^{\mathbb{Q}}(t \in [i; i+1]) = (i+1)\lambda^{\mathbb{Q}}_{i+1}(t) - i\lambda^{\mathbb{Q}}_i(t)$$

$$Var[\lambda^{\mathbb{Q}}(t)] = (i+1)^2\left(\lambda^{\mathbb{Q}}_{i+1}(0)2\upsilon^2_{i+1}\left(e^{-\frac{t}{2}} - e^{-t}\right) + \mu_{i+1}\upsilon^2_{i+1}\left(1 - e^{-\frac{t}{2}}\right)^2\right)$$

$$-i^2\left(\lambda^{\mathbb{Q}}_i(0)2\upsilon^2_{i+1}\left(e^{-\frac{t}{2}} - e^{-t}\right) + \mu_i \upsilon i^2\left(1 - e^{-\frac{t}{2}}\right)^2\right)$$

$$-2\rho_{i,(i+1)}i(i+1)\sqrt{\left(e^{-\frac{t}{2}} - e^{-t}\right)2(\lambda^{\mathbb{Q}}_i(0)\upsilon_i^2 + \lambda^{\mathbb{Q}}_{i+1}(0)\upsilon^2_{i+1}) + \left(1 - e^{-\frac{t}{2}}\right)^2(\mu_i\upsilon_i^2 + \mu_{i+1}\upsilon^2_{i+1})}$$

Using the parameters given in the appendix G for the Italian government bonds, for the first scheme 2.10.1.



((a)) Variance for Italy

((b)) Variance for France

For the French hazard rate $\lambda^{\mathbb{Q}}_{i,i+1}(t)$ it makes sense to have larger volatility for the i=3 , and i+1=4 (between the point 300 and 400) as the next presidential elections are in less than 4Y. With that kind of model you could adapt the volatility depending on the potential event that will happen, at least those that are planned, but with an unknown outcome. With such model, it is clear you can choose how to adjust Variance of the diffusion process on defined intervals.

3.8 The Expectation of the hazard rate $\lambda^{\mathbb{Q}}(t)$ for the first scheme CIR

Theorem 3.12. *Under this scheme, and with CIR $\lambda_i^{\mathbb{Q}}(t)$ the expectation for $\lambda(t)$ can be express as:*

$$\mathbb{E}[\lambda^{\mathbb{Q}}(t \in [i; i+1])] = (i+1)\left(\lambda^{\mathbb{Q}}_{i+1}(0)e^{-k_{i+1}t} + \mu_{i+1}(1 - e^{-k_{i+1}t})\right) - i\left(\lambda^{\mathbb{Q}}_i(0)e^{-k_i t} + \mu_i(1 - e^{-k_i t})\right)$$

3.9 Under the Third scheme

3.9.1 The Variance of $\lambda^{\mathbb{Q}}(t)$

Under the first scheme, $\lambda^{\mathbb{Q}}(t \in [i+\frac{j}{100}; i+\frac{j+1}{100}] = \lambda^{\mathbb{Q}}_{(i+\frac{j}{100}),(i+\frac{j+1}{100})}(t)$

$$\lambda^{\mathbb{Q}}(t) = 100\left(i+\frac{j+1}{100}\right)\lambda^{\mathbb{Q}}_{i+\frac{j+1}{100}}(t) - 100\left(i+\frac{j}{100}\right)\lambda^{\mathbb{Q}}_{i+\frac{j}{100}}(t)$$

with $\lambda^{\mathbb{Q}}_{i+\frac{j+1}{100}}(t) = \left(\frac{j+1}{100}\right)\lambda^{\mathbb{Q}}_{i+1}(t) + \left(1-\frac{j+1}{100}\right)\lambda^{\mathbb{Q}}_i(t)$ and $\lambda^{\mathbb{Q}}_{i+\frac{j}{100}}(t) = \left(\frac{j}{100}\right)\lambda^{\mathbb{Q}}_{i+1}(t) + \left(1-\frac{j}{100}\right)\lambda^{\mathbb{Q}}_i(t)$
Which implies:

$$\lambda^{\mathbb{Q}}(t) = 100\left(i+\frac{j+1}{100}\right)\left(\frac{j+1}{100}\lambda^{\mathbb{Q}}_{i+1}(t) + \left(1-\frac{j+1}{100}\right)\lambda^{\mathbb{Q}}_i(t)\right)$$
$$-100\left(i+\frac{j}{100}\right)\left(\frac{j}{100}\lambda^{\mathbb{Q}}_{i+1}(t) + \left(1-\frac{j}{100}\right)\lambda^{\mathbb{Q}}_i(t)\right)$$

Let's define $A = \left(i+\frac{j}{100}\right)\left(\frac{j}{100}\lambda^{\mathbb{Q}}_{i+1}(t) + \left(1-\frac{j}{100}\right)\lambda^{\mathbb{Q}}_i(t)\right)$
and $B = \left(i+\frac{i+1}{100}\right)\left(\frac{j+1}{100}\lambda^{\mathbb{Q}}_{i+1}(t) + \left(1-\frac{j+1}{100}\right)\lambda_i(t)\right)$
Then $Var[B-A] = 10000 * Var[A] + 10000 * Var[B] - 2 * 10000 * \rho_{AB}\sqrt{Var[A]Var[B]}$

$$Var[A] = \left(i+\frac{i+1}{100}\right)^2\Bigg[\left(\frac{j+1}{100}\right)^2 Var[\lambda^{\mathbb{Q}}_{i+1}(t)] + \left(1-\frac{j+1}{100}\right)^2 Var[\lambda^{\mathbb{Q}}_i(t)]$$
$$+2\rho_{i;i+1}\left(\frac{j+1}{100}\right)\left(1-\frac{j+1}{100}\right)\sqrt{Var[\lambda^{\mathbb{Q}}_i(t)]Var[\lambda^{\mathbb{Q}}_{i+1}(t)]}\Bigg]$$

$$Var[B] = \left(i+\frac{i}{100}\right)^2\Bigg[\left(\frac{j}{100}\right)^2 Var[\lambda^{\mathbb{Q}}_{i+1}(t)] + \left(1-\frac{j}{100}\right)^2 Var[\lambda^{\mathbb{Q}}_i(t)]$$
$$+2\rho_{i;i+1}\left(\frac{j}{100}\right)\left(1-\frac{j}{100}\right)\sqrt{Var[\lambda^{\mathbb{Q}}_i(t)]Var[\lambda^{\mathbb{Q}}_{i+1}(t)]}\Bigg]$$

The other that need to be calculated is ρ_{AB}:

$$\rho_{AB} = \frac{Cov[A,B]}{\sqrt{Var[A]Var[B]}}$$

$$Cov[A,B] = \left(i+\frac{j+1}{100}\right)\left(i+\frac{j}{100}\right)Cov\left[\frac{j+1}{100}\lambda_{i+1}(t)+\left(1-\frac{j+1}{100}\right)\lambda^{\mathbb{Q}}_i(t)\ ;\ \left(\frac{j}{100}\right)\lambda^{\mathbb{Q}}_{i+1}(t)+\left(1-\frac{j}{100}\right)\lambda^{\mathbb{Q}}_i(t)\right]$$

For a more readable formula let's define $\alpha = \left(i+\frac{j+1}{100}\right)\left(i+\frac{j}{100}\right)$

$$Cov[A;B] = \alpha\Bigg[\left(\frac{j+1}{100}\right)\left(\frac{j}{100}\right)Cov[\lambda^{\mathbb{Q}}_{i+1}(t);\lambda^{\mathbb{Q}}_{i+1}(t)] + \left(1-\frac{j+1}{100}\right)\left(1-\frac{j}{100}\right)Cov[\lambda^{\mathbb{Q}}_i(t);\lambda^{\mathbb{Q}}_i(t)]$$

$$+\left(\left(\frac{j+1}{100}\right)\left(1-\frac{j}{100}\right)+\left(1-\frac{j+1}{100}\right)\frac{j}{100}\right)Cov[\lambda_i^{\mathbb{Q}}(t);\lambda_{i+1}^{\mathbb{Q}}(t)]\Bigg]$$

$$Cov[A;B]=\alpha\left[\frac{j^2+j}{100^2}Var[\lambda_{i+1}(t)]+\left(1-\frac{2j+1}{100}+\frac{j^2+j}{100^2}\right)Var[\lambda_i^{\mathbb{Q}}(t)]\right.$$
$$\left.+\left(\frac{2j+1}{100}-\frac{2j^2+2j}{100^2}\right)\sqrt{Var[\lambda_{i+1}^{\mathbb{Q}}(t)]Var[\lambda_i^{\mathbb{Q}}(t)]}\right]$$

As I have $Var[\lambda_i^{\mathbb{Q}}(t)]$ and $Var[\lambda_{i+1}^{\mathbb{Q}}(t)]$, I have the ρ_{AB} and so I could compute the Variance of the hazard rate $\lambda^{\mathbb{Q}}(t)$ for any t. This allow me to create the following theorem:

Theorem 3.13. *If the hazard rate $\lambda_i^{\mathbb{Q}}(t)$ are modelled as CIR process, then $\lambda^{\mathbb{Q}}(t)$ under the following scheme:*

$$\lambda^{\mathbb{Q}}\left(t\in\left[i+\frac{j}{100};i+\frac{j+1}{100}\right]\right)=100\left(\left(i+\frac{j+1}{100}\right)\lambda^{\mathbb{Q}}_{i+\frac{j+1}{100}}(t)-\left(i+\frac{j}{100}\right)\lambda^{\mathbb{Q}}_{i+\frac{j}{100}}\right)$$

$\lambda^{\mathbb{Q}}_{i+\frac{j+1}{100}}(t)$ and $\lambda^{\mathbb{Q}}_{i+\frac{j+1}{100}}(t)$ are obtained trough the equivalent hazard rate surface, using linear approximation between the points of the surface that have been calibrated.
has for Variance:

$$10000*Var[A]+10000*Var[B]-2*10000*Cov[A,B]$$

Var[A], Var[B] and Cov[A,B] are given above.

3.9.2 Expectation of $\lambda^{\mathbb{Q}}(t)$

Theorem 3.14. *Under this model and assuming that $\lambda_i^{\mathbb{Q}}(t)$ are CIR processes, then the expectation of $\lambda(t)$ for every $j\in[0;99]$, for every $i\in\ [1;i_{max}]$, for every t $\in$[i$\frac{j}{100}$;i+$\frac{j+1}{100}$]:*

$$\mathbb{E}[\lambda^{\mathbb{Q}}(t)]=\frac{1}{100}\mathbb{E}[\lambda_i^{\mathbb{Q}}(t)]+\left(\frac{i}{100}+\frac{2j+1}{100^2}\right)\left(\mathbb{E}[\lambda_{i+1}^{\mathbb{Q}}(t)]-\mathbb{E}[\lambda_i^{\mathbb{Q}}(t)]\right)$$

with:

$$\mathbb{E}[\lambda_i^{\mathbb{Q}}(t)]=\left(\lambda_i^{\mathbb{Q}}(0)e^{-k_it}+\mu_i(1-e^{-k_it})\right)$$
$$\mathbb{E}[\lambda_{i+1}^{\mathbb{Q}}(t)]=\left(\lambda_{i+1}^{\mathbb{Q}}(0)e^{-k_{i+1}t}+\mu_{i+1}(1-e^{-k_{i+1}t})\right)$$

3.10 Impact of adding a jump process

First of all, the generation of the jump process is described in E.
In this section the hazard rate for the bonds with maturity iY is defined as below:

$$\lambda_i^{\mathbb{Q}}(t)=y_i(t)+J(t)$$

where J(t) is a pure jump process. The sequence of hazard rate, over the time t; the PDE that defines this process is:

$$\mathrm{d}\lambda_n^{\mathbb{Q}}(t) = k(\mu - \lambda_n^{\mathbb{Q}}(t))\mathrm{d}t + \upsilon\sqrt{\lambda_n^{\mathbb{Q}}}\mathrm{d}W(t) + \mathrm{d}J(t)$$

I will again use a Euler scheme for this model:

$$\lambda_i^{\mathbb{Q}}(t+1) = \lambda_i^{\mathbb{Q}}(t) + k(\mu_i - \lambda_i^{\mathbb{Q}}(t))h + \upsilon_i\sqrt{h\lambda_i^{\mathbb{Q}}(t)}Z_{t+1} + Y\mathbb{1}_{[thereisajumpbetweentandt+1]}$$

By adding a jump process to the diffusion process will increase the volatility if the remaining parameters are unchanged; you will need to adapt the volatility of the diffusion process.
As the default swaption market gives the implied volatility, as seen above, and the jumps process and the diffusion process are independent and I will simulate them as independent.The Variance of the hazard rate $\lambda_n^{\mathbb{Q}}$ will be:

$$Var[\lambda_n^{\mathbb{Q}}] > Var[diffusion\ process]$$

As the Jump process is defined, as in [8].
The estimation of time till the next jump is t_{jump} the volatility of the jump process is then:

$$J(t) = \sum_{i=0}^{t} \mathrm{d}J(i)$$

with $\mathrm{d}J(i) = e^{\upsilon Z_i} - 1$ if there is a jump at $(i+1)^{th}$ step, meaning it has occurred between the i^{th} step's time and the $(i+1)^{th}$ step's time.
with α the number of jumps in the interval $[t_i; t_{i+1}]$
$Z_i \sim \mathcal{N}$ (0,1) are independent to each others.

3.10.1 Under CIR model

Adding a jump process will have consequences on the PDE of the process, from Pr. Zheng lecture notes in Simulation methods [8]:

$$\mathrm{d}\lambda_n^{\mathbb{Q}}(t) = \mathrm{d}\lambda_n^{\mathbb{Q}c}(t) + \mathrm{d}J(t)$$

where $\lambda_n^{\mathbb{Q}n}(t)$ is a diffusion process.
As seen in [6.1], the PDE, under a CIR model with no jump is:

$$\mathrm{d}\lambda_{i,(i+1)}^{\mathbb{Q}}(t) = (i+1)\Bigg(k(\mu_{i+1} - \lambda_{i+1}^{\mathbb{Q}}(t)) + \upsilon_{i+1}\sqrt{\lambda_{i+1}^{\mathbb{Q}}(t)}\Bigg(\rho_{i,(i+1)}\mathrm{d}W_i(t) + \sqrt{1-\rho_{i,(i+1)}^2}\mathrm{d}W_i^{'}(t)\Bigg)\Bigg)$$
$$-i\Bigg(k(\mu_i - \lambda_i^{\mathbb{Q}}(t)) + \upsilon_i\sqrt{\lambda_i^{\mathbb{Q}}(t)}\mathrm{d}W_i(t)\Bigg) - (i+1)i\rho_{i,(i+1)}\upsilon_{i+1}\upsilon_i\sqrt{\lambda_i^{\mathbb{Q}}(t)\lambda_{i+1}^{\mathbb{Q}}(t)}\mathrm{d}t$$

Recalling that $\lambda_{i,(i+1)}^{\mathbb{Q}}(t) = (i+1)\lambda_{i+1}^{\mathbb{Q}}(t) - i\lambda_i^{\mathbb{Q}}(t)$

When adding a jump process to CDS spread, those jumps occur on events that make the market reevaluates the default probability at a smaller or higher value, the hike is repeated on all the $\lambda_n^{\mathbb{Q}}$, but not with the same intensity (for all the nY). To keep the correlation, I decided to affect the jump process amplitude as 1 for $\lambda_1^{\mathbb{Q}}(t)$ then for the other $\lambda_i^{\mathbb{Q}}(t)$ the magnitude is $\rho_{1,i}$ determined by the correlation matrix.

Remark 3.15. I can justify those coefficient by the fact that hazard rates for sovereign European are quite low, and the jumps for larger maturities are smaller, as the high default of probability is often diluted by the mean reversion feature that tends to lead the hazard rate to the mean reversion level on the long-term.

Remark 3.16. For higher hazard rate on the short-term, I would use other coefficient as the hazard rate on the short-term is affected, but as the hazard rates are already high, there are chances that on the 10Y for example, the hazard rate moves much less than $\rho_{1-10}\delta\lambda_1^{\mathbb{Q}}$.

Remark 3.17. One could argue that different coefficients than the correlation coefficients could change the correlation between the processes, I would say that the correlation matrix also depends on the length of the data, and that the jumps are not occurring every day, then the spread between the estimate of the correlation made with data and the correlation of the $\lambda_i^{\mathbb{Q}}$ created by the described simulation will likely be in an acceptable range. In other words, if you could accept the idea of finding correlation matrix with past data, the error made by generating the processes by this method should also be acceptable.

Using those remarks and hypothesis I can write the following property:
If:

$$\mathrm{d}\lambda_i^{\mathbb{Q}}(t) = R_i(t)\mathrm{d}t + \Gamma_i(t)\mathrm{d}W_i(t) + \mathrm{d}J_i(t)$$

$$\mathrm{d}\lambda_{i+1}^{\mathbb{Q}}(t) = R_{i+1}(t)\mathrm{d}t + \Gamma_{i+1}(t)\mathrm{d}W_{i+1}(t) + \mathrm{d}J_{i+1}(t)$$

with $\mathrm{d}[W_i, W_{i+1}](t) = \rho_{i,(i+1)}\mathrm{d}t$.
In Pr Zheng lecture notes [8], the quadratic correlation of $\lambda_i^{\mathbb{Q}}(t)$ and $\lambda_{i+1}^{\mathbb{Q}}(t)$ could be express the following way:

$$\mathrm{d}[\lambda_i^{\mathbb{Q}}, \lambda_{i+1}^{\mathbb{Q}}](t) = \rho_{i,(i+1)}\Gamma_i(t)\Gamma_{i+1}(t)\mathrm{d}t + \Delta J_i(t)\Delta J_{i+1}(t)$$

Using the above remarks:

$$\mathrm{d}[\lambda_i^{\mathbb{Q}}, \lambda_{i+1}^{\mathbb{Q}}](t) = \rho_{i,(i+1)}\Gamma_i(t)\Gamma_{i+1}(t)\mathrm{d}t + \rho_{1,i}\rho_{1,(i+1)}\Delta J_i(t)\Delta J_i(t)$$

$$\mathrm{d}\lambda_{i,i+1}^{\mathbb{Q}}(t) = \frac{\partial\lambda_{i,i+1}^{\mathbb{Q}}(t)}{\partial\lambda_i(t)}\mathrm{d}\lambda_i^{\mathbb{Q}}(t) + \frac{\partial\lambda_{i,i+1}^{\mathbb{Q}}(t)}{\partial\lambda_{i+1}^{\mathbb{Q}}(t)}\mathrm{d}\lambda_{i+1}^{\mathbb{Q}}(t) + \big((i+1)\rho_{1,i+1} - i\rho_{1,i}\big)\mathrm{d}J(t)$$

$$\mathrm{d}\lambda^{\mathbb{Q}}_{i,(i+1)}(t) = (i+1)\Big(k(\mu_{i+1} - \lambda^{\mathbb{Q}}_{i+1}(t)) + \upsilon_{i+1}\sqrt{\lambda^{\mathbb{Q}}_{i+1}(t)}\Big(\rho_{i,(i+1)}\mathrm{d}W_i(t) + \sqrt{1-\rho^2_{i,(i+1)}}\mathrm{d}W_i^{'}(t)\Big)\Big)$$
$$-i\Big(k(\mu_i - \lambda^{\mathbb{Q}}_{i}(t)) + \upsilon_i\sqrt{\lambda^{\mathbb{Q}}_{i}(t)}\mathrm{d}W_i(t)\Big) + \big((i+1)\rho_{1,i+1} - i\rho_{1,i}\big)\mathrm{d}J(t)$$

3.10.2 Under Exponential Vasicek model

Under the Exponential Vasicek model the PDE that describes $\lambda_{i,i+1}$ is:

$$\mathrm{d}\lambda^{\mathbb{Q}}_{i,i+1}(t) = (i+1)\ln(\lambda^{\mathbb{Q}}_{i+1}(t))\lambda^{\mathbb{Q}}_{i+1}(t)\big(k(\mu_{i+1} - \ln(\lambda^{\mathbb{Q}}_{i+1}(t)))\mathrm{d}t + \upsilon_{i+1}\mathrm{d}W_{i+1}(t) + \rho_{1,i+1}\mathrm{d}J(t)\big)$$
$$-i\ln(\lambda^{\mathbb{Q}}_{i}(t))\lambda^{\mathbb{Q}}_{i}(t)\big(k(\mu_i - \ln(\lambda^{\mathbb{Q}}_{i}(t)))\mathrm{d}t + \upsilon_i\mathrm{d}W_i(t) + \rho_{1,i}\mathrm{d}J(t)\big)$$
$$+\frac{1}{2}(i+1)(\ln(\lambda^{\mathbb{Q}}_{i+1}(t)))^2\lambda^{\mathbb{Q}}_{i+1}(t)(\upsilon^2_{i+1}\mathrm{d}t + \rho^2_{1,i+1}(\mathrm{d}J(t))^2)$$
$$-\frac{1}{2}i(\ln\lambda^{\mathbb{Q}}_{i}(t))^2\lambda^{\mathbb{Q}}_{i}(t)(\upsilon^2_{i}\mathrm{d}t + \rho^2_{1,i}(\mathrm{d}J(t))^2)$$

3.10.3 The jump diffusion process and its parameters

Keeping the Exponential Vasicek model for $\lambda^{\mathbb{Q}}_n(t)$, but adding a jump process the PDE verified by $\ln\lambda^{\mathbb{Q}}_n(t)$ is:

$$\mathrm{d}ln(\lambda^{\mathbb{Q}}_n(t)) = k(\mu_n - ln(\lambda^{\mathbb{Q}}_n(t)))\mathrm{d}t + \upsilon_n ln(\lambda^{\mathbb{Q}}_n(t))\mathrm{d}Z_t + \mathrm{d}J_t$$

So $\lambda^{\mathbb{Q}}$ is stochastic with jumps. The different analysis of past prices of bonds will give υ and $\lambda^{\mathbb{Q}}$ at time 0. I will use a Euler scheme to get those parameters. The time interval is divided, such as there are 100 points per year. So first I will run a lambda with this formula:

$$y_{i+1} = y_i + k(\mu_i - y_i) + \upsilon_i\sqrt{h}Z_{i+1} + YdN$$

$\lambda^{\mathbb{Q}}_n(t) = e^{y_t}$ So at each point i I have the value of $\lambda^{\mathbb{Q}}_n(i)$. Then I get the mean and variance and I have υ_n and μ_n the mean.

Then as I have the yield of the bonds I will adapt υ_n. To do that I take a bond, I price a risk bond with different value for υl ($\upsilon 1$ is the parameter of the jump process if there is a jump $\mathrm{d}J_t = e^{\upsilon 1 Z'_k}$-1, as explained in appendix E), as I have the yield of the bond, I get an estimation of υl by dichotomy. If my estimator is too large the bond with a coupon equals to the yield on the market will have a price under 100, reciprocally if the estimation is too small the bond with a coupon equals to the yield on the market will have a price over 100, as the jump will be lower.

Then I will be able to price any other bonds from this issuer, no matter the maturity. Here is an example of parameters for $\lambda^{\mathbb{Q}}_n$ for Italian bonds:

T	Mean	Var	$\lambda 0$	υ	μ	$\upsilon 1$	k
1Y	0.01719	0.000357	0.01740	1.0904	-4.6580	1	0.5
2Y	0.02371	0.0004808	0.03084	0.9185	-4.1637	0.983	0.5
3Y	0.02782	0.000529	0.03589	0.7871	-3.8917	0.9566	0.5
4Y	0.03031	0.0005298	0.04106	0.6705	-3.7210	0.9382	0.5
5Y	0.03191	0.0005283	0.04279	0.5864	-3.6167	0.92	0.5
10Y	0.03511	0.0003054	0.04642	0.3125	-3.3981	0.8628	0.5

Remark 3.18. As I don't allow the negative hazard rate, there is a skew for the under CIR model for the $\lambda_n^{\mathbb{Q}}(t)$, adding a jump process and not allowing the negative $\lambda_n^{\mathbb{Q}}(t)$ will increase the positive skew, as it is likely to put more value above the mean reversion level.
To compensate such effect the mean reversion level should be much less than the mean of the hazard rate.

3.10.4 Impact of the jump process on $\lambda_{i,i+1}^{\mathbb{Q}}(t)$

As seen in section 3.10.1 and 3.10.2 on $\lambda_{i,i+1}^{\mathbb{Q}}(t)$ which is also the behaviour of $\lambda^{\mathbb{Q}}(t)$ under the first scheme described 2.10.1, recalling that $\lambda_{i,i+1}^{\mathbb{Q}}(t)$ is the difference between (i+1)$\lambda_{i+1}^{\mathbb{Q}}(t)$ and i$\lambda_i^{\mathbb{Q}}(t)$.
In this section I am focusing my study on $\lambda_{i,i+1}^{\mathbb{Q}}(t)$ and so on $\lambda^{\mathbb{Q}}(t)$ under the first scheme.
The purpose of this section is to show the impact of describing the hazard rate as $\lambda_{i,i+1}^{\mathbb{Q}}(t)$ on the jump event. As $\lambda_i^{\mathbb{Q}}(t)$ and $\lambda_{i+1}^{\mathbb{Q}}(t)$ receive a jump, but a different amplitude, and the factors in front of each other are different in the difference, the amplitude of the normalized variation of $\lambda_{i,i+1}^{\mathbb{Q}}(t)$ will be higher or smaller than 1. 1 is the amplitude of the jump for the $\lambda_{0,1}^{\mathbb{Q}}(t)$ to normalized the value of the hike will repeat on $\lambda_{i,i+1}^{\mathbb{Q}}(t)$.

3.10.5 For the CIR model

the variation due to the jump of $\lambda_i^{\mathbb{Q}}(t)$ and $\lambda_{i+1}^{\mathbb{Q}}(t)$:

$$(i+1)\rho_{1,i+1} - i\rho_{1,i+1}$$

for the case 5,10 I use the relationship:

$$\frac{10\rho_{1,10} - 5\rho_{1,5}}{5}$$

Let's look at the value for the Italian hazard rate, with the correlation values in G.1.
Here is the amplitude of the repeated jump:

i,i+1	Amplitude
0,1	1
1,2	0.966
2,3	0.904
3,4	0.875
4,5	0.855
5,10	0.863

The jump repeats with a large magnitude on $\lambda^{\mathbb{Q}}_{i,i+1}(t)$, the first scheme has a small impact on the representation of the jump process model. The remaining jump is still relevant.

3.10.6 For the Exponential Vasicek model

For the Exponential Vasicek model, the formula is much longer, using an infinitesimal increase in time, and using a Euler scheme, the variation of $\lambda^{\mathbb{Q}}_{i,i+1}(t+h)$ with a jump (with amplitude 1) between t and t+δ t, then variation $\lambda^{\mathbb{Q}}_{i,i+1}(t+h) - \lambda^{\mathbb{Q}}_{i,i+1}(t)$ due to the jump is:

$$(i+1)\ln(\lambda^{\mathbb{Q}}_{i+1}(t))\lambda^{\mathbb{Q}}_{i+1}(t)\rho_{1,i+1} - i\ln(\lambda^{\mathbb{Q}}_{i}(t))\lambda^{\mathbb{Q}}_{i}(t)\rho_{1,i}$$

$$+\frac{1}{2}(i+1)(\ln(\lambda^{\mathbb{Q}}_{i+1}(t)))^2\lambda^{\mathbb{Q}}_{i+1}(t)\rho^2_{1,i+1} - \frac{1}{2}i(\ln\lambda^{\mathbb{Q}}_{i}(t))^2\lambda^{\mathbb{Q}}_{i}(t)\rho^2_{1,i}$$

Here are the results for Italy, with the correlation given on G.1 and $\lambda^{\mathbb{Q}}_i(t)$ in G. The $\lambda^{\mathbb{Q}}_i(t)$ I will use to calculate the variation due to the jump are the μ_i, meaning I consider a jump happening when $\lambda^{\mathbb{Q}}_i(t)$ is close to its mean reversion level.

i,i+1	Amplitude
0,1	1
1,2	0.071
2,3	0.048
3,4	0.037
4,5	0.025
5,10	0.017

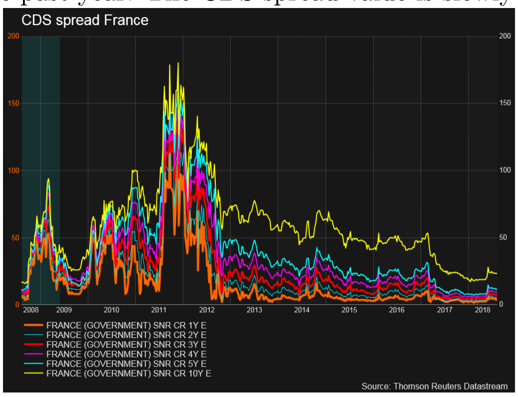
The amplitude is much lower than for the CIR model. Using the first scheme 2.10.1 with an Exponential Vasicek model will kill the jump effect on $\lambda_{i,i+1}(t)$, as a consequence there will not be any jumps outside the junction (when there is a transition for $\lambda(t)$, passing from $\lambda_{i,i+1}(t)$ to $\lambda_{i+1,i+2}(t)$) for the First scheme under Exponential Vasicek model.
Except for i=0 and i+1=1 there will not be any detectable jumps for the simulated path of $\lambda(t)$.

Remark 3.19. For the first scheme under a CIR model the behaviour is not the same as the jumps could happen at any time, even if the jump will probably be smaller for t ∈[5Y;10Y] than for t ∈[0;1Y].

Remark 3.20. For the scheme 2.10.3, $\lambda^{\mathbb{Q}}(t) = \lambda^{\mathbb{Q}}_{(i+\frac{j}{100}),(i+\frac{j+1}{100})}(t)$, using the analysis above, for the CIR model, the jump will still exist, but for the Exponential Vasicek model the jump process will not be visible on $\lambda^{\mathbb{Q}}(t)$.

3.11 Could the calibration be done through the past CDS spread/Hazard rate

As you could see in the appendix G.1, the values of the parameters are different from on simple estimate made with the past data. The first explanation is that the past data capture too much data. For example looking at the Southern European Sovereign bonds, the mean reversion level is near 0, at least for the past year. The CDS spread value is slowly going back to 0.



The choice of the parameters is subjective, moreover I didn't find any good estimation else that looking at the few past months to determine what should be the parameters in such market. The few jumps that have occurred could give an estimation of the amplitude and the repetition of the jump process.

4 Simulations

4.1 Without surface

To show that the market doesn't consider the risk of default the same way it does. A simplistic model would be to say, the hazard rate $\lambda^{\mathbb{Q}}(t)$ is stochastic and could be modelled by a CIR process or an Exponential Vasicek:

$$\mathrm{d}\lambda^{\mathbb{Q}}(t) = k(\mu - \lambda^{\mathbb{Q}}(t))\mathrm{d}t + \upsilon\sqrt{\lambda^{\mathbb{Q}}(t)}\mathrm{d}W(t)$$

or

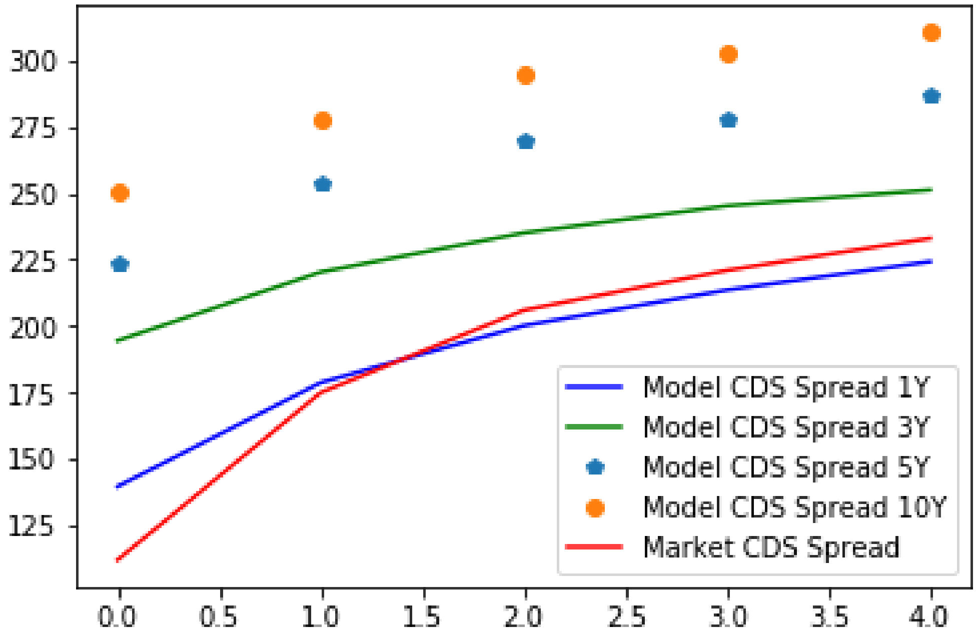
$$dy(t) = k(\mu - y(t))dt + \upsilon dW(t) \quad and \ \lambda^{\mathbb{Q}}(t) = e^{y(t)}$$

As I have shown before the variance of the process $\lambda^{\mathbb{Q}}(t)$ is quite volatile depending on the time, then it seems impossible to create a one-parameter model, with only one process, but a known shift could be use to get a value in adequation with the market.

4.1.1 CIR model

I have tried to generate $\lambda^{\mathbb{Q}}(t)$ as $\lambda_i^{\mathbb{Q}}(t)$ for different i, where $\lambda_i^{\mathbb{Q}}(t)$ follows a CIR PDE, using Euler scheme to compute the moves, and approximate the path of $\lambda_i^{\mathbb{Q}}(t)$. Then using a Monte Carlo method, I could get an estimation of the fair CDS spread.

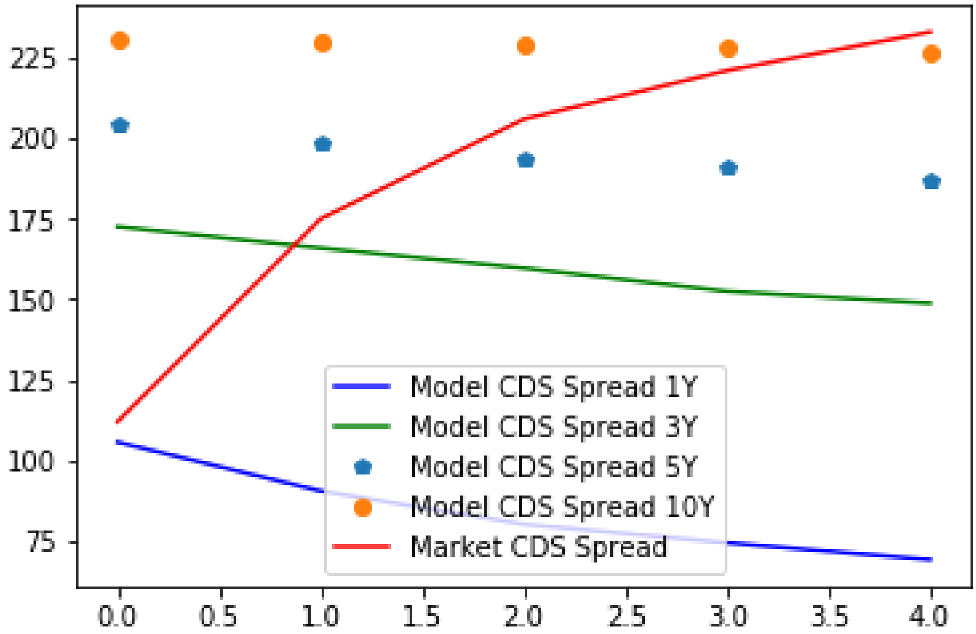
I have plotted the curves with the parameters given in the appendix G.1. I have generated the $\lambda_i^{\mathbb{Q}}(t)$ for i $\in$ $[1; 3; 5; 10]$ and for t $\in [0; 10]$.



None of the model fits well the CDS market curve, the 1Y curve is the closest, a deterministic function could help to fit the market curve. However such model shows this model is too simplistic, and there is no reason that the deterministic function should not evolve in time. Using $\lambda^{\mathbb{Q}}(t) = \lambda_1^{\mathbb{Q}}(t) + \Phi(t, T)$ where T is the maturity, and t the time.

4.1.2 Exponential Vasicek model

I have tried the same kind of fitting with an Exponential Vasicek model, but the result is worse than with the CIR model.



4.2 The problem of speed reversion for such model

Both models don't fit well, of course, there are different ways to make the curves fit the market. A smaller mean reversion speed parameter, with lower volatility as well as other mean reversion level, could force the curve to fit the market. At the same time, the hazard rate obtained doesn't make any sense as its behaviour is far from the moves I have observed in the market. We should keep in mind that the goal of this thesis is to find a fair representation of how the market is pricing the hazard rate, but it is definitely not to find the good value the $\lambda^{\mathbb{Q}}(t)$ should take to fit the market now, as soon as the market moves, the market should also follow.

4.3 Fitting without jumps

On the following graphics I have used the parameters given in the appendix G, I have plotted the CDS curve and an example of a path for each model (CIR and Exponential Vasicek) and each scheme for $\lambda^{\mathbb{Q}}(t)$. The maturities for the CDS spread are 1Y, 2Y, 3Y, 4Y, 5Y, 10Y. In the last I will look at the same maturities CDS spread to see if looking at not calibrated point make the model still valid or not.

On the CDS curve the abscissas are [1Y; 2Y;3Y;4Y;5Y;10Y]. Computing $\int_0^T \lambda^{\mathbb{Q}}(t)dt$ for T in [1;2;3;4;5;10] gives the CDS curve, with $\lambda^{\mathbb{Q}}(t)$ given by the scheme in the thesis.

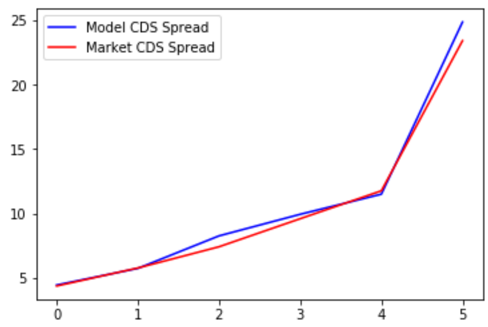
The number of paths per CDS spread is 1000 estimation.

I decided to adapt the different parameters for a realistic representation for the Italian, French and Spanish hazard rates. Whereas for the Portuguese hazard rates I used the market implied fitting over a long-term.

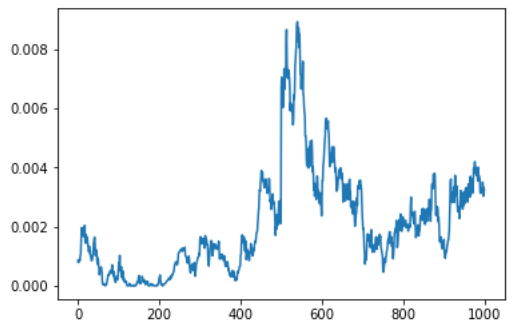
From Reuters data, I can extract one hazard rate per week as I have a CDS spread per week, the Variance of the process could be changed to get the volatility for 100 points per year:

$$\sqrt{\frac{52}{100}} Weekly\ Var[\lambda_n^{\mathbb{Q}}] = Var[\lambda_n^{\mathbb{Q}}\ for\ n\ points\ per\ year]$$

4.3.1 CIR first scheme

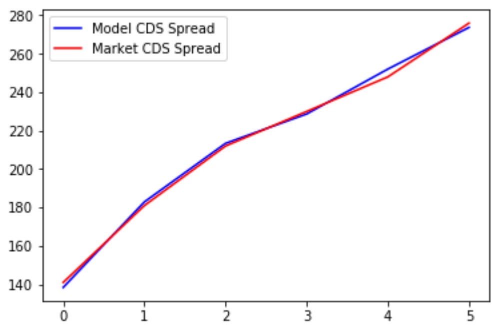


((c)) CDS curve

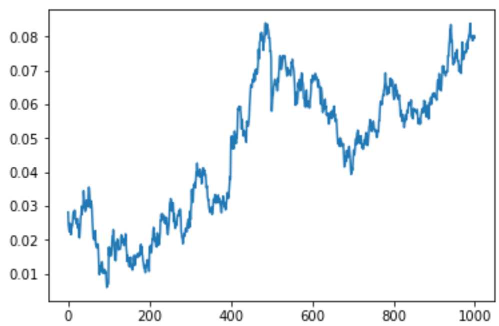


((d)) Example of a path

Figure 7: France

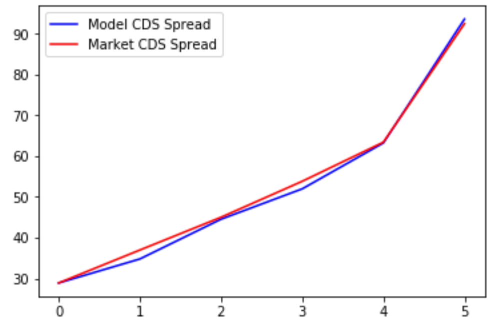


((a)) CDS curve

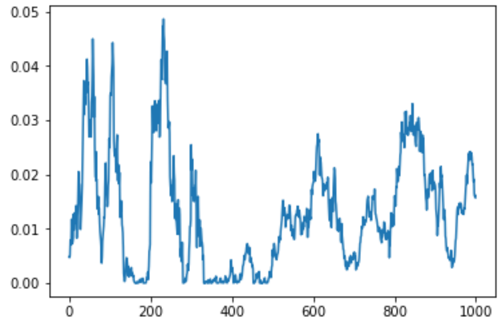


((b)) Example of a path

Figure 8: Italy

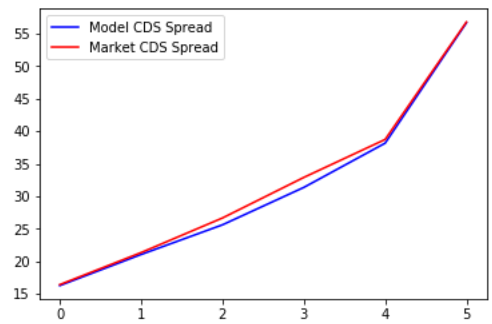


((a)) CDS curve

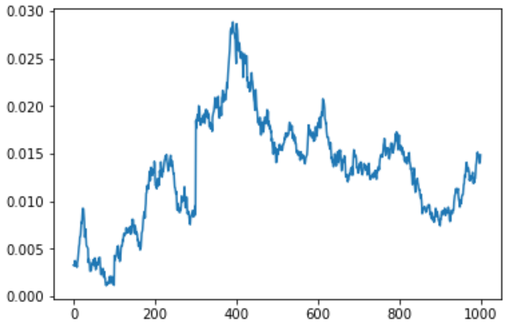


((b)) Example of a path

Figure 9: Portugal



((a)) CDS curve



((b)) Example of a path

Figure 10: Spain

After this first test, the historical average of volatility seems too high, as for the Portugal's curve I took the historical volatility. The CDS curve hits many times 0, at the same time the jumps are big and the moves are brutal with no transition time, and no period of stability, which lead to the fact that the volatility parameter is not adapted.

In addition with this scheme and model, there are some jumps, even if there is no jump process in the model, in fact, using a difference between $(i+1)\lambda^{\mathbb{Q}}_{i+1}(t)$ and $i\lambda^{\mathbb{Q}}_{i}(t)$. The jumps occur at the junction points, point number 500 for the French path, there is also smaller jump between the point 500-600. For the Italian curve, the only distinct jump is at the point 500. For the Spanish path, the jump occurs at the point 300.

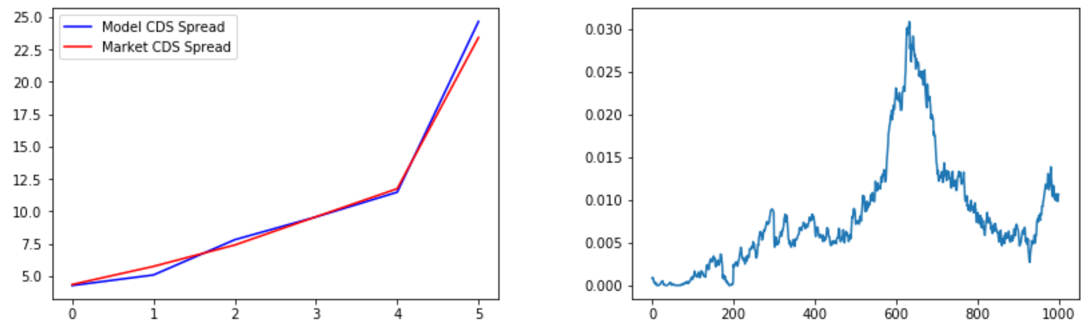
This behaviour is confirmed by the Portuguese path, with a jump at the point number 200.

There is not necessarily a jump at the junction but from what you could see it happens, often, in this case once per path. As there is 1000 points per path, there is one jump every 10 years, this might be a bit too small, as those hikes are not as high as the jumps that have occurred in the past market, with the Euro bonds crisis, and its replicas.

Remark 4.1. If there is an election in a major Euro-country, match the junction with the election date could be an excellent fitting. For example, the last Italian election has led to a jump, on every CDS spread.

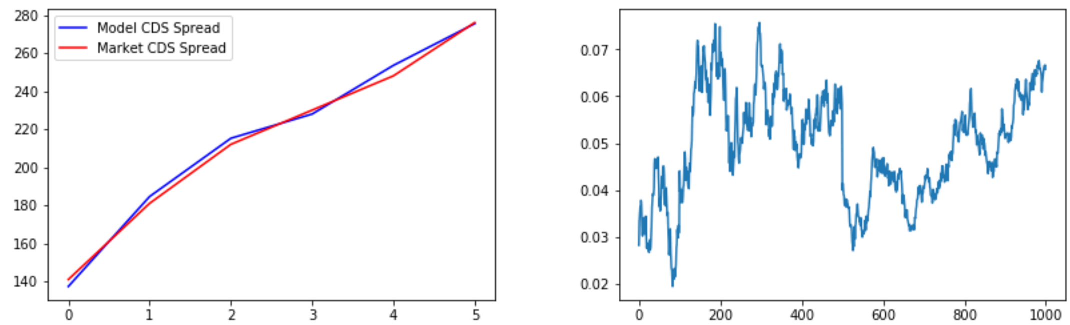
Remark 4.2. To go further, one could find the correlation between the CDS spread for the different countries, and then simulating the random shocks on the different countries. Such a proposal might need copulas introduction, and to model default probability as in [14]
Mathematically speaking, using a Monte Carlo method, there is no need for such correlation, as the CDS curve that will appear will be an expectation of the CDS spread, but that would make sense to compare the path to each other.

4.3.2 CIR second scheme



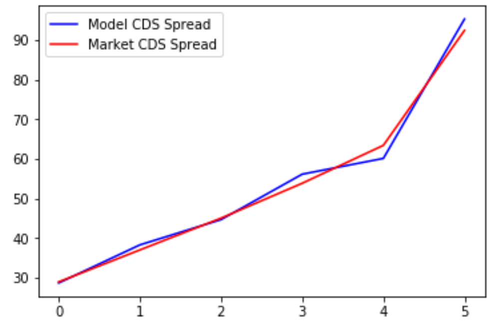
((a)) CDS curve ((b)) Example of a path

Figure 11: France

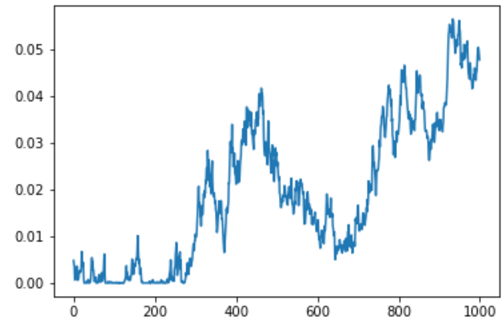


((a)) CDS curve ((b)) Example of a path

Figure 12: Italy

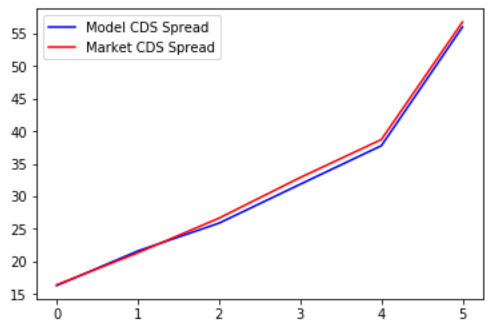


((a)) CDS curve

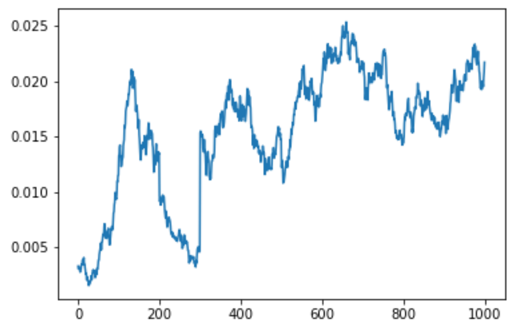


((b)) Example of a path

Figure 13: Portugal



((a)) CDS curve

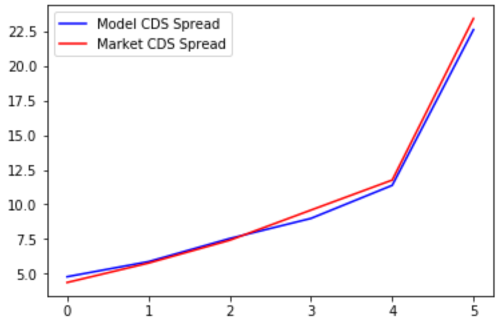


((b)) Example of a path

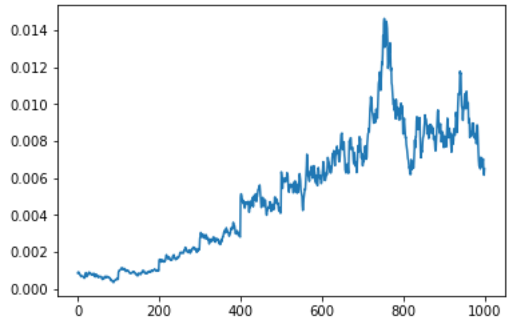
Figure 14: Spain

Remark 4.3. The jumps on junctions are still present as on the point number 300 for the Spanish path. By the way, the stagnation around 0 for the 2 first years of the simulation for the Portuguese path show that the volatility parameters are not appropriate.

4.3.3 ExpVas first scheme

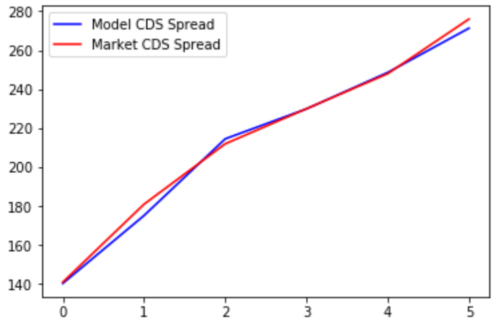


((a)) CDS curve

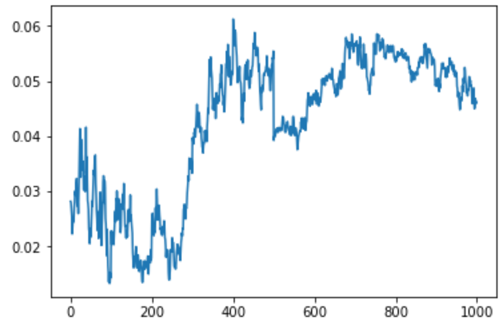


((b)) Example of a path

Figure 15: France

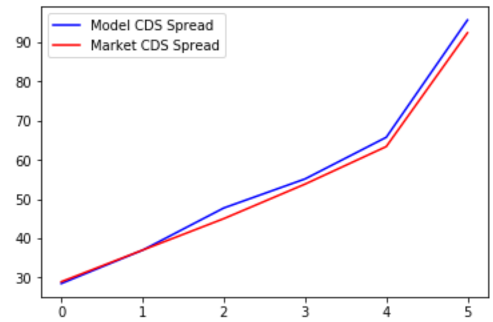


((a)) CDS curve

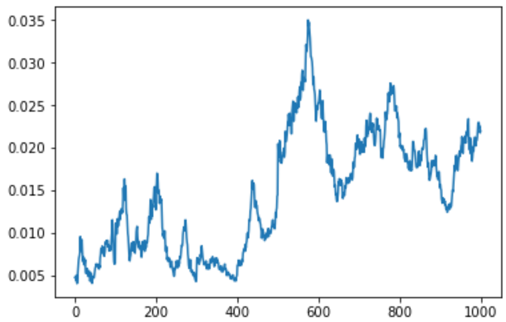


((b)) Example of a path

Figure 16: Italy

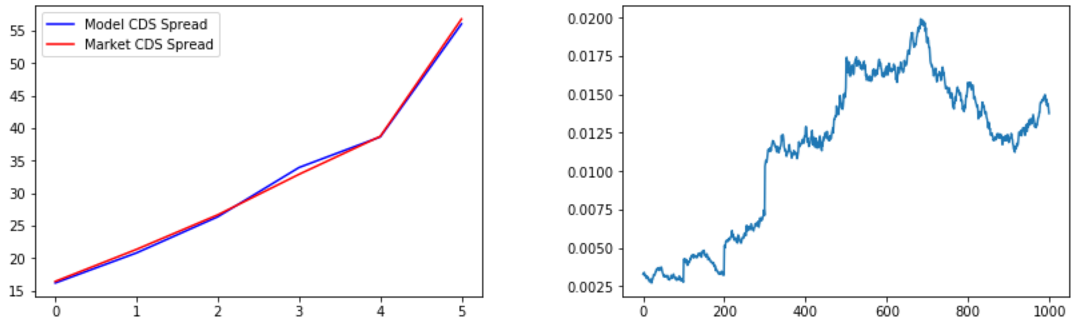


((a)) CDS curve



((b)) Example of a path

Figure 17: Portugal



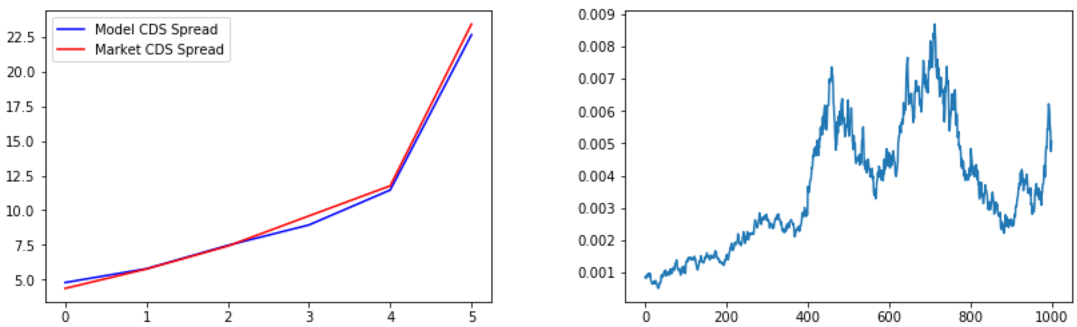
((a)) CDS curve

((b)) Example of a path

Figure 18: Spain

Remark 4.4. For the Exponential Vasicek model, the French and Spanish paths are not showing much volatility, and the $\lambda^{\mathbb{Q}}(t)$ stays in the neighbourhood of the mean. This is not the desired behaviour. For the French path it is clear for the beginning of the path, then the last volatility parameter (for the 10Y) looks like to be well calibrated, as some volatility appears. On the contrary, the Portuguese paths seem in adequation with what could happen, knowing the historical behaviour. However, it depends on the features we want for the paths.

4.3.4 Exp second scheme

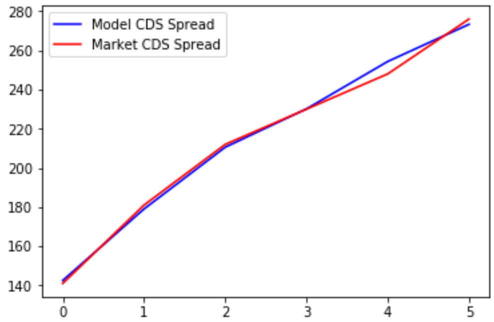


((a)) CDS curve

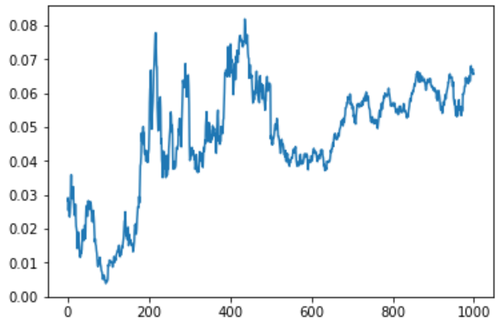
((b)) Example of a path

Figure 19: France

Remark 4.5.

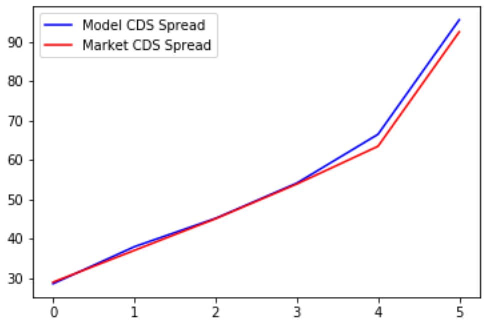


((a)) CDS curve

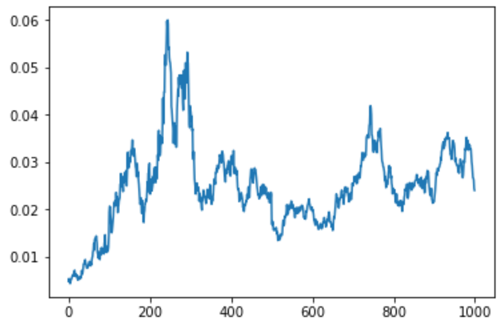


((b)) Example of a path

Figure 20: Italy

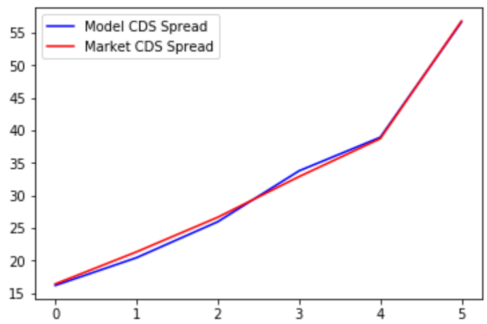


((a)) CDS curve

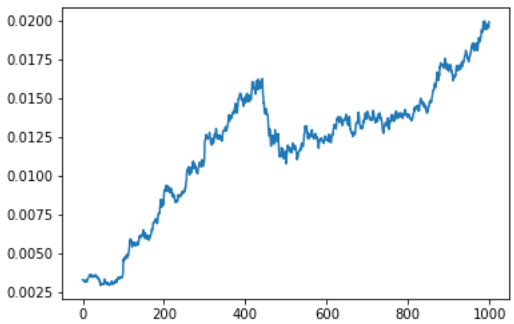


((b)) Example of a path

Figure 21: Portugal



((a)) CDS curve



((b)) Example of a path

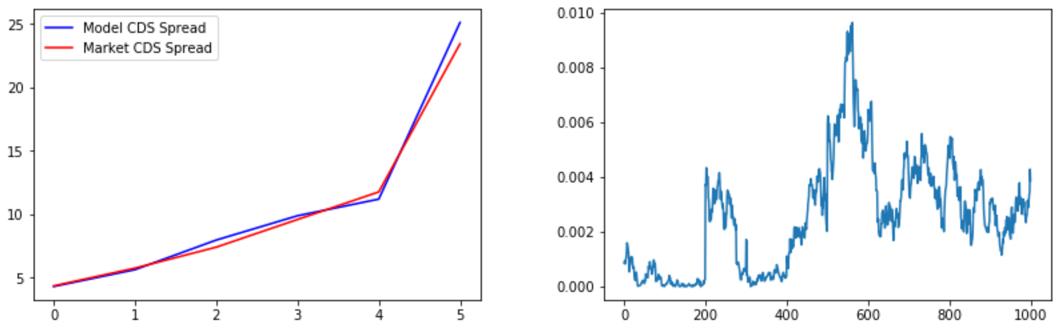
Figure 22: Spain

Remark 4.6. The previous remark on French and Spanish paths are confirmed. Building a dependency between the paths of different countries would make the identification of badly estimate parameters even easier.

Remark 4.7. The volatility parameters seems a better fit for the Portuguese case with Exponential Vasicek than for the CIR model, but I anticipated that as there were not the past historical estimate, using a simple average estimation.

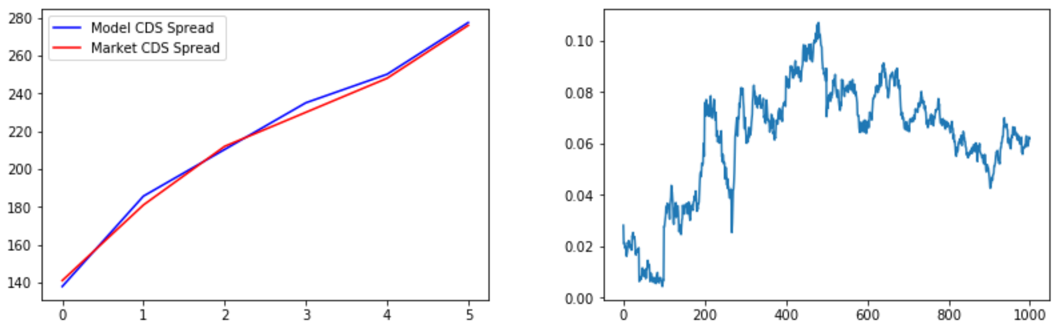
4.4 Fitting with jumps

4.4.1 CIR first scheme



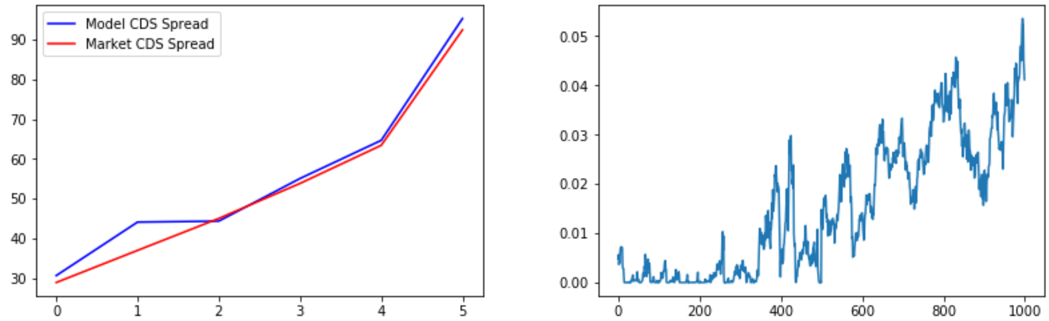
((a)) CDS curve ((b)) Example of a path

Figure 23: France



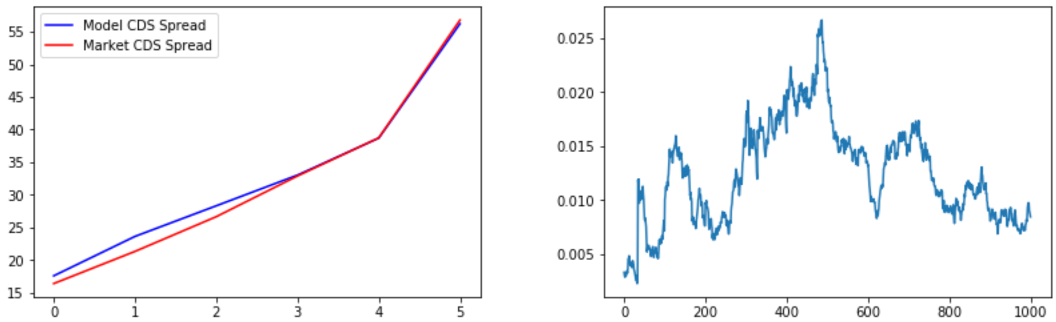
((a)) CDS curve ((b)) Example of a path

Figure 24: Italy



((a)) CDS curve ((b)) Example of a path

Figure 25: Portugal



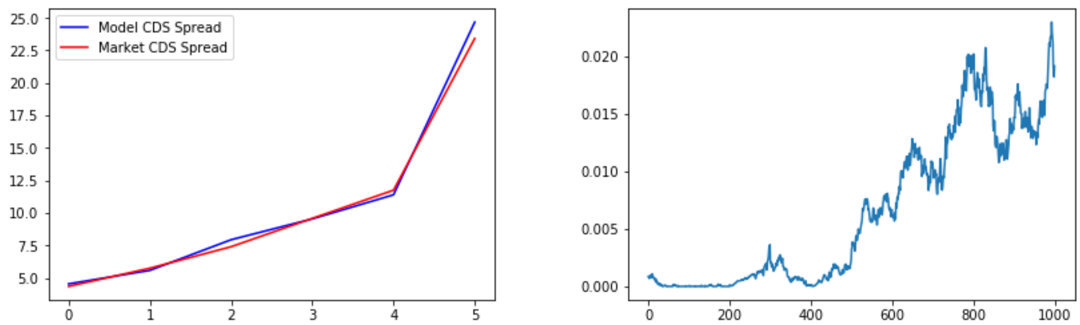
((a)) CDS curve ((b)) Example of a path

Figure 26: Spain

Remark 4.8. All the previous remarks are also accurate for the simulations adding a jump process. As I have used the same parameters for the jump diffusion process than the simulation without any jump. The jump process definitely adds some volatility to the global process.

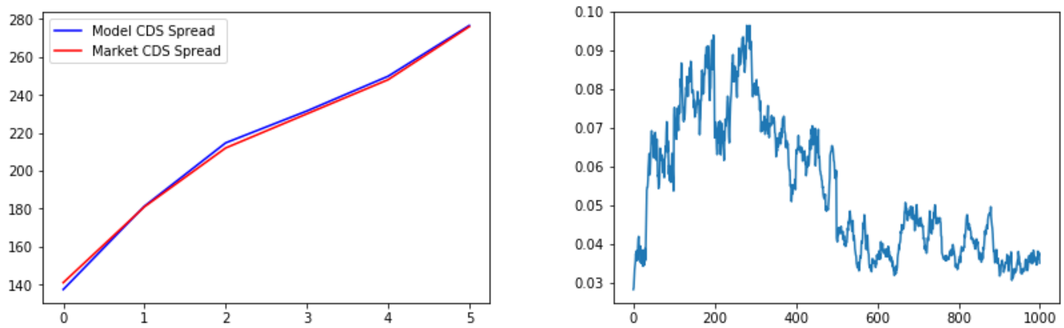
Remark 4.9. The fitting of the CDS curve is not impacted by adding a pure jump process for France and the Italy. The Italian CDS is quite high, and so adding a jump, even down will not bring the hazard rate to zero. If the jump process brings the computing of $\lambda^{\mathbb{Q}}(t)$ to the limit zero, then the distribution of the jump process has a positive skew. That is the reason why the Spanish and Portuguese CDS are not fitting the market, you need to make the mean reversion smaller, or reduce the variance of the diffusion process that reduces the positive skew due to the 0 limit as well as cutting down the high diffusion process values.

4.4.2 CIR second scheme



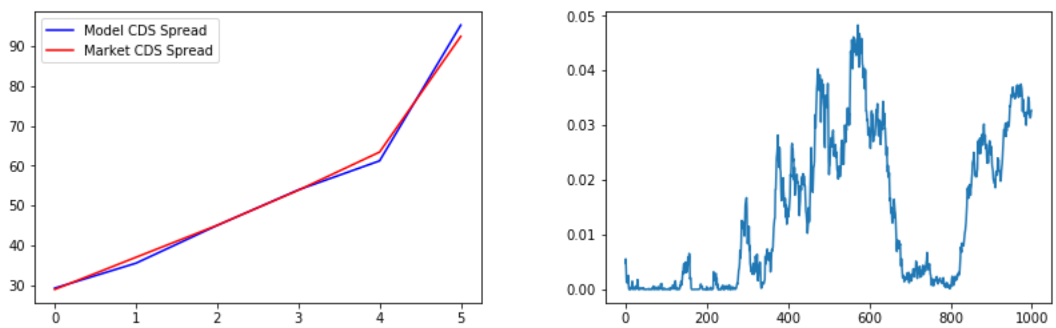
((a)) CDS curve ((b)) Example of a path

Figure 27: France



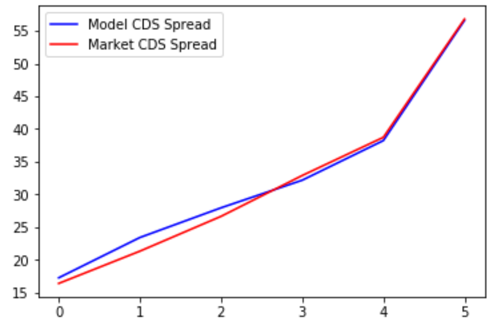
((a)) CDS curve ((b)) Example of a path

Figure 28: Italy

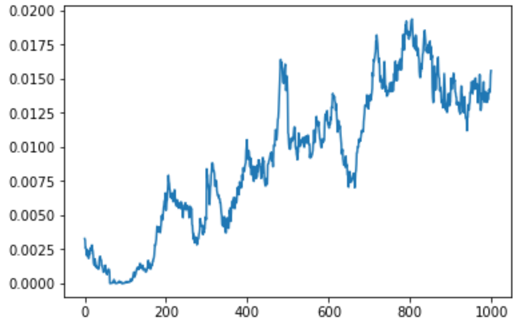


((a)) CDS curve ((b)) Example of a path

Figure 29: Portugal



((a)) CDS curve

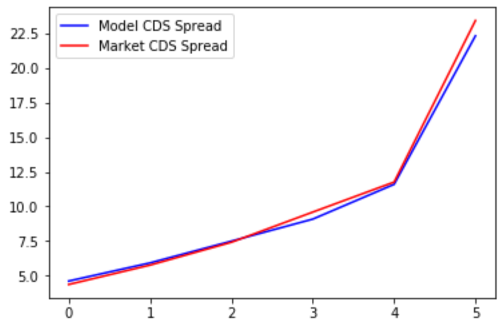


((b)) Example of a path

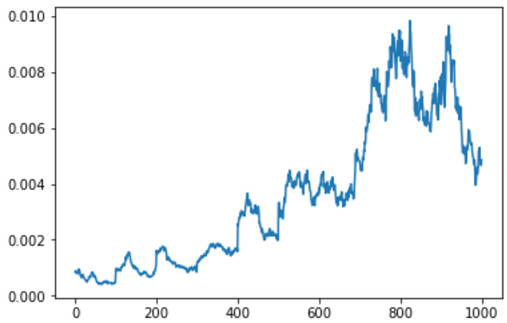
Figure 30: Spain

Remark 4.10. The problem of overestimating the CDS spreads for the 3 first years continues for Spain, but not for Portugal with this scheme.

4.4.3 ExpVas first scheme

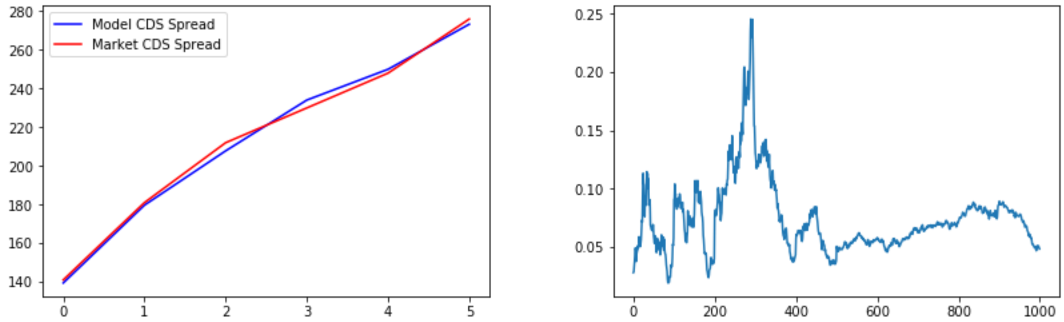


((a)) CDS curve



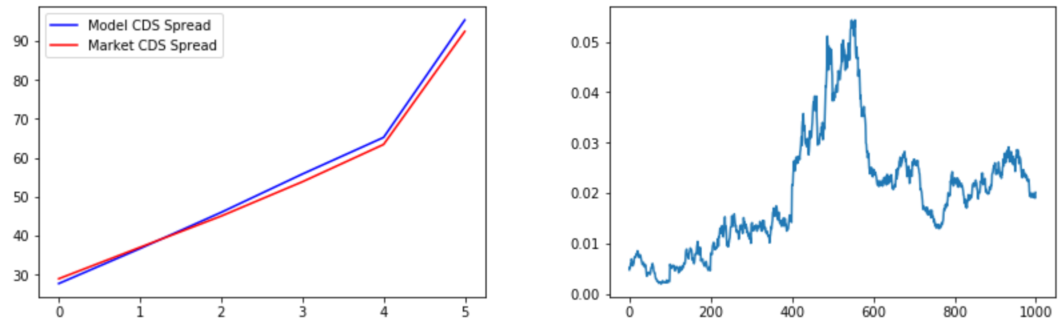
((b)) Example of a path

Figure 31: France



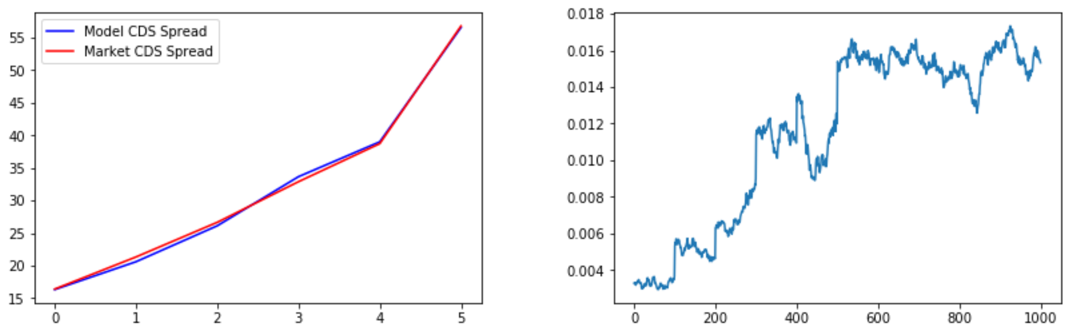
((a)) CDS curve ((b)) Example of a path

Figure 32: Italy



((a)) CDS curve ((b)) Example of a path

Figure 33: Portugal



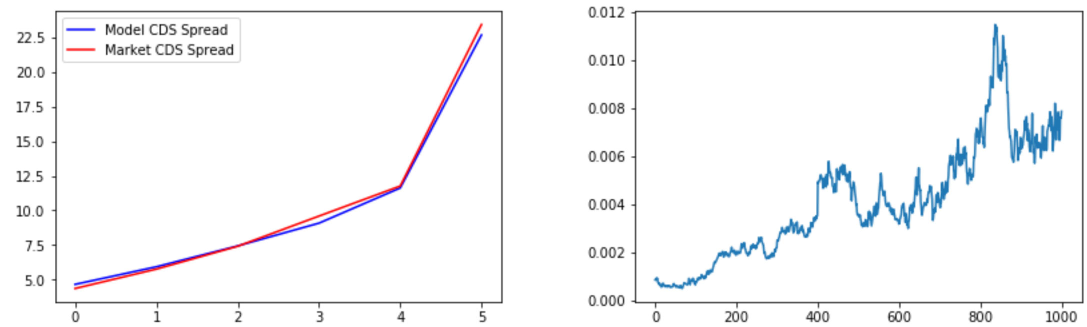
((a)) CDS curve ((b)) Example of a path

Figure 34: Spain

Remark 4.11. With the Exponential Vasicek model for $\lambda_n^{\mathbb{Q}}$, $\lambda^{\mathbb{Q}}(t)$ is much less volatile the under the CIR model. The periods of calm are more extended, as it could be observed on the market.

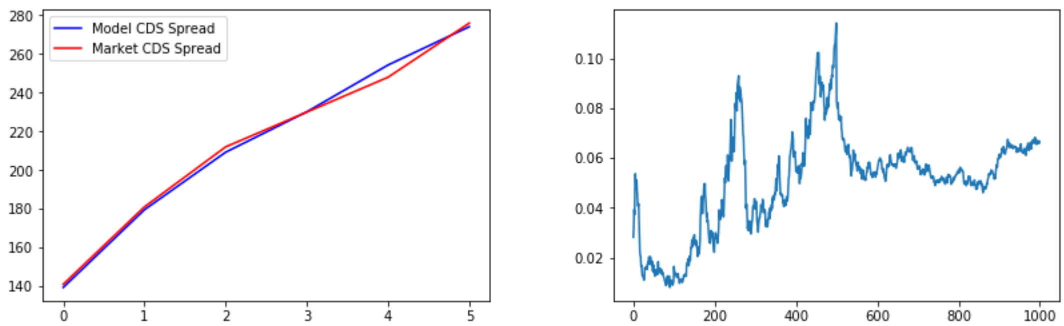
Remark 4.12. As seen in 3.10.2 the process $\lambda^{\mathbb{Q}}(t)$ is mainly driven by the diffusion process, the jump process doesn't have much influence. The significant volatility periods that look like jumps, are in fact the diffusion process with a large volatility parameter.

4.4.4 Exp second scheme



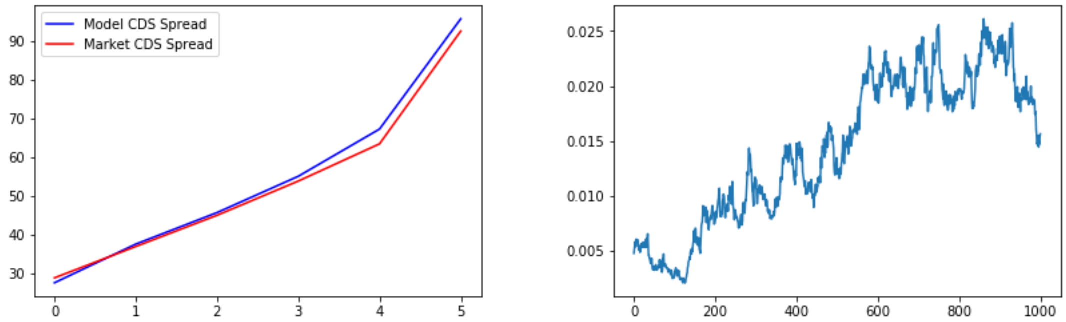
((a)) CDS curve ((b)) Example of a path

Figure 35: France



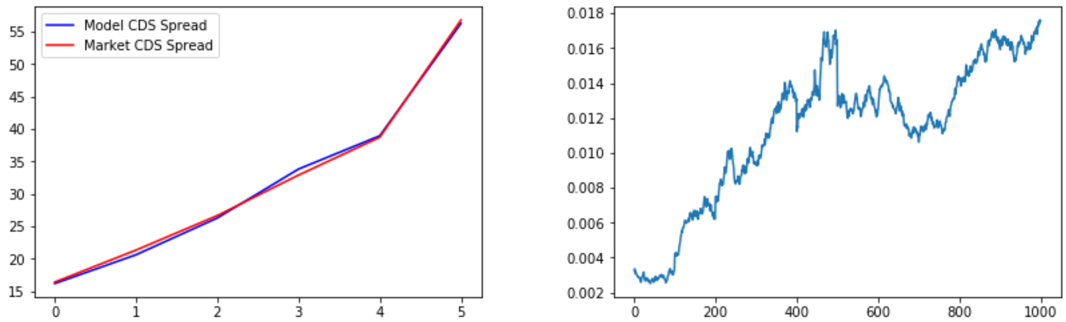
((a)) CDS curve ((b)) Example of a path

Figure 36: Italy



((a)) CDS curve ((b)) Example of a path

Figure 37: Portugal



((a)) CDS curve ((b)) Example of a path

Figure 38: Spain

Remark 4.13. As the second scheme seems highly volatile, and the first scheme not volatile enough, maybe a hybrid method, mixing both schemes could be a better scheme.

I want to make an observation remark:

Remark 4.14. The second scheme seems to need more paths to converge with the Monte Carlo method.

5 conclusion

This paper treats the hazard rate, extract from the bonds market as well as past CDS market data. However, the CDS spread market values can also gives these data. The representation I have done come back the essence of default risk and its pricing, but it is a way to get close the real market value of the risk and to model it.
I could also argue that the hazard rate surface extracted from the bonds/CDS market could help

to price new bonds emission by an issuer. Of course, the problem could be harder if there is a different degree of seniority of the debt issued. The newly issued debt could also raise the probability of default by adding some nominal to pay back in a longer term, but at the same time it gives leverage to the company and decreases the probability of default at a short-term by giving to this company fresh cash. During such events, the short-term CDS/hazard rate should be smaller whereas for the longer maturity the decreasing, if it happens, is smaller. The CDS spread curve will be steeper.

The parameters in the models need to be changed, a calibration by adjusting the hazard rate at time 0, and the mean reversion level should be done, to keep curves fitted.

Remark 5.1. I should make an important comment, I used the fact that the hazard rate for the day is fixed on the morning, nothing prevent you from saying that $\lambda^{\mathbb{Q}}(t \in [\frac{i}{100}; \frac{i+1}{100}]) = \lambda^{\mathbb{Q}}(\frac{2i+1}{200})$, in other words taking the hazard rate $\lambda^{\mathbb{Q}}(t)$ in the middle of the day, same for the end of the day. Fundamentally this does not change the results nor the fittings.

A Proof of the formula for $\lambda_n^{\mathbb{Q}}$(t)

Let's say the bond pays a yearly coupon i, and has a maturity of n years. The discounting factor $B^o(t) = \frac{1}{1+y}$, the first hypothesis will be that the bond default on the next day of coupon payment if it defaults. Using the expectation under the risk-neutral probability, I get:

$$P(0) = \sum_{j=1}^{n} \left(\frac{100\ i}{(1+y)^j} \right) + \frac{100\mathbb{Q}[\tau > n]}{(1+y)^n} + \int_0^n \frac{100\mathcal{R}^{\mathbb{Q}}}{r[\tau]} \mathrm{d}Q[\tau < u]$$

with P(0)=P(0,$n - years$)

$$= \sum_{j=1}^{n} \frac{100\ i}{(1+y)^j} e^{-j\lambda_n} + \frac{100 e^{-n\lambda_n^{\mathbb{Q}}}}{(1+y)^n} + + \int_0^n \frac{100\mathcal{R}^{\mathbb{Q}}}{r[\tau]} \mathrm{d}Q[\tau < u]$$

with $\mathcal{R}^{\mathbb{Q}} = 0.5$

$$P(0) = \sum_{j=1}^{n} \left(\frac{100i}{(1+y)^j} e^{-j\lambda_n^{\mathbb{Q}}} \right) + \frac{100 e^{-n\lambda_n^{\mathbb{Q}}}}{(1+y)^n} + 50\lambda_n^{\mathbb{Q}} \left(1 - \frac{e^{-n\lambda_n^{\mathbb{Q}}}}{(1+y)^n} \right) \frac{1}{\lambda_n^{\mathbb{Q}} \ln(1+y)}$$

B Appendix

B.1 Euler and Miller Scheme

For a standard SDE:

$$\mathrm{d}S(t) = a(S)\mathrm{d}t + b(S)\mathrm{d}W(t), 0 < t < T,$$

with W a standard Brownian motion.

The Euler scheme is:

$$S(t+h) = S(t) + a(S(t))h + b(S(t))\sqrt{h} Z_{k+1}$$

with k=$\frac{t}{h}$ where h is the δt between t+1 and t, and Z_{k+1} is a random variable with a Normal(0,1) distribution, the Z for different times are independent. For a simulation between 0 and T, with a δt=h, there will be n intervals, so n independent random Normal(0,1). The Miler scheme is:

$$S(t+h) = S(t) + a(S(t))h + b(S(t))\sqrt{h} Z_{k+1} + \frac{1}{2} b'(S(t)) b(S(t)) (Z_{k+1}^2 - 1)$$

with k=$\frac{t}{h}$
where h is the δt between t+1 and t, and Z_{t+1} is a random variable with a Normal(0,1) distribution, the Z for different times are independent.

B.2 Generation of a normal random variable with Box-Muller method

In this thesis, all the random variable with normal distribution are generated with Box-Muller method.

G. E. P. Box and M. E. MULLER in 1958 [7] have introduced this numerical method.

a, b are independent random Normal(0,1) variables:

$$Z = \sqrt{-2\log a}\cos(2\pi b)$$

Then Z is a random variable with distribution $\mathcal{N}(0,1)$.

C Schemes to estimate the daily hazard rate

C.1 First scheme

The hazard rate is highly dependent on the model we take. The first model I want to try is the stochastic model where the hazard rate rate on the period [i;i+1] will be determine by the PDE of the hazard rate $\lambda_{i;i+1}$ obtained by this equation:

$$\begin{aligned}
\lambda_i^{\mathbb{Q}}(t) &= k(\mu_i - \lambda_i^{\mathbb{Q}}(t))\mathrm{d}t + \upsilon_i\sqrt{\lambda_i^{\mathbb{Q}}(t)}\mathrm{d}W_i(t) \\
\lambda_{i+1}^{\mathbb{Q}}(t) &= k(\mu_{i+1} - \lambda_{i+1}^{\mathbb{Q}}(t))\mathrm{d}t + \upsilon_{i+1}\sqrt{\lambda_i^{\mathbb{Q}}(t)}\mathrm{d}W_{i+1}(t) \\
\mathcal{Q}(\tau < t_{i+1}) &= 1 - e^{\int_0^{t_{i+1}} \lambda^{\mathbb{Q}}(u)\mathrm{d}u}
\end{aligned}$$

As I have defined $\lambda_i^{\mathbb{Q}}$ such that $t_i\lambda_i^{\mathbb{Q}}(0) = \int_0^{t_i} \lambda^{\mathbb{Q}}(u\mathrm{d}u)$, I can write the integral as:

$$\begin{aligned}
\mathcal{Q}(\tau < t_{i+1}) &= 1 - e^{\int_0^{t_{i+1}} \lambda^{\mathbb{Q}}(u)\mathrm{d}u} \\
&= 1 - e^{t_i * \lambda_i^{\mathbb{Q}} + \int_{t_i}^{t_{i+1}} \lambda^{\mathbb{Q}}(u)\mathrm{d}u}
\end{aligned}$$

as I could consider the hazard rate for the period as the hazard rate $\lambda_{i,i+1}(t)$ generated by the previous equation.

$\mathrm{Q}(\tau < t_{i+1}) = 1 - e^{t_i\lambda_i^{\mathbb{Q}} + \int_{t_i}^{t_{i+1}} \lambda_{i,i+1}^{\mathbb{Q}}(u)\mathrm{d}u)}$

As I look at the daily hazard rate, the expression could be reduced to:

$\mathrm{Q}(\tau < t_{i+1}) = 1 - e^{-t_i\lambda_i^{\mathbb{Q}} - \frac{1}{n}\sum_{j=0}^{n-1} \lambda_{i,i+1}^{\mathbb{Q}}(t_i + \frac{j}{n})}$

So I can write that the short-term hazard rate for any t in the interval $[t_i; t_{i+1}]$ is:

$$t_{i+1}\lambda^{\mathbb{Q}}_{i+1}(t) = t_i\lambda^{\mathbb{Q}}_i(t) + (t_{i+1} - t_i)\lambda^{\mathbb{Q}}_{i;i+1}(t)$$

$$\lambda^{\mathbb{Q}}_{i,i+1}(t) = \frac{1}{t_{i+1} - t_i}\big(t_{i+1}\lambda^{\mathbb{Q}}_{i+1}(t) - t_i\lambda^{\mathbb{Q}}_i(t)\big)$$

with n the number of days in a year As in the model $\lambda^{\mathbb{Q}}_i$ is stochastic then $\lambda^{\mathbb{Q}}_{i;i+1}$ is and so the daily short-term hazard rate is.
It is easy to caliber the model for the $\lambda^{\mathbb{Q}}_i$ as soon as we consider the Black and Scholes model for default swap to get the volatility of $\lambda^{\mathbb{Q}}_i$. Then the rest is straightforward. To get on multiple period, you have to reiterate the process, by simulating the $\lambda^{\mathbb{Q}}_i$, then getting the hazard rate, by continuity of hazard rates, if there is no jump at the junction the initial condition will be obtained for the next interval thanks to the last value of the previous period.
For this representation, the volatility of the short-term hazard rate will be constant at each interval.

C.2 Second scheme

With the second model, I go even further, here the idea at the beginning is the same but the deposition of the period will be even harder, as I will decompose the period in $\frac{1}{100}$ long periods, and not anymore in 1 year, or more.

$$\mathbb{Q}(\tau < t_i + \frac{1}{100}) = 1 - e^{-t_i * \lambda^{\mathbb{Q}}_i - \int_{t_i}^{t_i + \frac{1}{100}} \lambda^{\mathbb{Q}}(u)\mathrm{d}u}$$

$$\mathbb{Q}(\tau < t_i + \frac{1}{100}) = 1 - e^{-(t_i + \frac{1}{100}) * \lambda^{\mathbb{Q}}_{i+\frac{1}{100}}}$$

on this interval $\lambda(t)$ will be consider constant.

$$\mathbb{Q}(\tau < t_i + \frac{1}{100}) = 1 - e^{-t_i * \lambda^{\mathbb{Q}}_i - \int_{t_i}^{t_i + \frac{1}{100}} \lambda^{\mathbb{Q}}(u)\mathrm{d}u}$$

with the hazard rate $\lambda^{\mathbb{Q}}_{i+\frac{1}{100}}$ is obtained by linear extension on the surface of hazard rate.
$\lambda^{\mathbb{Q}}_{i+\frac{1}{100}}(t) = \frac{99}{100}\lambda^{\mathbb{Q}}_i(t) + \frac{1}{100}\lambda^{\mathbb{Q}}_{i+1}(t)$
$\mathrm{e}^{-(i+\frac{1}{100})(\frac{99}{100}\lambda^{\mathbb{Q}}_i + \frac{1}{100}\lambda^{\mathbb{Q}}_{i+1})} = e^{-i\lambda_i - \frac{1}{100}\lambda^{\mathbb{Q}}(tin[i;i+\frac{1}{100}])}$

$(i + \frac{1}{100})(\frac{99}{100}\lambda^{\mathbb{Q}}_i + \frac{1}{100}\lambda^{\mathbb{Q}}_{i+1}) = i\lambda_i + \frac{1}{100}\lambda^{\mathbb{Q}}(tin[i; i + \frac{1}{100}])$
$\lambda^{\mathbb{Q}}(i + \frac{1}{100}) = \frac{(99-i)}{100}\lambda^{\mathbb{Q}}_i + (i + \frac{1}{100})\lambda^{\mathbb{Q}}_{i+1}$
Then the next hazard rate will be obtained by calculating the above $\lambda(i + \frac{1}{100})$ and taking this value:

$$\lambda^{\mathbb{Q}}(i + \frac{2}{100}) = \frac{(98 - i)}{100}\lambda^{\mathbb{Q}}_i + (i + \frac{2}{100})\lambda^{\mathbb{Q}}_{i+1} - \frac{1}{100}\lambda^{\mathbb{Q}}(i + \frac{1}{100})$$

for any j belonging to $[1; 99]$, we have:

$$s = \sum_{k=1}^{j-1} \lambda^{\mathbb{Q}}(i + \frac{k}{100}(estimatedattimei + \frac{j}{100}))$$

$$(i+\frac{j}{100})\lambda^{\mathbb{Q}}_{i+\frac{j}{100}} = i\lambda^{\mathbb{Q}}_i + \frac{1}{100}\lambda^{\mathbb{Q}}(i+\frac{j}{100}) + \frac{1}{100}s$$
$$\lambda^{\mathbb{Q}}(i+\frac{j}{100}) = (i+\frac{j}{100})(j\lambda^{\mathbb{Q}}_{i+1} + (100-j)\lambda^{\mathbb{Q}}_i) - 100i\lambda^{\mathbb{Q}}_i - s$$
$$\lambda^{\mathbb{Q}}(i+\frac{j}{100}) = (ij+\frac{j^2}{100})\lambda^{\mathbb{Q}}_{i+1} + ((i+\frac{j}{100})(100-j) - 100i)\lambda^{\mathbb{Q}}_i - s$$

here is a development to get an easy formula for s:

$$1 - e^{-(i+\frac{j-1}{100})\lambda^{\mathbb{Q}}_{i+\frac{j-1}{100}}} = 1 - e^{-i\lambda^{\mathbb{Q}}_i - \frac{1}{100}s}$$
$$s = 100(i+\frac{j-1}{100})(\frac{j-1}{100}\lambda^{\mathbb{Q}}_{i+1}(i+\frac{j}{100}) + \frac{100-j+1}{100}\lambda^{\mathbb{Q}}_i(i+\frac{j}{100}) - 100i\lambda^{\mathbb{Q}}_i(i+\frac{j-1}{100}))$$

with the λ_i and the $\lambda^{\mathbb{Q}}_{i+1}$ from the $(100i+j)^{th}$ surface obtained.
As I get a surface at each simulation, 100 simulation in a year, thanks to the previous surface and with a Euler scheme, the $\lambda^{\mathbb{Q}}_i$ and $\lambda^{\mathbb{Q}}_{i+1}$ are stochastic, and so the $\lambda(i+\frac{1}{100})$ are, but with volatility that changes as $\lambda^{\mathbb{Q}}_i$ and $\lambda^{\mathbb{Q}}_{i+1}$ don't have the same volatility. The volatility of the short-term hazard rate is changing depending on the parameter, but it is not a stochastic, but just a deterministic function. With this method you can notice that this volatility is linear between the nodes where I made the calibration.

C.3 Third scheme

Another way to consider the short-term hazard rate, and maybe the most natural way to get the hazard rate on a daily rate, the hazard rate will be directly the daily hazard rate. This way to estimate the short-term and then forward hazard rate is to estimate $\lambda^{\mathbb{Q}}_{i+\frac{j}{100}-i+\frac{j+1}{100}}$:
$Q(\tau < i+\frac{j+1}{100}) = 1 - e^{-(i+\frac{j+1}{100})\lambda^{\mathbb{Q}}_{i+\frac{j+1}{100}}}$
$Q(\tau < i+\frac{j+1}{100}) = 1 - e^{-(i+\frac{j}{100})\lambda^{\mathbb{Q}}_{i+\frac{j}{100}} - \frac{1}{100}\lambda^{\mathbb{Q}}_{i+\frac{j}{100}-i+\frac{j+1}{100}}}$
The short-term hazard rate for the daily period $[i+\frac{j}{100}; i+\frac{j+1}{100}]$ will be:
$\lambda^{\mathbb{Q}}_{i+\frac{j}{100},i+\frac{j+1}{100}} = (i+\frac{j+1}{100})\lambda^{\mathbb{Q}}_{i+\frac{j+1}{100}} - (i+\frac{j}{100})\lambda^{\mathbb{Q}}_{i+\frac{j}{100}}$
Using the surface of constant hazard rate and especially the segment between $\lambda^{\mathbb{Q}}_i(t)$ and $\lambda^{\mathbb{Q}}_{i+1}(t)$:

$$\begin{aligned}
\lambda^{\mathbb{Q}}_{i+\frac{j}{100},i+\frac{j+1}{100}} &= (i+\frac{j+1}{100})(\frac{j+1}{100}\lambda^{\mathbb{Q}}_{i+1} + \frac{100-j-i}{100}\lambda^{\mathbb{Q}}_i) - (i+\frac{j}{100})(\frac{j}{100}\lambda^{\mathbb{Q}}_{i+1} + \frac{100-j}{100}\lambda^{\mathbb{Q}}_i) \\
&= \lambda^{\mathbb{Q}}_{i+1}(i(\frac{j+1}{100} - \frac{j}{100}) + (\frac{j+1}{100})^2 - (\frac{j}{100})^2) + \lambda^{\mathbb{Q}}_i(-\frac{i}{100} + \frac{100-j-1}{100}\frac{j+1}{100} - \frac{j}{100}\frac{100-j}{100}) \\
&= \lambda^{\mathbb{Q}}_{i+1}(\frac{i}{100} + \frac{2j+1}{100^2}) + \lambda^{\mathbb{Q}}_i(-\frac{i}{100} - \frac{(2j+1-100)}{100^2}) \\
&= (\frac{i}{100} + \frac{2j+1}{100^2})\lambda^{\mathbb{Q}}_{i+1} - (\frac{i}{100} + \frac{2j+1-100}{100^2})\lambda^{\mathbb{Q}}_i
\end{aligned}$$

As a year in my notation is equivalent to 100 points, then the short-term hazard rate could be found with this formula: $\lambda(i+\frac{j}{100}) = \lambda^{\mathbb{Q}}(\text{for the } i^{th} \text{ year and } j^{th} \text{ day}) = (\frac{i}{100} + \frac{2j+1}{100^2})\lambda^{\mathbb{Q}}_{i+1}(i+$

$\frac{j}{100}) - (\frac{i}{100} + \frac{2j+1-100}{100^2})\lambda_i^{\mathbb{Q}}(i + \frac{j}{100})$ In the case the interval is longer than 1Y, for example 5Y as I will have to do for the interval [5Y;10Y]: the formula is a bit different:

$$\begin{aligned}
\lambda^{\mathbb{Q}}_{i+\frac{j}{100}->i+\frac{j+1}{100}} &= (i + \frac{j+1}{100})\lambda^{\mathbb{Q}}_{i+\frac{j+1}{100}} - (i + \frac{j}{100})\lambda^{\mathbb{Q}}_{i+\frac{j}{100}} \\
&= (i + \frac{j+1}{100})((\frac{j+1}{500})\lambda_{i+1} + (\frac{500-j-1}{500})\lambda_i^{\mathbb{Q}}) - (i + \frac{j}{100})((\frac{j}{500})\lambda^{\mathbb{Q}}_{i+1} + (\frac{500-j}{500})\lambda_i^{\mathbb{Q}}) \\
&= \lambda^{\mathbb{Q}}_{i+1}(\frac{ij+i}{500} + \frac{(j+1)^2}{50000} - \frac{ij}{500} - \frac{j^2}{50000}) + \lambda_i^{\mathbb{Q}}(-\frac{i}{500} + \frac{(j+1)(499-j)}{50000} - (i + \frac{j}{100})(\frac{500-j}{500})) \\
&= \lambda^{\mathbb{Q}}_{i+1}(\frac{i}{500} + \frac{2j+1}{50000}) + \lambda_i^{\mathbb{Q}}(-\frac{i}{500} + \frac{499-2j}{50000})
\end{aligned}$$

where j belongs to the interval [0;500]

For any interval length l,as I have 100 points per year:

$$\lambda^{\mathbb{Q}}_{i+\frac{j}{100},i+\frac{j+1}{100}} = \lambda^{\mathbb{Q}}_{i+1}(\frac{i}{100l} + \frac{2j+1}{10000l}) + \lambda_i^{\mathbb{Q}}(-\frac{i}{100l} + \frac{l-1-2j}{10000l})$$

D Details on Vasicek model

Definition D.1 (Static replication)**.** In the Vasicek model, the equivalent hazard rate on the period [0;T] is assumed to satisfy the stochastic differential equation:

$$\mathrm{d}\lambda_T^{\mathbb{Q}}(t) = k_T(\mu_T - \lambda_T^{\mathbb{Q}}(t))\mathrm{d}t + \upsilon_T \mathrm{d}W(t)$$

where W(t) is a standard Brownian motion under the risk-neutral probability Q.

Theorem D.2. *For any s and t such that 0≤ s ≤ t ≤ T :*$\lambda_T^{\mathbb{Q}}(t) = r(s)e^{-k_T(t-s)} + \mu_T(1 - e^{-k_T(t-s)}) + \upsilon_T \int_s^t e^{-k_T(t-u)}\mathrm{d}W(u)$

$\mathbb{E}[\lambda_T^{\mathbb{Q}}(t)|\mathcal{F}_f] = \lambda_T^{\mathbb{Q}}(s)e^{-k(t-s)} + \mu_T(1 - e^{-k(t-s)})$

$\mathbb{V}[\lambda_T^{\mathbb{Q}}(t)|\mathcal{F}_f] = \frac{\upsilon_T^2}{2k_T}(1 - e^{-2k(t-s)})$

Theorem D.3. *Under this model by analogy with the zero coupon bond with maturity T will have the value at time t:*

$$P(t,T) = A(t,T)e^{-r(t)B(t,T)}$$

where

$$B(t,T) = \frac{1 - e^{-k(T-t)}}{k}$$

and

$$A(t,T) = e^{(\mu - \frac{\upsilon^2}{2k^2})(B(t,T) - T + t) - \frac{\upsilon^2}{4k}B^2(t,T)}$$

for the hazard rates I have the value at time 0 of the $\lambda_T^{\mathbb{Q}}$ so I could get the an estimation of the probability of default on the period [0;T].

$$\mathbb{Q}(\tau < T) = 1 - e^{-\int_0^T \lambda^{\mathbb{Q}}(t)\mathrm{d}t}$$

$$\mathbb{E}[\mathbb{Q}(\tau < T)] = A(0,T)e^{-\lambda^{\mathbb{Q}}(0)B(0,T)}$$

$$B(0,T) = \frac{1-e^{-kT}}{k}$$

$$A(0,T) = e^{(\mu - \frac{v^2}{2k^2})(B(0,T)-T) - \frac{v^2}{4k}B^2(0,T)}$$

The main feature of the model I will use is the normal distribution. But the main problem is the positive probability of getting a negative hazard rate with this model. Another problem is that the skew is null, but in reality, the hazard rate has a positive skew.

E Simulation of a Jump process

This part has been defines in [8]. To generate a Jump process on a period with length T, I divide the timeline in interval with length $\frac{1}{100}$, I obtain 100T intervals and so 100T+1 point as the first point is known.

Definition E.1. A Geometric Brownian Motion Poisson Process is defined as:

$$\frac{\mathrm{d}X_t}{X_{t-}} = \mu \mathrm{d}t + v\mathrm{d}W(t) + \mathrm{d}J_t,$$

Using Euler Scheme you can get:

$$X_{t_{i+1}} = X_{t_i}(1 + \mu h + v\sqrt{h}Z_{i+1}),$$

$$X_{t_{i+1}} = X_{t_i}(1 + \mu h + v\sqrt{h}Z_{i+1} + (e^{v_J \tilde{Z}_k} - 1)),$$

where h is the length of th interval $[t_i; t_{i+1}]$, in the thesis it will be $\frac{1}{100}$.
Another scheme to estimate more appropriately $X_{t_{i+1}}$ from X_{t_i} with the following formula:

$$X_{t_{i+1}} = X_{t_i}e^{((\mu - \frac{1}{2}v^2)h + v\sqrt{h}Z_{i+1})},$$

$$X_{t_{i+1}} = X_{t_i}e^{((\mu - \frac{1}{2}v^2)h + v\sqrt{h}Z_{i+1} + v_J\tilde{Z}_k)},$$

F Calibration of the volatility parameter from the implied volatility of the default swaptions

Once the market implied volatility is given by the market, by using the Black and Scholes model on the market price of the default swaption on CDS with the maturity T, υ_T.

Remark F.1.

$$T\lambda_T(t) = \frac{\tilde{s}(t)}{1-R}$$

Another way to get this parameter is to approximate it to $\upsilon_T = \sqrt{\frac{2\Pi}{t_{exercise}}} \frac{\tilde{s}_T}{C_{t_{exercise},T}}$ where $\tilde{s}_T$ is the fair spread for the a CDS with maturity T, and $C_{t_{exercise},T}$ is the price of the option on the CDS with maturity T, and will be exercised or not at time $t_{exercise}$.
From the volatility of the CDS, I can get the volatility of the CDS spread, that will be defined as $\sqrt{Var[\tilde{s}(t)]}$ for the volatility at time t.

F.1 under CIR model

Definition F.2. In the Cox Ingersoll Ross model, the hazard rate is assumed to satisfy the stochastic differential equation:

$$\mathrm{d}\lambda_T^{\mathbb{Q}}(t) = k_T(\mu_T - \lambda_T^{\mathbb{Q}}(t))\mathrm{d}t + \upsilon_T\sqrt{\lambda_T^{\mathbb{Q}}(t)}\mathrm{d}W(t)$$

$$\mathrm{d}\lambda_i^{\mathbb{Q}}(t) = k(\mu_i - \lambda_i^{\mathbb{Q}}(t)) + \upsilon_i\sqrt{\lambda_i^{\mathbb{Q}}(t)}\mathrm{d}W_i(t)$$
$$\textit{with } 2k\mu_n > \upsilon_n^2 \textit{ and } 2k\mu_i > \upsilon_i^2$$

the expectation of the hazard rate under this model is:

$$\mathbb{E}[\lambda_i^{\mathbb{Q}}(t)|\mathcal{F}_0] = \lambda_i^{\mathbb{Q}}(0)e^{-kt} + \mu_i(1-e^{-kt})$$

$$\mathbb{Var}[\lambda_i^{\mathbb{Q}}(t)|\mathcal{F}_0] = \lambda_i^{\mathbb{Q}}(0)\frac{\upsilon_i^2}{2k}(e^{-kt} - e^{-2kt}) + \mu_i\frac{\upsilon_i^2}{2k}(1-e^{-kt})^2$$

as the only condition we have is the hazard rate now, by convention it is time t=0. then the mean of lambda.

Under the CIR model, the hazard rate $\lambda_T^{\mathbb{Q}}(t)$ will be defined by the following:

$$\mathrm{d}\lambda_T^{\mathbb{Q}}(t) = k_T(\mu_T - \lambda_T^{\mathbb{Q}}(t))\mathrm{d}t + \upsilon_T\sqrt{\lambda_{T(t)}^{\mathbb{Q}}}\mathrm{d}W(t)$$

$$Var[\lambda_T^{\mathbb{Q}}(t)|\mathcal{F}_\prime] = \lambda_T^{\mathbb{Q}}(0)\frac{\upsilon_T^2}{k_T}\left(e^{-k_T t} - e^{-2k_T t}\right) + \mu_T\frac{\upsilon_T^2}{2k_T}\left(1 - e^{-k_T t}\right)^2$$

$$Var[\lambda_T^{\mathbb{Q}}(t)] = Var[\frac{\tilde{s}(t)}{1-R}]$$

$$\sqrt{Var[\tilde{s}(t)|\mathcal{F}_0]} = (1-\mathcal{R}^{\mathbb{Q}})\sqrt{Var[\lambda_T^{\mathbb{Q}}(t)|\mathcal{F}_0]}$$

$$\sqrt{Var[\tilde{s}(t)|_{F0}]} = (1-\mathcal{R}^{\mathbb{Q}})\upsilon_T\sqrt{\frac{\lambda_T^{\mathbb{Q}}(0)}{k_T}(e^{-k_T t} - e^{-2k_T t}) + \frac{\mu_T}{2k_T}(1-e^{-k_T t})^2}$$

$$\upsilon_T = \frac{\sqrt{Var[\tilde{s}(t)|\mathcal{F}_\prime]}}{(1-\mathcal{R}^{\mathbb{Q}})\sqrt{\frac{\lambda_T(0)}{k_T}(e^{-k_T t} - e^{-2k_T t}) + \frac{\mu_T}{2k_T}(1-e^{-k_T t})^2}}$$

Theorem F.3. *If $\lambda_T^{\mathbb{Q}}(t)$ follows a CIR process then the fitting with the volatility calibration could be done with default swaption, assuming the CDS price are geometric Brownian motion, implied volatility:*

$$\upsilon_T = \frac{\sqrt{Var[\tilde{s}(t)|\mathcal{F}_\prime]}}{(1-\mathcal{R}^{\mathbb{Q}})\sqrt{\frac{\lambda_T^{\mathbb{Q}}(0)}{k_T}(e^{-k_T t} - e^{-2k_T t}) + \frac{\mu_T}{2k_T}(1-e^{-k_T t})^2}}$$

where k_T has been determined before, same for μ_T with the bond market. $\mathcal{R}$ is also predetermined. Then under the CIR model, $\lambda_T^{\mathbb{Q}}(t)$ is defined by the following PDE:

$$\mathrm{d}\lambda_T^{\mathbb{Q}}(t) = k_T(\mu_T - \lambda_T^{\mathbb{Q}}(t))\mathrm{d}t + \upsilon_T\sqrt{\lambda_{T(t)}^{\mathbb{Q}}}\mathrm{d}W(t)$$

F.2 Hypothesis to simplify this formula

In my thesis $(1-\mathcal{R}^{\mathbb{Q}})$=0.5 and using the fact that:

$$\lim_{t\to\infty} 4Var[\tilde{s}(t)] = \lim_{t\to\infty} Var[\lambda_T^{\mathbb{Q}}(t)]$$

$$\upsilon_T = \frac{2\sqrt{Var[\tilde{s}]}}{\sqrt{\frac{\mu_T}{2k_T}}} = 2\sqrt{\frac{2k_T Var[\tilde{s}]}{\mu_T}}$$

F.3 Under Exponential Vasicek model

The Exponential Vasicek model has many advantages, first it is not tractable, the hazard rate is positive, and the distribution of the hazard is log-normal, the rate is mean reverting, that is a really important feature.

A great feature of the exponential Vasicek model is that on the contrary to the CIR there is no condition on the parameter to satisfy the stability.

Definition F.4. In the Exponential Vasicek model, the stochastic process will be described by:

$$\lambda_n^{\mathbb{Q}}(t) = e^{y(t)}$$

$$\mathrm{d}y(t) = k(\mu - y(t))\mathrm{d}t + \upsilon \mathrm{d}W(t)$$

Where W(t) is a standard Brownian motion under the risk-neutral probability.

Remark F.5. On the contrary to the others models in this paper this model is not tractable.

AS stated in the definition, if $\lambda_T^{\mathbb{Q}}(t)$ follows a Exponential Vasicek model, then it solve the following PDE:

$$\lambda_T^{\mathbb{Q}}(t) = e^{y_T(T)}$$

$$\mathrm{d}y_T(t) = k_T(\mu_T - y_T(t))\mathrm{d}t + \upsilon_T \mathrm{d}W(t)$$

On the long-term the process $y_T(t)$ is equivalent to a normal random variable X$\sim \mathcal{N}(\mu_T, \upsilon_T^2)$. Moreover as $\lambda_T^{\mathbb{Q}}(t)$=$e^{y_T(t)}$, $\lambda_T^{\mathbb{Q}}(t)$ as a log-normal distribution, with the following properties:

$$Var[\lambda_T^{\mathbb{Q}}(t)] = (e^{Var[y_T(t)]} - 1)e^{2\mathbb{E}[\frown_{\mathbb{T}}(\approx)]+\mathbb{V}\partial\searrow[\frown_{\mathbb{T}}(\approx)]}$$

$$\mathbb{E}[\lambda_T^{\mathbb{Q}}(t)] = e^{\mathbb{E}[y_T(t)]+\frac{Var[y_T(t)]}{2}}$$

$$with\ market\ data: \quad Var[\tilde{s}|\mathcal{F}_0] = \frac{1}{(1-\mathcal{R})^2}Var[\lambda_T^{\mathbb{Q}}|\mathcal{F}_0]\ and\ Mean[\lambda_T^{\mathbb{Q}}]^2 = \frac{1}{(1-\mathcal{R})^2}(\ Mean[\tilde{s}])^2$$

By writing $\mathbb{E}[y_T] = \ln(2\mathbb{E}[\tilde{s}]) - \frac{Var[y_T]}{2}$
an so substituting in the $Var[\lambda_T^{\mathbb{Q}}]$ formula:

$$Var[\tilde{s}] = \frac{1}{4}(1-\mathcal{R})^2\left(e^{Var[y_T]} - 1\right)\left(e^{2\ln(2\mathbb{E}[\tilde{s}])-Var[y_T]+Var[y]}\right)$$

$$Var[\tilde{s}](e^{Var[y_T-1]})\mathbb{E}[\tilde{s}]^2$$

$$Var[y] = \ln\left(\frac{Var[\tilde{s}]}{\mathbb{E}[\tilde{s}]^2} + 1\right)$$

$$\upsilon_T = \sqrt{\ln\left(\frac{Var[\tilde{s}]}{\mathbb{E}[\tilde{s}]^2} + 1\right)}$$

Theorem F.6. *Under the exponential Vasicek model the parameter υ, which has a major impact on volatility of the process could be determined by the market data, and the following formula:*

$$\upsilon_T = \sqrt{\ln\left(\frac{Var[\tilde{s}]}{\mathbb{E}[\tilde{s}]^2} + 1\right)}$$

$$\upsilon_T = \sqrt{\ln\left(\frac{Var[\lambda_T^{\mathbb{Q}}|\mathcal{F}_\prime]}{Mean[\lambda_T^{\mathbb{Q}}]^2} + 1\right)}$$

$$in\ the\ thesis\ I\ take\ \mathcal{R}^{\mathbb{Q}} = \frac{1}{2}$$

$$as\ Var[\tilde{s}|\mathcal{F}_0] = 4Var[\lambda_T^{\mathbb{Q}}|\mathcal{F}_0]\ and\ Mean[\lambda_T^{\mathbb{Q}}]^2 = (2\ Mean[\tilde{s}])^2$$

$$(1-\mathcal{R})^2 Var[\lambda_T^{\mathbb{Q}}|\mathcal{F}_\prime] = Var[\tilde{s}|\mathcal{F}_\prime]$$

$$\upsilon_T = \sqrt{\ln\left(\frac{Var[\lambda_T^{\mathbb{Q}}|\mathcal{F}_\prime]}{e^{2\mu+\upsilon_T^2}} + 1\right)}$$

The $Var[\tilde{s}]$ could be determined b an analysis of the market, by looking at the past CDS spread volatility for the corresponding maturity.

$$\mathbb{E}[y_T(t)|\mathcal{F}_\prime] = y_0 e^{-k_T t} + \mu_T(1 - e^{-k_T t})$$

the value of $y_T(0)$ is the spot value of $\ln(\lambda_T^{\mathbb{Q}}(0))$.
μ_T is the mean reversion level implied by the analysis of the market and determined by the behaviour of the past market value of $\lambda_T^{\mathbb{Q}}$.
As

$$\upsilon_T = \sqrt{\ln\left(\frac{Var[\tilde{s}]e^{-\upsilon_T^2}}{(1-{}^{\mathbb{Q}}\mathcal{R})^2 e^{2\mu_T}} + 1\right)}$$

$$e^{\upsilon_T^2} = \frac{Var[\tilde{s}]e^{-\upsilon_T^2}}{(1-\mathcal{R}^{\mathbb{Q}})^2 e^{2\mu_T}} + 1$$

$$e^{2\upsilon_T^2} = \frac{Var[\tilde{s}]}{(1-\mathcal{R}^{\mathbb{Q}})^2 e^{2\mu_T}} + e^{\upsilon_T^2}$$

Let's solve the following equation, where X is the solution:

$$X^2 - X - \frac{Var[\tilde{s}]}{(1-\mathcal{R}^{\mathbb{Q}})^2 e^{2\mu_T}} = 0$$

As x=$e^{\upsilon_T^2}$, x$\geq$1, it remains one solution:

$$x = \frac{1 + \sqrt{1 + 4\frac{Var[\tilde{s}]}{(1-\mathcal{R}^{\mathbb{Q}})^2 e^{2\mu_T}}}}{2}$$

It follows:

$$\upsilon_T = \sqrt{\ln\left(\frac{1 + \sqrt{1 + 4\frac{Var[\tilde{s}]}{(1-\mathcal{R}^{\mathbb{Q}})^2 e^{2\mu_T}}}}{2}\right)}$$

As in my thesis the parameter $\mathcal{R}^{\mathbb{Q}} = \frac{1}{2}$, the formula is now:

$$\upsilon_T = \sqrt{\ln\left(\frac{1 + \sqrt{1 + 16Var[\tilde{s}]e^{-2\mu_T}}}{2}\right)}$$

As $\sqrt{1+16Var[\tilde{s}]e^{-2\mu_T}} > 1$, $\ln\left(\frac{1+\sqrt{1+16Var[\tilde{s}]e^{-2\mu_T}}}{2}\right)$ >0, everything is ok.

The mean reversion level l of the $\lambda_T^{\mathbb{Q}}$, will influence the y_T mean reversion level μ_T:

$$\mu_T = \ln(l) - \frac{\upsilon_T^2}{2}$$

G Parameters

The following parameters are the results over a period from 2007 to today, the variance is the daily variance.

For the CIR model for any n, with this data, I have stability as:

$$2k\mu_n > \upsilon_n^2$$

ITALY								
Market Data					CIR parameters		Exp. Vasicek parameters	
T	Mean[$\tilde{s}$]	Variance[$\tilde{s}$]	Mean[$\lambda_T^{\mathbb{Q}}$]	Var[$\lambda_T^{\mathbb{Q}}$]	μ_T	υ_T	μ_T	υ_T
1Y	0.008406	0.0000622	0.01681	0.0002486	0.01681	0.1216	-4.4014	0.7944
2Y	0.01060	0.0000691	0.02120	0.0002762	0.02120	0.1141	-4.0933	0.6921
3Y	0.01242	0.0000715	0.02484	0.0002858	0.02484	0.1073	-3.8856	0.6169
4Y	0.01358	0.0000676	0.02715	0.0002702	0.02715	0.09976	-3.7625	0.5588
5Y	0.01461	0.0000638	0.02923	0.0002552	0.02923	0.09344	-3.6632	0.5112
10Y	0.01679	0.00000469	0.03358	0.0001878	0.03358	0.07478	-3.4708	0.3925

Parameters to fit the CDS market							
Market Data			CIR parameters		Exp. Vasicek parameters		Jump parameter
T	CDS Spread in bps	$\lambda_T^{\mathbb{Q}}(0)$	μ_T	υ_T	μ_T	υ_T	υ_J
1Y	141	0.0282	0.026	0.1216	-4.6	1.0904	0.08
2Y	181	0.0362	0.0365	0.1141	-4.02	0.9185	0.08
3Y	212	0.0424	0.042	0.1073	-3.64	0.7871	0.08
4Y	230	0.0460	0.046	0.09976	-3.42	0.6705	0.08
5Y	248	0.0496	0.0505	0.09344	-3.25	0.5864	0.08
10Y	276	0.0552	0.055	0.07478	-3	0.3125	0.08

FRANCE								
Market Data					CIR parameters		Exp. Vasicek parameters	
T	Mean[$\tilde{s}$]	Variance[$\tilde{s}$]	Mean[$\lambda_T^{\mathbb{Q}}$]	Var[$\lambda_T^{\mathbb{Q}}$]	μ_T	υ_T	μ_T	υ_T
1Y	0.001721	0.00000320	0.003442	0.00001278	0.003442	0.06093	-6.0376	0.8554
2Y	0.002277	0.00000422	0.004554	0.00001688	0.004554	0.06088	-5.6895	0.7717
3Y	0.002899	0.00000551	0.005798	0.00002204	0.005798	0.06165	-5.4023	0.7101
4Y	0.003585	0.00000630	0.007170	0.00002518	0.007170	0.05926	-5.1372	0.6314
5Y	0.004217	0.00000736	0.008434	0.00002944	0.008434	0.05908	-4.9487	0.5885
10Y	0.006006	0.00000690	0.01201	0.00002760	0.01201	0.04794	-4.5096	0.4184

Parameters to fit the CDS market							
Market Data			CIR parameters		Exp. Vasicek parameters		Jump parameter
T	CDS Spread in bps	$\lambda_T^{\mathbb{Q}}(0)$	μ_T	υ_T	μ_T	υ_T	υ_J
1Y	4.26	0.000852	0.001	0.06	-6.9	0.65	0.01
2Y	5.76	0.001152	0.0011	0.045	-6.8	0.5	0.01
3Y	7.41	0.001481	0.0017	0.04	-6.57	0.45	0.01
4Y	9.59	0.001918	0.002	0.06	-6.4	0.4	0.01
5Y	11.76	0.002352	0.0023	0.04	-6.15	0.4	0.01
10Y	23.43	0.004686	0.005	0.045	-5.5	0.45	0.01

SPAIN								
Market Data					CIR parameters		Exp. Vasicek parameters	
T	Mean[$\tilde{s}$]	Variance[$\tilde{s}$]	Mean[$\lambda_T^{\mathbb{Q}}$]	Var[$\lambda_T^{\mathbb{Q}}$]	μ_T	υ_T	μ_T	υ_T
1Y	0.008300	0.00000584	0.01660	0.00002336	0.01660	0.1678	-4.1390	0.2852
2Y	0.01002	0.00000702	0.02004	0.00002806	0.02004	0.1673	-3.9438	0.2599
3Y	0.01137	0.000000752	0.02274	0.00003006	0.02274	0.1626	-3.8119	0.2377
4Y	0.01231	0.00000743	0.02462	0.00002470	0.02464	0.1553	-3.7242	0.1999
5Y	0.01310	0.00000729	0.02620	0.00002916	0.0262	0.1492	-3.6628	0.2040
10Y	0.01484	0.00000539	0.02968	0.00002154	0.02968	0.1205	-3.5294	0.1554

Parameters to fit the CDS market							
Market Data			CIR parameters		Exp. Vasicek parameters		Jump process
T	CDS Spread in bps	$\lambda_T^{\mathbb{Q}}(0)$	μ_T	υ_T	μ_T	υ_T	υ_J
1Y	16.4	0.00328	0.0031	0.06	-5.8	0.2852	0.05
2Y	21.33	0.004166	0.0043	0.055	-5.55	0.2599	0.05
3Y	26.65	0.00533	0.0051	0.05	-5.3	0.2377	0.05
4Y	32.89	0.006578	0.0063	0.048	-5.02	0.199	0.05
5Y	38.73	0.007746	0.0075	0.047	-4.88	0.2040	0.05
10Y	56.8	0.01136	0.0112	0.04	-4.5	0.1554	0.05

PORTUGAL								
Market Data					CIR parameters		Exp. Vasicek parameters	
T	Mean[$\tilde{s}$]	Variance[$\tilde{s}$]	Mean[$\lambda_T^{\mathbb{Q}}$]	Var[$\lambda_T^{\mathbb{Q}}$]	μ_T	υ_T	μ_T	υ_T
1Y	0.02518	0.001022	0.05036	0.004088	0.05038	0.2849	-3.4686	0.9798
2Y	0.02884	0.001167	0.05768	0.004668	0.05768	0.2849	-3.2912	0.9363
3Y	0.02980	0.0009687	0.05960	0.003875	0.05960	0.2549	-3.1889	0.8588
4Y	0.02976	0.0007601	0.05952	0.003040	0.05952	0.2260	-3.1312	0.7871
5Y	0.02989	0.0006289	0.05978	0.002516	0.05978	0.20515	-3.0836	0.7301
10Y	0.02941	0.0003569	0.05882	0.001428	0.05882	0.1556	-3.0060	0.5878

Parameters to fit the CDS market							
Market Data			CIR parameters		Exp. Vasicek parameters		Jump process
T	CDS Spread in bps	$\lambda_T^{\mathbb{Q}}(0)$	μ_T	υ_T	μ_T	υ_T	υ_J
1Y	28.92	0.004784	0.0064	0.25	-5.1	0.8	0.03
2Y	36.99	0.007398	0.0058	0.25	-5.1	0.75	0.03
3Y	45.02	0.009004	0.0082	0.22	-4.92	0.7	0.03
4Y	53.83	0.01077	0.0108	0.196	-4.72	0.63	0.03
5Y	63.41	0.01268	0.0130	0.175	-4.5	0.58	0.03
10Y	92.45	0.01849	0.0195	0.125	-4.05	0.43	0.3

G.1 Correlation data

$$Correlation_{Italy} = \begin{bmatrix} 1 & 0.9830 & 0.9566 & 0.9382 & 0.9200 & 0.8628 \\ 0.9830 & 1 & 0.9929 & 0.9832 & 0.9713 & 0.9233 \\ 0.9566 & 0.9929 & 1 & 0.9975 & 0.9914 & 0.9526 \\ 0.9382 & 0.9832 & 0.9975 & 1 & 0.9981 & 0.9668 \\ 0.9200 & 0.9832 & 0.9914 & 0.9981 & 1 & 0.9761 \\ 0.8628 & 0.9233 & 0.9526 & 0.9668 & 0.9761 & 1 \end{bmatrix}$$

$$Correlation_{France} = \begin{bmatrix} 1 & 0.9907 & 0.9703 & 0.9433 & 0.9020 & 0.8112 \\ 0.9907 & 1 & 0.9937 & 0.9777 & 0.9467 & 0.8670 \\ 0.9703 & 0.9937 & 1 & 0.9935 & 0.9725 & 0.9054 \\ 0.9433 & 0.9777 & 0.9935 & 1 & 0.9912 & 0.9381 \\ 0.9020 & 0.9467 & 0.9725 & 0.9912 & 1 & 0.9675 \\ 0.8112 & 0.8670 & 0.9054 & 0.9381 & 0.9675 & 1 \end{bmatrix}$$

$$Correlation_{Spain} = \begin{bmatrix} 1 & 0.9907 & 0.9742 & 0.9619 & 0.9477 & 0.9078 \\ 0.9907 & 1 & 0.9948 & 0.9880 & 0.9788 & 0.9468 \\ 0.9742 & 0.9948 & 1 & 0.9983 & 0.9936 & 0.9688 \\ 0.9619 & 0.9880 & 0.9983 & 1 & 0.9984 & 0.9797 \\ 0.9477 & 0.9788 & 0.9936 & 0.9984 & 1 & 0.9870 \\ 0.9078 & 0.9468 & 0.9688 & 0.9797 & 0.9870 & 1 \end{bmatrix}$$

$$Correlation_{Portugal} = \begin{bmatrix} 1 & 0.9948 & 0.9927 & 0.9860 & 0.9757 & 0.9360 \\ 0.9948 & 1 & 0.9966 & 0.9916 & 0.9838 & 0.9489 \\ 0.9927 & 0.9966 & 1 & 0.9983 & 0.9932 & 0.9656 \\ 0.9860 & 0.9916 & 0.9983 & 1 & 0.9981 & 0.9781 \\ 0.9360 & 0.9489 & 0.9656 & 0.9781 & 0.9877 & 1 \end{bmatrix}$$

$$L_{Italy} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.9830 & 0.1836 & 0 & 0 & 0 & 0 \\ 0.9566 & 0.2863 & 0.0543 & 0 & 0 & 0 \\ 0.9382 & 0.3320 & 0.0914 & 0.03452 & 0 & 0 \\ 0.9200 & 0.3646 & 0.1279 & 0.06405 & 0.01435 & 0 \\ 0.8628 & 0.4094 & 0.1848 & 0.1303 & 0.07492 & 0.1767 \end{bmatrix}$$

$$L_{France} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.9907 & 0.1361 & 0 & 0 & 0 & 0 \\ 0.9703 & 0.2382 & 0.0422 & 0 & 0 & 0 \\ 0.9433 & 0.3172 & 0.0630 & 0.0748 & 0 & 0 \\ 0.9020 & 0.3901 & 0.1035 & 0.1348 & 0.0736 & 0 \\ 0.8112 & 0.4654 & 0.1762 & 0.1894 & 0.1424 & 0.1953 \end{bmatrix}$$

$$L_{Spain} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.9960 & 0.1368 & 0 & 0 & 0 & 0 \\ 0.9742 & 0.2175 & 0.1509 & 0 & 0 & 0 \\ 0.9619 & 0.2569 & 0.0354 & 0.0866 & 0 & 0 \\ 0.9477 & 0.2925 & 0.0446 & 0.1165 & 0.0273 & 0 \\ 0.9078 & 0.3475 & 0.05858 & 0.1749 & 0.0786 & 0.1222 \end{bmatrix}$$

$$L_{Portugal} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.9948 & 0.1018 & 0 & 0 & 0 & 0 \\ 0.9927 & 0.0890 & 0.0814 & 0 & 0 & 0 \\ 0.9860 & 0.1054 & 0.1243 & 0.0353 & 0 & 0 \\ 0.9757 & 0.1294 & 0.1610 & 0.0682 & 0.0263 & 0 \\ 0.9360 & 0.1745 & 0.2568 & 0.1386 & 0.0292 & 0.0788 \end{bmatrix}$$

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