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Dividend Modelling and the Particle Method

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Declaration

The work contained in this thesis is my own work unless otherwise stated.
Signed James Leach - 05.09.2022

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Abstract

Dividends are a share of earnings that a company can pay to eligible shareholders. They may be paid annually, semi-annually, or quarterly and are often paid in cash (called "cash dividends"), but may also be in the form of additional shares (called "stock dividends"). This study focuses on the more common cash form that provides a challenging modelling problem.

Letting $S = (S_t)_{t\in\mathbb{R}_+}$ denote the share price process and $0 = \tau_0 < \tau_1 < \tau_2 ...$ be a series of dividend dates which we will infer as being both the ex-date and date of payment. On the dividend date τ_j a cash payment of d_j is paid to the holder of one share. In the absence of outside factors such as jump risk or tax-imbalances, the security price must drop an amount equal to d_j on the dividend payment. That is

$$S_{\tau_j} = S_{\tau_j^-} - d_j$$

where τ_j^- is the time just before the arrival of the dividend. Theoretically this can be seen by basic no-arbitrage arguments and is seen in practice at the open of trading on the ex-date.

This random drop in price is not captured in conventional equity models where dividends are often accounted for as an additional rate. Overlooking dividend risk is crucial, particularly in the current climate where the dividend payments of companies have fluctuated following the Covid-19 pandemic causing many trading desks to experience losses.

In the wake of this volatility, products incorporating dividends as an underlying have seen an increase in interest. Investors can now access a number of dividend products including futures, vanillas, swaps and more heavily structured products requiring the joint modelling of the share price process $S = (S_t)_{t\in\mathbb{R}}$ and dividend payments.

This thesis focuses on two methods to approach the joint modelling problem. Firstly, the blended dividends framework of H. Buehler [1] and then the particle method of J. Guyon and P. Henry-Labordere[2], [3]. The EURO STOXX 50 index is used as a running calibration example, after which we use calibrated models to price an example structured dividend product, the knock-in dividend swap with payoff $\mathbb{1}_{\min_{t\in[T_1,T_2]} S_t < B}\left(\sum_{\tau_j\in[T_1,T_2]} d_j\right)$.

Much of the first chapter follows the work of [1]; the second, [4]; and third, [3]. Concerted effort has been made to understand and add detail to the proofs of theoretical results, and to build implementations of these models.

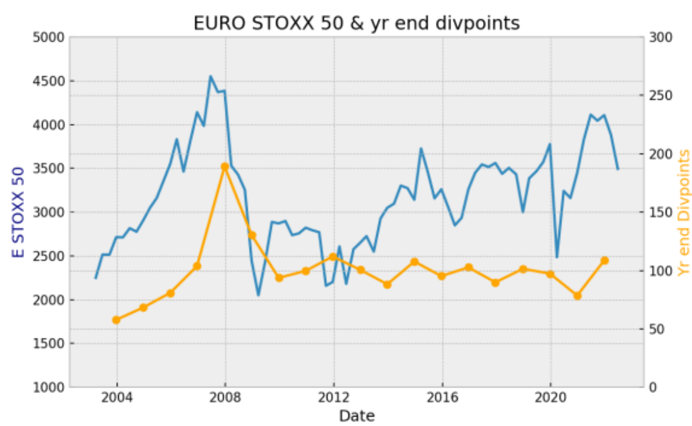

Figure 1: Plot exhibiting correlation between the EURO STOXX 50 index and annualised dividend payments.

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Chapter 1

Blended Dividends in Equity Modelling

As mentioned, the modelling of dividend products requires understanding of the joint distribution of the price process S and dividends. Weaker assumptions on this relationship will be detailed in later sections but for now we introduce the most referenced approach in practitioners literature, the 'blended dividends' framework introduced in [1]. In truth, there is no joint distribution in this model, as the dividend payment is assumed to be an affine function of the stock price. The simplicity of the assumption results in a tractable and rich framework which we will use as a benchmark. Further extensions of the blended framework have been made to give dynamics to the dividends [5], [6] though these approaches are not studied in detail here as the particle method discussed in later sections subsumes these approaches.

Throughout this project we assume the dividend is paid on the ex-date. Though this is not true in practice, the drop is observed at the open of the ex-date meaning mathematically this assumption is feasible. In practice, time scaling can be used to account for the life-cycle of a dividend payment. A brief introduction to dividends and their terminology is given in A.1.

1.1 The Blended Dividend Framework

We work under the usual stock and bank-account setting with stochastic basis $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $\mathbb{P}$ denotes the 'real-world' measure.

Using previous notation, the cash dividend d_j is assumed to be an affine function of the stock price. This structure is motivated by the economic observation that short term dividends are more predictable than longer term dividends that are thought to behave proportionally to the respective equity.

The dividend delivered at time τ_j is assumed to have cash value

$$d_j = S_{\tau_j^-}\beta_j + \alpha_j$$

where we refer to β_j as the proportional part of the dividend payment and α_j as the cash part (note the abuse of terminology, the whole dividend is still delivered as cash, though this is just the 'cash' part of the blended structure). Assuming the stock price only jumps due to dividends, we then have on ex-date τ_j that the share price must drop according to

$$S_{\tau_j} = S_{\tau_j^-}(1 - \beta_j) - \alpha_j \tag{1.1}$$

A consequence of this seemingly basic assumption is that the share process must conform to a particular structure which will be much of the focus of the following section.

For ease of notation we omit including credit risk in this model, but note a simple extension to include hazard rates is readily achievable. We assume interest and borrowing rates are deterministic and known in advance, denoting $r = (r_s)_{s \geq t}$ to be the interest rate process, $\mu = (\mu_s)_{s \geq t}$ to be the repurchase agreement rate and

$$P(t,T) := \exp^{-\int_t^T r_s ds}$$

to be the price of the T-maturity zero-coupon bond at time t.

The Forward

We start by deriving the fair price (strike) of the forward contract $F(t,T)$ where t is the current time and T is the maturity.

Assume that t falls between two dividend payments such that $\tau_{[t]} \leq t < \tau_{[t]+1}$ and at t we purchase η units of stock, financed by shorting ηS_t worth of bonds. Note that borrowers are not entitled to dividends and must pay them to the lender, so by holding stock we earn repo proceeds and dividends that at time $\tau_{[t]}$ are equal to

$$\eta \exp^{\int_t^{\tau_{[t]+1}} \mu_s ds}(\alpha_{[t]+1} + \beta_{[t]+1} S_{\tau^-_{[t]+1}} + S_{\tau_{[t]+1}}) = \eta \exp^{\int_t^{\tau_{[t]+1}} \mu_s ds}(\frac{\alpha_{[t]+1}}{1-\beta_{[t]+1}} + \frac{1}{1-\beta_{[t]+1}} S_{\tau_{[t]+1}})$$

At this point we reinvest the proportional proceeds in the stock and cash part to pay down our debt. The presumption of differentiating between case and proportional parts is a noted sticking point of the reasoning but is necessary for tractability. In practice, a blending scheme is chosen at the discretion of traders with a suitable choice of α_j such that the stock price has a negligible probability of becoming negative and that the long run proportional part may take precedence. In view of this, our assumption is reasonable in the long term.

By reinvesting all proceeds, at time T we hold

$$\frac{\eta \exp^{\int_t^T \mu_s ds}}{\prod_{j: t < \tau_j \leq T} (1-\beta_j)}$$

units of stock and

$$-\eta S_t \exp^{\int_t^T r_s ds} + \sum_{j: t<\tau_j \leq T} \eta \frac{\alpha_j}{\prod_{k \leq j} (1-\beta_k)} \exp^{\int_t^{\tau_j} \mu_s ds + \int_{\tau_j}^T r_s ds}$$
$$= \eta \exp^{\int_t^T r_s ds} \left(\sum_{j: t<\tau_j \leq T} \frac{\alpha_j}{\prod_{k \leq j} (1-\beta_k)} \exp^{\int_t^{\tau_j} (-r_s + \mu_s) ds} - S_t \right)$$

units of zero coupon bond. We choose $\eta = \prod_{j: t \leq \tau_j < T} (1-\beta_j) \exp^{-\int_t^T \mu_s ds}$ in order to deliver one unit of stock, so the time T bond position is

$$\exp^{\int_t^T (r_s - \mu_s) ds} \prod_{j: t<\tau_j \leq T} (1-\beta_j) \left(\sum_{j: t<\tau_j \leq T} (\alpha_j \exp^{\int_{\tau_j}^T (r_s - \mu_s) ds} \prod_{k: \tau_j < \tau_k \leq T} (1-\beta_k)) - S_t \right)$$

By nullifying this bond position we obtain the following succinct formula for the forward.

Proposition 1.1: Blended Dividend Forward

Defining the "proportional growth factor"

$$R(t,T) := \exp^{\int_t^T (r_s-\mu_s)ds} \prod_{j:t<\tau_j\leq T} (1-\beta_j)$$

the fair forward price in the blended dividends framework is given by

$$F(t,T) = R(t,T)S_t - \sum_{j:t<\tau_j\leq T} R(\tau_j,T)\alpha_j \tag{1.2}$$

We note the useful property that for $s < \tau_j < t$,

$$R(s,t) = \exp^{\int_s^{\tau_j}(r_u-\mu_u)du} \prod_{k:s<\tau_k\leq\tau_j} (1-\beta_k). \exp^{\int_{\tau_j}^{t}(r_u-\mu_u)du} \prod_{k:\tau_j<\tau_k\leq t} (1-\beta_k) = R(s,\tau_j)R(\tau_j,t)$$

and for convenience we define the abbreviations $R_T := R(0,T)$ and $F_T := F(0,T)$. We also note

$$F_T = \left(S_0 - \sum_{j:0<\tau_j\leq T} \frac{\alpha_j}{R_{\tau_j}}\right) R_T$$

making it apparent that the blended dividend forward can be seen as the time T-value of a correction to the current stock price and that this correction is equal to the present value of all future cash dividends. Intuitively this makes sense, as these future cash flows are (in this case, given no credit risk) guaranteed so the forward must be adjusted to reflect this. Similarly the proportional growth factor R is adjusted by the proportional shifts in stock price due to dividend payments.

Stock Representation

We now consider the implications for the stock price process given the form of the forward $F(t,T)$. Given that S must be non-negative in the absence of arbitrage, the forward $F(t,T)$ as in (1.2) for some maturity $T > t$ must also be non negative giving

$$R(t,T)S_t - \sum_{j:t<\tau_j\leq T} R(\tau_j,T)\alpha_j \geq 0$$

Assuming the dividend blended structure is well defined, $R(t,T)$ is strictly positive leading to the following result.

Proposition 1.2: Blended Dividend floor for the Stock price

The share price at time t cannot fall below floor D_t, that is

$$S_t \geq D_t \tag{1.3}$$

where process $D = (D_t)_{0\leq t}$, $D_t := \sum_{j:t<\tau_j} \frac{\alpha_j}{R(t,\tau_j)}$ is the present value of future cash dividend proportions.

This result leads us to an important observation in modelling in this framework. Given that there are future cash flows α_j, the floor process D brings a deterministic structure where the stock price, unlike preceding models such as Black-Scholes, is not entirely stochastic. In view of this, it makes sense to restrict modelling to the remaining random part of the stock process which we will see can be represented by a local martingale process X.

With the assumption that the stock price only jumps due to discrete cash dividends, the general 'No free lunch without vanishing risk' (NFLWVR) argument of [7] can be applied to the stock process between ex-dates. Taking the two ex-dates around t to be $\tau_{[t]} \leq t < \tau_{[t]+1}$, the we have NFLWVR if and only if there exists an equivalent measure $\mathbb{Q}$ to $\mathbb{P}$ and local $\mathbb{Q}$-martingale process $Y = (Y_s)_{s\in[0,\tau_{[t]+1})}$, such that

$$S_t = S_{\tau_{[t]}} \frac{Y_t}{Y_{\tau_{[t]}}} \exp^{\int_{\tau_{[t]}}^t (r_s-\mu_s)ds}$$

Using relation (1.1) repeatedly and that $Y_0 \equiv 1$ we find

$$\begin{aligned}
S_t &= (S_{\tau_{[t]}^-}(1-\beta_{[t]}) - \alpha_{[t]}) \frac{Y_t}{Y_{\tau_{[t]}}} \exp^{\int_{\tau_{[t]}}^t (r_s-\mu_s)ds} \\
&= (S_{\tau_{[t]-1}} \frac{Y_{\tau_{[t]}}}{Y_{\tau_{[t]-1}}} \exp^{\int_{\tau_{[t]-1}}^{\tau_{[t]}} (r_s-\mu_s)ds}(1-\beta_{[t]}) - \alpha_{[t]}) \frac{Y_t}{Y_{\tau_{[t]}}} \exp^{\int_{\tau_{[t]}}^t (r_s-\mu_s)ds} \\
&= S_{\tau_{[t]-1}} \frac{Y_t}{Y_{\tau_{[t]-1}}} \exp^{\int_{\tau_{[t]-1}}^t (r_s-\mu_s)ds}(1-\beta_{[t]}) - \alpha_{[t]} \frac{Y_t}{Y_{\tau_{[t]}}} \exp^{\int_{\tau_{[t]}}^t (r_s-\mu_s)ds} \\
&\dots \\
&= S_0 Y_t \exp^{\int_0^t (r_s-\mu_s)ds} \prod_{j:\tau_j\leq t}(1-\beta_j) - \sum_{j:\tau_j\leq t} \alpha_j \frac{Y_t}{Y_{\tau_j}} \exp^{\int_{\tau_j}^t (r_s-\mu_s)ds} \prod_{k:\tau_j<\tau_k\leq t}(1-\beta_k)
\end{aligned}$$

In view of no-arbitrage, local martingale Y must maintain the non-negative condition on the stock price process S. We first consider this condition in the simplified setting with $r \equiv 0$, $\mu \equiv 0$ and $\beta_j \equiv 0$ for all j, so have

$$S_t = S_0 Y_t - \sum_{j:\tau_j\leq t} \alpha_j \frac{Y_t}{Y_{\tau_j}}, \qquad S_t = S_{\tau_{[t]}} \frac{Y_t}{Y_{\tau_{[t]}}}, \qquad \text{for } t \in [\tau_{[t]}, \tau_{[t]+1})$$

As the share price cannot fall below the dividend floor (1.3), we have

$$S_t = S_{\tau_{[t]}} \frac{Y_t}{Y_{\tau_{[t]}}} \geq \sum_{j:\tau_j>t} \alpha_j$$

and hence local martingale $\frac{Y_t}{Y_{\tau_{[t]}}}$ is bounded below by some $l = \frac{\sum_{j:\tau_j>t}\alpha_j}{S_{\tau_{[t]}}} \leq \frac{\sum_{j:\tau_j>t}\alpha_j}{\sum_{j:\tau_k>[t]}\alpha_k} \leq 1$. Using the result that a local martingale M with unit mean that is bounded below by $l \in [0,1]$ can be written as $M_t = M_t^+(1-l) + l$ where M^+ is another non-negative unit mean local martingale, we write Y in terms of some non-negative local martingale X as

$$\frac{Y_t}{Y_{\tau_{[t]}}} = \frac{X_t}{X_{\tau_{[t]}}} \left(1 - \frac{\sum_{j:\tau_j>t}\alpha_j}{S_{\tau_{[t]}}}\right) + \frac{\sum_{j:\tau_j>t}\alpha_j}{S_{\tau_{[t]}}} \qquad \text{for } t \in [\tau_{[t]}, \tau_{[t]+1})$$

Using this representation and proceeding iteratively we find

$$S_t = S_0 \left(\frac{X_t}{X_0}\left(1 - \frac{\sum_{j:\tau_j>t}\alpha_j}{S_0}\right) + \frac{\sum_{j:\tau_j>t}\alpha_j}{S_0}\right) = X_t \left(S_0 - \sum_{k=1}^{\infty}\alpha_j\right) + \sum_{j:\tau_j>t}\alpha_j \qquad \text{for } t \in [0,\tau_1)$$

$$\begin{aligned}
S_{\tau_1} &= S_{\tau_1}\left(\frac{X_t}{X_{\tau_1}}\left(1 - \frac{\sum_{j:\tau_j>t}\alpha_j}{S_{\tau_1}}\right) + \frac{\sum_{j:\tau_j>t}\alpha_j}{S_{\tau_1}}\right) \\
&= \left(\frac{X_t}{X_{\tau_1}}\left(S_{\tau_1^-} + \alpha_1 - \sum_{j:\tau_j>t}\alpha_j\right) + \sum_{j:\tau_j>t}\alpha_j\right) \\
&= \left(\frac{X_t}{X_{\tau_1}}\left(X_{\tau_1^-}\left(S_0 - \sum_{k=1}^{\infty}\alpha_j\right) + \sum_{j:\tau_j>t}\alpha_j + \alpha_1 - \sum_{k=1}^{\infty}\alpha_j\right) + \sum_{j:\tau_j>t}\alpha_j\right) \\
&= X_t\left(S_0 - \sum_{k=1}^{\infty}\alpha_j\right) + \sum_{j:\tau_j>t}\alpha_j \qquad \text{for } t \in [\tau_1, \tau_2) \qquad \dots
\end{aligned}$$

Which leads to the general form

$$S_t = X_t \left(S_0 - \sum_{k=1}^{\infty} \alpha_j \right) + \sum_{j:\tau_j > t} \alpha_j \qquad \text{for } t \geq 0$$

In the case of nonzero interest rates, repo rates and proportional dividend parts we have the following.

Theorem 1.3: Blended Dividend Stock Price

In the absence of credit risk, all arbitrage-free stock price processes for S can be written in the form

$$S_t = (F_t - D_t)X_t + D_t \tag{1.4}$$

or alternatively when discounted,

$$\frac{S_t}{R_t} = S_0^* X_t + D_0 \tag{1.5}$$

where $X = (X_s)_{0 \leq s}$ is some non-negative local martingale and where:

- $\alpha_j^* := \frac{\alpha_j}{R_{\tau_j}}$ represent the growth rate-discounted time 0 value of a future cash dividend proportions α_j
- $S_0^* := S_0 - \sum_{j:0<\tau_j} \alpha_j^* = S_0 - D_0$ is the "inner spot" or part of equity value not due to the present value of future cash dividend parts

Proof. By repeating the arguments above with $\bar{S}_t := S_t / R_t$ the discounted stock process with respect to the growth-rate discount factor that pays dividends α_j^*, the desired result can be achieved. □

The previous theorem is the central result to this framework and allows for the separation of the "pure" stock process X from other characteristics of the stock. By construction X should not carry any dividend-paying structure and (though credit risk is not considered in this work) indirectly X is not required to incorporate credit risk. If X were to carry default risk and become zero it would not directly imply the termination of future cash dividend payments. Therefore X can be seen to purely model volatility risk and we can adjust the structure of accepted market practices.

European Options

As the representation (1.4) for S is an affine transformation of the "pure" process X we will see that pricing a European option on S translates to pricing on the pure process.

For a maturity $T \geq 0$ and strike $K \geq 0$, the market European call is given by

$$C(T, K) := P(0, T)\mathbb{E}^{\mathbb{Q}}[(S_T - K)^+]$$

We observe

$$\begin{aligned} C(T, K) &= P(0, T)\mathbb{E}^{\mathbb{Q}}[((F_T - D_T)X_T + D_T - K)^+] \\ &= P(0, T)\mathbb{E}^{\mathbb{Q}}[(F_T - D_T)(X_T - \frac{K - D_T}{F_T - D_T})^+] \\ &= P(0, T)(F_T - D_T)\mathbb{E}^{\mathbb{Q}}[(X_T - \frac{K - D_T}{F_T - D_T})^+] \\ &= P(0, T)(F_T - D_T)\mathbb{C}(T, \frac{K - D_T}{F_T - D_T}) \end{aligned}$$

where $\mathbb{C}(T, k) := \mathbb{E}^{\mathbb{Q}}[(X_T - k)^+]$ is the pure call with "relative strike" k. By rearranging in the opposing fashion we can find the reverse relation.

Proposition 1.4: Blended Dividend European Call

The price of a European call option on stock price S can be expressed as

$$C(T,K) = P(0,T)(F_T - D_T)\mathbb{C}(T, \frac{K - D_T}{F_T - D_T}) \tag{1.6}$$

and reversely,

$$\mathbb{C}(T,k) = \frac{1}{P(0,T)(F_T - D_T)} C(T, (F_T - D_T)k + D_T) \tag{1.7}$$

With $\mathbb{P}(T,k) := \mathbb{E}^{\mathbb{Q}}[(k - X_T)^+]$ and as X is assumed to be a martingale, we have the following by put-call parity

$$\mathbb{C}(T,k) - \mathbb{P}(T,k) = X_0 - k$$

which leads to similar formulae for the put and the usual put-call parity for the stock price process.

An interesting consideration from the call formula (1.6) is the value of calls with strike $K < D_T$. We know the 'dividend floor' D_T imposes a lower bound on the stock price so calls with such a strike must have a constant and deterministic value. Indeed, by considering the pure stock call with a negative strike is equal to a forward on pure process X, which by construction is $\equiv 1$, we see that the value of such a market call is equal to the discounted forward value $P(0,T)(F_T - D_T)$.

This feature highlights how some of the regular no-arbitrage conditions may be altered in the 'pure' setting. A summary of common no-arbitrage conditions are noted below

Result 1.5: No arbitrage constraints on pure process X

Assuming pure call prices $\mathbb{C}(T,k)$ are given for maturities $T \geq 0$ and strikes $k \geq 0$. Then X is a strictly positive martingale (and hence we have no arbitrage) if and only if

- The forward is preserved $\mathbb{C}(T,0) = F_T^X = 1$.
- Absence of strike arbitrage $\partial^2_{kk}\mathbb{C}(T,k) \geq 0$
- Calendar spreads are positive, $\partial_T \mathbb{C}(T,k) \geq 0$

Local and Implied Volatility

The formulas (1.6) and (1.7) give a way to translate market prices into prices on the "pure" stock process X. The marginals of X for quoted maturities T can then be recovered using the well-known formula

$$\mathbb{Q}(X_T = x) = \left.\frac{\partial^2 \mathbb{C}(T,k)}{\partial k^2}\right|_{k=x}$$

Assuming a unique solution to

$$\frac{dX_t}{X_t} = \sigma(t, X_t)dW_t \tag{1.8}$$

Dupire's [8] local volatility function

$$\sigma^2_{\mathrm{Dup}}(t,x) = \left.\frac{\partial_T \mathbb{C}(T,k)}{\frac{1}{2}k^2 \partial^2_{kk}\mathbb{C}(T,k)}\right|_{\substack{T=t\\k=x}} \tag{1.9}$$

can then be used to reprice the pure calls $\mathbb{C}(T,k) = \mathbb{E}^{\mathbb{Q}}[(X_T - k)^+]$ for all maturities T and pure strikes k. This then translates into the stock process S repricing observed market prices.

As in the conventional approach the implied volatility $\sigma_X(T,k)$ of 'pure' process X can be found by solving

$$\mathbb{BS}(\sigma_X(T,k), T, k) \equiv \mathbb{C}(T,k) \tag{1.10}$$

where $\mathbb{BS}(\sigma, T, K)$ is the regular Black-Scholes call formula defined in appendix A.2. Using this implied volatility gives the following relation to the market call price

$$C(T,K) = P(0,T)(F_T - D_T)\mathbb{BS}\left(\sigma_X(T, \frac{K - D_T}{F_T - D_T}), T, \frac{K - D_T}{F_T - D_T}\right)$$

1.2 Numerical Example

We now give a numerical example of implementing the blended dividends framework in a basic local volatility model (1.8) for 'pure process' X. This study provides a benchmark for later numerical studies and is also a useful insight into the blended dividend framework. Using result 1.5 and the 'pure' form of implied volatility it is possible to mark implied volatility freely in the via interpolation as in Gatheral [9]. We proceed similarly building a parameterised volatility surface and then using the well known implied volatility form of Dupire's local variance formula (1.9),

$$v_L = \frac{\partial_T w}{1 - \frac{y}{w}\partial_y w + \frac{1}{4}(-\frac{1}{4} - \frac{1}{w} + \frac{y^2}{w^2})(\partial_y w)^2 + \frac{1}{2}\partial_{yy} w} \tag{1.11}$$

where $w(y,T) \equiv \sigma_X(T,k)^2 T$ is the Black-Scholes total implied variance and $y = \ln \frac{k}{F_T^X}$.

The Data

EURO STOXX 50 Index Options (OESX) data provided by Eurex with settlement prices for 12 August 2022 is used for numerical studies throughout this report, with index dividend futures (FEXD) and options on dividends (OEXD) data used in later sections. The spot price of EURO STOXX 50 on 12 August 2022 was 3776.81. Dividend payments on EURO STOXX 50 ETFs occur quarterly and are paid in cash form. The dividend is quoted in points of the index after calculating the gross dividends paid by index constituents adjusted by their weighting. The figures below show the distributions of historical dividend payments in terms of ratio to the spot price before ex-date and as the raw number of dividend points that are paid. The inclusion of this figure is to provide reasonable estimates for a blending scheme.

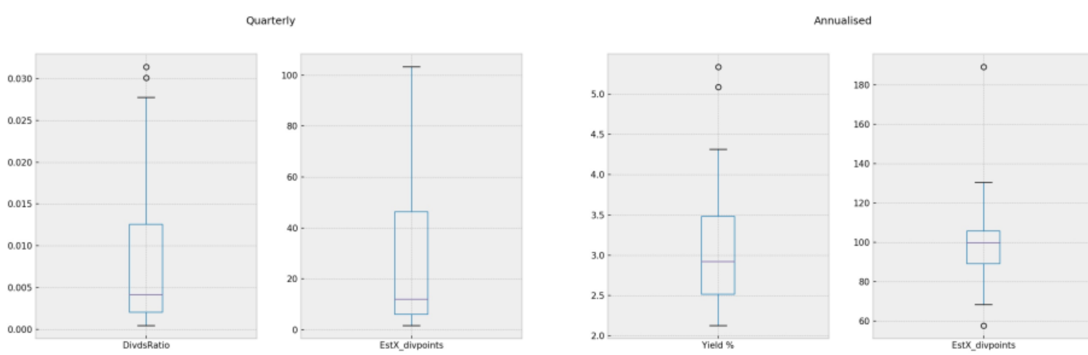

Figure 1.1: Distribution of EURO STOXX 50 dividend payments, quarterly and annually

Method

Building a robust forward curve and volatility surface without access to industry resources is in itself a challenging problem. To achieve reasonable estimates, SSVI surfaces of [10] are used. A constant rate of $r = 0.005$ is assumed and the repo rate μ is optimized for each maturity to ensure convergence of call and put implied volatilities and define a 'proxy' forward curve. The approach taken is summerised below

Algorithm 1.6: Calibration of local volatility blended dividend model

1. Clean raw exchange data - remove outliers, change maturities to year fraction convention, etc.
2. Use regular Black-Scholes formula (A.2) with $r = 0.005$ and find constant repo rates μ using the following constraint recursively through maturities

$$\text{argmin}_{\mu_t \in [T_{i-1}, T_i]} \left(\sum_K \sigma_{S,\text{ call}}(T_i, K) - \sigma_{S,\text{ put}}(T_i, K) \right)$$

   μ is then defined to be a piecewise constant function with steps at maturities, interpolate $r - \mu$ with a Cubic Spline and store to be used as a forward curve.
3. With approximation to the forward curve, define a blending scheme α_j, β_j then use (1.7) to compute 'pure' call and put prices for pure strikes k. Apply transformation $y = \ln k$ to get log-moneyness strikes, solve analogue of (1.10) to get 'pure' implied vols $\sigma_X(T, y)$ in terms of log-moneyness y.
4. Fit power-law SSVI parameterization with parameters $\gamma, \eta, \sigma, \rho$ to find approximation $w_{\text{SSVI}}(y, T; \gamma, \eta, \sigma, \rho) \approx w(y, T)$, check static no arbitrage conditons for parameters.
5. Compute Monte Carlo call prices $\mathbb{C}(T, y) = \mathbb{E}^{\mathbb{Q}}[(X_T - e^y)^+]$ using (1.11) and w_{SSVI}, interchange into implied volatilities and optimize parameters $\gamma, \eta, \sigma, \rho$.
6. Store blending scheme, forward curve and $SSVI$ parameters. Simulate X and interchange to S using (1.4) for pricing.

Calibration

Dividend payments are modelled to be paid quarterly, starting with $\alpha_{\tau_1} = 25$, $\beta_{\tau_1} = 0$ (100% cash, 0% proportional). This scheme then changes linearly to have $\alpha_{\tau_j} = 0$, $\beta_{\tau_j} = 0.0075$ for $j \geq 21$, that is cash parts become zero and proportional parts become the historical mean after 5 years.

The table below summarises the calibration to the market implied volatilities for a number of maturities.

Implied variance (%)	Sep 2022	Dec 2022	Mar 2023	Jun 2023	Sep 2023	Dec 2023
Avg. $\lvert$Mkt – MC$\rvert$	0.59	0.75	0.73	1.43	1.76	1.90
Max. $\lvert$Mkt – MC$\rvert$	1.12	1.29	1.32	2.30	3.16	3.46
Implied variance (%)	**Jun 2024**	**Dec 2024**	**Dec 2025**	**Dec 2026**	**Dec 2028**	**Dec 2030**
Avg. $\lvert$Mkt – MC$\rvert$C	1.96	2.01	2.12	2.07	1.93	2.05
Max. $\lvert$Mkt – MC$\rvert$	3.57	3.74	3.83	3.85	3.72	3.84

Table 1.1: Implied volatility repricing

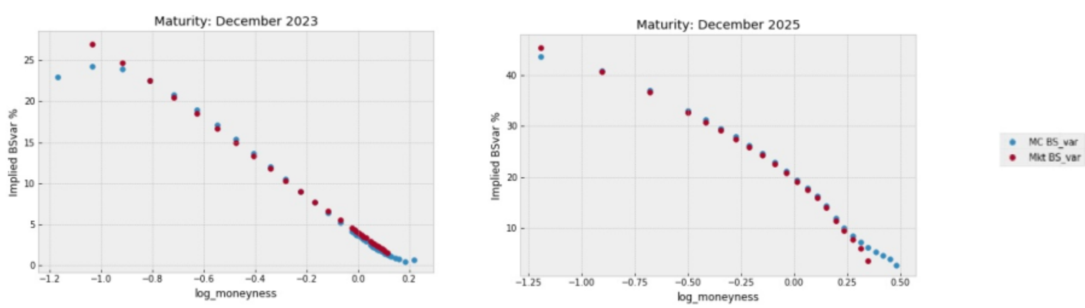

Figure 1.2: Repriced volatility surfaces using Monte Carlo for December 2023 December 2025 maturities

Pricing

Pricing of the Knock-in dividend products with payoff $\mathbb{1}_{\min_{t\in[T_1,T_2]} S_t<B}\left(\sum_{\tau_j\in[T_1,T_2]} d_j\right)$ are computed by Monte Carlo simulation.

Maturity Boundary	3yr 90%	3yr 80%	5yr 90%	5yr 80%
MC Price	51.52	23.77	71.15	33.86
MC 0.95 CI ±	3.53	2.95	5.46	5.50

Table 1.2: Knock-in Dividend pricing

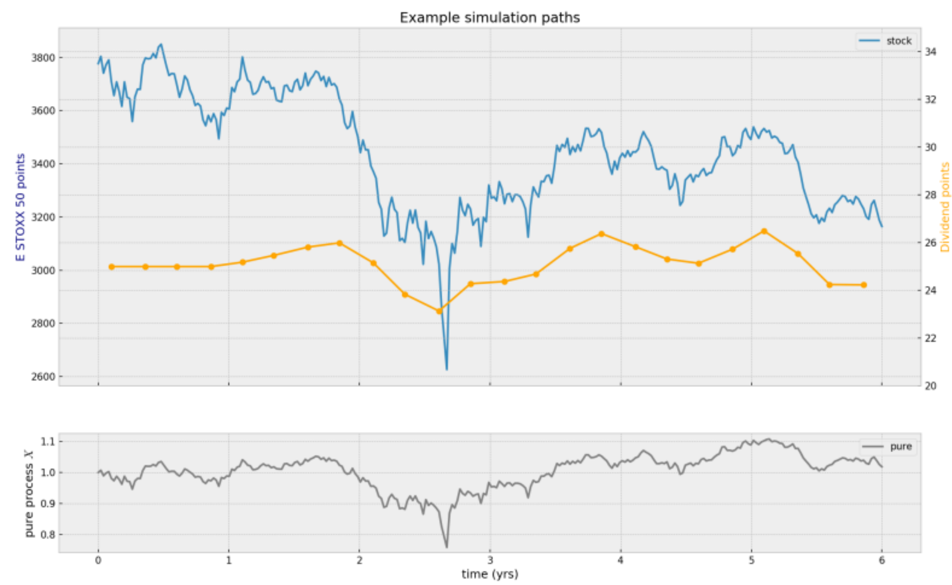

Figure 1.3: Example path of a blended dividend scheme

Remarks

It is noted that the blended dividend approach is less prone to numerical instabilities by separating stock price characteristics like credit risk and the dividend paying structure from the pure process X. Indeed this was found to be the case with successful calibration volatilites up to the error of the calibrated SSVI surface.

Chapter 2

LSVM and the particle method

Dupire-type local volatility models like (1.9) of the previous section are the standard reference for pricing exotic trades in equity modelling. However, there are downfalls to this type of model. Mostly notably exotic option risks not captured by model including volatility-of-volatility risk, forward smile risk, and spot/vol correlation risk. A local volatility model is a special case of a stochastic volatility model (which in turn is a 'hybrid' model) and the generalisation to these models is necessary to have more control when pricing exotic trades. Though, the cost of moving to hybrid modelling is that the method of simulation and calibration must be updated to account for more complex dynamics.

In this section we first introduce a general local stochastic volatility model (LSVM) and discuss the theoretical underpinning of it's calibration. We introduce McKean-Vlasov SDEs and the particle method from a theoretical standpoint before giving an practical implementation in a basic LSVM setting with Heston dynamics and without discrete dividends. The inclusion of dividends will come in the final chapter.

2.1 Calibration of Local Stochastic Volatility models

We define the general local stochastic volatility model (LSVM) as:

$$\frac{dS_t}{S_t} = (r_t - \mu_t)dt + v_t\sigma(t, S_t)dW_t^S, \qquad dv_t = a^v(t, v_t)dt + b^v(t, v_t)dW_t^v \qquad d\langle W^S, W^v\rangle_t = \rho_t^{S,v}dt \tag{2.1}$$

where v_t is a (possibly multi-factor) stochastic volatility term with it's own dynamics. As usual, r represents the interest rate process and μ is repo rate, that may or may not be inclusive of dividend yield, depending on modelling assumptions. The inclusion of $\sigma(t, S_t)$ is again crucial and is what defines this model as 'local'. Vanilla products are highly liquid and are important in the hedging of exotic products, therefore our models must be able to reprice (or calibrate to) these options. It is well-known that stochastic volatility models generate smiles of implied volatilities but the local function $\sigma(t, S_t)$ ensures that we have an 'infinity' of parameters to attain a good calibration.

The calibration condition

Dupire's original formula (1.9) is built on the idea of finding a local volatility function $\sigma(t, S)$: $\mathbb{R}^2_{\geq 0} \to \mathbb{R}$ that exists, is unique and reprices observed market marginals given some dynamics of the volatility process such as (1.8). The extension to stochastic volatility models changes the dynamics of the volatility process and calls for a new representation of the local volatility derived by Dupire in [11]. As these models are of the family of hybrid models we will refer to this new term as the 'hybrid' local volatility to distinguish from the usual (now fully written to include drift terms) Dupire local volatility given by

$$\sigma_{\text{Dupire}}(t, s)^2 = \left.\frac{\frac{\partial C}{\partial T}(T, K) + K(r_T - \mu_T)\frac{\partial C}{\partial K}(T, K) + \mu_T C(T, K)}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}(T, K)}\right|_{\substack{T=t \\ K=s}} \tag{2.2}$$

We will see that the new hybrid local volatility $\sigma_{\text{Hybrid}}(t,s)$ can be written as a combination of $\sigma_{\text{Dupire}}(t,S)$ with additive or multiplicative adjustment terms. These adjustment terms can be written as conditional expectations of the processes associated to the hybrid model. Though later numerical studies will not include stochastic interest rates, we give a more general result due to [4] for a hybrid model including stochastic rates. That is, in addition to the local stochastic volatility dynamics of (2.1), the interest rate process r satisfies:

$$dr_t = a^r(t,r_t) + b^r(t,r_t)dW_t^r, \qquad d\langle W^S, W^r\rangle_t = \rho_t^{S,r}dt, \qquad d\langle W^r, W^v\rangle_t = \rho_t^{r,v}dt \tag{2.3}$$

and define discount factor $D_{0,T} := \exp(-\int_0^T r_s ds)$. We refer to this model as the "stochastic interest rate - local stochastic volatility model" (SIR-LSVM), that, to calibrate to observed market marginals, has a hybrid local volatility given by the following result.

Proposition 2.1: Local volatility of LSV-IR models

The LSV-IR model defined above with local volatility $\sigma_{\text{Hybrid}}(t,S)$, is exactly calibrated to market smiles for all $(t,S) \in \mathbb{R}_{\geq 0}$ if

$$\sigma_{\text{Hybrid}}(t,s)^2 = \left. \frac{\sigma_{\text{Dupire}}(T,K)^2 - \frac{\mathbb{E}^{\mathbb{Q}}[(r_T - r_T^0)D_{0,T}\mathbb{1}_{\{S_T>K\}}]}{\frac{1}{2}K\frac{\partial^2 C}{\partial K^2}(T,K)}}{\mathbb{E}^{\mathbb{Q}}[D_{0,T}v_T^2 \mid S_T = K]/\mathbb{E}^{\mathbb{Q}}[D_{0,T} \mid S_T = K]} \right|_{\substack{T=t\\K=s}} \tag{2.4}$$

and $r_T^0 := -\partial_T \ln \mathbb{E}^{\mathbb{Q}}[\exp(-\int_0^T r_s ds)]$.

Proof. (Due to [11], [4]) Define $\mathcal{P}(t,s) := D_{0,t}(s-K)^+$ as the discounted call payoff, $\theta(x) = \frac{d}{dx}(x)^+$ to be the Heaviside step function about 0 and $\delta(x) = \frac{d}{dx}\theta(x)$ the dirac delta function. Since $\mathcal{P}$ is not smooth for all S we apply the Ito-Tanaka formula taking derivatives to be understood in the sense of distributions, giving

$$\begin{aligned} d\mathcal{P}(T,S_T) &= \partial_t \mathcal{P}(T,S_T)dT + \partial_s \mathcal{P}(T,S_T)dS_T + \frac{1}{2}\partial_s^2(T,S_T)\mathcal{P}dS_T dS_T \\ &= -r_T D_{0,T}(S_T-K)^+dT + D_{0,T}\theta(S_T-K)((r_T-\mu_T)S_T dT + v_t\sigma(T,S_T)S_T dW_T^S) \\ &\qquad + \frac{1}{2}D_{0,T}\delta(S_T-K)v_T^2\sigma(T,S_T)^2 dT \\ &= r_T D_{0,T}\mathbb{1}_{\{S_T>K\}}KdT - D_{0,T}\mathbb{1}_{\{S_T>K\}}\mu_T S_T dT + \frac{1}{2}K^2 D_{0,T}\delta(S_T-K)v_T^2\sigma(T,S_T)^2 dT \\ &\qquad + D_{0,T}\mathbb{1}_{\{S_T>K\}}v_t\sigma(T,S_T)S_T dW_T^S \end{aligned}$$

We apply $\mathbb{E}^{\mathbb{Q}}[.]$ to both sides. By an analogue of the dominated convergence theorem $\mathbb{E}^{\mathbb{Q}}[d\mathcal{P}(T,S_T)] = d\mathbb{E}^{\mathbb{Q}}[\mathcal{P}(T,S_T)]$ and assuming the diffusion term is a true martingale, we have

$$\begin{aligned} d\mathbb{E}^{\mathbb{Q}}[\mathcal{P}(T,S_T)] = \mathbb{E}[r_T D_{0,T}\mathbb{1}_{\{S_T>K\}}KdT - &D_{0,T}\mathbb{1}_{\{S_T>K\}}\mu_T S_T dT \\ &+ \frac{1}{2}K^2 D_{0,T}\delta(S_T-K)v_T^2\sigma(T,S_T)^2 dT] \end{aligned}$$

Observing the left-hand side is equal to $dC(T,K) = \partial_T C(T,K)dT$ as this is the difference of two strike K market calls over a short time period dT, by substituting and dividing through by dT, the above yields

$$\partial_T C(T,K) = K\mathbb{E}^{\mathbb{Q}}[r_T D_{0,T}\mathbb{1}_{\{S_T>K\}}] - \mu_T \mathbb{E}^{\mathbb{Q}}[D_{0,T}S_T\mathbb{1}_{\{S_T>K\}}] + \frac{1}{2}K^2\sigma(T,S_T)^2\mathbb{E}^{\mathbb{Q}}[D_{0,T}\delta(S_T-K)v_T^2]$$

We recall the well-known properties $\partial_K C(T,K) = -\mathbb{E}^{\mathbb{Q}}[D_{0,T}\mathbb{1}_{\{S_T>K\}}]$, $\partial_K^2 C(T,K) = \mathbb{E}^{\mathbb{Q}}[D_{0,T}\delta(S_T-K)]$ (proof in A.3 and note

$$\begin{aligned} C(T,K) = \mathbb{E}^{\mathbb{Q}}[D_{0,T}(S_T-K)\theta(S_T-K)] &= \mathbb{E}^{\mathbb{Q}}[D_{0,T}S_T\theta(S_T-K)] - K\mathbb{E}^{\mathbb{Q}}[D_{0,T}\theta(S_T-K)] \\ &= \mathbb{E}^{\mathbb{Q}}[D_{0,T}S_T\mathbb{1}_{\{S_T>K\}}] - K\partial_K C(T,K) \\ &\implies \mathbb{E}^{\mathbb{Q}}[D_{0,T}S_T\mathbb{1}_{\{S_T>K\}}] = C(T,K) - K\partial_K C(T,K) \end{aligned}$$

Using these relations and properties of the conditional expectation we have,

$$\partial_T C(T,K) = K\mathbb{E}^{\mathbb{Q}}[D_{0,T}(r_T - r_T^0)\mathbb{1}_{\{S_T>K\}}] - r_T^0 K\partial_K C(T,K) - \mu_T(C(T,K) - K\partial_K C(T,K)) + \frac{1}{2}K^2\sigma(T,K)^2\partial_K^2 C(T,K)\frac{\mathbb{E}^{\mathbb{Q}}[D_{0,T}v_T^2 \mid S_T = K]}{\mathbb{E}^{\mathbb{Q}}[D_{0,T} \mid S_T = K]}$$

Rearranging we find

$$\begin{aligned}\sigma(T,K)^2 &= \frac{\partial_T C(T,K) - K(r_T^0 - \mu_T)\partial_K + C(T,K)\mu_T C(T,K) - K\mathbb{E}^{\mathbb{Q}}[D_{0,T}(r_T - r_T^0)\mathbb{1}_{\{S_T>K\}}]}{\frac{1}{2}K^2\partial_K^2 C(T,K)\frac{\mathbb{E}^{\mathbb{Q}}[D_{0,T}v_T^2|S_T=K]}{\mathbb{E}^{\mathbb{Q}}[D_{0,T}|S_T=K]}} \\ &= \frac{\mathbb{E}^{\mathbb{Q}}[D_{0,T} \mid S_T = K]}{\mathbb{E}^{\mathbb{Q}}[D_{0,T}v_T^2 \mid S_T = K]}\left(\sigma_{\text{Dupire}}(T,K)^2 - \frac{\mathbb{E}^{\mathbb{Q}}[(r_T - r_T^0)D_{0,T}\mathbb{1}_{\{S_T>K\}}]}{\frac{1}{2}K\frac{\partial^2 C}{\partial K^2}(T,K)}\right)\end{aligned}$$

which gives the desired result. □

Using the above result but assuming deterministic rates, we have the following general local stochastic volatility model that will be used for the remaining part of this thesis

$$\frac{dS_t}{S_t} = (r_t - \mu_t)dt + v_t\sigma_{\text{Hybrid}}(t,S_t)dW_t^S, \qquad D_{t,T} := e^{-\int_t^T r_s ds} \tag{2.5}$$

$$dv_t = a^v(t,v_t)dt + b^v(t,v_t)dW_t^v, \qquad \langle W^S, W^v\rangle_t = \rho_t^{S,v}dt$$

$$\sigma_{\text{Hybrid}}(t,s)^2 = \left.\frac{\sigma_{\text{Dupire}}(T,K)^2}{\mathbb{E}^{\mathbb{Q}}[v_T^2 \mid S_T = K]}\right|_{\substack{T=t\\K=s}} \tag{2.6}$$

For simplicity we do not consider discrete dividends at this point but will add this notion in later sections. In order for market marginals to be calibrated we require the local volatility to have the form (2.6), which differs from the regular Dupire local volatility as the denominator depends on joint density $p(t,s,v')$ of (S_t, v_t) under measure $\mathbb{Q}$. Writing the conditional expectation in terms of density p,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[v_T^2 \mid S_T = K] &= \int v'^2 p(T,v' \mid s = K)dv' \\ &= \int v'^2\frac{p(T,K,v')}{p(T,K)}dv' \\ &= \frac{1}{p(T,K)}\int v'^2 p(T,K,v')dv' \\ &= \frac{\int v'^2 p(T,K,v')dv'}{\int p(T,K,v')dv'}\end{aligned}$$

Hence we can rewrite the local volatility function

$$\sigma_{\text{Hybrid}}(t,s,p) = \sigma_{\text{Dupire}}(T,K)\left.\sqrt{\frac{\int p(T,K,v')dv'}{\int v^2 p(T,K,v')dv'}}\right|_{\substack{T=t\\K=s}}$$

The dependence of the local volatility on joint density p means the SDE for the stock (2.5) is an example of a McKean-Vlasov SDE, a group of SDEs where the drift and volatility coefficients depend on the law of the process itself. This type of SDE introduces added complexity to the calibration procedure which cannot be completed using typical Ito diffusion methods. Before continuing to find solution processes to this local volatility condition, we give a brief introduction to McKean-Vlasov SDEs, some associated results and introduce the particle method, a method for simulating processes satisfying McKean-Vlasov SDEs.

2.2 A brief introduction to McKean-Vlasov SDEs and the particle method

Before introducing McKean-Vlasov SDEs, we note the Fokker-Planck or Forward Kolmogorov equation

Result 2.2: n-dimensional Fokker-Planck equation

Let $n, m \in \mathbb{N}$, if n-dimensional random process $\boldsymbol{X} = (\boldsymbol{X}_t)_{t\geq 0}$ is described by SDE

$$d\boldsymbol{X}_t = \boldsymbol{\mu}(t, \boldsymbol{X}_t)dt + \boldsymbol{\sigma}(t, \boldsymbol{X}_t)d\boldsymbol{W}_t$$

where $\boldsymbol{\mu}(t, \boldsymbol{X}_t)$ is an n-dimensional vector and $\boldsymbol{\sigma}(t, \boldsymbol{X}_t)$ is an $n \times m$ matrix and $\boldsymbol{W}_t$ is an m-dimensional standard Brownian motion, then the probability density $p(t, \boldsymbol{x})$ for $\boldsymbol{X}_t$ satisfies the Fokker-Planck equation

$$\frac{\partial p(t,\boldsymbol{x})}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}[\mu_i(t,\boldsymbol{x})p(t,\boldsymbol{x})] + \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}[D_{ij}(t,\boldsymbol{x})p(t,\boldsymbol{x})] \tag{2.7}$$

with drift vector $\boldsymbol{\mu} = (\mu_1, ..., \mu_n)$ and diffusion tensor $\boldsymbol{D} = \frac{1}{2}\boldsymbol{\sigma}\boldsymbol{\sigma}^T$, that is

$$D_{ij}(t,\boldsymbol{x}) = \frac{1}{2}\sum_{k=1}^{m} \sigma_{ik}(t,\boldsymbol{x})\sigma_{jk}(t,\boldsymbol{x})$$

McKean-Vlasov SDEs

McKean-Vlasov SDEs were first studied by Henry McKean in 1966 [12] as a proposed class of SDEs satisfying differential equations for particle system equations in statistical mechanics, such as the Vlasov equation.

Assuming the stochastic basis $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and the same notation for dimensions as above, a McKean-Vlasov equation for n-dimensional process $\boldsymbol{X}$ is an SDE in which the drift and volatility depend not only on the current value of random variable $\boldsymbol{X}_t$ but also on the probability distribution $\mathbb{P}_t$ of $\boldsymbol{X}_t$:

$$d\boldsymbol{X}_t = \boldsymbol{\mu}(t, \boldsymbol{X}_t, \mathbb{P}_t)dt + \boldsymbol{\sigma}(t, \boldsymbol{X}_t, \mathbb{P}_t)d\boldsymbol{W}_t, \qquad \mathbb{P}_t = \mathrm{Law}(\boldsymbol{X}_t), \qquad \boldsymbol{X}_0 \in \mathbb{R}^n \tag{2.8}$$

Under conditions detailed in [13], existence and uniqueness for a process satisfying (2.8) can be proved if the vector drift and diffusion functions $\boldsymbol{\mu}(t, \boldsymbol{x}, \mathbb{P}_t)$ and $\boldsymbol{\sigma}(t, \boldsymbol{x}, \mathbb{P}_t)$ are both Lipschitz continuous functions of vector $\boldsymbol{x}$ and a linear growth condition with respect to the Wasserstein distance, a metric that can be interpreted as a cost of moving between measures, is satisfied. We do not note these results here as in many cases relating to hybrid modelling, the Lipschitz condition is not satisfied. More discussion on the existence and uniqueness of our particular model (2.6) is detailed in the next section.

By rewriting the Fokker-Planck equation (2.7), probability density function $p(t, \boldsymbol{x})$ of $\boldsymbol{X}_t$ is the solution to

$$\frac{\partial p(t,\boldsymbol{x})}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}[\mu_i(t,\boldsymbol{x},\mathbb{P}_t)p(t,\boldsymbol{x})] + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left[\sum_{k=1}^{m} \sigma_{ik}(t,\boldsymbol{x},\mathbb{P}_t)\sigma_{jk}(t,\boldsymbol{x},\mathbb{P}_t)p(t,\boldsymbol{x})\right] \tag{2.9}$$

with initial condition $\lim_{t\to 0} p(t, \boldsymbol{x}) = \delta(\boldsymbol{x} - \boldsymbol{X}_0)$. This equation is nonlinear in $p(t, .)$ as $\mu_i(t, \boldsymbol{x}, \mathbb{P}_t)$ and $\sigma_{ik}(t, \boldsymbol{x}, \mathbb{P}_t)$ now depend on the unknown density $p(t, .)$, hence the reasoning why this group of SDE are often refered to as 'nonlinear' McKean-Vlasov SDEs.

Standard approaches use (2.9) to calibrate hybrid local volatility models. These methods entail defining a time grid and initial distribution condition on $p(t_0, .)$, then repeating the following for time steps $...t_l < t_{l+1}...$:

- Write the Fokker-Planck equation with $\mu_i(t_l, \boldsymbol{x}, \mathbb{P}_{t_l})$ and $\sigma_{ik}(t_l, \boldsymbol{x}, \mathbb{P}_{t_l})$ defined using the previous step, solve for joint density $p(t_l, .)$ using PDE methods.
- Going forward in time one step, use joint density $p(t_l, .)$ to compute the drift and diffusion terms by numerical integration. Use these as terms as $\mu_i(t_{l+1}, \boldsymbol{x}, \mathbb{P}_{t_{l+1}})$ and $\sigma_{ik}(t_{l+1}, \boldsymbol{x}, \mathbb{P}_{t_{l+1}})$.
- Repeat previous steps until last required calibration date.

This approach is effective for hybrid models using a low dimension of driving Brownian motions but there are some notable drawbacks. Solving the associated PDE is not generally possible for higher dimensions ($m > 3$) and time performance is of order

$$O\left(\text{number of time steps} \times \prod_{i=1}^{m} \text{number of space steps for dimension } i\right)$$

which also deteriorates with an increase of dimension. With this in mind we turn our attention to an alternative method, the particle method.

The particle method

Introduced in [2], the particle method is a more direct Monte Carlo approach to simulating the McKean-Vlasov SDE (2.8). Consider the LSV-IR calibration condition (2.4), the computation of three expectations $\mathbb{E}^{\mathbb{Q}}[(r_T - r_T^0)D_{0,T}\mathbb{1}_{\{S_T>K\}}]$, $\mathbb{E}^{\mathbb{Q}}[D_{0,T}v_T^2 \mid S_T = K]$ and $\mathbb{E}^{\mathbb{Q}}[D_{0,T} \mid S_T = K]$ is required. For Monte Carlo pricing we require the simulation of paths r_T^i, v_T^i and S_T^i. The particle method idea is then, instead of simulating path-by-path could we start many simulations at once, then empirically estimate these expectations at each time step? It turns out that this approach is indeed reasonable and leads to simultaneous calibration and simulation of processes. The term particle is used to describe the simultaneous simulation paths as at each time step the dynamics of simulation i is affected by the distribution of $N-1$ other simulations. Therefore this method can be seen as a collection of interacting particles, hence why the particle term is used.

Estimating the McKean drift and diffusion coefficients in this way consists of replacing the law $\mathbb{P}_t$ with the empirical distribution

$$\mathbb{P}_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\boldsymbol{X}_t^{i,N}}$$

where $\boldsymbol{X}^{i,N} = (\boldsymbol{X}_t^{i,N})_{t\geq 0}$ for $1 \leq i \leq N$ are the random 'particle' processes simulating $\boldsymbol{X}$ and N is the number of simulating particles. The particle processes $\boldsymbol{X}^{i,N}$ are then described by the classical-type SDE,

$$d\boldsymbol{X}_t^{i,N} = \boldsymbol{\mu}(t, \boldsymbol{X}_t^{i,N}, \mathbb{P}_t^N)dt + \boldsymbol{\sigma}(t, \boldsymbol{X}_t^{i,N}, \mathbb{P}_t^N)d\boldsymbol{W}_t^i, \qquad \text{Law}(\boldsymbol{X}_0^{i,N}) = \mathbb{P}_0$$

Note the above represents $N \times \mathbb{R}^n$-dimensional SDEs (one for each particle), $\{W_t^i\}_{1\leq i\leq N}$ are N independent m-dimensional Brownian motions and $\mathbb{P}_t^N$ is a random measure that is the source of interaction between one particle and the $N-1$ remaining particles. The inclusion of measure $\mathbb{P}_t^N$ also illustrates the need to simulate by 'time-slice' and not by path as the knowledge of one particle position $\boldsymbol{X}_t^{i,N}$ at time t is not enough to determine the dynamics of that particle at a later time $t + \delta t$.

Theoretically, this method is only valid if there is some notion of convergence. In fact, convergence of the empirical measure $\mathbb{P}_t^N$ to the true measure $\mathbb{P}_t$ is due to the *chaos propagation property* proved for McKean-Vlasov SDEs in [13], details of this theory are not explored here. Though, in short, if at $t = 0$, the $\boldsymbol{X}_0^{i,N}$ are independent particles, then as $N \to \infty$, for any fixed $t > 0$, $\boldsymbol{X}_t^{i,N}$ are asymptotically independent and their empirical measure $\mathbb{P}_t^N$ converges in distribution toward the true measure $\mathbb{P}_t$. Practically, for any bounded continuous function $f \in C(\mathbb{R}^n)$,

$$\frac{1}{N}\sum_{i=1}^{N} f(\boldsymbol{X}_t^{i,N}) \xrightarrow[N\to\infty]{\text{L}^1} \int_{\mathbb{R}^n} f(\boldsymbol{x})p(t,\boldsymbol{x})d\boldsymbol{x}$$

where $p(t,.)$ is the fundamental solution to Fokker-Planck equation (2.9).

The importance of this result is that the particle method is the first 'exact' calibration method, in that it is the first method where convergence to the true volatility function increases with computational effort.

2.3 Calibration of a LSVM via the particle method

We return to the calibration of the LSVM (2.1) with deterministic rate and local volatility (now with shortened notation) given by

$$\sigma_{\text{Hyb}}(t,s,p) = \sigma_{\text{Dup}}(t,s)\sqrt{\frac{\int p(t,s,v')dv'}{\int v'^2 p(t,s,v')dv'}}$$

As discussed in the previous section, existing results on the uniqueness and existence of a solution to the McKean SDE (2.8) require Lipschitz continuity of the drift and diffusion coefficients. This is not satisfied for our considered model as the quotient of integrals cannot be bounded as required. Therefore the existence of a process satisfying the local volatility condition for arbitrary arbitrage-free volatility surfaces is not obvious. The question of existence and uniqueness is challenging and open, though there have however been numerical studies on the dependence of volatility-of-volatility which is seen as a critical parameter.

We now apply the particle method, replacing the true risk-neutral marginal distribution $\mathbb{Q}_t$ with empirical distribution $\mathbb{Q}_t^N$. Assuming Heston dynamics and redefining one-dimensional Brownian motions for the stock and volatility denoted W, B respectively, under $\mathbb{Q}_t^N$ the N interacting particles have dynamics,

$$\begin{aligned} dS_t^{i,N} &= (r_t-\mu_t)S_t^{i,N}dt + v_t^{i,N}\sigma(t,S_t^{i,N},\mathbb{Q}_t^N)dW_t^i, \qquad && d\langle W,B\rangle_t = \rho_t^{S,v}dt \\ d(v^{i,N})_t^2 &= \alpha(\theta-(v^{i,N})_t^2)dt + \sigma_v v_t^{i,N}dB_t^i && \alpha,\theta,\sigma_v\in\mathbb{R},\ 2\alpha\theta\geq\sigma_v^2 \end{aligned} \tag{2.10}$$

with,

$$\sigma(t,s,\mathbb{Q}_t^N) = \sigma_{\text{Dup}}(t,s)\sqrt{\frac{\int p_N(t,s,v')dv'}{\int v'^2 p_N(t,s,v')dv'}} = \sigma_{\text{Dup}}(t,s)\sqrt{\frac{\sum_{i=1}^N \delta(S_t^{i,N}-s)}{\sum_{i=1}^N (v_t^{i,N})^2\delta(S_t^{i,N}-s)}}$$

Here the local volatility $\sigma(t,s,\mathbb{Q}_t^N)$ is not properly defined as the empirical densities p_N give Dirac masses in the conditional expectation as $\mathbb{Q}_t^N$ is atomic with atoms at particle observations $S_t^{i,N}$. We smooth the density p_N by replacing the Dirac functions with a regularizing kernel $\delta_{t,N}(.)$ to obtain the Nadaraya-Watson estimator, so defining

$$\sigma_N(t,s,\mathbb{Q}_t^N) = \sigma_{\text{Dup}}(t,s)\sqrt{\frac{\sum_{i=1}^N \delta_{t,N}(S_t^{i,N}-s)}{\sum_{i=1}^N (v_t^{i,N})^2\delta_{t,N}(S_t^{i,N}-s)}} \tag{2.11}$$

By replacing the local volatility in (2.10) with σ_N we can simulate paths of S.

The Regularising Kernel

We take $\delta_{t,N}(x) = \frac{1}{h_{t,N}}K(\frac{x}{h_{t,N}})$ where K is a fixed, symmetric kernel with variable bandwidth $h_{t,N}$ that tends to 0 as $N\to\infty$. In numerical studies we compare two possibilities for the kernel K,

- Gaussian: $K(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$
- Epanechnikov: $K(x) = \frac{3}{4}(1-x^2)\mathbb{1}_{|x|\leq 1}$

The Kernel density estimate is known to depend critically on the bandwidth $h_{t,N}$. Estimators are found by minimising the asymptotic mean integrated squared error of the Nadaraya-Watson estimator adding a factor of $N^{-1/5}$. Taking $\hat{\sigma}$ to be the sample standard deviation, Silverman's well known 'rule of thumb' for bandwidth selection is given by

$$\hat{h}_{RT} = \left(\frac{4}{3}\right)^{1/5} N^{-1/5}\hat{\sigma} \approx 1.06N^{-1/5}\hat{\sigma} \tag{2.12}$$

In [2] the authors consider a financial analogue, without sample standard deviation to save computational time. The suggested bandwidth is

$$h_{t,N} = \kappa S_0\sigma_{\text{VS},t}\sqrt{\max(t,t_{\min})}N^{-\frac{1}{5}}$$

where $\sigma_{VS,t}$ is the variance swap volatility at maturity t, $t_{\min} = \frac{1}{4}$ and $\kappa\approx 1.5$.

Acceleration Techniques

Assuming $\sigma_{\text{Dup}}(t,s)$ is precomputed before carrying out the particle method (say by SSVI parametric approximation or by a combination of finite difference and cubic spline interpolation) then the remaining problem is to calculate hybrid local volatility (2.11) is calculating the Nadaraya-Watson estimator part that is $O(N^2)$ calculations. Typical choices for the number of particles are $> 10^3$ so the problem of calculating the estimator efficiently arises.

There are a couple of ideas to improve the efficiency of such a calculation, which are noted below:

- **Computing on a smaller grid:** the estimator does not have to be calculated as required for all inputs s. Assuming at time step t_i you have an array of particle positions $[S_{t_i}^{1,N}, ..., S_{t_i}^{N,N}]$. Define $s_{\min}$, $s_{\max}$ as either the respective minimum and maximum of the array up to some threshold α, that is the $\lfloor \alpha N \rfloor$-th minimum or maximum value. (costly to compute, though it depends if a sorting method is used). Alternatively, one could use the condition

$$\mathbb{Q}(S_{t_i} < s_{\min}) = \mathbb{Q}(S_{t_i} > s_{\max}) = 1 - \alpha$$

  which may be estimated using the kernel of the previous step or by the prices of digital options. The estimator can then be computed on a grid $G_{T,s} = [s_{\min}, ..., s_{\max}]$ of length $N_s << N$ and be interpolated on the interior of the array and extrapolated for values no in interval $[s_{\min}, s_{\max}]$.

- **Disregarding simulations:** similar to choosing the significant grid points, simulation values $S_{t_i}^{j,N}$ that are far from s can be disregarded using the following criteria:

$$\delta_{t_i,N}(S_{t_i}^{j,N} - s) \leq \eta \tag{2.13}$$

- **Sorting simulations:** [4] suggest sorting the simulations, which does increase efficiency of computing a grid and disregarding simulations, though this adds a computational cost of $O(N \times \log(N))$.

- **Restricting the summation range:** Assume some time step T for ease of notation. If we want to sum only over significant values we can invert the bound (2.13) for each simulation $S_T^{i,N}$ and argument s giving

$$|\delta_{T,N}^{-1}(\eta)| < S_T^{i,N} - s < |\delta_{T,N}^{-1}(\eta)|$$
$$S_T^{i,N} - |\delta_{T,N}^{-1}(\eta)| < s < S_T^{i,N} + |\delta_{T,N}^{-1}(\eta)|$$

  We can then define minimum $\underline{j}_T^{i,N}(s)$ and maximum $\overline{j}_T^{i,N}(s)$ indices satisfying

$$S_T^{i,N} - |\delta_{T,N}^{-1}(\eta)| \leq \underline{j}_T^{i,N}(s) < s \leq \overline{j}_T^{i,N}(s) < S_T^{i,N} + |\delta_{T,N}^{-1}(\eta)|$$

  that is simulation $S_T^{i,N}$ only contributes to the kernel functions of arguments

$$\underline{j}_T^{i,N}(s) \leq s \leq \overline{j}_T^{i,N}(s)$$

  Calculating these indices for each particle pair is too computationally costly but defining $\underline{j}_T(s)$ and $\overline{j}_T(s)$ to be the smallest and largest indices for which $\delta_{T,N}(S_T^{i,N} - s) \leq \eta$ for at least one i then we can use the approximation

$$\sigma_N(t,s,\mathbb{Q}_t^N) \approx \sigma_{\text{Dup}}(t,s)\sqrt{\frac{\sum_{i=\underline{j}_T(s)}^{\overline{j}_T(s)} \delta_{t,N}(S_t^{i,N} - s)}{\sum_{i=\underline{j}_T(s)}^{\overline{j}_T(s)} (v_t^{i,N})^2 \delta_{t,N}(S_t^{i,N} - s)}}$$

  using this computation we have can achieve an order $O(N \times \log(N_s))$ computation where $N_s < N$.

- **Equally spaced arguments:** note further improvements to order $O(N)$ are possible if the input grid for s is equally space, though they are not detailed here.

The Basic Particle Method Algorithm

We now have the required tools to set out a general algorithm for the particle method in the case of a LSVM.

Algorithm 2.3: Heston LSVM calibration with the particle method

1. Define parameters $N >> 0$, $\rho_t^{S,v} \in [-1,1]$ and $\alpha, \theta, \sigma_v \in \mathbb{R}$, such that $2\alpha\theta \geq \sigma_v^2$ (by historical estimation in some cases).
2. Set up a grid of time steps $[0, ..., T_{\max}]$ where $T_{\max}$ is the final market observed maturity for vanilla options and set $\sigma_N(t,s) = \frac{\sigma_{\text{Dup}}(0,s)}{v_0}$ where v_0 is the initial value of the volatility process.
   (i) Simulate N processes $\{S_t^{i,N}, v_t^{i,N}\}_{i=1,\dots,N}$ using an Euler or Milstein scheme between $[t_{k-1}, t_k]$ according to:
   $$dS_t^{i,N} = (r_t - \mu_t)S_t^{i,N}dt + v_t^{i,N}\sigma_N(t_{k-1}, S_t^{i,N})dW_t^i$$
   $$d(v^{i,N})_t^2 = \alpha(\theta - (v^{i,N})_t^2)dt + \sigma_v v_t^{i,N}dB_t^i$$
   (ii) Calibrate (compute) local volatility
   $$\sigma_N(t, s, \mathbb{Q}_t^N) = \sigma_{\text{Dup}}(t,s)\sqrt{\frac{\sum_{i=1}^N \delta_{t,N}(S_t^{i,N} - s)}{\sum_{i=1}^N (v_t^{i,N})^2\delta_{t,N}(S_t^{i,N} - s)}}$$
   using any combination of acceleration techniques. Store the local volatility as cubic interpolators and extrapolate flat outside the interval $[s_{\min}, s_{\max}]$. Set $\sigma_N(t,s) = \sigma_N(t_k, s)$ for all $t \in [t_k, t_{k+1}]$.
3. Iterate previous two steps until $T_{\max}$

(In practice step 2 (i) is easily parallelizable using multi-threading or a GPU.)

2.4 Numerical Example

We use the EURO STOXX 50 Index Options (OESX) data introduced in chapter 1 and the algorithm set out above to calibrate to vanilla options. An estimate for the forward curve is again used by the repo optimisation technique detailed in the previous chapter.

Calibration

The table below summarises the calibration error for a particle method approach with $N = 5000$ particles, the sorting technique approach for kernel estimation with a Gaussian kernel and 'rule of thumb' bandwidth selection.

Implied variance (%)	Sep 2022	Dec 2022	Mar 2023	Jun 2023	Sep 2023	Dec 2023
Avg. |Mkt − MC|	2.36	2.44	4.56	3.81	3.75	3.94
Max. |Mkt − MC|	3.08	4.74	8.85	6.92	6.96	6.72
Implied variance (%)	**Jun 2024**	**Dec 2024**	**Dec 2025**	**Dec 2026**	**Dec 2028**	**Dec 2030**
Avg. |Mkt − MC|	4.47	4.24	4.18	3.76	3.61	3.52
Max. |Mkt − MC|	5.88	5.59	4.71	4.95	4.26	4.35

Table 2.1: Implied volatility repricing for Heston LSVM using the particle method

Remarks

1. Calibration estimates were heavily affected by the Nadaraya-Watson estimator. Realistic results were only found when using a bandwidth estimator using standard deviation which mitigated computation time saved by any acceleration techniques.

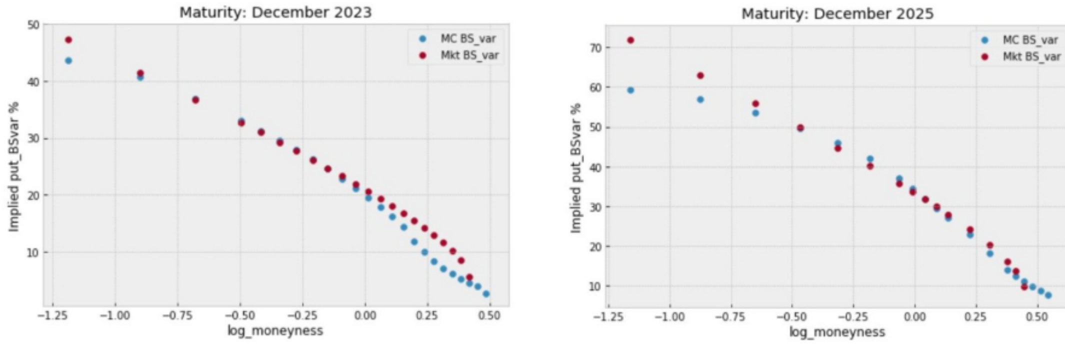

Figure 2.1: Repriced volatility surfaces using particle method paths for December 2023 December 2025 maturities

2. Volatility parameters were also found to be crucial with convergence of the above example using $\alpha = 0.02$, $\theta = 0.1$ and $\sigma_v = 0.01$.

The following plots show the distribution of the stock and volatility process at the two maturities used for pricing in the previous section, December 2025 and December 2027 for the model described above.

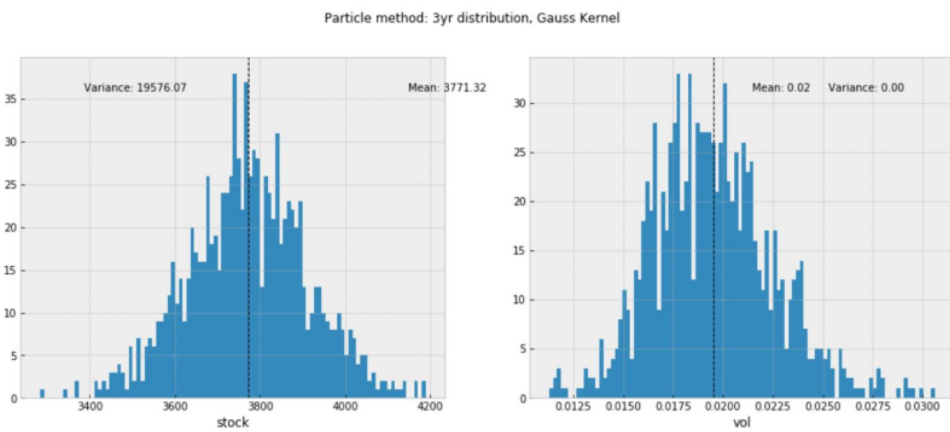

Figure 2.2: Stock and volatility process marginals for December 2025 maturity using particle method

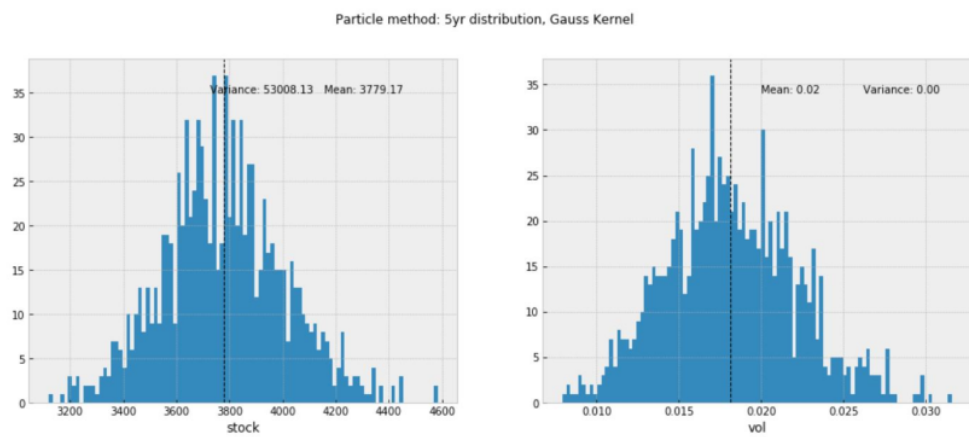

Figure 2.3: Stock and volatility process marginals for December 2027 maturity using particle method

Chapter 3

Modelling with stochastic dividends

Much of the content of this chapter follows the work of [3] where the authors set out a general framework to calibrate a LSVM with a Markov functional type model for the dividends.

3.1 Markov Representation of Dividend futures

We start by modelling the dividend forward curve by considering a Markov-functional approach. This type of approach is commonly used in fixed income where the matching date of underlying rates and maturities makes their inclusion a natural choice. The approach was first introduced to equity markets by Carr and Madan [14] and consists of assuming that some functional f maps a driving process X to the states of the underlying. If f is known at some future maturity T, then the volatility smile for this maturity can be calibrated and the dynamics of f generate smiles for $t < T$.

Let $(D_{t_i})_{t_i \in \tau}$ be defined on the filtered risk-neutral probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{Q})$, where D_{t_i} is an $\mathcal{F}_t$-measurable random variable representing the dividend to be paid at time t_i and τ is a (countable) set of dividend times.

We define process $H_t := \mathbb{E}^{\mathbb{Q}}\left[D_{t_i} \mid \mathcal{F}_t\right]$ as the dividend forward curve and for brevity we drop the notation on measure and conditional expectation to $\mathbb{E}_t\left[.\right]$ for the rest of this section. Note for $t \geq t_i$, $H_t = D_{t_i}$ and by definition H_t is a martingale.
(Proof: let $0 \leq s < t$, by the tower property $\mathbb{E}_s\left[H_t\right] = \mathbb{E}_s\left[\mathbb{E}_t\left[D_{t_i}\right]\right] = \mathbb{E}_s\left[D_{t_i}\right] = H_s$).

The crucial assumption of this model is that the dynamics of H_t are given by the following functional form:

$$H_t = \mathbb{E}_t\left[D_{t_i}\right] = f_i(t, X_t) \tag{3.1}$$

where X_t is a low-dimensional Ito process. Again note by the measurability of D_{t_i} that we have the boundary condition $f_i(t_i, X_{t_i}) = D_{t_i}$. Our goal is to recover $f_i(t, .)$ from calibration to dividend vanilla options.

We take X_t to be an Ornstein-Uhlenbeck process with dynamics $dX_t = -kX_t dt + dB_t$ and $X_0 = 0$. Some useful Ornstein-Uhlenbeck relations that will be used are given in the appendix A.4. With these dynamics we have the relation,

$$\mathbb{E}_t\left[D_{t_i}\right] = \mathbb{E}_t\left[f_i(t_i, X_{t_i})\right] = \mathbb{E}_t\left[f_i(t_i, X_t e^{-k(t_i - t)} + \int_t^{t_i} e^{-k(t_i - s)} dB_s\right]$$

hence the functional relation depends uniquely on the boundary condition $f_i(t_i, .)$ at ex-date t_i.

We would like to recover a parametrisation for f using this boundary condition, so we assume a no-arbitrage parametric cdf $F^{\vec{\lambda},i}$ for D_{t_i} (i.e. $\mathbb{E}\left[\mathbb{1}_{D_{t_i} < K}\right] \equiv F^{\vec{\lambda},i}(K)$) to define the functional inverse.

We can now specify a relation of the inverse of f which in turn specifies our model. Recalling $X_{t_i} \sim \mathcal{N}(0, \frac{1-e^{-2kt_i}}{2k})$,

$$\begin{aligned} F^{\vec{\lambda},i}(K) &= \mathbb{E}\left[\mathbb{1}_{D_{t_i}<K}\right] = \mathbb{Q}(D_{t_i} < K) \\ &= \mathbb{Q}(f_i(t_i, X_{t_i}) < K) \\ &= \mathbb{Q}\left(X_{t_i} < f_i^{-1}(t_i, K)\right) \\ &= \mathbb{Q}\left(Z < \sqrt{\frac{2k}{1-e^{-2kt_i}}} f_i^{-1}(t_i, K)\right) \\ &= \Phi\left(\sqrt{\frac{2k}{1-e^{-2kt_i}}} f_i^{-1}(t_i, K)\right) \end{aligned}$$

Hence we have the relation

$$f_i^{-1}(t_i, K) = \sqrt{\frac{1-e^{-2kt_i}}{2k}}.\Phi^{-1}(F^{\vec{\lambda},i}(K)) \tag{3.2}$$

and as the standard normal quantile function and cdf $F^{\vec{\lambda},i}$ are increasing in K we have that f is well defined on the domain of strikes.

3.2 Calibration to dividend Vanillas

Vanilla options on realised dividends are readily quoted in the market and, depending on trading venue and underlying, can have enough liquidity to build a volatility surface for calibration. Two future dates T_1, T_2 are set (typically $T_2 = T_1 + 1$ year) and a call option can be defined as

$$\begin{aligned} C^{T_1,T_2}(K) &\equiv \mathbb{E}\left[\Big(\sum_{T_1<t_i<T_2} D_{t_i} - K\Big)^+\right] \\ \Big(&= \mathbb{E}\left[\Big(\sum_{T_1<t_i<T_2} f_i(t_i, X_{t_i}) - K\Big)^+\right]\Big) \end{aligned}$$

In our case we will stay with EURO STOXX 50 as an example. As mentioned previously, dividends on this index are paid quarterly with historically the most significant dividend payment in the final quarter of the calendar year due to the business cycles of index constituents. For that reason we make the following modelling assumption that one dividend payment D_{t_i} falls between the maturities T_{i-1} and T_i instead of a sum of random variables. With t_i to be the proceeding ex-date before T_i we rewrite

$$C^{T_i}(K) \equiv \mathbb{E}\left[(D_{t_i} - K)^+\right] = \mathbb{E}\left[(f_i(t_i, X_{t_i}) - K)^+\right]$$

As touched on in [3], dividend options can be illiquid for out of the money strikes. This leads the authors to use a low-dimensional SVI parametrisation. We proceed similarly, by assuming cdf $F^{\vec{\lambda},i}(K)$ comes from a no-arbitrage SVI parametrization $\vec{\lambda} = \{\text{SVI parameters}\}$ of the Black-Scholes implied variance $\sigma^2_{\text{SVI}}(K)t_i$. By the well-known marginal distribution relation (proof in appendix A.4) we have $\frac{\partial C^{T_i}(K)}{\partial K} = F_i(K) - 1$ where F_i is the cdf of the underlying at time T_i. With $f_i(0,0) = \mathbb{E}_0[D_{t_i}]$ the dividend forward at $t = 0$ we find the relation

$$\begin{aligned} F^{\vec{\lambda},i}(K) &= 1 + \frac{\partial C^{T_i}(K)}{\partial K} \\ &= 1 + \frac{\partial}{\partial K}\text{BS\_call}(\sigma^2_{\text{SVI}}(K)T_i, K, f_i(0,0)) \end{aligned}$$

where,

$$\begin{aligned} &\text{BS\_call}(v, k, f; r, T) = e^{-rT}\left[\Phi(d_+)f - \Phi(d_-)K\right] \\ &d_+ = \frac{1}{\sqrt{v}}\left(\log\left(\frac{f}{k}\right) + \frac{v}{2}\right), \qquad d_- = d_+ - \sqrt{v} \end{aligned}$$

is the Black-Scholes call formula with variance v, strike k, forward f, rate r and maturity T. We introduce the following change of parameters for tractability:

- log forward-moneyness $y := \log(\frac{K}{f_i(0,0)})$
- Black-Scholes implied variance $w_{\text{SVI}}(y, T_i) := \sigma^2_{\text{SVI}}(f_i(0,0)\exp^y)T_i = \sigma^2_{\text{SVI}}(K)T_i$, which is directly given by the SVI parametrisation. Note the subscript SVI notation is dropped during the following calculation for brevity.

So continuing the calculation (relevant BlackScholes relations are given in the appendix A.4) with the new variables, we have

$$
\begin{aligned}
F^{\vec{\lambda},i}(K) &= 1 + \frac{\partial \text{BS}}{\partial v}(w, K, f_i(0,0))\frac{\partial w}{\partial y}\frac{\partial y}{\partial K} + \frac{\partial \text{BS}}{\partial k}(w, K, f_i(0,0)) \\
&= 1 + e^{-rT}\phi(d_+)\frac{f_i(0,0)}{2\sqrt{w}}.\frac{\partial w}{\partial y}.\frac{1}{K} - e^{-rT}\Phi(d_-) \\
&= 1 + e^{-rT}\left(\frac{e^{-y}}{2\sqrt{w}}.\phi(d_+)\frac{\partial w}{\partial y}(y, T_i) - \Phi(d_-)\right) \qquad (3.3)
\end{aligned}
$$

now with

$$
d_+ = \frac{1}{\sqrt{w}}\left(-y + \frac{w}{2}\right), \qquad d_- = d_+ - \sqrt{w}
$$

Relations (3.2) and (3.3) fully define, via the inverse, functional $f_i(t_i, .)$ for $t_i \in (T_{i-1}, T_i]$. As the maturities create non-overlapping intervals $(T_{i-1}, T_i]_{i=1,...n}$ we can specify functionals $f_i(t_i, .)$ for all dividend dates up to final quoted maturity T_n. We now set out the calibration procedure and document an implementation with EURO STOXX data.

Algorithm 3.1: Markov-functional calibration on dividend vanillas

1. Define non-overlapping market quoted maturities $(T_i)_{i=1,...,n}$ and parameters for the underlying driving process X. Choose a suitable SVI parametrization and make an initial guess for all sets of parameters $\vec{\lambda}_i$ such that they satisfy associated no arbitrage conditions.
2. For a maturity T_i, calculate market implied vols $\sigma_{\text{Div\_Mkt}}(K_j)$ for quoted (and liquid) strikes $(K_j)_{j=0,...,m}$. Using (3.2) and (3.3) compute and store the pairs $(f_i^{-1}(t_i, K_j), K_j)$ for all K_j. The domain of the functional is dependant on driver process parameters and $\vec{\lambda}_i$. Hopefully, (dependant on parameter choice) this relationship should give a linear/shallow cubic relationship. Interpolate and pre-compute a grid to quickly map from inverted values to the strike space.
3. Simulate N ($> 10,000$) paths X_{t_i}, the Ornstein-Uhlenbeck choice allows this to be done exactly. Using these paths, compute call prices $C^{T_i}(K_j)$ for all strikes K_j by Monte Carlo.
4. Use relation $C^{T_i}(K_j) \equiv \text{BS\_call}(\sigma^2_{\text{Div\_MC}}(K_j)T_i, K_j, f_i(0,0))$ to compute the Monte-Carlo implied volatilities $\sigma^2_{\text{Div\_MC}}(K_j)$. Optimize the SVI parameters $\vec{\lambda}_i$ according to a penalisation function, say least squares:

$$
\vec{\lambda}_i^* = \arg\min_{\vec{\lambda}_i}\left(\sum_{j=0,...,m}(\sigma_{\text{Div\_MC}}(K_j) - \sigma_{\text{Div\_Mkt}}(K_j))^2\right)
$$

5. Collect optimal parameters $\vec{\lambda}_i^*$, the functional $f_i(t_i, .)$ is now defined for all maturities $i = 1, ..., n$ and can be computed using the pre-computed grid map.

Numerical Example

We use EURO STOXX 50 options on dividends (OEXD) data and dividend futures (FEXD) with maturity T as a proxy for the spot price $f_i(0,0)$. OEXD option maturities are quoted for the final quarter of each calendar year and take the annual sum of that years dividends as an underlying, hence why our assumption of a single dividend random variable is reasonable in this setting. As mentioned above, option prices are illiquid for some out of the money options, so we restrict our sample size to liquid strikes, that is strikes that have seen trading activity in the last business week and have log-moneyness in the range [-0.5, 0.25]. Similar to previous examples we take $r = 0$ and optimize the repo rate μ to match implied volatilities for call and put options. The deterministic repo rate μ is then interpolated and stored as a proxy to a forward curve.

We take the mean reversion rate of the ornstein-uhlenbeck process to be $= 1.0$, and initial value $= 0.0$. Similar to the above algorithm, we first fit a Heston SSVI surface to the computed volatilities of all maturities and then fix maturity T and optimize parameters whilst maintaining no arbitrage conditions. The results are summarized below with Heston parameter tuples representing (γ, σ, ρ) in the notation of [10].

Maturity	Dec 2022	Dec 2023	Dec 2025	Dec 2027
Heston SVI params	(1.72, 0.25, -1.70)	(1.72, 0.25, -1.70)	(1.79, 0.25, -2.16)	(2.03, 0.24, -2.81)
Avg. $\lvert\text{Mkt} - \text{MC}\rvert$	0.33	0.41	1.08	4.12
Max. $\lvert\text{Mkt} - \text{MC}\rvert$	2.55	2.78	4.37	5.66

Table 3.1: Markov functional Calibration to OEXD

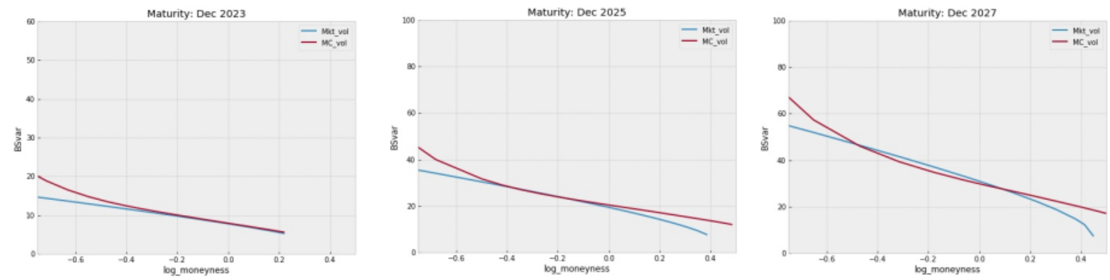

Figure 3.1: Plot of market implied variance compared to the implied variance of Monte Carlo simulation prices for call options using the optimal SVI parameters.

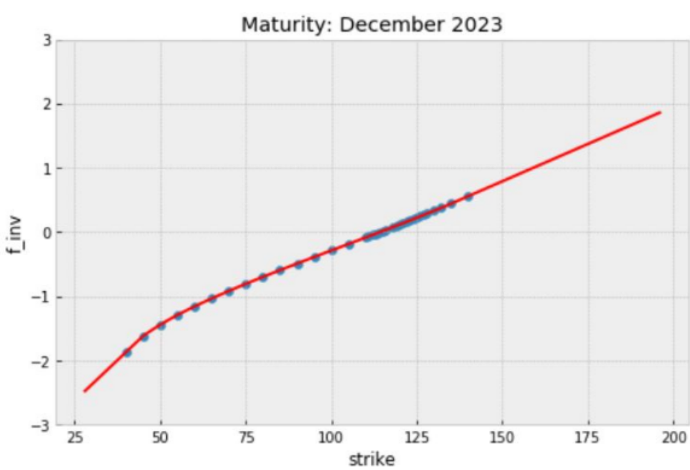

Figure 3.2: Example of interpolating the relationship between $f_i(t_i, .)^{-1}$ and strikes K_j for December 2023 maturity. Given the choice of Orstein-Uhlenbeck parameters, by considering the mapping of y axis to x axis you can appreciate what random dividend values are generated. For example, the long term mean 0. maps to ≈ 110 dividend points which is inline with historical expectations of (2.3)

3.3 LSVMs with stochastic dividends

We now define the following dynamics of the underlying S_t with dividend jumps of amount D_{t_i} at time t_i. Under risk-neutral measure $\mathbb{Q}$:

$$dS_t = r_t S_t dt + \sigma_t dW_t, \qquad S_{t_i^+} = S_{t_i^-} - D_{t_i}, \qquad d\langle W, B\rangle = \rho_{S,D} dt$$

with r_t the deterministic rate, σ_t the stochastic volatility and B is the Brownian motion associated with the dividend process. Extensions to stochastic rates are achievable given the efficiency of the particle method but for simplicity we focus only on the stochastic volatility framework in this study. Assume we have pre-computed market no-arbitrage volatility curves interpolated for all strikes $K \in \mathbb{R}_+$. The dynamics of volatility σ_t are then to be defined to match Vanilla options with maturities $T_1, ..., T_{\mathcal{N}}$ and defined for all strikes, that is

$$e^{-\int_0^{T_i} r_s ds}\mathbb{E}^{\mathbb{Q}}[(S_{T_i} - K)^+] = C^{\mathrm{mkt}}(T_i, K) \qquad K \in \mathbb{R}_+,\ i = 1, ..., \mathcal{N}$$

Note these maturities are different to the dividend option maturities of the previous section. In practice maturities of regular vanilla options are much more frequent than their dividend counterpart and are chosen to not coincide with a dividend ex-date t_i.

In the blended dividend setting of section 1.1 the framework allowed us to calibrate a volatility surface on the smooth 'pure' stock process instead of the actual stock process with blended dividend jumps. Using analogous no-arbitrage conditions for the pure setting we could then interpolate the volatility between maturities to build a surface for local volatility modelling. This is not possible in the case of stochastic discrete dividends. We instead require interpolated call prices on ex-dividend dates to satisfy the condition

$$e^{-\int_0^{t_i} r_s ds}\mathbb{E}^{\mathbb{Q}}[(S_{t_i^-} - K)^+] = e^{-\int_0^{t_i} r_s ds}\mathbb{E}\left[(S_{t_i^+} - K)^+\right] = C^{mkt}(t_i^+, K)$$

which depends not only on the risk-neutral marginal distribution of $S_{t_i^-}$, but on the joint distribution $(S_{t_i^-}, D_{t_i})$.

We denote $\mathbb{Q}_i^{\mathrm{mkt}}$ to be the marginal implied from T_i-Vanillas, by the well-know relation (appendix A.4)

$$\mathbb{Q}_i^{mkt}(K) \equiv \frac{\partial^2}{\partial K^2}\mathbb{E}^{\mathbb{Q}}[(S_{T_i} - K)^+] = e^{\int_0^{T_i} r_s ds}\partial_K^2 C^{mkt}(T_i, K)$$

In order to mark our model to the market and subsequently define the local volatility σ_t, we work towards constructing a continuous process that has the same marginals as the stock process at maturities T_i. Define $S_t^{(n)}$ to be the price of the forward with maturity T_n. Noting that we are in a similar setting to that of blended dividends but now with time t cash parts $\alpha_i = \mathbb{E}_t^{\mathbb{Q}}[D_{t_i}]$ and no proportional parts $\beta_i = 0$, by the result 1.2 we have the forward price

$$S_t^{(n)} = S_t e^{\int_t^{T_n} r(s)ds} - \sum_{t<t_i<T_n} e^{\int_{t_i}^{T_n} r_s ds}\mathbb{E}_t\left[D_{t_i}\right] \qquad t \in [0, T_n],\ n = 1, ..., N \tag{3.4}$$

By construction $S_t^{(n)}$ is a continuous martingale, which can be seen as it is a linear combination of dividend forward martingale processes $\mathbb{E}_t\left[D_{t_i}\right]$. We have the boundary condition $S_{T_n}^{(n)} = S_{T_n}$ and it has dynamics

$$dS_t^{(n)} = e^{\int_t^{T_n} r_s ds}\sigma_t dW_t - \sum_{t<t_i<T_n} e^{\int_{t_i}^{T_n} r_s ds} d\mathbb{E}_t\left[D_{t_i}\right], \qquad t \in [0, T_n] \tag{3.5}$$

For completeness we let σ_D^i denote the volatility vector of the t_i-dividend future and W^D a multi-dimensional Brownian motion, defining

$$d\mathbb{E}_t[D_{t_i}] = \sigma_D^i dW_t^D \tag{3.6}$$

to specify the model generally. In our case, in the previous section we have defined a Markov-functional form for the dividend dynamics so have $W^D := B$ to be a one-dimensional Brownian motion and that $\sigma_D^i = \partial_x f_i(t_i, X_{t_i})$.

Calibration to Vanillas

We now approach the problem of calibrating a local stochastic volatility model with general dividend dynamics 3.6. Given the discontinuities of dividend payments, we alter the modelling problem to find continuous processes sharing the same market marginals at observed maturities. From this a local volatility function for the stock process can be constructed. In short, we take the following steps:

1. specify a continuous process $\bar{S}^{(n)}$ with local volatility $\sigma^{(n)}_{\text{Dup}}(t, \bar{S}^{(n)}_t)$ that has the same marginals as the T_n-forward process $S^{(n)}_t$ at T_{n-1} and the market marginal $\mathbb{Q}^{\text{mkt}}_n$ at T_n.
2. specify local volatility of the stock price σ_t as a function of $\sigma^{(n)}_{\text{Dup}}$

Building a local volatility $\sigma^{(n)}_{\text{Dup}}$

Consider the pair of consecutive future maturities $\{T_{n-1}, T_n\}$ and assume that the local volatility σ_t is already specified for $t \le T_{n-1}$, meaning our model is already calibrated to vanilla options for proceeding maturities $T_1, < ... < T_{n-1}$. We now consider the problem of how we might determine the dynamics on the future interval $[T_{n-1}, T_n]$. Given our model, the stock process S_t jumps on ex-dividend dates, meaning we cannot build a well-defined volatility surface directly to derive local dynamics in the usual Dupire-like way.

Instead we consider the forward $S^{(n)}_t$ that is a continuous martingale with the boundary property $S^{(n)}_{T_n} = S_{T_n}$. Given this condition we notice that if we were able to simulate the forward process $S^{(n)}_t$ for $t \in [T_{n-1}, T_n]$ then the simulated T_n maturity call prices must match the market prices, that is

$$C^{\text{model}}_n(T_n, K) \equiv e^{-\int_0^{T_n} r_s ds}\mathbb{E}^{\mathbb{Q}}[(S^{(n)}_{T_n} - K)^+] \equiv C^{\text{mkt}}(T_n, K)$$

for all strikes $K \in \mathbb{R}_+$. We will see that simulating the forward is indeed possible, so we also compute the T_{n-1} maturity calls in an effort to construct a volatility surface over this interval for a process without jumps. Computing T_{n-1} maturity call options of process $S^{(n)}$ for a large grid of strikes K, we have

$$\begin{aligned} C^{\text{model}}_n(T_{n-1}, K) &\equiv e^{-\int_0^{T_{n-1}} r_s ds}\mathbb{E}^{\mathbb{Q}}[(S^{(n)}_{T_{n-1}} - K)^+] \\ &\equiv e^{-\int_0^{T_{n-1}} r_s ds}\mathbb{E}^{\mathbb{Q}}\left[\left(S_{T_{n-1}} e^{\int_{T_{n-1}}^{T_n} r(s)ds} - \sum_{T_{n-1}<t_i<T_n} e^{\int_{t_i}^{T_n} r_s ds}\mathbb{E}_{T_{n-1}}[D_{t_i}] - K\right)^+\right] \end{aligned}$$

Using $C^{\text{model}}_n(T_{n-1}, K)$ and $C^{\text{model}}_n(T_n, K) = C^{\text{mkt}}(T_n, K)$, which are known respectively from a Monte-Carlo computation and the market value of T_n-Vanillas, we can derive the Black-Scholes implied volatilities. The volatilites of the respective maturities T_{n-1} and T_n are defined as $\sigma(T_{n-1}, K)$ and $\sigma(T_n, K)$ and respectively solve

$$C^{\text{model}}_n(T_\alpha, K) = \text{BS\_call}(\sigma(T_\alpha, K)^2 T_\alpha, K, S^{(n)}_0), \qquad \alpha = \{n-1, n\}$$

By linearly interpolating the variances $\sigma(T_{n-1}, K)^2 T_{n-1}$ and $\sigma(T_n, K)^2 T_n$, this specifies an implied volatility surface for all $t \in [T_{n-1}, T_n]$ by

$$\sigma(t, K)^2 t \equiv (\sigma(T_n, K)^2 T_n - \sigma(T_{n-1}, K)^2 T_{n-1})\frac{(t - T_{n-1})}{T_n - T_{n-1}} + \sigma(T_{n-1}, K)^2 T_{n-1} \tag{3.7}$$

Using the interpolation lemma of [10], assuming that both smiles are free of butterfly arbitrage and $\sigma(T_n, K)^2 T_n \ge \sigma(T_{n-1}, K)^2 T_{n-1}$ for all strikes $K \in \mathbb{R}_+$ (which can be checked numerically) then there exists an interpolation such that the interpolated volatility surface is free of static arbitrage for all $T_{n-1} < t < T_n$. If this holds we can then specify a process $\bar{S}^{(n)}$:

$$d\bar{S}_t = \sigma^{(n)}_{\text{Dup}}(t, \bar{S}^{(n)}_t)dW_t, \qquad t \in [T_{n-1}, T_n]$$

$$\sigma^{(n)}_{\text{Dup}}(t, K)^2 \equiv \frac{\partial C(t, K)}{\frac{1}{2}K^2 \partial^2_K C(t, K)} \tag{3.8}$$

where $C(t,K) = \text{BS\_call}(\sigma(t,K)^2 t, K, S_0)$ for $T_{n-1} < t < T_n$. By construction $\bar{S}$ has the same marginals as the forward at the end points of the interval, that is $\bar{S}^{(n)}_{T_{n-1}} = S^{(n)}_{T_{n-1}}$ and therefore $\bar{S}_{T_n} \sim \mathbb{Q}^{\text{mkt}}_n$. In conclusion, our model is calibrated to equity vanilla prices with maturity T_n if $S^{(n)}_t$ and $\bar{S}^{(n)}_t$ have the same marginals for all $t \in [T_{n-1}, T_n]$.

Building σ_t from $\sigma^{(n)}_{\mathbf{Dup}}$

By [11], the processes $S^{(n)}_t$ and $\bar{S}^{(n)}_t$ have the same marginals for all $t \in [T_{n-1}, T_n]$, so we have the calibration condition $S^{(n)}_{T_n} = S_{T_n} \sim \mathbb{Q}^{\text{mkt}}_n$, if and only if the following equation holds:

Proposition 3.2

$S_{T_n} \sim \mathbb{Q}^{\text{mkt}}_n$ for all $n = 1, ..., N$ if and only if σ_t satisfies:

$$\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{d\langle \ln S^{(n)}\rangle_t}{dt}\right) \mid S^{(n)}_t = K\right] = \sigma^{(n)}_{\text{Dup}}(t,K)^2, \qquad t \in [T_{n-1}, T_n] \tag{3.9}$$

By taking $\sigma_t = a_t \sigma(t, S^{(n)}_t)$ for all $t \in [T_{n-1}, T_n]$ with a_t a multi-factor stochastic volatility model, by applying Ito's formula and using dynamics (3.5) we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{d\langle \ln S^{(n)}\rangle_t}{dt}\right) \mid S^{(n)}_t = K\right] &= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{dt}\left(\frac{1}{(S^{(n)})^2}(dS^{(n)})^2 + O((dS^{(n)})^3)\right) \mid S^{(n)}_t = K\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{K^2 dt}\left(e^{\int_t^{T_n} r_s ds} a_t \sigma(t,K) dW_t - \sum_{t<t_i<T_n} e^{\int_{t_i}^{T_n} r_s ds} \sigma^i_D dW^D_t\right)^2 \mid S^{(n)}_t = K\right] \\
&= \frac{1}{K^2}\left(\sigma^2(t,K) e^{2\int_t^{T_n} r_s ds} I_0(t,K) - 2\sigma(t,K) I_1(t,K) + I_2(t,K)\right)
\end{aligned}$$

with

$$\begin{aligned}
I_0(t,K) &\equiv \mathbb{E}^{\mathbb{Q}}[a_t^2 \mid S^{(n)}_t = K] \\
I_1(t,K) &\equiv \rho_{\text{SDiv}} e^{\int_t^{T_n} r_s ds} \sum_{t<t_i<T_n} e^{\int_{t_i}^{T_n} r_s ds} \mathbb{E}^{\mathbb{Q}}[a_t \sigma^i_D \mid S^{(n)}_t = K] \\
I_2(t,K) &\equiv \sum_{t<t_i,t_j<T_n} e^{2\int_{t_i}^{T_n} r_s ds} \mathbb{E}^{\mathbb{Q}}[\sigma^i_D, \sigma^j_D \mid S^{(n)}_t = K]
\end{aligned}$$

The above is a second-order algebraic equation in $\sigma(t,K)$ that can be solved giving:

$$\sigma(t,K) = \frac{I_1(t,K) + \sqrt{\Delta(t,K)}}{e^{2\int_t^{T_n} r_s ds} I_0}$$

with $\Delta(t,K) \equiv I_1^2(t,K) + e^{2\int_t^{T_n} r_s ds} I_0(t,K)\left(K^2 \sigma_{\text{Dup}}(t,K)^2 - I_2(t,K)\right)$.

Here we need to assume that $\Delta(t,K) \geq 0$ (which can be checked numerically). Finally, the dynamics of S_t read

$$dS_t = r_t S_t dt + \sigma(t, S^{(n)}_t) a_t dW_t, \qquad S_{t_i^+} = S_{t_i^-} - D_{t_i}, \qquad \forall t \in [T_{n-1}, T_n] \tag{3.10}$$

We recognise that (3.10) is a McKean-Vlasov SDE studied in the previous section and the particle method can be used to jointly simulate and calibrate such a model.

The Particle Method

We define the system composed of N processes $(S^{i,N}_t, X^{i,N}_t, a^{i,N}_t)_{i=1,\dots,N}$ by

$$dS^{i,N}_t = r_t S^{i,N}_t dt + \sigma_N(t, S^{i,N,(n)}_t) a^{i,N}_t dW^{i,N}_t \qquad \text{for all} \qquad t \in [T_{n-1}, T_n] \tag{3.11}$$

$$dX^{i,N}_t = -k X^{i,N}_t dt + dW^{i,N}_t, \qquad d\langle W^{i,N}_t, B^{i,N}\rangle_t = \rho_{\text{SDiv}} dt \tag{3.12}$$

where

$$S_t^{i,N,(n)} = S_t^{i,N} e^{\int_t^{T_n} r_s ds} - \sum_{t<t_j<T_n} e^{\int_{t_j}^{T_n} r_s ds} f_j(t, X_t^{i,N})$$

the Brownian motions $(W_t^{i,N}, B_t^{i,N})$ and $(W_t^{j,N}, B_t^{j,N})$ are independent for all $i \neq j$ and $\sigma_N(t,K)$ is defined by

$$\sigma_N(t,K) = \frac{I_1^N(t,K) + \sqrt{I_1^N(t,K)^2 + e^{2\int_t^{T_n} r_s ds} I_0^N(t,K)\left(K^2 \sigma_{\mathrm{Dup}}^{(n)}(t,K)^2 - I_2^N(t,K)\right)}}{e^{2\int_t^{T_n} r_s ds} I_0^N(t,K)} \quad (3.13)$$

$I_0^N(t,K)$, $I_1^N(t,K)$ and $I_2^N(t,K)$ are the empirical measure $\mathbb{Q}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{(S_t^{i,N}, a_t^{i,N}, X_t^{i,N})}$ analogues of $I_0(t,K)$, $I_1(t,K)$ and $I_2(t,K)$ obtained using the regularizing kernel $\delta_{t,N}$. Explicitly they are given by

$$I_0^N(t,K) \equiv \frac{\sum_{i=1}^N (a_t^{i,N})^2 \delta_{t,N}(K - S_t^{i,N})}{\sum_{i=1}^N \delta_{t,N}(K - S_t^{i,N})}$$

$$I_1^N(t,K) \equiv \rho_{\mathrm{SDiv}} e^{\int_t^{T_n} r_s ds} \sum_{t<t_i<T_n} e^{\int_{t_i}^{T_n} r_s ds} \frac{\sum_{i=1}^N (\sigma_D^i a_t^{i,N}) \delta_{t,N}(K - S_t^{i,N})}{\sum_{i=1}^N \delta_{t,N}(K - S_t^{i,N})}$$

$$I_2^N(t,K) \equiv \sum_{t<t_i,t_j<T_n} e^{2\int_{t_i}^{T_n} r_s ds} \frac{\sum_{i=1}^N (\sigma_D^i \sigma_D^j) \delta_{t,N}(K - S_t^{i,N})}{\sum_{i=1}^N \delta_{t,N}(K - S_t^{i,N})}$$

We now set out the full algorithm of [3], to calibrate a LSVM with stochastic dividends

Algorithm 3.3: Match equity Vanilla prices

1. From T_1-Vanillas, compute the local volatility $\sigma_{\mathrm{Dup}}^{(1)}(t,K)$ using Dupire's local volatility formula (3.8)
2. Then compute the local volatility $\sigma(.,.)$ as given in Equation (3.13) for all $t \in [0, T_1]$ using the particle method. This consists in: **Building** $\sigma(t,.)$**:**
   (a) Set $k = 1$ and set $\sigma(0, S_0) = \sigma_{\mathrm{Dup}}^{(1)}(0, S_0)$ for all $t \in [0, \frac{k}{M}T_1]$.
   (b) Simulate N processes $\{S_t^{i,N}, a_t^{i,N}, X_t^{i,N}\}_{1 \le i \le N}$ from t_{k-1} up to $t_k \equiv \frac{k}{M}T_1$ using a discretization scheme for SDEs (3.11), say, an Euler or Milstein scheme.
   (c) Compute the local volatility $\sigma_N(t,K)$ from Equation (3.13).
   (d) Set $k := k+1$. Iterate steps (ii) and (iii) up to the maturity T_1.
3. Compute T_1 call options on $S_{T_1}^{(2)} = S_{T_1} e^{\int_{T_1}^{T_2} r_s ds} - \sum_{T_1<t_i<T_2} e^{\int_{t_i}^{T_2} r_s ds} \mathbb{E}_{T_1}^{\mathbb{Q}}[D_{t_i}]$ for all K (in a grid):
$$\mathbb{E}^{\mathbb{Q}}[(S_{T_1}^{(2)} - K)^+] \approx \frac{1}{N}\sum_{i=1}^N (S_{T_1}^{i,N,(2)} - K)^+$$
4. From the implied volatility of $S_{T_1}^{(2)}$ at T_1 and the market implied volatility at T_2, compute the local volatility $\sigma_{\mathrm{Dup}}^{(2)}(t,K)$ as outlined in the 'building a local volatility' subsection. Then compute the local volatility $\sigma(t,K)$ (3.3) in the interval $[T_1, T_2]$ using the particle method (see (i)-(iv)). Compute the implied volatility of $S_{T_2}^{(3)} = S_{T_1} e^{\int_{T_2}^{T_3} r_s ds} - \sum_{T_2<t_i<T_3} e^{\int_{t_i}^{T_1} r_s ds} \mathbb{E}^{\mathbb{Q}}[D_{t_i}]$.
5. Iterate up to $T_{\mathcal{N}}$

3.4 Numerical Example

We use the EURO STOXX 50 Index Options (OESX) data for calibrating vanilla options and the calibrated dividend model discussed earlier in the section. As in the previous chapter, convergence of realistic results were only found using a bandwidth requiring sample standard deviation which lengthened computation time, ultimately meaning less particles and time steps than desired were used. The following results are from a model with the following parameters:

Number of particles N	5000
Number of timesteps	250/yr
α, θ, σ_v	0.02, 0.1, 0.01
Kernel	Gauss
Bandwidth selection	rule of thumb (2.12)
Dividend model	as in sect 3.2

Table 3.2: Stochastic dividend LSVM parameters

Calibration

Implied variance (%)	Sep 2022	Dec 2022	Mar 2023	Jun 2023	Sep 2023	Dec 2023
Avg. |Mkt − MC|	3.52	4.67	3.20	3.59	3.66	3.38
Max. |Mkt − MC|	4.21	7.61	4.38	5.07	4.02	5.77
Implied variance (%)	Jun 2024	Dec 2024	Dec 2025	Dec 2026	Dec 2028	Dec 2030
Avg. |Mkt − MC|	4.94	4.27	4.10	4.05	3.68	3.55
Max. |Mkt − MC|	3.57	5.31	7.53	6.94	6.30	6.47

Table 3.3: Implied volatility repricing

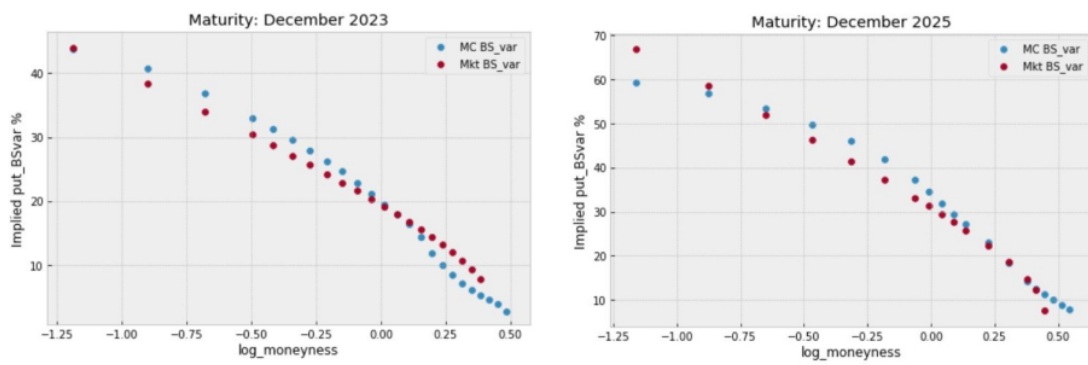

Figure 3.3: Plot of market implied variance compared to the implied variance of particle simulations for two maturities, December 2023 and December 2025.

Pricing

Prices for dividend knock-in product are computed using the simulated particle paths. It is interesting to note that prices for all combinations of maturity and barrier are increased from the blended dividend framework results of chapter 1. This is in part due to the added volatility of the particle model used.

Maturity Boundary	3yr 90%	3yr 80%	5yr 90%	5yr 80%
MC Price	57.64	29.02	74.26	35.68
MC 0.95 CI ±	3.11	2.48	4.98	5.66

Table 3.4: Knock-in Dividend pricing for stochastic dividend LSVM

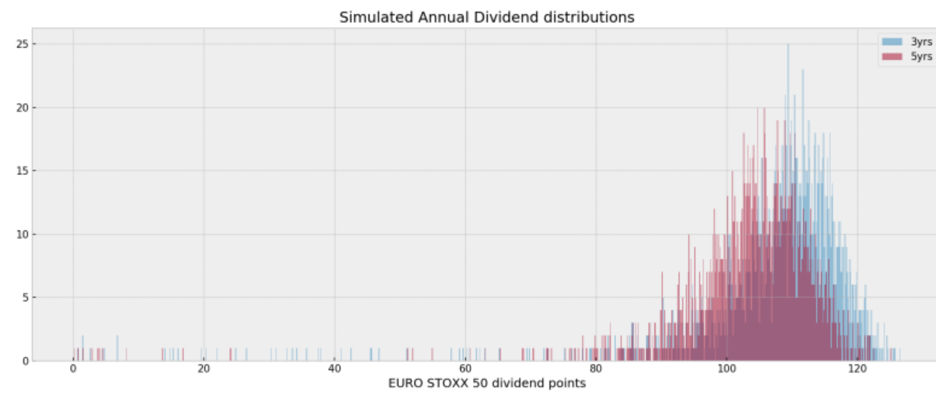

Figure 3.4: Distribution of simulated dividends for the 3yr and 5yr knock-in swap.

Conclusion

This thesis aimed to introduce frameworks for modelling dividends for the involvement of pricing structured products.

The 'blended' framework of H. Buehler was introduced in Chapter 1 and recognized for its tractability and rich structure. The approach with a basic local volatility model for the pure process X was found to be efficient for repricing observed smiles. However, limitations included the dependence on defining a blended structure and the requirement to extend the model in order to give dynamics to the dividends, which is necessary to accommodate the risks associated to pricing a structured product.

The 'particle method' of J. Guyon and P. H-Labordere was then introduced as a more universal approach to the hybrid modelling required for pricing such trades. The theoretical motivations of the method were discussed in Chapter 2 and a framework relevant to dividend modelling was derived in Chapter 3.

Numerical experiments using EURO STOXX 50 as a running example were noted at the end of each chapter. Though numerics were affected by using a 'proxy' forward curve, the structure of both frameworks was appreciated. The calibration of the 'blended dividend' model in Chapter 1 was only limited to the optimization of the associated SSVI surface. Relevant implied volatilities and prices were generated to show a reasonable fit of the model and give estimates of pricing the 'Knock-in' dividend swap. Despite the implementation of the particle method being heavily affected by bandwidth selection, reasonable numerics were achieved and the adequacy of using such an approach to general structured product modelling was proven.

Appendix A

A.1 Dividends terminology

There are some important dates to be aware of in the life-cycle of a dividend payment. The board of directors or issuer of the security (ETFs can also pay dividends) announce the details of a dividend payment on the 'declaration date'. The 'ex-dividend date' is then the trading date on which new buyers of the security cannot receive the dividend. Typically on the day following the ex-date, a company will check their records for shareholders eligible for the dividends on what is called the 'date of record' and the dividend is then transferred to eligible holders on the 'date of payment' that is typically a week later.

The drop in share price equal to the dividend amount can be seen by basic no-arbitrage arguments - say if the drop is less than d_j, buy the stock on the close before the ex-date, pocket dividend d_j then resell the position at the open for a profit. The reverse operation could be executed in the case of a larger drop.

Other important points of note are that the shot seller of a security is obligated to pay dividends to the lender and that dividend payments exhibit seasonality - the annual frequency that a dividend may be paid is specific to that company or issuer but the most substantial payments typically come in the semi-annual quarters of the calendar year. For example EURO STOXX 50 dividend payments occur quarterly but the semi-annual quarter payments are historically $5\times$ larger than first quarter and third quarter dividend payments - note this relationship has fluctuated in times of economic stress, such as 2008-09 and 2020-21.

A.2 The Black-Scholes formula

$$\begin{aligned} \text{BS\_call}(v, k, f; r, T) &= e^{-rT}\left[\Phi(d_+)f - \Phi(d_-)K\right] \\ d_+ &= \frac{1}{\sqrt{v}}\left(\log\left(\frac{f}{k}\right) + \frac{v}{2}\right), \qquad d_- = d_+ - \sqrt{v} \end{aligned}$$

A.3 Local Volatility marginal distributions

$$\begin{aligned} \frac{\partial C}{K} &= P(t,T)\int_K^\infty \frac{\partial}{\partial K} p(T,s)ds \\ &= -P(t,T)\int_K^\infty p(T,s)ds \end{aligned}$$

$$\begin{aligned} \frac{\partial C}{K} &= -P(t,T)[p(s,T)]_{s=K}^{s=\infty} \\ &= P(t,T)p(T,K)ds \end{aligned}$$

A.4 Ornstein-Uhlenbeck formulae

(Proof:*(finding formulae for X_t & X_{t_i})*

$$\begin{aligned} d(X_t e^{kt}) &= kX_t e^{kt}dt + e^{kt}dX_t \\ &= kX_t e^{kt}dt + e^{kt}(-kX_t dt + dW_t^X) \\ &= e^{kt}dW_t^X \end{aligned}$$

Integrating over:

$$\begin{aligned} [0,t] : X_t e^{kt} - X_0 e^{k.0} = \int_0^t e^{ks}dW_s^D \rightarrow X_t = \int_0^t e^{-k(t-s)}dW_s^X \\ [t,t_i] : X_{t_i}e^{kt_i} - X_t e^{kt} = \int_t^{t_i} e^{ks}dW_s^D \rightarrow X_{t_i} = X_t e^{-k(t_i-t)} + \int_t^{t_i} e^{-k(t_i-s)}dW_s^X \end{aligned}$$

Note by ito integral properties and ito isometry,

$$\begin{aligned} \mathbb{E}\left[X_{t_i}\right] &= 0 \\ \mathbb{V}ar\left(X_{t_i}\right) &= \int_0^{t_i} e^{-2k(t_i-s)}ds = \frac{1-e^{-2kt_i}}{2k} \end{aligned}$$

So under $\mathbb{Q}$, $X_{t_i} \sim \sqrt{\frac{1-e^{-2kt_i}}{2k}}.Z$ where $Z \sim \mathcal{N}(0,1)$.)

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