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Implementing an IR-FX Model for CVA Calculation

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Declaration

The work contained in this thesis is my own work unless otherwise stated.

Signature and date:

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Abstract

This thesis focuses on the development and implementation of a one-factor Hull White model for modelling Interest Rates (IR) and Foreign Exchange (FX) with the overall aim of computing CVA on portfolios of derivatives in multiple different currencies. Specifically, a one-factor model is assumed for each stochastic factor considered, for example a foreign bond or an FX spot rate, with non-zero correlation modelled between the factors.

We calibrate respective interest rates (foreign and domestic) to co-terminal swaptions and provide detail on the calibration procedure before showcasing results. Due to time constraints, the respective FX rates are not calibrated to FX options but instead the volatility of each FX diffusion is user specified. Following on from this, we implement the diffusion for each of the stochastic factors to build the necessary curves required for the CVA calculation. Finally, we present results obtained when calculating CVA for a portfolio of different flavours of Interest Rate Swaps (IRSs) and analyse the structure of the Expected Positive Exposure (EPE).

The implementation was carried out in Python in an object-oriented framework, integrated within the current pricing library developed by the Mazars Quantitative Solutions Team, which we refer to as the Valuation Platform (hereafter, *VP*). Since this is a proprietary library, the code will not be included in this report. We do however, outline some of the algorithms that the code is based on.

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Introduction

Credit Valuation Adjustments (hereafter *CVA*) have become increasingly important since the financial crisis as credit risk can no longer be considered to be negligible. CVA represents the discount an investor obtains on a trade with a default-risky counterparty. Indeed, the higher the perceived likelihood of default of the counterparty, the greater the discount the investor should obtain.

As shown in [1, Slide 837], the calculation of CVA on a particular financial product adds an extra level of optionality to the payoff; the impact of this increase in complexity is that products that were once model independent become model dependent in the underlying market. The first section of this thesis will establish the stochastic calculus framework that determines the evolution of the fundamental quantities. The scope of products in the portfolios that we wish to consider include derivatives in multiple currencies for example interest rate swaps, cross currency swaps, FX options, IR options etc., and therefore for full generality we will model foreign exchange (FX) as well as interest rate (IR) risk drivers.

The framework we choose to model interest rate dynamics is a Hull White 1-factor model, so that, when considering a portfolio of N currencies. As we will see, this will result in the diffusion of $(2N - 1)$ correlated stochastic factors; we have (N) interest rate factors and $(N - 1)$ FX spot rates. The choice of such a model hinges on its tractability [1, Slide 274] as well as the fact that it allows for negative interest rates, a desirable property in today's financial climate. Such a model is poorly adapted to price products which main risk factor is a spread between two tenors of the curve such as CMS spread options due to the fact that it only allows for a parallel shift in the yield curve [2, page 19]. This of course limits the scope of the products in the portfolio that we wish to consider, however due to its simplicity in implementation and calibration, we still favour it over a multi-factor approach.

The internal derivatives pricing library developed by Mazars' Quant Team, the Valuation Platform (VP) already has the ability to price simple vanilla products using market data downloaded from Bloomberg, but it does not have a robust diffusion model implementation. A longer-term goal of this project is the full integration of the diffusion model with the current VP Pricers, so that the "Market Data" becomes the diffused market data simulated via our model. Due to the complex structure of the library, this is a non-trivial task that will require more time to be properly implemented. The calibration of the model to relevant market data needs to be implemented with enough generality to cope with any currency, and any choice of market data. In particular, the exact instruments that we calibrate to depends on the portfolio we wish to calculate the CVA on. The implementation will be carried out in `Python` using an object-oriented architecture.

Literature Review

The Hull White framework for modelling interest rates is a well studied one, with a wide range of books and papers available dating back to 1990 when the model was first introduced by John Hull and Alan White. Most frequently, we see papers defining the short rate $r(t)$ as a stochastic differential equation (SDE), with its corresponding mean reversion and volatility parameters, for example in [1] and [3]. Since the short rate is not observable in the market, the framework we will be developing in is proposing an SDE for the zero coupon bond in terms of the short rate. From a practitioner's point of view, working in a framework with market instruments (bonds) as opposed to mathematical concepts (short rate) is favourable, although the two are completely equivalent, see for example [4, page 2]. In fact, we could only find one or two papers, namely [4] that use the Hull White model in a Heath-Jarrow-Morton (HJM) framework with the bond modelled by an SDE. Thus, we derive formulas satisfied by the domestic and foreign bonds, and FX Spot rate, from scratch.

Similarly, there is a wealth of articles describing the calibration process of the Hull White 1-factor model. We found the most useful to be Gurrieri, Nakabayashi and Wong's paper *Calibration Methods of Hull White Model*, which presented different parameterisations for the instantaneous volatility and subsequent calibration results. From this, we gained inspiration for extending the list of co-terminal swaptions used to calibrate on, when there are relatively few exact co-terminals quoted in the market. Whilst we have not found in any literature details of the exact calibration procedure for our setup, which assumes a piecewise linear instantaneous volatility for the bonds and FX spot rate, this paper provided a strong base of the understanding of the basic concepts.

For more general mathematical finance tools, such as change of measure, Itô's Lemma, and formulation of CVA, we relied heavily on Brigo and Mercurio's comprehensive book entitled *Interest Rate Models - Theory and Practice*. A lot of the understanding for the drift-freezing approximation for pricing a swaption analytically was also obtained from here.

Whilst a simple and well-studied model, efficient and robust implementation is a lengthy and challenging task. In particular, I had to skip the calibration of the FX model to FX options which I will come back at a later phase beyond the internship.

Chapter 1

Defining the Model

1.1 Mathematical Preliminaries

1.1.1 Notation

Denote by $B_d(t,u)$ the time$-t$ price of a zero coupon bond (ZCB) in the domestic currency which pays 1 at time u for $(0 \leq t \leq u)$, and $B_f(t,u)$ the corresponding bond in the foreign currency. As is standard in the literature, we denote by $f(\cdot,\cdot)$ the instantaneous forward rate function, $r(\cdot)$ the short rate defined by $f(x,x) = r(x)$. We also denote the FX Spot rate at time$-t$ by $X(t)$. Further notation such as instantaneous volatility and correlation will be defined at a later stage.

We state the change of numeraire formula which will be fundamental in the derivation of the swaption price as well as the CVA calculation. For Itô's formula, the rigorous mathematical results can be found in [5, Theorem 31.1], which we will use but not state.

1.1.2 Change of Numeraire

The discounted price of asset X is a martingale under the numeraire associated to the bond. Explicitly, if we denote by X_t the price of X at time t and $\mathbb{E}^B$ the expected value under the numeraire associated with the bond B, it follows that

$$\begin{aligned} X_t &= \mathbb{E}_t^B\left[B(t)\frac{X(T)}{B(T)}\right], \quad \mathbb{Q}^B \\ &= \mathbb{E}_t^H\left[B(t,H)\frac{X(T)}{B(T,H)}\right], \quad \mathbb{Q}^H \\ \implies \frac{X_0}{B(0,H)} &= \mathbb{E}^H\left[\frac{X(T)}{B(T,H)}\right] \end{aligned} \tag{1.1.1}$$

where in the second line we change numeraire from the one associated with the bond to the one associated with the *domestic* $H-$forward measure $B(t,H)$ as illustrated in [1, slide 256] and in the final line we set $t=0$ and use the fact that $B(t,H)$ is $\mathcal{F}_0$-measurable.

1.2 Domestic Interest Rate Dynamics

Firstly, we define a framework for the interest rate dynamics. We will be modelling both "domestic" and "foreign" interest rates, as well as the FX spot rate between each combination of two currencies. We will use a one-factor model for each of the stochastic factors due to its simplicity. This is inline with how most banks model interest rates for their CVA framework up to this date. It is also outlined in [1, Slide 212] through principal component analysis of historical data that a one factor explains up to 76% of the total variation, and as much as 92% for the UK Market. Assuming of course that we do not intend to price spread dependent products, where a multi-factor model is required, the higher accuracy obtained by a 2-factor model is not significant enough to outweigh a more technical implementation and calibration procedure, in particular the correlation between the two factors. We therefore choose a Hull White 1-factor model expressed in an HJM framework, which we will explicitly define in the following sections.

Traditionally when defining a Hull White model for interest rates, an SDE for the short rate is chosen. Since we are working within an HJM framework, we will define an SDE for the bond in terms of the short rate, ultimately leading to an expression for the bond price that will not include explicit short rate dependence, but only implicit dependence through the other parameters.

It is important to note that our model will, overall, consist of $(2N-1)$ factors if we have N currencies including the domestic currency, as mentioned in the Introduction. For notational convenience, this section will write a subscript f to denote the "foreign currency", which in reality will be $f_1, \dots, f_{N-1}$. The corresponding FX spot rates given by X_{f_i,f_j} for $i \neq j; \quad i,j \in \{1,\dots,N-1\}$ and $X_{f_k,d}$ for $k \in \{1,\dots,N-1\}$ for the rates between foreign currencies f_i and f_j and foreign and domestic currencies f_k and d respectively, will be denoted by "X". It will become clear on the context which specific currency is being referred to at any one time.

Consequently, the measure change from the risk-neutral measure to the *$H-$forward measure* is in fact a $2N-1$-dimensional measure change which uses the multi-dimensional version of Girsanov's theorem, [6, Theorem 5.4.1]. This is equivalent to $2N-1$ one-dimensional measure changes, as long as the correlation between the factors is correctly expressed. Since we are using a single subscript f to denote the foreign currencies, we will simply illustrate 3 separate one-dimensional measure changes; the domestic and foreign bonds and the FX Spot rate.

Proposition 1.2.1 (Price of the domestic bond). *Consider a given probability space $(\Omega, \mathcal{F}, \mathbb{Q}^d)$ supporting a Brownian motion $(W_t^d)_{t\geq 0}$. Let $B_d(s,T)$ denote the value of a bond at time $s, 0 \leq s \leq T$ that matures at time $T > s$. Let $r_d : \mathbb{R} \to \mathbb{R}$ and $\sigma_d(\cdot, T) : \mathbb{R} \to \mathbb{R}$ be functions defining the short rate and the instantaneous volatility corresponding to the domestic bond. Assuming the bond price satisfies the SDE given by*

$$\frac{dB_d(s,T)}{B_d(s,T)} = r_d(s)\mathrm{d}s + \sigma_d(s,T)\mathrm{d}W_s^d, \quad B_d(t,t) = 1 \text{ for all } 0 \leq t \leq T, \quad \mathbb{Q}^d \tag{1.2.1}$$

under the H-forward measure, we have the following dynamics:

$$B_d(t,T) = \frac{B_d(0,T)}{B_d(0,t)}\exp\bigg\{\int_0^t (\sigma_d(s,T) - \sigma_d(s,t))\mathrm{d}W_s^{d,H} \\ - \frac{1}{2}\int_0^t (\sigma_d(s,T) - \sigma_d(s,t))(\sigma_d(s,T) - 2\sigma_d(s,H) + \sigma_d(s,t))\mathrm{d}s\bigg\} \quad (1.2.2)$$

Proof.

The proof uses a simple application of Itô's Lemma to the logarithm of the bond price before integrating and finally exponentiating. We then use the fact that $B_d(t,t) = 1$ to eliminate the short rate dependence. Indeed,

$$\begin{aligned}
\mathrm{d}\left[\log(B_d(s,T))\right] &= \frac{1}{B_d(s,t)}\mathrm{d}B_d(s,T) - \frac{1}{2B_d(s,T)^2}\mathrm{d}[B_d(s,T)] \\
&= \left[r_d(s) - \frac{1}{2}\sigma_d^2(s,T)\right]\mathrm{d}s + \sigma_d(s,T)\mathrm{d}W_s^d \\
\implies \log(B_d(t,T)) - \log(B_d(0,T)) &= \int_0^t \left[r_d(s) - \frac{1}{2}\sigma_d^2(s,T)\right]\mathrm{d}s + \int_0^t \sigma_d(s,T)\mathrm{d}W_s^d \quad (1.2.3) \\
\implies B_d(t,T) &= B_d(0,T)\exp\left\{\int_0^t \left[r_d(s) - \frac{1}{2}\sigma_d^2(s,T)\right]\mathrm{d}s + \int_0^t \sigma_d(s,T)\mathrm{d}W_s^d\right\} \quad (1.2.4)
\end{aligned}$$

We now use the assumption that $B_d(t,t) = 1$. Substituting this into (1.2.4) yields:

$$\begin{aligned}
1 = B_d(t,t) &= B_d(0,t)\exp\left\{\int_0^t \left[r_d(s) - \frac{1}{2}\sigma_d^2(s,t)\right]\mathrm{d}s + \int_0^t \sigma_d(s,t)\mathrm{d}W_s^d\right\} \\
\implies \frac{1}{B_d(0,t)} &= \exp\left\{\int_0^t \left[r_d(s) - \frac{1}{2}\sigma_d^2(s,t)\right]\mathrm{d}s + \int_0^t \sigma_d(s,t)\mathrm{d}W_s^d\right\} \quad (1.2.5)
\end{aligned}$$

Using this along with some simple algebra in (1.2.5) enables us to write:

$$\begin{aligned}
B_d(t,T) &= \frac{B_d(0,T)}{B_d(0,t)}\exp\left\{\int_0^t \left(\sigma_d(s,T) - \sigma_d(s,t)\right)\mathrm{d}W_s^d - \frac{1}{2}\int_0^t \left(\sigma_d^2(s,T) - \sigma_d^2(s,t)\right)\mathrm{d}s\right\} \\
&= \frac{B_d(0,T)}{B_d(0,t)}\exp\left\{\int_0^t \left(\sigma_d(s,T) - \sigma_d(s,t)\right)\mathrm{d}W_s^d - \frac{1}{2}\int_0^t \left(\sigma_d(s,T) - \sigma_d(s,t)\right)\left(\sigma_d(s,T) + \sigma_d(s,t)\right)\mathrm{d}s\right\} \quad (1.2.6)
\end{aligned}$$

Define the measure

$$\mathbb{Q}^{d,H}(A) := \mathbb{E}^{\mathbb{Q}^d}\left[Z_1(T)\mathbb{1}_A\right]$$

where

$$Z_1(t) := \exp\left\{\int_0^t \sigma_d(s,H)\mathrm{d}W_s^{d,H} - \frac{1}{2}\int_0^t \sigma_d^2(s,H)\mathrm{d}s\right\}$$

An application of Girsanov's Theorem [6, Theorem 5.2.3] gives us that under $\mathbb{Q}^{d,H}$,

$$W_t^{d,H} = W_t^d - \int_0^t \sigma_d(s,H)\mathrm{d}s\mathrm{d}t, \quad \mathbb{Q}^{d,H} \quad (1.2.7)$$

is a $\mathbb{Q}^{d,H}$-Brownian motion. Substitution into (1.2.6) gives

$$\begin{aligned} B_d(t,T) =&\frac{B_d(0,T)}{B_d(0,t)}\exp\bigg\{\int_0^t(\sigma_d(s,T)-\sigma_d(s,t))dW_s^{d,H} \\ &-\frac{1}{2}\int_0^t(\sigma_d(s,T)-\sigma_d(s,t))(\sigma_d(s,T)-2\sigma_d(s,H)+\sigma_d(s,t))ds\bigg\}, \quad \mathbb{Q}^{d,H} \end{aligned} \tag{1.2.8}$$

□

Now, we need to be able to generate the whole bond curve, that is, the time$-t$ bond price for any maturity $M > t$. Since we are in a one-factor model, this should be possible via the generation of only one Gaussian random variable. Indeed, we first notice that the expression derived in (1.2.8) is lognormally distributed. Our assumption that σ_d is a deterministic function means that we know the distribution of the stochastic integral term inside the exponential; details of the calculation can be found in [7, page 2]. Indeed,

$$\begin{aligned} I_t := \int_0^t(\sigma_d(s,T)-\sigma_d(s,t))dW_s^{H_d} &\sim N(0,\int_0^t(\sigma_d(s,T)-\sigma_d(s,t))^2\mathrm{d}s) \\ &\sim f(d;t,T)N(0,1) \end{aligned} \tag{1.2.9}$$

where $f(d;t,T) := \sqrt{\int_0^t(\sigma_d(s,T)-\sigma_d(s,t))^2\mathrm{d}s}$. Since the second integral inside the exponential is entirely deterministic, the full term inside the exponential is normally distributed with mean equal to the deterministic integral. Thus, we have lognormality. It is important to note that from (1.2.9), the fact that the function f encapsulates the maturity means that the whole curve can be generated using one draw from a $N(0,1)$ distribution.

1.3 Foreign Interest Rate and FX Dynamics

Having defined the framework for the domestic bond, we now consider the foreign IR dynamics as well as the FX spot rate. Analogously to the domestic case, we will formulate the foreign IR dynamics by explicitly writing an SDE for the foreign bond in terms of the foreign short rate r_f. Upon solving, as in the domestic case we eliminate the explicit dependence on the short rate.

Proposition 1.3.1 (FX Rate and Foreign IR Dynamics). *Consider a given probability space $(\Omega,\mathcal{F},\mathbb{Q}^X)$ supporting a Brownian motion $(W_t^X)_{t\geq 0}$ and a probability space $(\Omega,\mathcal{F},\mathbb{Q}^f)$ supporting a Brownian motion $(W_t^f)_{t\geq 0}$. Denote the correlation between W_t^X and W_t^f by $\rho_{X,f}$, so that $[W^X,W^f]_t = \rho_{X,f}t$. Let X_t denote the price of the FX spot rate at time-t, r_d as in Proposition 1.2.1, and $r_f:\mathbb{R}\to\mathbb{R}$ be the function representing the foreign short rate. Finally, let $\sigma_t^X:\mathbb{R}\to\mathbb{R}$, $\sigma_f(\cdot,T):\mathbb{R}\to\mathbb{R}$ be deterministic functions representing the instantaneous volatility of X and the foreign bond respectively. Assume the following SDEs for the dynamics of the FX spot rate and foreign bond.*

$$\frac{\mathrm{d}X_t}{X_t} = (r_d(t)-r_f(t))\,\mathrm{d}t + \sigma_X(t)\mathrm{d}W_t^X, \quad \mathbb{Q}^{\mathbb{X}} \tag{1.3.1}$$

$$\frac{\mathrm{d}B_f(t,T)}{B_f(t,T)} = [r_f(t)-\rho_{X,f}\sigma_f(t,T)\sigma_X(t)]\,\mathrm{d}t + \sigma_f(t,T)\mathrm{d}W_t^f, \quad B_f(t,t)=1 \text{ for all } 0\leq t\leq T, \quad \mathbb{Q}^f \tag{1.3.2}$$

Then, the following equations hold under the H-forward measure

$$
\begin{aligned}
X(t) = \frac{X(0)B_f(0,t)}{B_d(0,t)} \exp\Big\{ & \frac{1}{2}\int_0^t \left(\sigma_d^2(s,t) - \sigma_f^2(s,t) - \sigma_X^2(s)\right) \mathrm{d}s + \rho_{f,d}\int_0^t \sigma_d(s,H)\sigma_f(s,t)\mathrm{d}s \\
& - \rho_{X,f}\int_0^t \sigma_f(s,t)\sigma_X(s)\mathrm{d}s + \rho_{X,d}\int_0^t \sigma_d(s,H)\sigma_X(s)\mathrm{d}s \\
& - \int_0^t \sigma_d(s,H)\sigma_d(s,t)\mathrm{d}s + \int_0^t \sigma_X(s)\mathrm{d}W_s^{X,H} \\
& + \int_0^t \sigma_f(s,t)\mathrm{d}W_s^{f,H} - \int_0^t \sigma_d(s,t)\mathrm{d}W_s^{d,H} \Big\}
\end{aligned} \tag{1.3.3}
$$

$$
\begin{aligned}
B_f(t,T) = \frac{B_f(0,T)}{B_f(0,t)} \exp\Big\{ & \int_0^t \rho_{f,d}(s)\sigma_d(s,H)\left[\sigma_f(s,T) - \sigma_f(s,t)\right] \mathrm{d}s \\
& - \int_0^t \left\{\rho_{X,f}\sigma_X(s)(\sigma_f(s,T) - \sigma_f(s,t))\right\} \mathrm{d}s \\
& - \frac{1}{2}\int_0^t \left[\sigma_f^2(s,T) - \sigma_f^2(s,t)\right] \mathrm{d}s \\
& + \int_0^t \left(\sigma_f(s,T) - \sigma_f(s,t)\right) \mathrm{d}W_s^{f,H} \Big\}
\end{aligned} \tag{1.3.4}
$$

Proof.

Solving the second SDE in the system (1.3.2) by using Itô's Lemma (solution is analogous to the steps involved in obtaining (1.2.4)). We obtain

$$
B_f(t,T) = B_f(0,T)\exp\left\{\int_0^t \left(r_f(s) - \rho_{X,f}\sigma_f(s,T)\sigma_X(s) - \frac{1}{2}\sigma_f^2(s,T)\right)\mathrm{d}s + \int_0^t \sigma_f(s,T)\mathrm{d}W_s^f\right\} \tag{1.3.5}
$$

Again using the fact that $B_f(t,t) = 1$, we have

$$
\frac{1}{B_f(0,t)} = \exp\left\{\int_0^t \left(r_f(s) - \rho_{X,f}\sigma_f(s,t)\sigma_X(s) - \frac{1}{2}\sigma_f^2(s,t)\right)\mathrm{d}s + \int_0^t \sigma_f(s,t)\mathrm{d}W_s^f\right\} \tag{1.3.6}
$$

so that

$$
\begin{aligned}
B_f(t,T) = \frac{B_f(0,T)}{B_f(0,t)} \exp\Big\{ & - \int_0^t \left\{\rho_{X,f}\sigma_X(s)(\sigma_f(s,T) - \sigma_f(s,t))\right\} \mathrm{d}s \\
& - \frac{1}{2}\int_0^t \left[\sigma_f^2(s,T) - \sigma_f^2(s,t)\right] \mathrm{d}s \\
& + \int_0^t \left(\sigma_f(s,T) - \sigma_f(s,t)\right) \mathrm{d}W_s^f \Big\}
\end{aligned} \tag{1.3.7}
$$

Now we apply Girsanov's theorem to derive an equation satisfied by the bond under the H-forward measure. Define the measure

$$
\mathbb{Q}^{f,H}(A) := \mathbb{E}^{\mathbb{Q}^f}\left[Z_2(T)\mathbb{1}_A\right]
$$

where

$$
Z_2(t) := \exp\left\{\rho_{f,d}\int_0^t \sigma_d(s,H)\mathrm{d}W_s^f - \frac{\rho_{f,d}^2}{2}\int_0^t \sigma_d^2(s,H)\mathrm{d}s\right\}
$$

An application of Girsanov's Theorem [6, Theorem 5.2.3] gives us that under $\mathbb{Q}^{f,H}$,

$$W_t^{f,H} = W_t^f - \rho_{f,d} \int_0^t \sigma_d(s,H)\mathrm{d}s\mathrm{d}t, \quad \mathbb{Q}^{f,H} \tag{1.3.8}$$

is a $\mathbb{Q}^{f,H}$-Brownian motion. Expressed in differential notation, this is equivalent to

$$\mathrm{d}W_s^{f,H} = \mathrm{d}W_s^f - \rho_{f,d}\sigma_d(s,H)\mathrm{d}s$$

So that substituting into (1.3.7), we obtain

$$\begin{aligned} B_f(t,T) = \frac{B_f(0,T)}{B_f(0,t)} \exp\Bigg\{ & \int_0^t \rho_{f,d}\sigma_d(s,H)\left[\sigma_f(s,T) - \sigma_f(s,t)\right]\mathrm{d}s \\ & - \int_0^t \{\rho_{X,f}\sigma_X(s)(\sigma_f(s,T) - \sigma_f(s,t))\}\,\mathrm{d}s \\ & - \frac{1}{2}\int_0^t \left[\sigma_f^2(s,T) - \sigma_f^2(s,t)\right]\mathrm{d}s \\ & + \int_0^t (\sigma_f(s,T) - \sigma_f(s,t))\,\mathrm{d}W_s^{f,H} \Bigg\} \end{aligned} \tag{1.3.9}$$

Similarly, we can solve the SDE for $X(t)$ and obtain an equation under the $H-$forward probability measure. Applying Itô's Lemma to $\log(X(t))$, integrating and exponentiating gives

$$X(t) = X(0)\exp\left\{\int_0^t \left[r_d(s) - r_f(s) - \frac{1}{2}\sigma_X^2(s)\right]\mathrm{d}s + \int_0^t \sigma_X(s)\mathrm{d}W_s^X\right\} \tag{1.3.10}$$

So that, with a measure defined in a similar way to (1.3.8) with the $\rho_{X,d}$ replacing $\rho_{f,d}$, using Girsanov's theorem so that $W_t^{X,H} = W_t^X - \rho_{X,d}\int_0^t \sigma_d(s,H)\mathrm{d}s\mathrm{d}t$ is a $\mathbb{Q}^{X,H}$-Brownian motion, under this measure,

$$X(t) = X(0)\exp\left\{\int_0^t \left[r_d(s) - r_f(s) + \rho_{X,d}\sigma_d(s,H)\sigma_X(s) - \int_0^t \frac{1}{2}\sigma_X^2(s)\right]\mathrm{d}s + \int_0^t \sigma_X(s)\mathrm{d}W_s^{X,H}\right\} \tag{1.3.11}$$

By combining (1.2.5) and (1.3.6) re-written under the domestic H-forward measure, we can remove the short rate dependence. Explicitly, we have

$$\begin{aligned} \exp\left\{\int_0^t (r_d(s) - r_f(s))\,\mathrm{d}s\right\} = \frac{B_f(0,t)}{B_d(0,t)}\exp\Bigg\{ & \frac{1}{2}\int_0^t \left(\sigma_d^2(s,t) - \sigma_f^2(s,t)\right)\mathrm{d}s + \rho_{f,d}\int_0^t \sigma_d(s,H)\sigma_f(s,t)\mathrm{d}s \\ & - \rho_{X,f}\int_0^t \sigma_f(s,t)\sigma_X(s)\mathrm{d}s - \int_0^t \sigma_d(s,H)\sigma_d(s,t)\mathrm{d}s \\ & + \int_0^t \sigma_f(s,t)\mathrm{d}W_s^{f,H} - \int_0^t \sigma_d(s,t)\mathrm{d}W_s^{d,H} \Bigg\} \end{aligned} \tag{1.3.12}$$

And thus (1.3.11) becomes:

$$X(t) = \frac{X(0)B_f(0,t)}{B_d(0,t)} \exp\left\{\frac{1}{2}\int_0^t \left(\sigma_d^2(s,t) - \sigma_f^2(s,t) - \sigma_X^2(s)\right) \mathrm{d}s + \rho_{f,d}\int_0^t \sigma_d(s,H)\sigma_f(s,t)\mathrm{d}s \right.$$
$$- \rho_{X,f}\int_0^t \sigma_f(s,t)\sigma_X(s)\mathrm{d}s + \rho_{X,d}\int_0^t \sigma_d(s,H)\sigma_X(s)\mathrm{d}s$$
$$- \int_0^t \sigma_d(s,H)\sigma_d(s,t)\mathrm{d}s + \int_0^t \sigma_X(s)\mathrm{d}W_s^{X,H}$$
$$\left. + \int_0^t \sigma_f(s,t)\mathrm{d}W_s^{f,H} - \int_0^t \sigma_d(s,t)\mathrm{d}W_s^{d,H}\right\} \tag{1.3.13}$$

□

To ease notation, we have referred to the previous time-step as "0". When it comes to diffusing the curves and FX spots in practice, if we are diffusing at each point in the grid $t_i, i \in \{0, \dots, N\}, t_0 := 0, t_N := T$, the "0" will of course be replaced by the previous point in the grid.

As mentioned previously, in reality we will not have 3 factors, we will have precisely $2N-1$ factors, with correlations described in (3.1.2).

Chapter 2

IR Calibration

We will calibrate the IR models to swaptions denominated in the currency of the IR model we are calibrating. We use "co-terminal" swaptions, which are swaptions which have the same maturity, but increasing expiries and thus decreasing tenor. The primary reason for this is that the CVA of a single swap is a collection of multiple co-terminal swaptions as we will see later on.

In practice, the user should be able to specify the length of the maturity of the co-terminal swaptions, since the calibration will depend on the products within the portfolio considered. For example, the CVA on a portfolio of FX forwards with the longest maturing contract in 30 years should be calculated under a model whose interest rates are calibrated to co-terminal 30Yr swaptions. Here, we implement calibration to ATM swaptions, but with enough generality to adapt to swaptions that are, for example, 5% out-of-the-money (OTM). This will be linked to an excel file for which the user can specify precisely which swaptions to calibrate to.

Before we calibrate, we must first price a European swaption in the framework we developed in the previous chapters, which we do hereafter. Following this, we will discuss the calibration approach.

2.1 Instantaneous Volatility Parameterisation

We first need to assume a form for the deterministic, instantaneous volatility function σ[1] depending on both the maturity T and the "diffusion lag" t. The parameterisation for σ will be the integral of a piecewise constant function (and therefore will be piecewise linear), which differs from [3, 6.4.1, page 223] which assumes a piecewise constant volatility function. The former creates a continuous, strictly increasing function for the volatility, since the piecewise constant parameters will be greater than zero. For the numerical implementation and model calibration, we will assume a structure for volatility as follows:

$$\sigma(t,T) = \lambda(t)(G(T) - G(t)) \tag{2.1.1}$$
$$G(t) = \int_0^t g(s)\mathrm{d}s$$

where $\lambda(\cdot)$ is the (deterministic) mean reversion parameter, and $g(\cdot)$ is chosen to be piecewise constant, which is defined as follows.

[1]In the following, 'σ' encapsulates both the foreign and domestic instantaneous volatility dynamics.

Piecewise constant function g

Fix a set of swaptions of increasing expiries that are to be calibrated to. Without loss of generality, let the current time be 0 and define a grid $\mathbf{T}^{g_i} := [T_0, T_1, \ldots, T_N]$, where T_0 is the expiry of the shortest-expiring swaption, T_N is the (constant) maturity for each of the co-terminal swaptions, and T_i is the expiry of the i^{th} swaption.

The function $g : [0, \infty) \to \mathbb{R}_{>0}$ is defined as follows (note the flat extrapolation where $t < T_1$ and $t > T_{N-1}$:

$$g(t) = \begin{cases} g_0 & 0 < t < T_1 \\ g_i & T_i \leq t < T_{i+1}, \quad i \in \{1, \ldots, N-2\} \\ g_{N-1} & T_{N-1} \leq t \end{cases} \tag{2.1.2}$$

Furthermore, we choose a constant mean reversion parameter, that is, $\lambda(t) := \lambda$. Thus, (2.1.1) becomes

$$\begin{aligned} \sigma(t, T) &= \lambda \int_t^T g(s)\mathrm{d}s \\ &= \lambda \left[g_l(T_{l+1} - t) + \sum_{s=l+1}^{m-1} g_s(T_{s+1} - T_s) + g_m(T - T_m) \right] \end{aligned} \tag{2.1.3}$$

where $l := \max_k \{T_k : t > T_k\}$ and $m := \min_k \{T_k : T < T_{k+1}\}$

2.2 Analytical Formula for a European Swaption

A (receiver) swaption is a contract giving the right to enter a (receiver) IRS. Since we have derived formulas satisfied by the bond, it will be useful to formulate the swaption's payoff in terms of the bond, so that we have no dependency on the short rate. In the below, we use the notation $B_k(s_1, s_2)$ to denote the value of a bond *in currency* k, $k \in \{d, f_1, \ldots, f_{N-1}\}$ at time s_1 maturing at s_2, and $r_k(\cdot)$ the corresponding short rate function as seen in earlier sections. We will of course be calibrating our model to swaptions of each currency; the notation chosen here allows enough generality to show this.

Proposition 2.2.1 (European Receiver Swaption). *Denote the current time by* t*. Consider an IRS contract where the first reset date is* T_F*, maturing at time* T_N*, with legs indexed at* $T_i = \sum_{s=1}^{i} \alpha_s$*,* $\alpha_s := T_s - T_{s-1}, T_0 := T_F$*. As usual, let* $D(t, T_F) := \exp\left(-\int_t^{T_f} r_k(s)\mathrm{d}s\right)$*. Then the time-t price of a European receiver swaption can be written as:*

$$S_t = \mathbb{E}_t^k \left[\exp\left(-\int_t^{T_f} r_k(s)\mathrm{d}s\right) \left(\sum_{i=1}^{N} a_i B_k(T_F, T_i) - a_0\right)^+ \right] \tag{2.2.1}$$

where $a_i := K\alpha_i, \quad a_N := 1 + K\alpha_N, \quad a_0 := 1$

Proof.
Using [1, Slide 109], we know that the discounted payoff of a receiver IRS is given by

$$IRS^{\text{Rec}}_{\text{DiscPayoff}} = D(t,T_F)\left[\sum_{i=1}^{N} B_k(T_F,T_i)\alpha_i(K - F(T_F;T_{i-1},T_i))\right] \tag{2.2.2}$$

where K is the fixed rate, and $F(T_F;T_{i-1},T_i)$ is the forward Linear rate prevailing at time T_F for the expiry T_{i-1} and maturity T_i. By definition, we have that [1, Slide 104] $F(T_F;T_{i-1},T_i) = \frac{1}{\alpha_i}\left(\frac{B_k(T_F,T_{i-1})}{B_k(T_F,T_i)} - 1\right)$. Thus, substituting into (2.2.2), we get

$$\begin{aligned} IRS^{\text{Rec}}_{\text{DiscPayoff}} &= D(t,T_F)\left[\sum_{i=1}^{N} B_k(T_F,T_i)\alpha_i\left(K - \left(\frac{1}{\alpha_i}\left(\frac{B_k(T_F,T_{i-1})}{B_k(T_F,T_i)} - 1\right)\right)\right)\right] \\ &= D(t,T_F)\sum_{i=1}^{N}\left(K\alpha_i B_k(T_F,T_i) - B_k(T_F,T_{i-1}) + B_k(T_F,T_i)\right) \end{aligned}$$

We now notice that summing over the first two terms leads to a telescoping sum, and so we have $\sum_{i=1}^{N} B_k(T_F,T_i) - B_k(T_F,T_{i-1}) = B_k(T_F,T_N) - B_k(T_F,T_0)$. Finally, we obtain

$$IRS^{\text{Rec}}_{\text{DiscPayoff}} = D(t,T_F)\left[\sum_{i=1}^{N} K\alpha_i B_k(T_F,T_i) + B_k(T_F,T_N) - B_k(T_F,T_0)\right] \tag{2.2.3}$$

Now, by definition a swaption is an option on the IRS. Thus, taking the positive part of the payoff (2.2.3) and using that $D(t,T_F) > 0$ (so can be taken out of the positive part) before finally taking the risk neutral expectation, we get:

$$S_t = \mathbb{E}_t^k\left[(IRS^{\text{Rec}}_{\text{Payoff}})^+\right] = \mathbb{E}_t^k\left[\exp\left(-\int_t^{T_f} r_k(s)\mathrm{d}s\right)\left(K\sum_{i=1}^{N} a_i B_k(T_F,T_i) - a_0\right)^+\right]$$

Finally, we note that $B_k(T_F,T_0) = B_k(T_F,T_F) = 1$, and we can define $a_i := K\alpha_i$ for $1 \le i \le N-1$ and $a_0 = 1, a_N = (1 + K\alpha_N)$ to get exactly (2.2.1). □

Two common practices to proceed from here are to decompose (2.2.1) using Jamshidian's decomposition coupled with a one-dimensional numerical root solver, or use a drift-freezing approximation. We will briefly outline the former, and give a detailed explanation of the latter since this is the one we will implement. The reason we choose to implement the latter is mainly for its speed; it was shown in [4] that in using an analytical approximation instead of Jamshidian's decomposition, we can expect a performance reduction of around 20 times.

2.2.1 Jamshidian's Decomposition

We give a short summary on how one can use Jamshidian's decomposition to express an option on a portfolio of assets into a portfolio of options. This is only applicable in a 1-factor interest rate model such as the one we have here. We first note that we can write the value of a bond $B_k(T_F,T_i,r_k(\cdot))$ as a function of short rate with the short rate appearing with a negative sign in the exponential. This can be achieved by writing the dynamics for the short rate $r_k(\cdot)$ and using that $B_k(t,T) := \mathbb{E}_t^k\left[\exp\left\{\int_t^T r_k(s)\mathrm{d}s\right\}\right]$, see for example, [3, Eq 3.39, page 75]. We follow the steps discussed in [8]. The proof will hinge on the fact that the function $B_k(t, T, \cdot)$ can be written as a monotonically decreasing function of the short rate $r_k(\cdot)$.

Firstly, we find a value r_k^* such that

$$\sum_{i=1}^{N} K a_i B_k(T_F, T_i; r_k^*) = a_0$$

via a numerical solver. The existence and uniqueness of the solution to the above comes from the fact that $B_k(T_F, T_i; r_k^*)$ is monotonically decreasing, and thus the sum is, since all the weights Ka_i are positive. Set

$$\kappa_i := B_k(T_F, T_i; r_k^*) \tag{2.2.4}$$

Then, rewriting the swaption payoff, the term inside the positive part in (2.2.1) becomes

$$\begin{aligned}(IRS^{\text{Rec}}_{\text{Payoff}})^+ &= \left(\sum_{i=1}^{N} K a_i B_k(T_F, T_i; r_k(T_F)) - \sum_{i=1}^{N} K a_i \kappa_i\right)^+ \\ &= \left(\sum_{i=1}^{N} K a_i (B_k(T_F, T_i; r_k(T_F)) - \kappa_i)\right)^+\end{aligned}$$

Once again, the assumption of a monotonically decreasing bond price means that the expression inside the positive part is non-negative if and only if $r_k \geq r_k^*$, and the same is true for each term in the sum again using (2.2.4). Consequently,

$$(IRS^{\text{Rec}}_{\text{Payoff}})^+ = \sum_{i=1}^{N} K a_i (B_k(T_F, T_i; r_k(T_F)) - \kappa_i)^+ \tag{2.2.5}$$

Thus, we have reduced the swaption payoff to a sum of call options on the zero coupon bond, with strikes κ_i. To calculate the price (conditional expected value of the discounted payoff) one can apply the usual Black formula.

Not only is this method slower, as previously highlighted, but the approximation yields very accurate results, as will be demonstrated later in the calibration results.

2.2.2 Modified Black Formula for the Swaption

An alternative approach to Jamshidian's decomposition involves approximating the sum of lognormals as a lognormal random variable, via a drift freezing procedure. This is a standard practice and as detailed in [3, 8.7], in usual situations is a good approximation. From this, we obtain a Black formula as an analytical approximation for the swaption price.

Proposition 2.2.2 (Swaption Price under the T_F-forward measure)**.**

Under the T_F-measure, we can write the swaption price (2.2.1) *as the following:*

$$S_t = B_k(t,T_F)\mathbb{E}_t^{k,T_F}\left[(H_{T_F} - a_0)^+\right]$$
$$H_t := \sum_{i=1}^{N} a_i \frac{B_k(t,T_i)}{B_k(t,T_F)} \tag{2.2.6}$$

Proof.
If we let $B(s) := \exp\left(\int_0^s r_k(s)\right)$ be the numeraire associated with the standard risk neutral expectation $\mathbb{E}_s^k$, then we can write (2.2.1) as

$$S_t = \mathbb{E}_t^k\left[\frac{B(t)}{B(T_F)}\left(\sum_{i=1}^{N} a_i B_k(T_F,T_i) - a_0\right)^+\right]$$

Changing the measure to the one associated with the numeraire $B_k(t,T_F)$ (noting this is indeed a numeraire since by definition it is the price of a zero coupon bond), and using [1, Slide 256], we have, denoting by $\mathbb{E}^{k,T_F}$ expectation under this measure,

$$\begin{aligned}
S_t &= \mathbb{E}_t^{k,T_F}\left[\frac{B_k(t,T_F)}{B_k(T_F,T_F)}\left(\sum_{i=1}^{N} a_i B_k(T_F,T_i) - a_0\right)^+\right] \\
&= B_k(t,T_F)\mathbb{E}_t^{k,T_F}\left[\left(\sum_{i=1}^{N} a_i \frac{B_k(T_F,T_i)}{B_k(T_F,T_F)} - \frac{a_0}{B_k(T_F,T_F)}\right)^+\right] \\
&= B_k(t,T_F)\mathbb{E}_t^{k,T_F}\left[(H_{T_F} - a_0)^+\right]
\end{aligned} \tag{2.2.7}$$

Where in the second line we used the fact that $B_k(t,T_F)$ is $\mathcal{F}_t$-measurable and in the final line we used the fact that $B_k(T_F,T_F) = 1$ and hence we omit it as a multiplier of a_0. □

Proposition 2.2.3 (H_t is a $\mathbb{Q}^{k,T_F}$-martingale)**.**
H_t is a martingale under the $\mathbb{Q}^{k,T_F}$-measure, and satisfies the following SDE

$$\frac{\mathrm{d}H_t}{H_t} = \Sigma(t)\mathrm{d}W_t^{k,T_F}, \quad \mathbb{Q}^{k,T_F}$$
$$\Sigma(t) := \lambda\left(\frac{\sum_{i=1}^{N} a_i(G(T_i) - G(T_F))B_k(t,T_i)}{\sum_{i=1}^{N} a_i B_k(t,T_i)}\right) \tag{2.2.8}$$

where we assume the volatility parameterisation in (2.1.1)

Proof.
Define $Z^i(t) := \frac{B_k(t,T_i)}{B_k(t,T_F)}$. Then [1, Slide 255] tells us that under the measure $\mathbb{Q}^{k,T_F}$, $Z^i(t)$ is a martingale. Thus, we claim that it follows the following dynamics:

$$\mathrm{d}Z^i(t) = Z^i(t)\left[\sigma_k(t,T_i) - \sigma_k(t,T_F)\right]\mathrm{d}W_t^{k,T_F}, \quad \mathbb{Q}^{k,T_F}$$

Proving this begins with a simple application of Itô's lemma. Under the standard risk neutral measure $\mathbb{Q}^k$,

$$\begin{aligned}
\mathrm{d}Z^i(t) &= \mathrm{d}\left(\frac{1}{B_k(t,T_F)}\right)B_k(t,T_i) + \mathrm{d}B_k(t,T_i)\frac{1}{B_k(t,T_F)} \\
&= \left((\dots)\mathrm{d}t - \frac{\sigma_k(t,T_F)}{B_k(t,T_F)}\mathrm{d}W_t^k\right)B_k(t,T_i) + \frac{1}{B_k(t,T_F)}\left((\dots)\mathrm{d}t + \frac{\sigma_k(t,T_i)}{B_k(t,T_i)}\mathrm{d}W_t^k\right) \\
&= (\dots)\mathrm{d}t + Z^i(t)\left[\sigma_k(t,T_i) - \sigma_k(t,T_F)\right]\mathrm{d}W_t^k \qquad (2.2.9)
\end{aligned}$$

Since $Z^i(t)$ is a $\mathbb{Q}^{k,T_F}$-martingale, the claim holds since the SDE must be driftless and the diffusion coefficient is invariant to the change of measure by [1, Slide 39]. We now note that $H_t = \sum_{i=1}^N a_i Z^i(t)$ and so by linearity

$$\begin{aligned}
\frac{\mathrm{d}H_t}{H_t} &= \frac{\sum_{i=1}^N a_i \mathrm{d}Z^i(t)}{\sum_{i=1}^N a_i Z^i(t)} \\
&= \frac{\sum_{i=1}^N a_i \frac{B_k(t,T_i)}{B_k(t,T_F)}\left[\sigma_k(t,T_i) - \sigma_k(t,T_F)\right]\mathrm{d}W_t^{k,T_F}}{\sum_{i=1}^N a_i \frac{B_k(t,T_i)}{B_k(t,T_F)}} \\
&= \frac{\sum_{i=1}^N a_i B_k(t,T_i)\left[\sigma_k(t,T_i) - \sigma_k(t,T_F)\right]\mathrm{d}W_t^{k,T_F}}{\sum_{i=1}^N a_i B_k(t,T_i)}
\end{aligned}$$

Proposition 2.2.3 follows by substituting in the assumed model for the volatility. □

Now, the fact that $B_k(t,T_i)$ is a time dependent process means that the dynamics of H_t are not lognormal. A good approximation to (2.2.8) is to freeze the time dependency to time $u < t$ where the bond price is known (to remove the randomness in Σ creating a deterministic function) to give the following approximated dynamics for H_t

$$\begin{aligned}
\frac{\mathrm{d}H_t}{H_t} &= \hat{\Sigma}(t;u)\mathrm{d}W_t^{k,T_F}, \quad \mathbb{Q}^{k,T_F} \\
\hat{\Sigma}(t;u) &:= \lambda\left(\frac{\sum_{i=1}^N a_i(G(T_i) - G(T_F))B_k(u,T_i)}{\sum_{i=1}^N a_i B_k(u,T_i)}\right) \qquad (2.2.10)
\end{aligned}$$

Black's formula

We now use this approximation to form an expression for the price of a swaption.

$$\begin{aligned}
\mathrm{d}[\log H_s] &= \frac{1}{H_s}\mathrm{d}H_s - \frac{1}{2H_s^2}\mathrm{d}[H,H]_s \\
&= -\frac{1}{2}\hat{\Sigma}^2(s;0)\mathrm{d}s + \hat{\Sigma}(s;0)\mathrm{d}W_s^{k,T_F} \\
\implies H_t &= H_0 \exp\left\{-\frac{1}{2}\int_0^t \hat{\Sigma}^2(s;0)\mathrm{d}s + \int_0^t \hat{\Sigma}(s;0)\mathrm{d}W_s^{k,T_F}\right\} \qquad (2.2.11)
\end{aligned}$$

Computing $\mathbb{E}_t^{k,T_F}\left[(H_{T_F} - a_0)^+\right]$ becomes a straightforward task to give a Black formula for the swaption. Firstly, we note that H_t is indeed a lognormal random variable since $\hat{\Sigma}(s;u)$ is deterministic due to the drift freezing. In the usual manner, we derive the mean and variance of the term inside the exponent, and following exactly the steps involved in [1, Slide 266], finally, we arrive at:

$$S_t(T_F;T_N) = B_k(t,T_F)\left[H_t\Phi(d_1) - a_0\Phi(d_2)\right]$$
$$d_{1,2} = \frac{\frac{H_t}{a_0} \pm \frac{1}{2}\int_t^{T_F}\hat{\Sigma}^2(s;t)\mathrm{d}s}{\sqrt{\int_t^{T_F}\hat{\Sigma}^2(s;t)\mathrm{d}s}} \tag{2.2.12}$$

where $\Phi(\cdot)$ denotes the normal CDF. In the calibration procedure, we will set $t = 0$.

2.3 Multi-curve Framework

2.3.1 General Setting

The classical approach to interest rate modelling assumes the Libor rate is "risk-free" meaning that default events are so unlikely that they are considered irrelevant. The crisis of 2008 has redefined how banks and other institutions value products; banks are now considered "risky" and the risk-free Libor assumption is flawed, and so a new approach is needed to take this into account.

The modern multi-curve framework as the name suggests uses a single discounting curve, typically built from the Overnight Index Swap (OIS) rate as the "risk-free" discounting curve. Then, depending on the underlying, a rate is built out of liquid vanilla IR products with varying maturities to get the full curve which is used as the forward rate.

The VP uses a multi-curve setup, and so we will want to incorporate this into our diffusion model framework. For swaption pricing, we demonstrate the approach below. Again, we were unable to find verifications for the formulas to follow in the literature for approximate swaption prices in the multi-curve framework; as such we will derive them explicitly.

It is also important to note that, as Libor decomissioning is approaching at least for some currencies, this post-crisis Libor framework can be easily adapted to cope with the new risk free rate, without the need to redevelop a whole new swaption pricing framework. Therefore, the below will still be a useful basis for when the new risk free rate is used.

Denote by the superscript "disc" the discounting curve, and by "fix" the forward curve in the following. We must first specify the dynamics of the two curves, which are in (2.3.1) and (2.3.2) below, under the common risk neutral measure. We deliberately choose the instantaneous volatility to be equal, with the drifts equal to the corresponding short rate.

$$\frac{\mathrm{d}B_k^{\text{disc}}(t,T)}{B_k^{\text{disc}}(t,T)} = r_k^{\text{disc}}(t)\mathrm{d}t + \sigma_k(t,T)\mathrm{d}W_t^k \tag{2.3.1}$$
$$\frac{\mathrm{d}B_k^{\text{fix}}(t,T)}{B_k^{\text{fix}}(t,T)} = r_k^{\text{fix}}(t)\mathrm{d}t + \sigma_k(t,T)\mathrm{d}W_t^k \tag{2.3.2}$$

Following exactly the same calculations used to obtain (1.2.6), we obtain:

$$B_k^j(t,T) = \frac{B_k^j(0,T)}{B_k^j(0,t)} \exp\left\{ \int_0^t (\sigma_k(s,T) - \sigma_k(s,t))\, \mathrm{d}W_s^k - \frac{1}{2}\int_0^t \left(\sigma_k^2(s,T) - \sigma_k^2(s,t)\right) \mathrm{d}s \right\},$$
$$j \in \{\mathrm{disc}, \mathrm{fix}\} \tag{2.3.3}$$

The fact that we model both curves with the same instantaneous volatility means that we can write a simple formula for the ratio of the two. Since the exponential term is the same for both the discount and forward curve (note there is no j appearing in (2.3.3)), some simple algebra yields the following important ratio:

$$\frac{B_k^{\mathrm{fix}}(t,T)}{B_k^{\mathrm{disc}}(t,T)} = \frac{B_k^{\mathrm{fix}}(0,T) B_k^{\mathrm{disc}}(0,t)}{B_k^{\mathrm{disc}}(0,T) B_k^{\mathrm{fix}}(0,t)} \tag{2.3.4}$$

Now we use the expressions obtained when using the fact that $B_k^j(t,t) = 1$ and $B_k^j(T,T) = 1$ for $j \in \{\mathrm{disc}, \mathrm{fix}\}$, i.e. (1.2.5), in order to find expressions for $B_k^j(0,t)$ and $B_k^j(0,T)$. Thus, we obtain the following expression for (2.3.4):

$$\frac{\exp\left\{\int_0^T \left(r_k^{\mathrm{disc}}(s) - \frac{1}{2}\sigma_k^2(s,T)\right) \mathrm{d}s + \int_0^T \sigma_k(s,T) \mathrm{d}W_s^k\right\} \exp\left\{\int_0^t \left(r_k^{\mathrm{fix}}(s) - \frac{1}{2}\sigma_k^2(s,t)\right) \mathrm{d}s + \int_0^t \sigma_k(s,t) \mathrm{d}W_s^k\right\}}{\exp\left\{\int_0^T \left(r_k^{\mathrm{fix}}(s) - \frac{1}{2}\sigma_k^2(s,T)\right) \mathrm{d}s + \int_0^T \sigma_k(s,T) \mathrm{d}W_s^k\right\} \exp\left\{\int_0^t \left(r_k^{\mathrm{disc}}(s) - \frac{1}{2}\sigma_k^2(s,t)\right) \mathrm{d}s + \int_0^t \sigma_k(s,t) \mathrm{d}W_s^k\right\}}$$
$$= \exp\left\{ - \int_t^T \left[r_k^{\mathrm{fix}}(s) - r_k^{\mathrm{disc}}(s)\right] \mathrm{d}s \right\}$$
$$=: \exp\left\{ - \int_t^T \mu(s) \mathrm{d}s \right\}, \quad \mu(s) := r_k^{\mathrm{fix}}(s) - r_k^{\mathrm{disc}}(s) \tag{2.3.5}$$

2.3.2 Multi-curve swaption price

We now derive a formula for the price of a swaption in this setup. Denote by $F^{\mathrm{fix}}(T_F; T_{i-1}, T_i)$ the forward Linear rate prevailing at time T_F for the expiry T_{i-1} and maturity T_i, but here, derived under the forward curve with dynamics (2.3.2). Let $D(t_1,t_2) := \exp\left\{ - \int_{t_1}^{t_2} r_k^{\mathrm{disc}}(s) ds \right\}$

Proposition 2.3.1 (Swaption Price in multi-curve framework). *Let S_t^{MC} denote the price of a European receiver swaption in the multi-curve framework. Assuming the dynamics in* (2.3.1) *and* (2.3.2), *the following equation is satisfied by S_t^{MC}.*

$$S_t^{MC} = \mathbb{E}_t^k \left[\exp\left(- \int_t^{T_F} r_k^{disc}(s) \mathrm{d}s \right) \left(\sum_{i=1}^N b_i B_k^{disc}(T_F, T_i) - b_0 \right)^+ \right] \tag{2.3.6}$$
$$b_i := 1 + K\alpha_i - \exp\left(- \int_{T_{i-1}}^{T_i} \mu(s) \mathrm{d}s \right), \quad i \in \{1, \dots, N-1\}$$
$$b_N := 1 + K\alpha_N$$
$$b_0 := \exp\left(- \int_{T_0}^{T_1} \mu(s) \mathrm{d}s \right)$$

Proof.

Since discounting is done with the discounting curve in the multi-curve framework, we have that the discounted payoff of the swaption is

$$S^{\mathrm{MC}}_{\mathrm{DiscPayoff}} = D(t,T_F)\left[\sum_{i=1}^{N} B_k^{\mathrm{disc}}(T_F,T_i)\alpha_i(K - F^{\mathrm{fix}}(T_F;T_{i-1},T_i))\right]^+ \tag{2.3.7}$$

We need to first find an expression for the forward linear rate $F^{\mathrm{fix}}(t;T,S)$. The spot-Linear rate using the forward curve dynamics [1, Slide 91] is $L^{\mathrm{fix}}(t,T) = \frac{1}{\tau}\left(\frac{1}{B_k^{\mathrm{fix}}(t,T)} - 1\right)$ where $\tau := \frac{1}{T-t}$. We now use this to find an expression for the price of a Forward Rate Agreement (FRA) with strike K as follows.

$$\begin{aligned}
FRA^{\mathrm{fix}}(t,T,S,K) &= \mathbb{E}_t\left[D(t,S)\tau K - D(t,S)\tau L^{\mathrm{fix}}(T,S)\right] \\
&= \tau K B_k^{\mathrm{disc}}(t,S) - \mathbb{E}_t\left[D(t,T)D(T,S)\tau L^{\mathrm{fix}}(T,S)\right] \\
&= \tau K B_k^{\mathrm{disc}}(t,S) - \mathbb{E}_t\left[D(t,T)\tau L^{\mathrm{fix}}(T,S)B_k^{\mathrm{disc}}(T,S)\right] \\
&= \tau K B_k^{\mathrm{disc}}(t,S) - \mathbb{E}_t\left[D(t,T)\left(\frac{1}{B_k^{\mathrm{fix}}(t,T)} - 1\right)B_k^{\mathrm{disc}}(T,S)\right] \\
&= \tau K B_k^{\mathrm{disc}}(t,S) - \mathbb{E}_t\left[D(t,T)\frac{B_k^{\mathrm{disc}}(T,S)}{B_k^{\mathrm{fix}}(T,S)}\right] + \mathbb{E}_t\left[D(t,T)B_k^{\mathrm{disc}}(T,S)\right] \\
&= \tau K B_k^{\mathrm{disc}}(t,S) - \exp\left\{-\int_T^S \mu(s)\mathrm{d}s\right\}B_k^{\mathrm{disc}}(t,T) + B_k^{\mathrm{disc}}(t,S) \qquad (2.3.8)
\end{aligned}$$

Finding the strike K such that (2.3.8) is null will give an expression for the forward linear rate. Doing so gives, for $i \in \{1,\ldots,N\}$,

$$F^{\mathrm{fix}}(T_F;T_{i-1},T_i) = \frac{1}{\alpha_i}\left[\exp\left\{\int_{T_{i-1}}^{T_i}\mu(s)\mathrm{d}s\right\}\frac{B_k^{\mathrm{disc}}(T_F,T_{i-1})}{B_k^{\mathrm{disc}}(T_F,T_i)} - 1\right] \tag{2.3.9}$$

Using this we can derive an equivalent expression for the multi-curve version of (2.2.1)

$$\begin{aligned}
S^{\mathrm{MC}}_{\mathrm{DiscPayoff}} &= D(t,T_F)\left[\sum_{i=1}^{N} B_k^{\mathrm{disc}}(T_F,T_i)\alpha_i\left(K - \frac{1}{\alpha_i}\left[\exp\left\{-\int_{T_{i-1}}^{T_i}\mu(s)\mathrm{d}s\right\}\frac{B_k^{\mathrm{disc}}(T_F,T_{i-1})}{B_k^{\mathrm{disc}}(T_F,T_i)} - 1\right]\right)\right]^+ \\
&= D(t,T_F)\left[\sum_{i=1}^{N} B_k^{\mathrm{disc}}(T_F,T_i)\left(1 + K\alpha_i - \exp\left\{-\int_{T_{i-1}}^{T_i}\mu(s)\mathrm{d}s\right\}\frac{B_k^{\mathrm{disc}}(T_F,T_{i-1})}{B_k^{\mathrm{disc}}(T_F,T_i)}\right)\right]^+ \\
&= D(t,T_F)\Bigg[\sum_{i=1}^{N} B_k^{\mathrm{disc}}(T_F,T_i)\left(1 + K\alpha_i - \exp\left\{-\int_{T_{i-1}}^{T_i}\mu(s)\mathrm{d}s\right\}\right) \\
&\quad + \sum_{i=1}^{N}\left(\exp\left\{-\int_{T_{i-1}}^{T_i}\mu(s)\mathrm{d}s\right\}\left(B_k^{\mathrm{disc}}(T_F,T_i) - B_k^{\mathrm{disc}}(T_F,T_{i-1})\right)\right)\Bigg]^+
\end{aligned} \tag{2.3.10}$$

Now that we have a telescoping sum with the final term in (2.3.7), we can simplify

$$\begin{aligned}
&\sum_{i=1}^{N}\left(\exp\left\{-\int_{T_{i-1}}^{T_i}\mu(s)\mathrm{d}s\right\}\left(B_k^{\mathrm{disc}}(T_F,T_i) - B_k^{\mathrm{disc}}(T_F,T_{i-1})\right)\right) \\
&= \exp\left\{-\int_{T_{N-1}}^{T_N}\mu(s)\mathrm{d}s\right\}B_k^{\mathrm{disc}}(T_F,T_N) - \exp\left\{-\int_{T_0}^{T_1}\mu(s)\mathrm{d}s\right\}B_k^{\mathrm{disc}}(T_F,T_0) \qquad (2.3.11)
\end{aligned}$$

So that, on substitution into (2.3.7), noting that $B_k^{\text{disc}}(T_F, T_0) = 1$ and taking expectation under the corresponding measure to get the price, defining $b_i, \quad i \in \{0, \ldots, N\}$ as in the proposition statement, we get precisely (2.3.6). □

Since (2.3.6) is analogous to (2.2.1) with coefficients b_i rather than a_i (note that $r_k \equiv r_k^{\text{disc}}$ and $B_k^{\text{disc}}(t_1, t_2) \equiv B_k(t_1, t_2)$), following an identical drift-freezing procedure to the single curve swaption price and using Black's formula gives the following formula for the price of a swaption expiring at time$-T_F$ and maturing at time$-T_N$

$$\begin{aligned} S_t^{\text{MC}} &= B_k^{\text{disc}}(t, T_F)\left[H_t^{\text{MC}}\Phi(d_1^{\text{MC}}) - b_0\Phi(d_2^{\text{MC}})\right] \\ d_{1,2}^{\text{MC}} &:= \frac{\frac{H_t^{\text{MC}}}{b_0} \pm \frac{1}{2}\int_t^{T_F}[\hat{\Sigma}^{\text{MC}}(s;t)]^2 \mathrm{d}s}{\sqrt{\int_t^{T_F}[\hat{\Sigma}^{\text{MC}}(s;t)]^2 \mathrm{d}s}} \end{aligned} \tag{2.3.12}$$

where

$$H_t^{\text{MC}} := \sum_{i=1}^{N} b_i \frac{B_k^{\text{disc}}(t, T_i)}{B_k^{\text{disc}}(t, T_F)} \tag{2.3.13}$$

$$\hat{\Sigma}^{MC}(s;t) := \lambda\left(\frac{\sum_{i=1}^{N} b_i(G(T_i) - G(T_F))B_k^{\text{disc}}(t, T_i)}{\sum_{i=1}^{N} b_i B_k^{\text{disc}}(t, T_i)}\right) \tag{2.3.14}$$

2.3.3 Algorithm for determining optimal parameters

In order to optimally calculate the g_is, we can use a sequential "bootstrapping" approach, starting with the swaption with expiry T_{N-1} to calculate g_{N-1}, using this optimal g_{N-1} to calculate g_{N-2} and so on. Whilst a typical bootstrapping algorithm in the literature iterates forward in time, the presence of the $G(T_i) - G(T_F)$ in our swaption formula (2.2.1) means that starting with the earliest expiring swaption depends on all the parameters g_i, rather than just one. Thus we must start from the end and iterate backwards in order to obtain a series of equations increasing in the number of unknowns.

We first present a generic algorithm outlining the overall approach and the objective function to minimise in Algorithm 1, before providing finer details in Algorithm 2. We refer to the analytical swaption price with expiry T_i and maturity T_j as $S^A(T_i, T_j)$, and $S^M(T_i, T_j)$ as the corresponding market price. To simplify notation in the following, recall the grid $\mathbf{T}^{g_i}$ from section 2.1, where where each point on the grid is the expiry of the chosen swaptions. The algorithms below will refer the T_i in this grid, rather than the payment times of each individual swaption as this section has used so far.

Algorithm 1: Volatility Calibration

Result: $\mathbf{g}$, dictionary of g_i parameters

$\mathbf{g} \leftarrow \{\}$;

$g[N-1] \leftarrow argmin_{g[N-1]}(S^M(T_{N-1}, T_N) - S^A(T_{N-1}, T_N; g[N-1]))^2$;

for *i=N-2,...,2,1* **do**

  $g[i] \leftarrow argmin_{g[i]}(S^M(T_i, T_N) - S^A(T_i, T_N; g[N-1], \dots, g[i+1]))^2$;

end

$g[0] \leftarrow g[1]$;

$g[N] \leftarrow g[N-1]$;

A more detailed algorithm is presented in the below.

Algorithm 2: Volatility Calibration in detail

Result: Dictionary of volatility parameters

$g \leftarrow \{\}$;

for i = *N-1, ..., 2,1* **do**

  $g[i] \leftarrow \text{argmin}_{g_i}(B_k(0,T_i)[H_0(i)\Phi(d_1(i)) - \Phi(d_2(i))] - S^M(T_i, T_N))^2$;

end

$g[0] \leftarrow g[1]$;

$g[N] \leftarrow g[N-1]$;

The parameters used in Algorithm 2 are described as follows. In the simpler single curve framework, recall that we have that the time zero price of a swaption expiring at T_i, and maturing at time T_N and freezing the drift at time zero, from (2.2.12),

$$S_0^A(T_i, T_N) = B_k(0, T_i)\left[H_0\Phi(d_1(i)) - \Phi(d_2(i))\right], \qquad d_{1,2}(i) := \frac{H_0(i) \pm \frac{1}{2}T_{i-1}\hat{\Sigma}^2(i)}{\sqrt{T_{i-1}\hat{\Sigma}^2(i)}}$$

$$\hat{\Sigma}(i) := \frac{\lambda}{\sum_{j=0}^{M} a_j B_k(0,t_j)}\left[g_{\mathbf{ind}(i)} \sum_{j=0}^{\mathbf{ind}(i)} \lambda a_j (t_j - T_i) B_k(0,t_j) + \sum_{j=\mathbf{ind}(i)}^{M} a_j (G(t_j) - G(T_i)) B_k(0,t_j)\right]$$

$$H_0(i) := \sum_{j=1}^{M} a_i \frac{B_k(0,t_j)}{B_k(0,T_i)}, \qquad a_0 := 1, \qquad a_j := K\alpha_j, \qquad a_M := 1 + K\alpha_M$$

where $t_j, \quad j \in \{0, \dots, M\}$, is the swaption's payment schedule. Since this clearly depends on the swaption, it would be more correct to write $t_j =: t_j^i$ however we omit the superscript for the sake of clarity. For each swaption in the loop, t_0 is the expiry of the swaption; by construction will be equal to one of the T_is, and $t_M := T_N$ is (approximately) constant across the swaptions. The function $\mathbf{ind}(i)$ returns the index on the grid $\mathbf{T}^{g_i}$ such that $\mathbf{ind}(i) = \{i : t_0 = T_i\}$. Note that we have split $\hat{\Sigma}^2$ to two functions; the first where the g parameter is unknown, and the second containing the known g parameters. We also note that having a constant, rather than time-dependent mean reversion parameter λ results $\hat{\Sigma}$ is constant in s, so that the integral term in (2.2.12) is simply the length of the integral multiplied by $\hat{\Sigma}$.

The procedure for the implementation is described below.

- Observe swaption implied volatilities in the market for all maturities, tenors and strikes. In general, ATM swaptions are the most liquid and hence make the most sense to calibrate to. However for full generality we will keep the "money-ness" of the swaption as a user input for example, swaptions that are 1% OTM.
  - Read in the swaption implied volatility surface from Bloomberg
  - Select the specific swaptions that the user wishes to calibrate to. The choice should depend on the portfolio for which the user wants to calculate CVA
  - Reconstruct the swaption prices using the implied volatilities, the expiry, tenor and strike of the swaption
- Read in a user-specified mean reversion, typically of the order of 1%. Doing so ensures that we are, at each stage of the calibration, solving an equation for one[2] unknown, which will be a constant g_i, as outlined in Algorithm 2.
- Implement Algorithm 2.

It is also worth noting that the bootstrapping algorithm described can be unstable [9]. To remediate this, we also perform simultaneous calibration to minimise the following objective function:

$$\min_{g_0,\dots,g_{N-1}} \sum_{\text{Exp,Ten}} (S^A(g_0,\dots,g_{N-1};\text{Exp,Ten}) - S^M(\text{Exp,Ten}))^2$$

Since in either case, overall we have the same number of unknowns as equations, we expect the optimal solutions to coincide.

2.3.4 FX Options

In this section, we demonstrate the calibration procedure in order to calibrate the piecewise-constant $\sigma_X(\cdot)$. As in the calibration of the IR volatility for the foreign and domestic bonds, we assume a piecewise-constant volatility over a grid determined by the calibration instruments. Firstly, we need to derive an analytical price for an FX Option in our framework. Ultimately, we will have a formula for the analytical FX Option price in terms of σ_d, σ_f and σ_X. At this stage, we have already calibrated our model to swaptions of corresponding currencies to determine functions σ_f and σ_d, so that calibration of σ_X again entails solving for one unknown at each maturity. Thus, it makes sense to use a bootstrapping algorithm, as in the previous calibrations.

Implementing an efficient FX Option Pricer would require more time, and so we do not carry out the calibration. Nevertheless, we give the full procedure and note that this will be completed in the short term.

[2]Indeed, calibrating the mean reversion parameter would involve using another product, for example caplets, otherwise the model would overfit since it would be solving a single equation in two unknowns. For simplicity, we keep it fixed, but include enough generality to specify a different parameter per currency.

It can be shown [10, 2.18, page 5] that under the $T-$forward measure, the numeraire of which is the domestic bond maturing at time$-T$, $B_d(t,T)$, the FX Forward rate is

$$F(t,T) = \frac{X(t)B_f(t,T)}{B_d(t,T)}$$

Now, since $\frac{X_t}{B_f(t,T)}$ is a tradeable asset, under the T-forward measure in the domestic currency (with associated Brownian motion $\mathbb{Q}^{d,T}$), $F(t,T)$ is a martingale [1, 'Fact One', Slide 255], since $B_d(\cdot,T)$ is the numeraire with which the measure $\mathbb{Q}^{d,T}$ is associated.

We can now use Itô's Lemma to derive the SDE satisfied by $F(t,T)$ under this measure. Since we know that the process will have zero drift as it's a martingale so we only need to consider it's diffusion coefficient. Indeed,

$$\begin{aligned}\mathrm{d}F(t,T) &= \mathrm{d}\left[\frac{X(t)B_f(t,T)}{B_d(t,T)}\right] \\ &= \mathrm{d}\left[\frac{X(t)}{B_d(t,T)}\right]B_f(t,T) + \mathrm{d}\left[B_f(t,T)\right]\frac{X(t)}{B_d(t,T)} + \mathrm{d}\left[X, \frac{1}{B_d(\cdot,T)}\right]_t\end{aligned} \tag{2.3.15}$$

Another application of Itô's rule in the first term in (2.3.15) gives

$$\mathrm{d}\left[\frac{X(t)}{B_d(t,T)}\right] = \mathrm{d}\left[X(t)\right]\frac{1}{B_d(t,T)} + \mathrm{d}\left[\frac{1}{B_d(t,T)}\right]X(t) + \mathrm{d}\left[X, \frac{1}{B_d(\cdot,T)}\right]_t \tag{2.3.16}$$

Now, the quadratic covariation terms in (2.3.15) and (2.3.16) contribute solely to the drift of the process, which we know to be zero and hence we can ignore them. We have dynamics for each of the processes above, except for the "$\mathrm{d}\left[\frac{1}{B_d(t,T)}\right]$" term, the dynamics for which can be easily derived using Itô's Lemma. Before substituting in the dynamics for the relevant processes, we first make sure that all of the processes are written in terms of the $T-$fwd measure and it's respective Brownian motions. Since the diffusion coefficient is unchanged in a change of measure, and we can ignore the drift component, changing to the $T-$fwd measure here simply becomes substituting the Brownian motions to the corresponding $T-$fwd Brownian motion. As an example, in the dynamics of $B_d(t,T)$ (written explicitly in (1.2.1)), changing the measure from the standard risk neutral measure $\mathbb{Q}^d$ to the $T-$fwd measure defined by (1.2.7) (with $H := T$), (1.2.1) becomes

$$\frac{dB^d(s,T)}{B^d(s,T)} = (r_d(s) - \sigma_d^2(s,T))\mathrm{d}s + \sigma_d(s,T)\mathrm{d}W_s^{T,d} \quad \mathbb{Q}^{d,T}$$

and we see that the only change with respect to the diffusion component is the change of Brownian motion.

Indeed, from (2.3.15) we obtain

$$\begin{aligned}\mathrm{d}F(t,T) &= \frac{X(t)B_f(t,T)}{B_d(t,T)}\left((\dots)\mathrm{d}t + \sigma_X(t)\mathrm{d}W_t^{X,T} - \sigma_d(t,T)\mathrm{d}W_t^{d,T}\right) \\ &\quad + \frac{X(t)B_f(t,T)}{B_d(t,T)}\left((\dots)\mathrm{d}t + \sigma_f(t,T)\mathrm{d}W_t^{f,T}\right) \\ \implies \frac{\mathrm{d}F(t,T)}{F(t,T)} &= \sigma_X(t)\mathrm{d}W_t^{X,T} + \sigma_f(t,T)\mathrm{d}W_t^{f,T} - \sigma_d(t,T)\mathrm{d}W_t^{d,T}, \quad \mathbb{Q}^T\end{aligned} \tag{2.3.17}$$

Now, if $X_1, \dots, X_N$ are N correlated random normal variables, with correlation between X_i and

X_j for $i, j \in \{1, \dots, N\}, i \neq j$ equal to $\rho_{i,j}$, then $X := \sum_{i=1}^N X_i$ is normally distributed with $\mathbb{E}[X] = \sum_{i=1}^N \mathbb{E}[X_i]$ and $\text{Var}[X] = \sum_{i=1}^N \text{Var}[X_i] + \sum_{i=1}^N \sum_{j \neq i}^N \text{Cov}(X_i, X_j)$. Thus, we can represent (2.3.17) as:

$$\begin{aligned}\frac{\mathrm{d}F(t,T)}{F(t,T)} = \Big(&\sigma_X^2(t) + \sigma_f^2(t,T) + \sigma_d^2(t,T) + 2\rho_{X,f}\sigma_X(t)\sigma_f(t,T) \\ &+ 2\rho_{X,d}\sigma_X(t)\sigma_d(t,T) - 2\rho_{f,d}\sigma_f(t,T)\sigma_d(t,T)\Big)^{\frac{1}{2}} \mathrm{d}W_t^T, \quad \mathbb{Q}^T \\ &:= \Sigma_{\text{FWD}}(t,T)\mathrm{d}W_t^T, \quad \mathbb{Q}^T\end{aligned}$$

Now, solving this is simply another application of Itô's Lemma on the logarithm of $F(t,T)$, to yield:

$$F(t,T) = F(0,T)\exp\left\{-\frac{1}{2}\int_0^t \Sigma_{\text{FWD}}^2(s,T)\mathrm{d}s + \int_0^t \Sigma_{\text{FWD}}(s,T)\mathrm{d}W_s^T\right\} \tag{2.3.18}$$

Calibration of the model consists of fitting to FX Options of a fixed strike and varying maturities. We first need to derive the price of an FX Option in this model. An FX Option with strike K, FX Spot rate $X(t)$ and maturity T has the following price at time$-t$

$$\text{FX}_t^{\text{opt}} = B_d(t,T)\mathbb{E}_t^{d,T}\left[(F(T,T) - K)^+\right] \tag{2.3.19}$$

where $\mathbb{E}^{d,T}$ denotes expectation under the $T-$forward measure. Using (2.3.18), we compute an explicit formula for the FX Option price in terms of Σ_{FWD}. Indeed, we follow the procedure outlined in [1, Slide 266]. Denote the term inside the exponent in (2.3.18) by $I(t,T)$. The distribution of $I(t,T)$ is Gaussian, since it is the stochastic integral of a deterministic function against a Brownian motion, and so its distribution is fully determined by its mean and variance. It is clear that

$$\begin{aligned}\mathbb{E}^{d,T}[I(t,T)] &= -\frac{1}{2}\int_0^t \Sigma_{\text{FWD}}^2(s,T)\mathrm{d}s := \text{M}_{\text{FWD}}(t,T) \\ \text{Var}^{d,T}[I(t,T)] &= \text{Var}\left[\int_0^t \Sigma_{\text{FWD}}(s,T)\mathrm{d}W_s^T\right] \\ &= \int_0^t \Sigma_{\text{FWD}}^2(s,T)\mathrm{d}s := V_{\text{FWD}}^2(t,T)\end{aligned} \tag{2.3.20}$$

by Itô's Isometry in the calculation of the variance. Thus,

$$F(t,T) = F(0,T)\exp\left\{\text{M}_{\text{FWD}}(t,T) + \text{V}_{\text{FWD}}(t,T)Z\right\}$$

where $Z \sim N(0,1)$. Thus after calculating the expectation in (2.3.19) explicitly, we finally arrive at:

$$\begin{aligned}\text{FX}_t^{\text{opt}}(\text{Strike} = K, X, T) &= B_d(t,T)\left[F(t,T)\Phi(d_1) - K\Phi(d_2)\right] \\ d_{1,2} &= \frac{\frac{F(t,T)}{K} \pm \frac{1}{2}\int_t^T \hat{\Sigma}_{\text{FWD}}^2(s,T)\mathrm{d}s}{\sqrt{\int_s^T \hat{\Sigma}_{\text{FWD}}^2(s,T)\mathrm{d}s}}\end{aligned} \tag{2.3.21}$$

Now we have the necessary ingredients to begin the calibration procedure. As per the previous case, we first assume that $t = 0$, we fix a particular strike to calibrate on (usually this will be ATM options), and we consider options with varying maturities.

We assume that the volatility function satisfies

$$\sigma_X(s) = \lambda^X(s) \int_0^s g^X(u)\mathrm{d}u$$

for some piecewise-constant function $g^X : [0, \infty) \to [0, \infty)$, and mean reversion $\lambda^X(\cdot)$ which again will be chosen to be constant, $\lambda^X(\cdot) \equiv \lambda^X$. We follow exactly the same steps detailed in Algorithm 1 to obtain the optimal g_i^X parameters. Similarly to the previous case, choosing the parameterisation of σ_X in this way gives a strictly increasing, continuous function for the volatility.

As mentioned previously, we will not implement this calibration methodology; this will be done at a later date. Instead, we will assume a function for g^X and simulate the FX spot as per (1.3.13). In particular, given the calibration results in the following section, we will assume the user input for g parameterisation is

$$g^X(u) = 0.1, \quad u \geq 0$$

Using this rather than simply specifying a fixed value for σ_X means that the correct infrastructure is already in the code, so that when it comes to writing the FX Option calibrator, the diffusion is automatic.

2.4 Calibration Results

In this section, we present the results obtained from the calibration. The algorithms were all written in `Python`, integrated within the VP. Hereafter, we consider 4 currencies; USD, EUR, GBP and CHF, so we will present a total of four calibration results. All market data is downloaded from Bloomberg as of 30$^{\text{th}}$ June 2021, which includes, for each currency,

- Swaption volatilities for ATM swaptions, and
- Zero Coupon Bonds for all maturities

We also note that for simplicity, we only implement a "single curve" calibration. To adopt a multi-curve approach, one would require market data for the additional curve at the relevant time. Then, using (2.3.4), calculate $\exp\left\{-\int_t^T \mu(s)\mathrm{d}s\right\}$. Once this is obtained, the calibration follows an identical procedure as Algorithm 1, but with formulas (2.3.12).

Denote S_E for the expiry of swaption S and S_T for the tenor. We also choose to calibrate to "20Yr" co-terminal swaptions. In practice, calibrating to swaptions that are "exactly" co-terminal, meaning that the expiry and tenor add up to precisely 20 years in this case, doesn't result in enough data points. For example, for USD Swaptions where typically the whole surface is quoted (which is not the case with CHF for example), precisely 4 swaptions satisfy $S_E + S_T = 20\text{Yr}$. As shown in [9, Table 1, page 17], a simple solution is to choose swaptions such that this condition is satisfied to within a specified tolerance, i.e. such that

$$|S_E + S_T - 20\text{Yr}| < \text{Tolerance} \tag{2.4.1}$$

	Tenor											
Expiry	1Yr	2Yr	3Yr	4Yr	5Yr	6Yr	8Yr	9Yr	10Yr	12Yr	15Yr	20Yr
1Mo	80.71	76.86	62.75	60.85	60.02	54.20	48.45	46.90	45.73	43.06	40.82	39.34
2Mo	90.30	82.04	67.34	64.55	60.77	55.60	49.79	48.06	46.78	43.43	41.12	40.26
3Mo	100.74	88.15	71.62	64.86	60.88	55.69	50.27	48.66	47.44	44.70	42.31	40.49
6Mo	100.63	84.50	70.22	62.86	58.05	53.45	48.38	46.84	45.49	43.06	40.88	**39.25**
9Mo	101.32	76.36	66.48	60.05	54.96	51.42	46.94	45.30	43.87	41.69	39.69	38.09
1Yr	94.06	71.85	63.39	57.53	52.48	49.05	44.96	43.72	42.66	40.69	38.83	**37.30**
18Mo	70.72	59.49	55.52	51.07	48.00	45.59	42.54	41.50	40.59	39.02	37.50	**36.18**
2Yr	60.39	54.61	50.27	46.95	44.67	42.91	40.56	39.66	38.96	37.61	36.21	35.23
3Yr	51.06	47.25	44.39	42.35	40.84	39.80	38.13	37.53	37.04	36.03	34.97	34.14
4Yr	45.46	42.94	41.03	39.65	38.51	37.85	36.57	36.13	35.74	34.93	**33.93**	33.35
5Yr	41.17	39.67	38.61	37.78	37.01	36.29	35.43	35.06	34.72	34.09	**33.21**	32.59
6Yr	38.56	38.01	37.37	36.54	35.76	35.22	34.47	34.12	33.83	33.23	**32.52**	32.03
7Yr	37.05	36.95	36.33	35.56	34.98	34.57	33.73	33.42	33.12	**32.55**	32.03	31.57
8Yr	36.11	35.74	35.04	34.57	34.22	33.68	32.98	32.71	32.46	**31.87**	31.48	31.09
9Yr	34.85	34.27	33.96	33.69	33.45	32.98	32.38	32.13	31.88	**31.55**	31.14	30.71
10Yr	33.28	33.29	33.16	32.99	32.84	32.48	31.90	31.65	**31.40**	31.25	30.79	30.29
15Yr	31.01	31.06	30.77	30.50	30.29	30.41	30.45	30.41	30.35	30.08	29.46	29.90
20Yr	32.10	31.81	31.43	31.21	31.05	30.99	30.80	30.72	30.61	30.93	30.77	30.87
25Yr	31.48	32.24	31.72	31.34	31.18	31.91	33.65	34.11	34.40	33.81	33.61	35.43
30Yr	44.85	45.27	43.91	43.14	41.95	41.45	40.10	40.15	40.18	41.15	42.10	43.48

Table 2.1: USD ATM swaption volatilities as of June 30^{th} 2021

We choose the tolerance to be 2Yr, which adds several points as illustrated in Table 2.1. As mentioned previously, the first stage in the calibration is to calculate the market price, given the implied volatility drawn from the market since our derived analytical formulas are for the swaption price not the volatility. Implied volatilities collected from Bloomberg can be either "Normal" or "Black" volatilities based on whether the model chosen for the swaptions in the Swap Market Model used by Bloomberg is normal or lognormal. We implement functionality for both cases for full generality. Figures 2.1 and 2.2 show the results for the USD case, using the data from Table 2.1. Highlighted in yellow in Table 2.1 are the swaptions used in the calibration for USD. Of course, we will only be using a portion of the data; in particular points roughly along the counter-diagonal due to the co-terminal requirement. We also note the lack of available data for some currencies; Figures 2.4 and 2.3 show that quotes are only available along the counter-diagonal. This is an added benefit of calibrating to co-terminal swaptions, since typically these are more liquid and more quotes are observable.

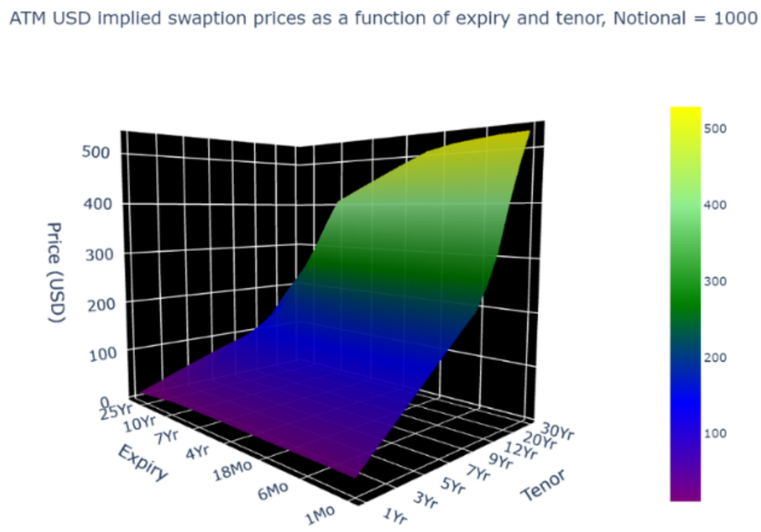

Figure 2.2: ATM USD swaption implied prices obtained from the implied volatilities in Table 2.1

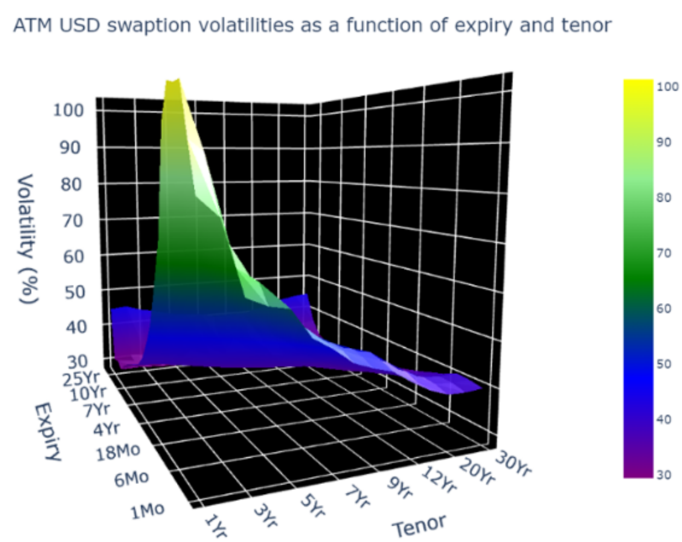

Figure 2.1: ATM USD swaption implied volatilities across observed tenors and expiries

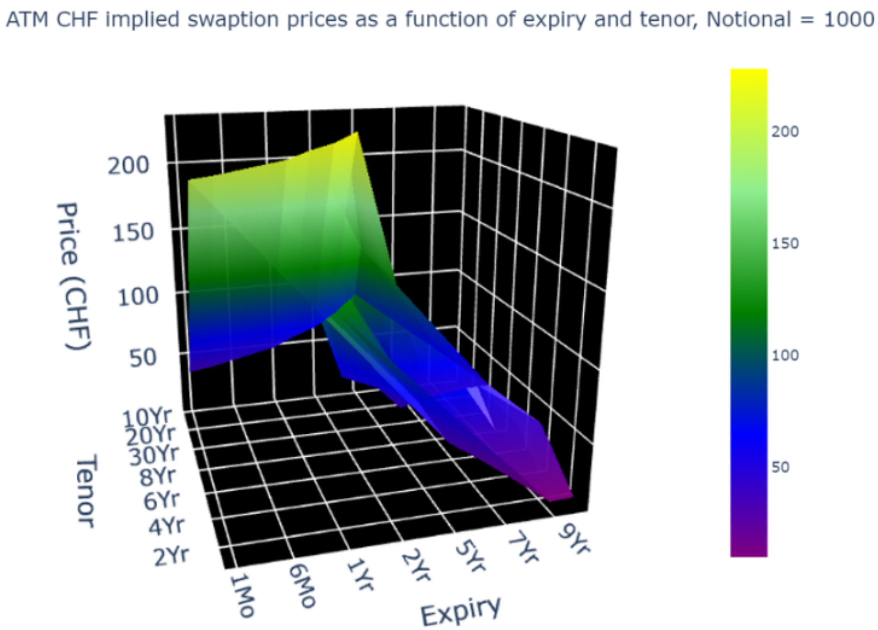

Figure 2.4: ATM CHF swaption implied prices across observed tenors and expiries

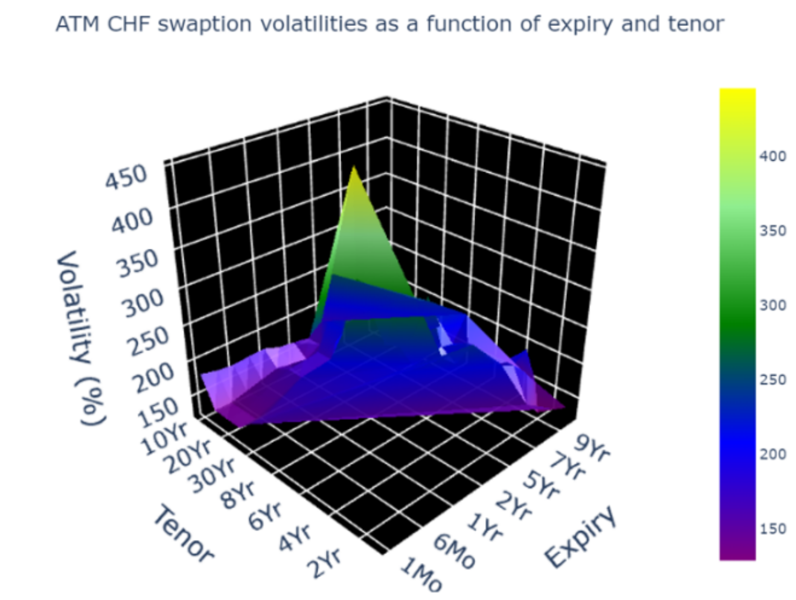

Figure 2.3: ATM CHF swaption implied volatilities across observed tenors and expiries

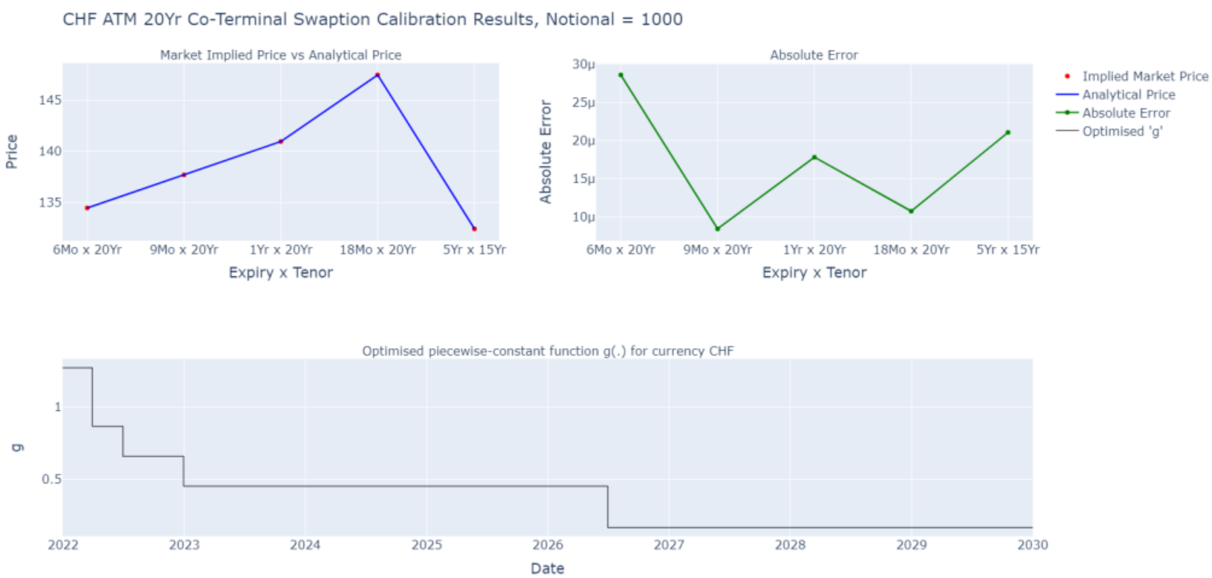

Figure 2.5: Swaption calibration results for CHF

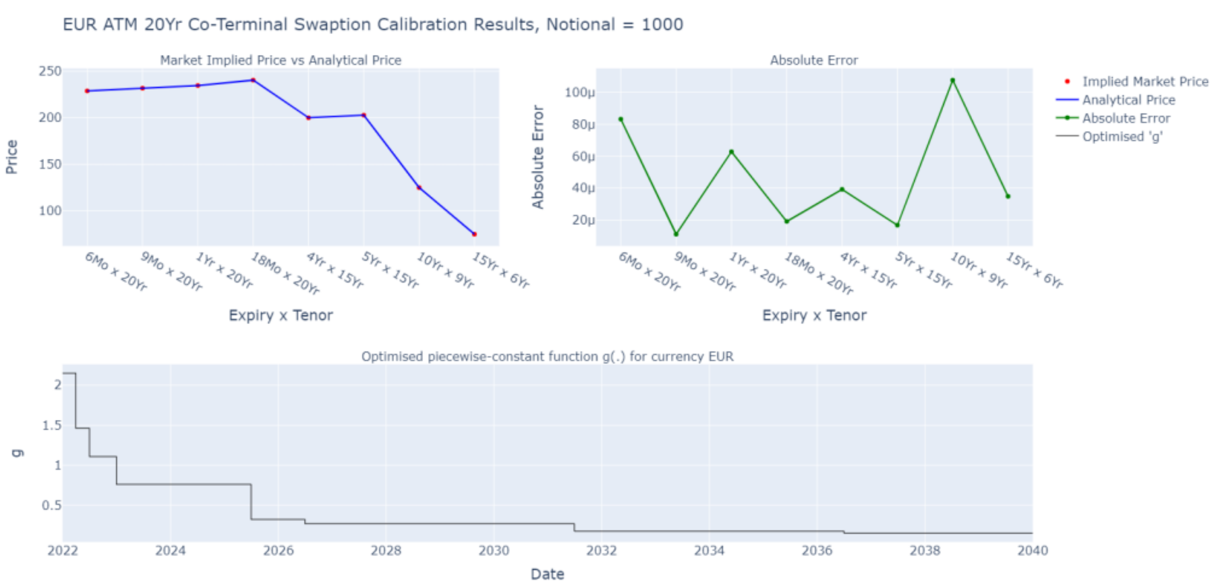

Figure 2.6: Swaption calibration results for EUR

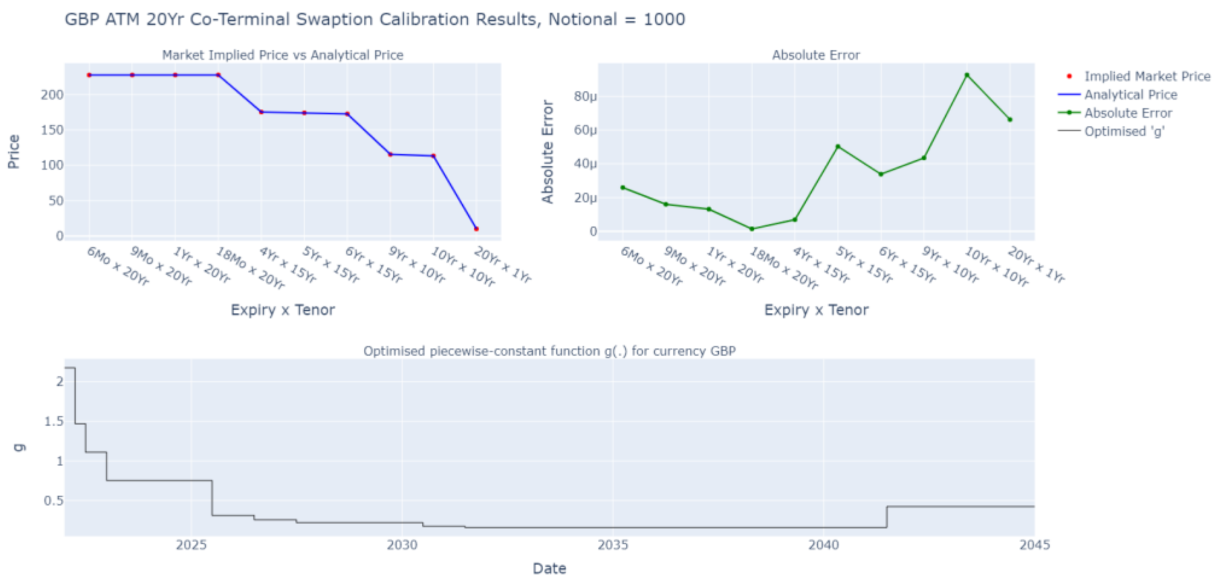

Figure 2.7: Swaption calibration results for GBP

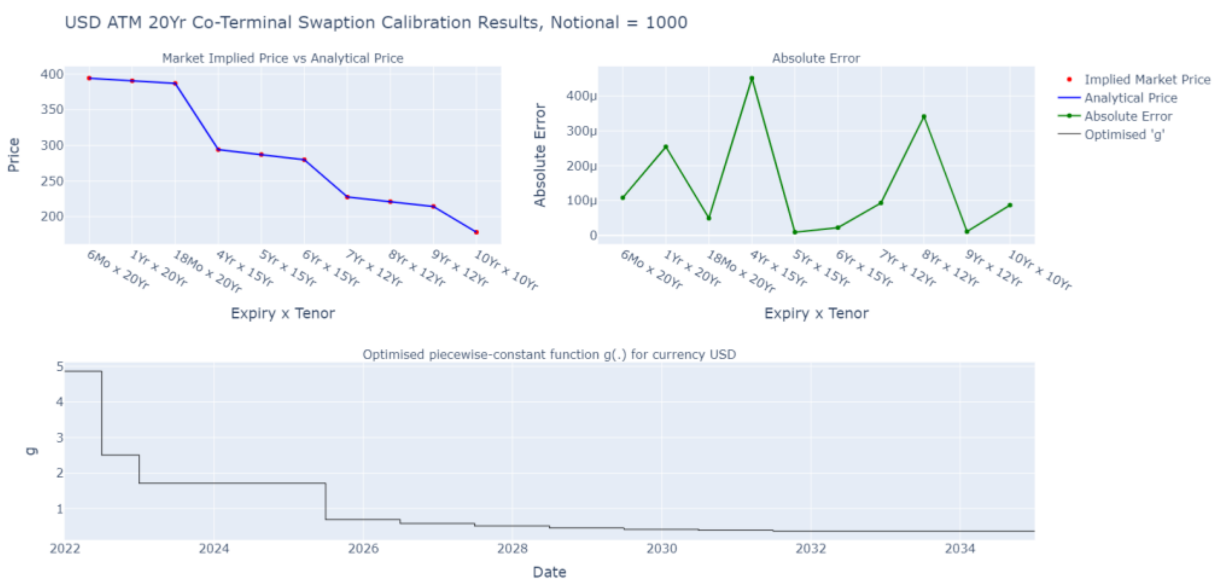

Figure 2.8: Swaption calibration results for USD

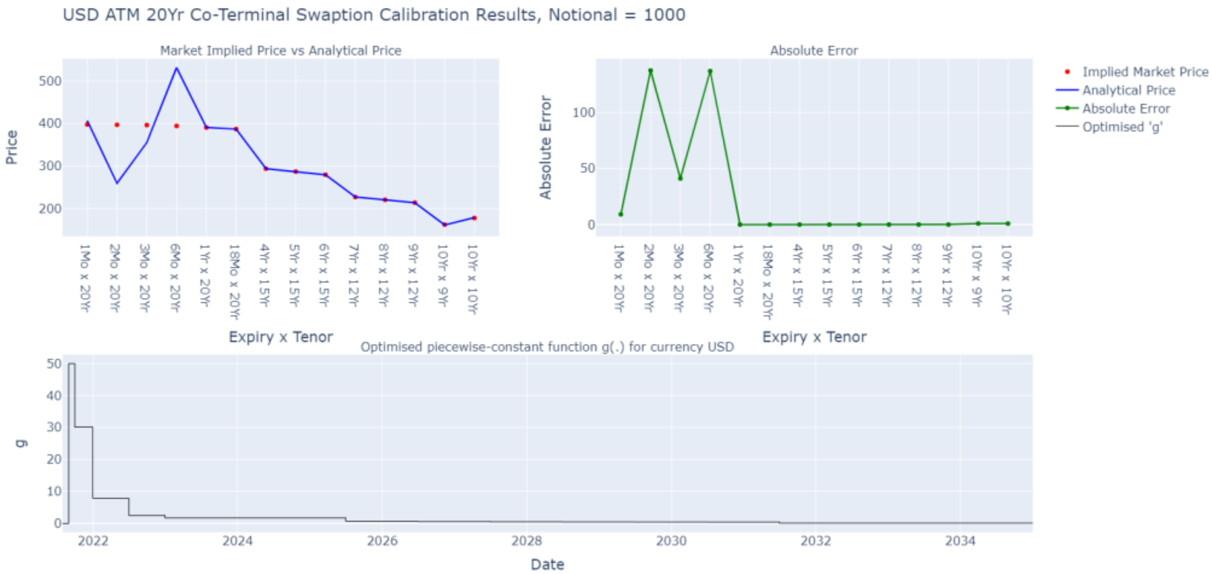

Figure 2.9: Swaption calibration results for USD *before* instrument selection.

In general, we were able to achieve almost perfect fit for each currency, given the chosen swaptions. We would expect this, since by design, we are iteratively solving for one g_i per swaption, or in the simultaneous calibration case solving for N parameters with N equations, if we have N swaptions. To illustrate the fit, we plot the absolute error function, $|S^A_{E,T} - S^M_{E,T}|$ for $(E,T) \in (\mathcal{E}, \mathbb{T})$, the set of chosen expiries and tenors, where $S^A_{E,T}, S^M_{E,T}$ are the analytical and market implied prices respectively, for expiry E and tenor T. Errors for 3 of the currencies were of the order 10^{-5}, with USD errors of the order 10^{-4}. The total time for calibration was approximately $2.59s$ on a machine running a 1.7GHz CPU, giving an average of around $0.65s$ per currency. Note also that the mean reversion is assumed fixed at 1% for each currency.

Using all the swaptions available that satisfy (2.4.1) can result in, at times, a poor fit. As a reference, Figure 2.9 illustrates this; the calibration error is much higher for multiple swaptions with small expiries. A possible reason for this is that swaptions with very short expiries are much less liquid [9, page 17], so it may be the case that there exists arbitrage within the market that would require negative piecewise-constant parameters g_i for our model to fit with zero error. In order to remediate this, a careful selection of swaptions was chosen through trial and error until the calibration error was sufficiently small[3].

Each function $g(\cdot)$ is displayed as the bottom graph in each of the above figures. We observe that this is in general a decreasing function, with more frequent "jumps" near the beginning due to the gaps in swaption expiries being smaller. The one exception to this is the GBP calibration results, whereby the optimised parameter for the final swaption expiry is indeed larger than the penultimate parameter, shown in Figure 2.7. This is likely due to the inclusion of the 20Yr x 1Yr swaption, which is not present in the other calibrations in order to reduce the error as discussed in the paragraph above.

Ultimately, we need to use the calibrated g_i parameters for the diffusion stage. By design, each currency has its own grid for which the piecewise-constant parameters are defined. Thus, for each currency we will need to store the g_i parameters as well as the corresponding grid. Note that for times larger than the latest expiry and for times smaller than the first expiry, we will extrapolate the last and first g_i parameters accordingly. For the implementation of the integrals that define the FX spot rate in equation (1.3.13), in particular the integral of the product of two sigma functions, special care needs to be taken. By construction, each $\sigma_k(\cdot,\cdot), k \in \{d, f_1, \dots, f_N\}$ is a piecewise-linear function, and so it's integral can be easily implemented as the sum of trapeziums, the height of each being the corresponding difference in time on the grid. The integration is exact, and will result in a large time-improvement over a numerical method. However, the product of two sigmas of different currencies is still piecewise-linear, but only over the grid specified by the union of the two individual grids. Thus, the union of all the time grids must first be calculated and a mapping defined between the union and each individual grid before the integration step.

[3] i.e. errors which have an approximate order of magnitude of no greater than 10^{-4}

Chapter 3

Diffusion

The purpose of this report is to create an efficient platform in which one can value CVA for cross currency products. The most challenging part of this is the implementation of the diffused curves under the common $H-$forward measure using the formulas developed in Chapter 2, once the models are calibrated to market data. Upon creating the diffused curves, we will illustrate how one calculates the Expected Positive Exposure for a vanilla IRS contract, before presenting numerical results for the EPE of a small portfolio of 4 vanilla IRS contracts, each denominated in a different currency.

3.1 Correlated Gaussian Generation

In this section we describe how to generate the correlated Gaussian random variates. In previous sections, we derived bond prices and FX spot rates all under the $H-$forward measure. In practice, this enables us to simply generate the correlated Brownian motions at each diffusion lag, and use them without the need to implement a "change of measure" function that would adjust the drift according to the measure in which the generated random variate is indeed normally distributed. We choose H in the following way, denoting by $\mathcal{P}$ the portfolio of instruments for which our diffusion model is used, and P_i the i^{th} instrument in the portfolio, and $\mathbf{mat}(\cdot)$ the maturity function of the instrument:

$$H \geq \max_{i \in \mathcal{P}} \{\mathbf{mat}(P_i)\}$$

The reason for using this measure is outlined in [3, 2.5, page 38]; primarily it is used for convenience in deriving analytical prices.

Correlation

Having set up our measure for which $W^{X,H}$, $W^{f,H}$ and $W^{d,H}$ are Brownian motions, we outline the procedure for creating a "cube" of random variates to be used in the diffusion. First, recall that $W^{X,H}$ is the Brownian motion under the $H-$forward measure corresponding to the FX Spot rate, $W^{f,H}$ and $W^{d,H}$ are the Brownian motions corresponding to the foreign and domestic bonds under the same measure. Define the instantaneous correlation between the Brownian motions as in (3.1.1).

$$
\begin{aligned}
\mathrm{d}W^{f_i,H}(t)\mathrm{d}W^{d,H}(t) &= \rho_{f_i,d}\mathrm{d}t \\
\mathrm{d}W^{X,H}_{f_i,d}(t)\mathrm{d}W^{d,H}(t) &= \rho_{FX(f_i,d),d}\mathrm{d}t \\
\mathrm{d}W^{X,H}_{f_i,d}(t)\mathrm{d}W^{f_j,H}(t) &= \rho_{FX(f_i,d),f_j}\mathrm{d}t \\
\mathrm{d}W^{X,H}_{f_i,d}(t)\mathrm{d}W^{X,H}_{f_j,d}(t) &= \rho_{FX(f_i,d),FX(f_j,d)}\mathrm{d}t
\end{aligned} \tag{3.1.1}
$$

where the subscript i on f denotes the i^{th} foreign currency. For $N-1$ foreign currencies, we represent this correlation structure as a $(2N-1)$ square matrix which will be useful from an implementation perspective, which is illustrated below. Clearly, the diagonals are one, and the matrix is symmetric.

$$
\begin{array}{cccccccccc}
IR_d & IR_{f_1} & IR_{f_2} & \cdots & IR_{f_{N-1}} & FX_{f_1 d} & FX_{f_2 d} & \cdots & FX_{f_{N-1} d} & \\
\end{array}
$$

$$
\boldsymbol{\rho} = \left[\begin{array}{ccccccccc}
1 & \rho_{f_1,d} & \rho_{f_2,d} & \cdots & \rho_{f_{N-1},d} & \rho_{FX(f_1,d),d} & \rho_{FX(f_2,d),d} & \cdots & \rho_{FX(f_{N-1},d),d} \\
 & 1 & \rho_{f_2,f_1} & \cdots & \rho_{f_{N-1},f_1} & \rho_{FX(f_1,d),f_1} & \rho_{FX(f_2,d),f_1} & \cdots & \rho_{FX(f_{N-1},d),f_1} \\
 & & 1 & \cdots & \rho_{f_{N-1},f_2} & \rho_{FX(f_1,d),f_2} & \rho_{FX(f_2,d),f_2} & \cdots & \rho_{FX(f_{N-1},d),f_2} \\
 & & & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
 & & & & 1 & \rho_{FX(f_1,d),f_{N-1}} & \rho_{FX(f_2,d),f_{N-1}} & \cdots & \rho_{FX(f_{N-1},d),f_{N-1}} \\
 & & & & & 1 & \rho_{FX(f_2,d),FX(f_1,d)} & \cdots & \rho_{FX(f_{N-1},d),FX(f_1,d)} \\
 & & \textbf{Sym.} & & & & 1 & \cdots & \rho_{FX(f_{N-1},d),FX(f_2,d)} \\
 & & & & & & & \cdots & \vdots \\
 & & & & & & & & 1
\end{array}\right]
\begin{array}{l}
IR_d \\ IR_{f_1} \\ IR_{f_2} \\ \vdots \\ IR_{f_{N-1}} \\ FX_{f_1 d} \\ FX_{f_2 d} \\ \vdots \\ FX_{f_{N-1} d}
\end{array}
\tag{3.1.2}
$$

In the implementation, we will treat this matrix as given, set to some arbitrary values by the user such that $\boldsymbol{\rho}$ is positive definite. By [11, Exercise 8, page 11], there exists a lower diagonal matrix L such that $\boldsymbol{\rho} = LL^T$. Ideally, these correlations would be estimated from historical data based on a time series of zero coupon bonds and FX rates. In such case, one would then transform the matrix using for example, [12] so that a Cholesky decomposition is always possible. Specifying constant correlations between the FX factors does not necessarily guarantee that options on cross currency products will be correctly repriced [13, page 4]. To remediate this, a time-dependent instantaneous correlation structure will need to be implemented in order to get the extra degrees of freedom needed, for example by parameterising the instantaneous correlation between each FX factor as a piecewise constant function.

Cholesky Decomposition

Since we are in a 1-factor model, for each IR factor we will generate the whole curve using one Gaussian random variate, which will be the random variate corresponding to the particular factor. Thus, we must first generate, at each diffusion lag, a random vector of correlated standard normal variates. The algorithm defined in [11, page 11] demonstrates how to generate a single vector of correlated Gaussian random variables. Using Python's `numpy` package, we can easily and efficiently generate M vectors to form a $(2N-1)$ x M matrix. The `linalg` package of `numpy` will find the Cholesky decomposition of the correlation matrix which is what the `Cholesky` function in Algorithm 3 refers to.

Algorithm 3: Correlated Gaussian generation

Result: $\mathbf{X}$, $(2N-1)$ X M, $M :=$ number of simulations
Input $CorrelationMatrix \quad \rho$;
$L \leftarrow Cholesky(\rho)$;
$DiffusionCube \leftarrow \{\}$;
for *lag in DiffusionLags* **do**
  $Z \leftarrow Normal(size = (2N-1)\text{ X } M, mean = 0, variance = 1)$;
  $DiffusionCube[lag] = L \cdot Z$;
end

3.2 Creating the diffused curves

Suppose we have a portfolio of IRS for which we wish to calculate the expected positive and negative exposure (EPE/ ENE) to ultimately calculate CVA and DVA. In order to do so, The goal is to calculate EPE and ENE over various points in time which we will refer to as *diffusion lags*. In order to have the necessary inputs to do so, we need simulated values for each factor at each lag.

For simulation, we want to be able to generate the bond price exactly in the sense that there will be no discretisation. This is possible since we know that the bond is lognormally distributed; removing discretisation will greatly speed up the diffusion process and removes the need for an Euler or Milstein scheme.

3.2.1 Bond and FX Spot distribution

We firstly need to derive the distribution of the domestic bond, the foreign bonds and the FX spot rate in order to use the already-generated correlated Gaussian random variables. Here, we will use the same notation as in Chapter 1 and only consider one foreign currency. In practice, we will have the structure as detailed in Section 3.1, with $N-1$ foreign currencies and the corresponding correlation structure. Indeed, suppose we have drawn N_d, N_f, N_X such that

$$\begin{pmatrix} N_d \\ N_f \\ N_X \end{pmatrix} \sim \mathbf{N}(\mathbf{0}, \Sigma), \quad \mathbf{0} := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Sigma := \begin{pmatrix} 1 & \rho_{f,d} & \rho_{X,d} \\ \rho_{f,d} & 1 & \rho_{X,f} \\ \rho_{X,d} & \rho_{X,f} & 1 \end{pmatrix}$$

Consider the term inside the exponential of (1.2.2), which is Gaussian by the discussion in (3.2.1). Since a Gaussian distribution is fully characterised by its first two moments, by calculating the mean and variance, we automatically have the whole distribution. Indeed, suppose we are at time t_i. Consider the value of a domestic bond at time $t_j > t_i$, that matures at time $m > t_j$. Denoting $I^d(t_i, t_j, m; H) := \int_{t_i}^{t_j} (\sigma_d(s,m) - \sigma_d(s,t_j)) \mathrm{d}W_s^{d,H} - \frac{1}{2}\int_{t_i}^{t_j} (\sigma_d(s,m) - \sigma_d(s,t_j))(\sigma_d(s,m) - 2\sigma_d(s,H) + \sigma_d(s,t_j)) \mathrm{d}s$ and working under the $H-$forward measure

$$\begin{aligned} \mathbb{E}_{t_i}^{d,H}\left[I^d(t_i,t_j,m;H)\right] &= -\frac{1}{2}\int_{t_i}^{t_j} (\sigma_d(s,m) - \sigma_d(s,t_j))(\sigma_d(s,m) - 2\sigma_d(s,H) + \sigma_d(s,t_j)) \mathrm{d}s =: M_d \\ \mathbb{V}_{t_i}^{d,H}\left[I^d(t_i,t_j,m;H)\right] &= \int_{t_i}^{t_j} (\sigma_d(s,m) - \sigma_d(s,t_j))^2 \mathrm{d}s =: V_d^2 \end{aligned} \tag{3.2.1}$$

Where the mean calculation uses the fact that the expectation of a deterministic function against a stochastic integral is zero, and the variance calculation uses Itô's isometry. Thus, $I^d(t_i, t_j, m; H) \stackrel{\Delta}{=} M_d + V_d N_d$. Analogously, we can calculate the distribution of the exponent of the foreign bond under the $H-$forward measure:

$$
\begin{aligned}
I^f(t_i, t_j, m; H) &\stackrel{\Delta}{=} M_f + V_f N_f \qquad (3.2.2)\\
M_f &:= \int_{t_i}^{t_j} \rho_{f,d}\sigma_d(s,H)\left[\sigma_f(s,m) - \sigma_f(s,t_j)\right] \mathrm{d}s \\
&\quad - \int_{t_i}^{t_j} \left\{\rho_{X,f}\sigma_X(s)(\sigma_f(s,m) - \sigma_f(s,t_j))\right\} \mathrm{d}s \\
&\quad - \frac{1}{2}\int_{t_i}^{t_j} \left[\sigma_f^2(s,m) - \sigma_f^2(s,t_j)\right] \mathrm{d}s \\
V_f^2 &:= \int_{t_i}^{t_j} \left(\sigma_f(s,m) - \sigma_f(s,t_j)\right)^2 \mathrm{d}s
\end{aligned}
$$

Finally, we calculate the distribution of the exponent in the FX Spot:

$$
\begin{aligned}
I^{FX}(t;T) &\stackrel{\Delta}{=} M_{FX} - V_{FX_d}N_d + V_{FX_f}N_f + V_{FX_{f,d}}N_X \qquad (3.2.3)\\
M_{FX} &:= \frac{1}{2}\int_{t_i}^{t_j} \left(\sigma_d^2(s,t_j) - \sigma_f^2(s,t_j) - \sigma_X^2(s)\right) \mathrm{d}s + \rho_{f,d}\int_{t_i}^{t_j} \sigma_d(s,H)\sigma_f(s,t_j)\mathrm{d}s \\
&\quad - \rho_{X,f}\int_{t_i}^{t_j} \sigma_f(s,t_j)\sigma_X(s)\mathrm{d}s + \rho_{X,d}\int_{t_i}^{t_j} \sigma_d(s,H)\sigma_X(s)\mathrm{d}s - \int_{t_i}^{t_j} \sigma_d(s,H)\sigma_d(s,t_j)\mathrm{d}s \\
V_{FX_d}^2 &:= \int_{t_i}^{t_j} \sigma_d^2(s,t_j)\mathrm{d}s, \quad V_{FX_f}^2 := \int_{t_i}^{t_j} \sigma_f^2(s,t_j)\mathrm{d}s, \quad V_{FX_{f,d}}^2 := \int_{t_i}^{t_j} \sigma_X^2(s)\mathrm{d}s
\end{aligned}
$$

Using `numpy` arrays, in practice N_d, N_f, N_X are vectors, the length of each being the number of Monte Carlo simulations, specified by the user. This greatly increases the speed since transformations of `numpy` arrays are in general very efficient. The following algorithm describes how to simulate the full IR curve (i.e. for each maturity) and each FX Spot, for each diffusion lag. *Factors* is a list of factors that are to be diffused, *Lags* is the list of diffusion lag in years, *TimeZeroData* is the data corresponding to curves at time 0, *Cube* is the cube of correlated standard Gaussian random variables described in Section 3.1.

It is also important to note that for each factor, in order to calculate the value of at the $(i+1)^{th}$ diffusion lag, we use the value of the factor derived at the i^{th} diffusion lag as the "time-0" curve. Therefore we have tractability across the diffusion lags. An alternative would be to simply start from time-0 when diffusing at each scenario, with longer scenarios using more Brownian motion components. Since both methods follow the same distribution, on average they are equivalent.

Algorithm 4 refers to descriptive functions such as *GetFacMean*. The purpose of this to see a more holistic view of the algorithm. In reality, these are comprised of several smaller functions to incorporate the various piecewise integrals observed in (3.2.1), (3.2.2) and (3.2.3). The condition *if not expired* refers to the fact that the maturity of the bond must be greater than or equal to the diffusion lag for the bond to exist. We will therefore have a reducing number of maturities for the bonds at each diffusion lag. We interpolate each bond curve linearly if a requested maturity is not in the set of available maturities.

Algorithm 4: Curve diffusion

Input: *Factors*, *Lags*, *TimeZeroData*, *Cube*;

Result: **DiffusedFactors**, nested dictionary of simulations with 3 levels:

- *Level 1*: Diffusion Lag
- *Level 2*: Diffused factor
- *Level 3*: Maturity (only applies to IR factors)

```
for i, lag in length(Lags), Lags do
    LagDiff ← Lags[i+1] - Lags[i] ;
    for fac in Factors do
        if "IR" in fac then
            means ← [GetFacMean(LagDiff, m) for m in maturities if not expired];
            vols ← [GetFacVol(LagDiff, m) for m in maturities if not expired];

            for m in maturities do
                BondRatio ← GetBondRatio(Lags[i-1], Lags[i], m, fac);
                X ← means[m] + vols[m] * Cube[lag][fac];
                DiffusedFactors[lag][fac][m] ← BondRatio * e^X;
            end
        end
        else
            FXRatio ← GetFXRatio(Lags[i-1], Lags[i], fac);
            mean ← GetFXMean(LagDiff) ;
            dvol, fvol, fxvol ← GetFXVol(d), GetFXVol(IRfac), GetFXVol(FXfac);

            X ← mean + dvol * Cube[lag][d]+ fvol * Cube[lag][IRfac]+ fxvol *
             Cube[lag][FXfac] ;

            DiffusedFactors[lag][fac][m] ← FXRatio * e^X
        end
    end
end
```

3.2.2 Unit Testing

Having implemented the diffusion, we will need to make sure that our results are reliable and as we expect. For this, we will calculate the empirical standard deviation (as the square root of the variance) using our simulations and compare to the "theoretical" standard deviation for both the foreign and domestic bonds and the FX Spot rates. Due to log-normality, the easiest way to do so is consider the logarithm of the bonds and the spot rates, rather than deriving complicated expressions for the variance of the bond directly. As an example, consider (1.2.3). It is clear that, denoting the standard deviation by *Std*,

$$\mathrm{Std}^{d,H}[\log(B_d(t,T))] = \sqrt{\int_0^t \sigma_d(s,T)^2 \mathrm{d}s} \tag{3.2.4}$$

Similar formulas can be obtained for the foreign bond as well as the FX spot rate standard deviation. This is known exactly via the parameters obtained through calibration (hence is labelled *theoretical* results), and so we can compare to the empirical calculations. We will do so at the 1-year diffusion lag which is sufficient time to allow for substantial variance, and we will only consider one maturity per bond. This was chosen at random and we expect that the proportionate differences between the empirical and theoretical results should be similar across maturities. The results shown in Table 3.1 were obtained in approximately 6.85 seconds, using 50,000 iterations of 58 diffusion lags, with seven factors considered.

Factor	**Theoretical log-std (%)**	**Empirical log-std (%)**	**Absolute Error (%)**
$\mathrm{USD}_{\mathrm{IR}}$ (dom.)	8.2070	8.2103	0.0303
$\mathrm{GBP}_{\mathrm{IR}}$	3.6236	3.6341	0.0105
$\mathrm{EUR}_{\mathrm{IR}}$	3.6346	3.6425	0.0079
$\mathrm{CHF}_{\mathrm{IR}}$	2.1641	2.1718	0.0077
$\mathrm{USD} - \mathrm{GBP}_{\mathrm{FX}}$	31.4783	31.4992	0.0209
$\mathrm{USD} - \mathrm{EUR}_{\mathrm{FX}}$	31.4915	31.5017	0.0102
$\mathrm{USD} - \mathrm{CHF}_{\mathrm{FX}}$	32.4194	32.2863	0.1331

Table 3.1: Empirical versus theoretical standard deviations for each factor diffused at 1 year

We see very large variation across the FX components; this is because the model was not calibrated to FX Options and the "default" values of parameters g^X chosen are clearly much larger than what each corresponding optimised value would be had the model been calibrated. In general, we see the empirical results are very close to the theoretical results over 50,000 iterations, with a maximum absolute error of the order 10^{-4} (10^{-2} %). Given that the drift component is simply a function of the volatility since we are in an HJM framework, this also gives us confidence that the corresponding drifts are accurate.

Whilst increasing the number of iterations is one way to reduce the variance (by the Strong Law of Large Numbers), we could also consider using a control variate method or antithetic sampling, detailed in [11, Chapter 4]. This would reduce the empirical variance without the additional memory consumption required when further increasing the number of simulations.

Chapter 4

CVA Calculation

We conclude this report with an illustration of how to calculate the CVA of a given portfolio using our diffusion model. We will introduce the theory behind the CVA calculation, including the stripping of default probabilities from CDS curves, before presenting numerical results for the CVA of 4 vanilla Interest Rate Swaps, each denominated in different currencies.

4.1 CVA under Default Bucketing

Let R be the recovery rate, τ be the default time of the counterparty and denote by V_t the residual net present value (NPV) of the contract at time t, i.e. if $\Pi(t,T)$ is the discounted payoff at time-t for maturity T, then $V_t := \mathbb{E}_t[\Pi(t,T)]$. As usual, denote the stochastic discount factor, $\exp\left\{-\int_t^T r(s)ds\right\}$, by $D(t,T)$. The *time-t* unilateral CVA (UCVA$_t$) can be written [1, slide 833] as:

$$\mathrm{UCVA}_t = (1-R)\mathbb{E}_t\left[\mathbf{1}_{\{\tau<T\}} D(t,\tau)(V_\tau)^+\right] \tag{4.1.1}$$

where expectation is under the standard risk neutral measure. For numerical application, we will divide the interval $[0,T]$ into n sub-intervals with grid points T_i, $i \in \{0,1,\dots,n-1\}$. In practice this grid will coincide with the grid of diffusion lags as seen in the previous section. Since the intervals are disjoint by construction, the time of default τ must fall in precisely one interval. This allows us to write (4.1.1) as the following, using [1, slide 880]

$$\begin{aligned}
\mathrm{UCVA}_t &= (1-R)\mathbb{E}_t\left[\sum_{i=0}^{n-1} \mathbb{1}_{\{\tau\in(T_i,T_{i+1}]\}} D(t,\tau)(V_\tau)^+\right] \\
&= (1-R)\sum_{i=0}^{n-1} \mathbb{E}_t\left[\mathbb{1}_{\{\tau\in(T_i,T_{i+1}]\}} D(t,\tau)(V_\tau)^+\right] \\
&\approx (1-R)\sum_{i=0}^{n-1} \mathbb{E}_t\left[\mathbb{1}_{\{\tau\in(T_i,T_{i+1}]\}} D(t,T_i)(V_{T_i})^+\right] \\
&= (1-R)\sum_{i=0}^{n-1} \left[\mathbb{Q}(\tau>T_i) - \mathbb{Q}(\tau>T_{i+1})\right]\mathbb{E}_t\left[D(t,T_i)(V_{T_i})^+\right] \\
&=: (1-R)\sum_{i=0}^{n-1} \left[\mathbb{Q}(\tau>T_i) - \mathbb{Q}(\tau>T_{i+1})\right]\cdot \mathrm{EPE}_t(T_i)
\end{aligned} \tag{4.1.2}$$

where $\text{EPE}_t(T_i)$ denotes the time-T_i "Expected Positive Exposure" given we are at time$-t$. The "approximation" step assumes defaults can only happen at the end of the intervals which is reasonable with a fine enough grid spacing. This approximation allows us to decouple the τ-dependence of the discounted payoff with the indicator function, so that we can write the expectation of a product as the product of expectations to get (4.1.2). The other big assumption is that we assume that the various risk factors (IR, FX) and the credit component are independent. In some instances this is not a good approximation in which case the wrong way risk and right way risk element needs to be modelled.

4.2 Calculating Default Probability

As demonstrated in (4.1.2), there are two main components to calculating the CVA of a portfolio; the EPE and the default probabilities. In this section we show how to obtain the latter in a model-independent way from the counterparty's CDS curve. In this section, we will remove the currency dependence on the bond and short rate. In practice, this will be in the same currency as the CDS is denominated, with the exact same calculations since the formulas is model independent.

Under the (standard) assumption that the stochastic discount factors $D(t, T)$ are independent of the default time τ for all possible $0 < t < T$, pricing a CDS simply involves static no-arbitrage arguments for each of the two legs. We state without proof the formula for the price of a CDS; a detailed proof can be found in [3, 21.3.5].

Briefly setting up the notation for the components of a CDS as in [3, 21.3.5], we denote the protection buyer by A, the protection seller by B, and the risky counterparty by C. Supposing C defaults at time $\tau \in [T_a, T_b]$, $T_b > T_a$, B pays A the loss given default, LGD, multiplied by the notional. As compensation for this insurance in the event of default, A pays to B a protection premium rate R at times $T_{a+1}, \ldots, T_b$ or until default occurs. Denoting $\alpha_i := T_i - T_{i-1}$, the running CDS discounted payoff **to B** at time $0 < T_a$ is

$$\Pi(0; a, b) := D(0, \tau)(\tau - T_{\beta(\tau)-1}) R \mathbb{1}_{\{T_a < \tau < T_b\}} + \sum_{i=a+1}^{b} D(0, T_i)\alpha_i R \mathbb{1}_{\{\tau > T_i\}} - \mathbf{1}_{\{T_a < \tau \leq T_b\}} D(0, \tau) LGD$$

where $T_{\beta(\tau)}$ is the first T_i following τ.

Proposition 4.2.1 (CDS Price)**.**
The price of a CDS contract as seen from 'B' at time 0 is given by

$$\mathbb{E}\left[\Pi(0; a, b)\right] = \textbf{\textit{PremLeg}}_{0,b}(R; B(0, \cdot), \mathbb{Q}(\tau > \cdot)) - \textbf{\textit{ProtLeg}}_{0,b}(LGD; B(0, \cdot), \mathbb{Q}(\tau > \cdot))$$

where

$$\textbf{\textit{PremLeg}}_{0,b}(R; B(0, \cdot), \mathbb{Q}(\tau > \cdot)) := R\left[\int_{T_a}^{T_b} B(0, t)(T_{\beta(t)-1} - t)\mathrm{d}_t\mathbb{Q}(\tau \geq t) + \sum_{i=a+1}^{b} B(0, T_i)\alpha_i \mathbb{Q}(\tau \geq T_i)\right]$$

$$\textbf{\textit{ProtLeg}}_{0,b}(LGD; B(0, \cdot), \mathbb{Q}(\tau > \cdot)) := LGD \int_{T_a}^{T_b} B(0, t)\mathrm{d}_t\mathbb{Q}(\tau \geq t)$$

This is a model independent formula, since no assumption was made regarding the IR dynamics. Note here, that price of the bond at time zero $B(0,\cdot)$ the bond in the currency that the CDS is quoted in. Before outlining the algorithm we use to strip the curve, we first make the simplifying assumption that the protection leg pays at the end of each accrual period, and that for the premium leg, defaults occur midway during each payment period with the accrual payment made at the end of the period. This known as the *JP Morgan method*, and is detailed in [14, page 8]. These assumptions reduce the integrals in Proposition 4.2.1 to summations forming a basis for the bootstrapping algorithm which we will briefly outline. Under these assumptions, we can write the value of each leg as the following. Details of the calculations are in [14, pages 8-9], so we just present the main results.

$$\mathbf{ProtLeg}_{0,b}(LGD; B(0,\cdot), \mathbb{Q}(\tau > \cdot)) = LGD \sum_{i=a+1}^{b} B(0,T_i)(\mathbb{Q}(\tau > T_{i-1}) - \mathbb{Q}(\tau > T_i)) \quad (4.2.1)$$

and similarly, the present value of the premium leg as

$$\mathbf{PremLeg}_{0,b}(R; B(0,\cdot), \mathbb{Q}(\tau > \cdot)) = R \sum_{i=a+1}^{b} \alpha_i B(0,T_i) \frac{(\mathbb{Q}(\tau > T_{i-1}) + \mathbb{Q}(\tau > T_i))}{2} \quad (4.2.2)$$

Now, assuming $T_a = 0$, we observe in the market at time 0, the fair value of $R = R_{0,b}^{\text{mkt MID}}$ is the value such that the corresponding CDS contract has no value, i.e. the PV of the premium and default legs are identical. Given that the time-0 bond prices $B(0,\cdot)$ can be easily obtained from market data as well as the time-0 default probabilities, with a standard assumption on $LGD = 60\% = 1 - Rec$, the algorithm becomes clear and is as follows.

Algorithm 5: Stripping Default Probabilities from the CDS Curve

---

Result: **PD**, list of default probabilities; **PD[i]** $:= \mathbb{Q}(\tau > T_i)$

PD $\leftarrow \{\}$;

for i=a+1,...,b **do**

    if i=a+1 **then**

        $P_{i-1} \leftarrow 0$;

        $Q_{i-1} \leftarrow 0$;

    else

        $C \leftarrow \frac{1}{B(0,T_i)(2LGD + R\alpha_i)}$;

        $P_{i-1} \leftarrow C \cdot LGD \sum_{j=a+1}^{i-1} B(0,T_j)(PD[j-1] - PD[j])$;

        $Q_{i-1} \leftarrow 0.5R \cdot C \sum_{j=a+1}^{i-1} \alpha_j B(0,T_j)(PD[j-1] - PD[j])$;

    $PD[i] = P_{i-1} - Q_{i-1} + \frac{(2LGD - R\alpha_i)PD[i-1]}{2LGD + R\alpha_i}$

end

In Algorithm 5, we take $R := R_{0,b}^{\text{mkt MID}}$, and $PD[a]$ is known as an input from the market. The algorithm was derived simply by splitting up the sum around the i^{th} point in (4.2.1) and (4.2.2) and rearranging in terms of the one unknown at each step.

The Valuation Platform to which the algorithms present here are being implemented already has functionality to strip default probabilities from CDS curves, so we do not need to build our own. We now finalise the report by presenting results for the EPE and corresponding CVA for a small portfolio consisting of 4 IRS contracts, each with the same counterparty and payment schedule properties but denominated in different currencies.

4.3 Numerical Results

The goal is to calculate CVA on a portfolio level containing products of multiple currencies. We will not, however, be able to benchmark this analytically since the EPE and hence CVA is a complex FX-IR hybrid. However, a simple unit test we can do is check that we are indeed able to reproduce the EPE for each swap at a product-level, since this reduces to pricing a swaption. We have not had time to do this as of yet; this will be done in the near future. Before the code is able to be put in production, we would need to develop several other unit tests.

The only remaining unknown in the CVA calculation (4.1.2) is the EPE at each time T_i in the grid of diffusion lags, given that we have the algorithm for calculating $\mathbb{Q}(\tau > T_i)$ for all i. In the case of a single IRS contract, the EPE is exactly the value of the corresponding swaption due to the added optionality. In this case, we would leverage the analytical swaption pricer implemented during the calibration section. Since we want to calculate the EPE over the whole portfolio, the added FX component means that we resort to Monte Carlo to do so.

We now consider calculating the CVA on a portfolio of receiver Interest Rate Swaps, the details of which are shown below in Table 4.1.

Product type	**Fixed currency**	**Float currency**	**Payment frequency**	**Start Date**	**Maturity**
IRS (Rec.)	USD	USD	Quarterly	30/06/2021	20Yr
IRS (Rec.)	GBP	GBP	Quarterly	30/06/2021	20Yr
IRS (Rec.)	EUR	EUR	Quarterly	30/06/2021	20Yr
IRS (Rec.)	CHF	CHF	Quarterly	30/06/2021	20Yr

Table 4.1: Portfolio used in EPE and CVA computations

We assume a notional of 100,000 for each contract, *denominated in each local currency.* We take the counterparty to be JP Morgan (JPM), arbitrarily. Define the "Valuation Date" to be 30$^{\text{th}}$ June 2021, since this was the date used in the calibration of the IR models. We choose the diffusion lags to be quarterly up until 10 years from the Valuation Date, from which they will be every 6 months up to 15 years, and finally from 15 years to 20 years, they will be annual. This gives a total number of 55 points in the grid. Note that we use this grid for both the diffusion and the CVA default bucketing grid, as mentioned previously.

4.3.1 Default Probabilities

Here we show the calculated default probabilities for JPM, stripped from the CDS curve as of the Valuation Date. We were able to utilise the CDS curve stripping algorithm already implemented in the pricing library and so we just illustrate the results. We present the cumulative probabilities to outline the general shape; interval probabilities of default are easily calculated from this and it is the interval probabilities that will be needed in the CVA calculation.

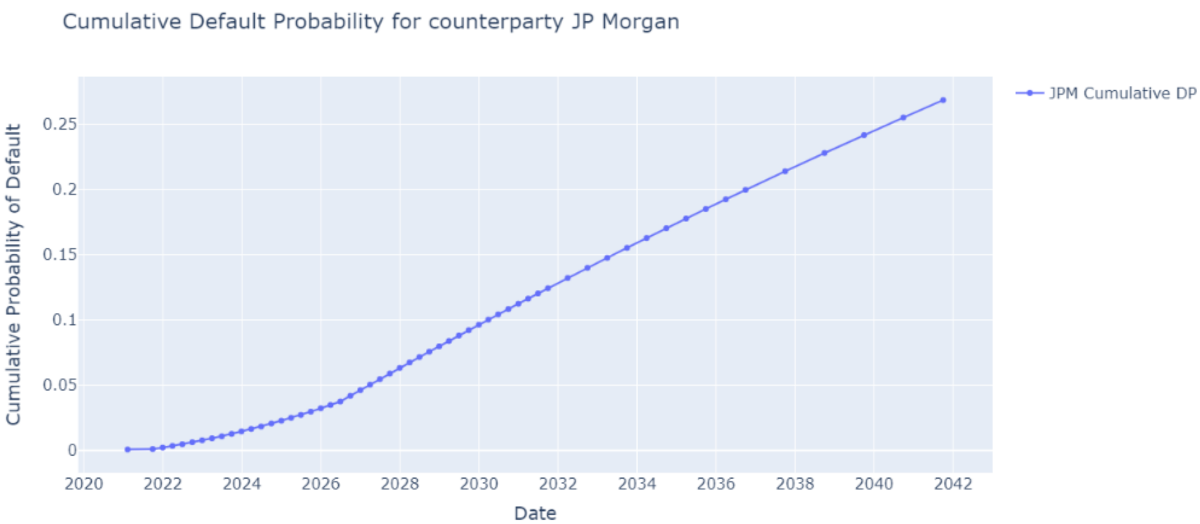

Figure 4.1: Cumulative default probabilities for counterparty JPM

4.3.2 Expected Positive Exposure

Figure 4.2: Expected positive exposure for the portfolio of IRS contracts.

The shape observed in Figure 4.2 is as we would expect for the EPE of a portfolio of IRS contracts.

Firstly, consider a single swap. The swap starts at the money, so the EPE will be zero. Then, as rates move stochastically it is more likely that the optionality on the swap in the EPE calculation is to be exercised, resulting in a higher EPE. Finally, we would expect to see it reducing towards the maturity of the swap as the number of remaining cash flows reduces to zero. On a portfolio of 4 vanilla swaps, we expect the overall shape of the EPE to be very similar, with more variation due to the FX component as each swap is in a different currency.

4.3.3 Numerical CVA Calculation

Once we have calculated the EPE for the portfolio, calculating the CVA becomes a straightforward application of the formula derived in (4.1.2):

CVA (USD)	0.9052

Table 4.2: CVA Calculation for the portfolio considered above

In simple products such as an IRS where the stochastic factors that we are able to diffuse (namely foreign and domestic zero-coupon bonds and FX Spot) are all that is needed to compute the CVA, the model developed is sufficient. In more complex products where there are other factors that would ideally be stochastic, such as spreads (tenor basis or cross currency basis) and credit (to account for wrong/ right way risk), care needs to be taken as to how these factors would evolve (stochastically) over time.

Conclusion

The aim of this paper was to design and implement an IR-FX Hull-White diffusion model that can be used to calculate the exposure profile of various portfolios of cross-currency products. The primary objective has therefore been reached; we have showcased results in the simple case where the portfolio consists of simple vanilla IRSs in different currencies, and attained a profile for the EPE with a shape as we would roughly expect. We also completed some basic unit-testing to ensure that the model is correctly implemented, although further robust testing is required for example by comparing prices obtained with another model.

Having first established the stochastic environment, we provided a detailed section on the calibration procedure including results. It was extremely useful to be able to use genuine market data from Bloomberg; for this I am grateful to the Mazars Quant Team. This also highlighted some of the difficulties in practice, a prime example being the lack of data for some currencies. Before finally implementing the diffusion and discussing CVA calculation, we derived distributions for the diffused factors, and discussed correlated Gaussian generation. We are still in the stage of fully integrating parts of the project within the current pricing library to utilise analytical pricing formulas where possible, to be used for CVA.

There are several areas where this project can be developed further in a greater time frame. The two most notable are implementing an FX Option pricer to complete the calibration process, and historical calibration of the correlation matrix between factors. The program has been designed to accommodate these extensions without significant refactoring. One could also generalise the calibration to allow for an optimal time-dependent mean reversion parameter, by additionally calibrating to caps. The CVA model should also account for collateral to capture the margin period of risk. Finally, this framework can be leveraged to account for other common xVAs, in particular DVA and FVA.

Appendix A

Code Flowchart

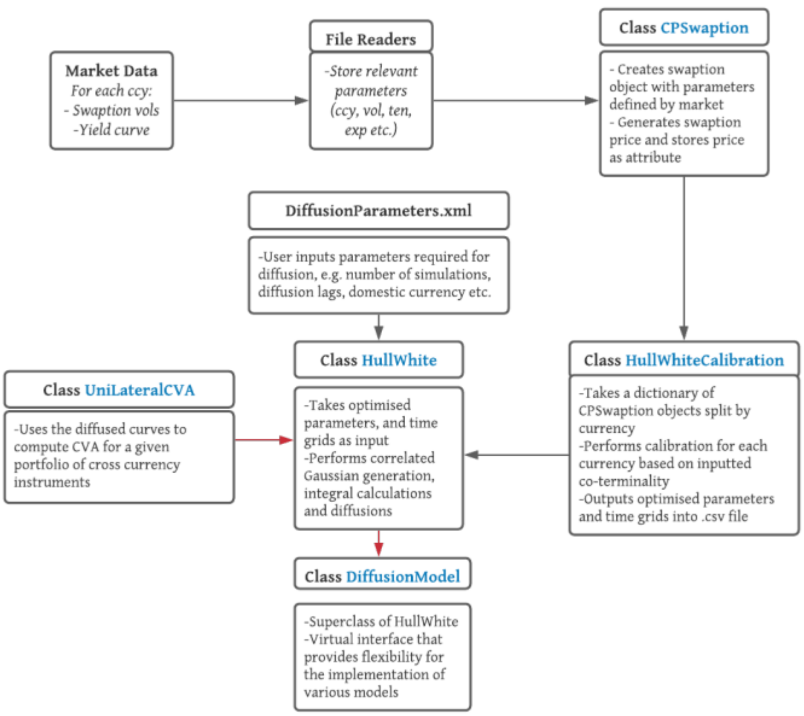

Figure A.1: Flowchart describing the overall structure of the code. Arrows in grey represent input/ output direction, arrows in red represent class inheritance. The direction is such that A $\rightarrow$ B $\equiv$ "A inherits from B"

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