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Delta Hedging Convertible Bonds with Credit Risk

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Declaration

The work contained in this thesis is my own work unless otherwise stated.

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Abstract

Convertible arbitrage is a popular hedge fund strategy that has brought considerable profits to convertible arbitrageurs over the past two years. Dynamic delta hedging is an essential part of this strategy, through which the delta-hedged position becomes insensitive to underlying stock price movements, but will be exposed to the issuer's credit risk and interest rate risk. Therefore, Arbitragers may want to consider to hedge credit risk in addition to equity exposure. CDS are widely-used for hedging credit due to their low basis risk and cost-effectiveness, but lack of availability and liquidity issues prevent them from being applicable to every convertible bond as a general hedging tool. Moreover, in modern default studies, scholars have pointed out that credit risk and equity are correlated, but there has been no systemically well-developed model integrating this correlation. Therefore, we want to explore alternatives to CDS that reflect the credit-equity correlation in the credit-hedging process.

In this thesis, we propose equity as an alternative instrument, demonstrate its correlation with credit risk, and introduce a delta hedging strategy adjusted for this correlation. Additionally, we compare the hedging effectiveness of equity and CDS by developing hypothetical hedged portfolios and back-testing in our universe.

keywords: Convertible arbitrage, Dynamic delta hedging, Credit hedging, Credit-equity correlation.

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Introduction

The convertible bond market saw another heyday in 2020, with a rapid expansion in market size as corporations rushed to shore up balance sheets in the wake of the COVID-19 pandemic, and elevated equity volatility made convertibles a relatively attractive instrument. However in 2022, increased geopolitical tension, rising inflation and more restrictive central bank policies caused both interest rates and credit spreads to widen, while equity prices fell. Broad credit indices spiked, with the Markit CDX North America High Yield Index reaching 584.661 bps[1] on 16^{th} June, and the Markit iTraxx Europe Crossover index hitting 628.154 bps[1] on 14^{th} July. The convertible bond market has been more volatile with increasing sensitivity to credit risk as falling equity prices cause convertibles to become more 'bond-like'. Most convertible bond issuers are classified as non-investment grade as their credit ratings are evaluated to be lower than BBB-, which puts their investors under significant credit risk. [1]

One simple way that investors hedge credit risk is by buying CDS protection on underlying fixed income securities to transfer credit exposure to a third party, but the CDS market has become much smaller and less liquid over time, so it is harder for investors to hedge the credit risk of convertible bonds. Alternative credit hedging methods are thereby very important, and my thesis explores one of these: using equity to hedge convertible bonds with credit risk.

Dynamic delta hedging is a trading strategy aimed at reducing the risk associated with the price movement of the underlying equity, by simultaneously taking an offsetting position in the underlying stock and adjusting positions constantly to keep the portfolio delta-neutral. While the Black Scholes model has been widely used to compute delta for dynamic delta hedging, there has been no systemically well-developed model for delta hedging with credit-equity correlation incorporated, especially for hybrid securities that are sensitive to, and correlated with, both credit and equity markets, such as the hybrid securities convertible bonds.

The valuation of convertible bonds depends on four main factors, interest rate dynamics, underlying stock price evolution, volatility of the stock price and the credit risk of the underlying company. [2] Because of the hybrid features, convertible bond investors are exposed to both the credit risk inherent in the straight bond portion and the market volatility of the share price of the underlying, as it impacts the value of the embedded conversion option. The number of shares that need to be traded to offset the exposure to the stock is defined as the hedge ratio, the size of which is determined by delta. Delta measures the sensitivity of a security's price to its underlying stock price. When a position in a convertible bond is delta hedged, the exposures remaining are to credit, volatility and interest rates. The presence of credit risk brings considerable uncertainty to the valuation of convertible bonds. Hence for securities that are credit-sensitive, other hedging strategies can be used to reduce the volatility of a convertible bond position, in addition to the Black-Scholes delta hedging.

There are plenty of strategies with various hedging instruments that are widely used to control and reduce credit exposure, such as hedging directly by buying a CDS protection on the same reference entity, using ASCOTs, straight bonds, or hedging by utilizing the correlation between the underlying companies and some companies covered by CDX index. Although these strategies and traditional credit hedging methods enable investors to offset more risk than delta hedging alone, one problem is that they all ignore the credit-equity link. It's worth highlighting that the correlation of credit with equities has picked up in the last decade and remained relatively high

[1] Data source: Bloomberg Finance L.P.

over the past few years. [3] Considering that convertible bonds are susceptible to both equity and credit risk, and the correlation between credit and equity is not negligible, the performance of convertible bonds can neither be fully attributed to movements in equity or credit markets, nor can it be considered a simple addition of the effects under two risks. Also, we are not convinced that the reduced volatility has reached the minimum, or that there is no opportunity to reduce it further. We believe that a credit-adjusted delta will effectively capture this correlation and deliver better results by reducing the variance of the delta-neutral portfolios to a lower level.

The purpose of this thesis is to present an alternative credit hedging method as proposed above and incorporate credit-equity correlation into the classic dynamic delta hedging strategy to decrease the convertible bond volatility associated with underlying share price movement and changes in creditworthiness. By comparing its results with the classic delta hedging strategy and various credit hedging strategies that are widely used in the industry, we can explore under which circumstances it effectively reduces the variance of the hedged portfolio.

The rest of the thesis is organized as follows. In Chapter 2, the capital structure model for companies with debt and equity is introduced. In Chapter 3, different valuation models are presented for convertible bonds, such as the structural model and reduced-from model, and two numerical approaches to solve the valuation problems are provided. In Chapter 4, various approaches that are used to hedge credit exposure either directly or through correlation are introduced and compared, relevant hedging instruments involved are CDS, ASCOT, straight bonds, CDS index and equity. In Chapter 5, potential functional relationships between credit and equity are discussed, such as a linear functional form, exponential functional form, and other functional forms incorporating more variables. In Chapter 6, the effectiveness of this new hedging method is analysed by using back-testing, where both unhedged and hedged portfolios of different strategies are developed to assess and compare the effectiveness of hedges in terms of volatility reduction for convertible bond issuers with CDS or with no CDS contracts. In Chapter 7, the conclusion and limitations of the new approach proposed in this paper are discussed.

Chapter 1

Convertible Bonds

Convertible bonds are a typical category of hybrid security, equipped with features from both equity and fixed income. The conversion option makes convertible bonds more complicated by adding features from financial derivatives. They can be regarded as corporate bonds with additional conversion rights, as bondholders are entitled to convert the bonds into equity at some specific number called the conversion ratio, which is specified in the prospectus at issuance. The conversion option not only allows bondholders to obtain exposure to stock price appreciation but also makes investors face limited downside risk assuming the company does not default, as the value of a convertible bond decreases less than its underlying share price drops because of the positive convexity features.

1.1 Valuation of Convertible Bonds

Taking European convertible bonds as an example, the value of a convertible bond at maturity is the greater of its conversion value and the face value of a straight bond of the same seniority and features, i.e.

$$V(S,T) = \max\{C_r S, FV\}$$

Alternatively, convertible bonds can be viewed as straight bonds with face value N and call options on the conversion value with strike price N. The options will be exercised when the conversion value exceeds N, i.e.

$$V(S,T) = N + \max\{C_r S - FV, 0\}$$

From the perspective of put options, holding a convertible bond is equivalent to taking a long position in the underlying stock and a long position in the underlying conversion value put option, the strike price of which is equal to the bond principal N, i.e.

$$V(S,T) = C_r S + \max\{FV - C_r S, 0\}$$

1.2 Categories of Convertible Bonds

Convertible bonds can be divided into many sub-categories according to different dimensions. For example, by the type of conversion. Voluntary convertible bonds are one of the products we are most familiar with, where bondholders are able to convert at their own discretion. On the contrary, mandatory convertible bonds are forced to convert into common equity at some specified range of conversion ratios at expiry. For some more exotic products, such as contingent convertible bonds (CoCo), the conversion is automatically triggered when certain conditions are satisfied. In this case, the payoff of the bonds becomes path-dependent on the underlying stock price evolution, which makes the valuation of such bonds more difficult. Again, we will focus more on voluntary convertible bonds.

The next dimension is the restriction around conversion rights. Just like ordinary options, there are European convertible bonds, where conversions only happen at bond maturity, American convertible bonds, where the bondholders have the discretion to exercise the conversion options at any time before or at maturity, and Bermudan convertible bonds, where conversions occur on specific dates before maturity, or exactly at maturity. In this thesis, American convertible bonds are our main interest.

1.3 Why Are Convertible Bonds Attractive Investments?

Convertible bonds attract different types of investors because of their hybrid nature, which enables investors to access the potential unlimited equity appreciation with limited exposure to downside losses, making convertible bonds a competitive investment product for investors. Moreover, convertible bonds are less correlated with other asset classes, thus adding them to a portfolio may increase the expected returns without adding more risk due to diversification effects.

If we divide investors into different categories based on their investment preferences and products of interest, convertible bonds are quite attractive for different investors for different reasons. The long-only convertible bond investors are the main type of investors, who dominate the market, their mandate is to invest in convertible bonds and not to hedge, as this gives positive convexity, equity-like exposure on the upside, but bond-like exposure on the downside. For hedge funds and arbitrageurs, if there is a price difference between convertible bonds, the underlying stock and other related securities, they can benefit from exploiting the difference by buying low and selling high, which is known as convertible arbitrage. For fixed-income investors, adding a convertible bond position to their portfolio allows them to have equity exposure without deviating from a client's investment mandate, such as investing in fixed-income products only. As a result, it may bring a higher return to the investment portfolio. [4, 4.3]

On the other hand, for bond issuers, convertible bonds can be a cheaper source of financing compared to corporate bonds because of the lower coupon rate, which effectively reduces the cost of capital, especially for those growth companies that are capital-intensive. The lower coupon rate is compensated by embedded conversion rights at the holders' direction. Typically, issuing new shares increases the total number of shares outstanding, thereby diluting the rights of existing shareholders and causing share prices to fall upon announcement. The options embedded in convertible bonds are able to delay the conversion time, effectively delaying the dilution of earnings. Therefore, there would be a smaller announcement effect on the share price if financing by issuing convertible bonds instead of issuing additional shares. What's more, issuing convertible bonds can effectively help businesses save taxes due to the tax shield that comes with debt interest payments. [4, 1.2.2]

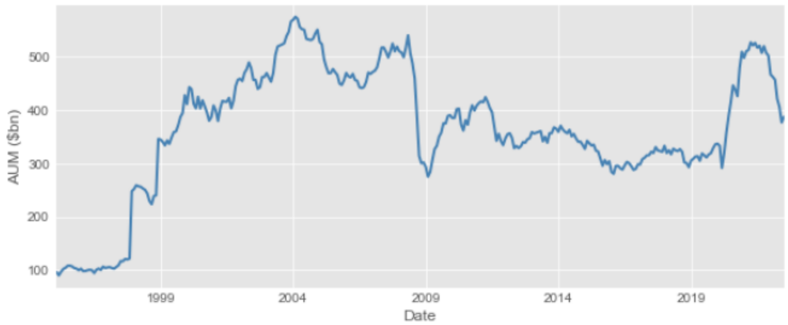
1.4 The Convertible Bond Market

The convertible bond market has been expanding since the 1990s. Prior to the financial crisis in 2008, the convertible bond market grew steadily, with Asset Under Management (AUM) peaking at around \$580 billion[2] in 2004 and global issuance volumes peaking at \$167 billion[2] in 2001.

During the financial crisis in 2008, due to the contagion effect caused by Mortgage-Backed Securities (MBS), convertible markets suffered a devastating blow, with AUM reducing from \$540 billion[2] to around \$270 billion[2] in 9 months. What's worse, there was almost no new issuance in the year following the financial crisis from late 2008 to early 2009. The situation did not improve in the following two years. Starting in 2013 however, resilience kept showing in the convertible bond market because investors have been attracted by the record-low interest rate in the financial market.

It was not until 2012 that signs of recovery eventually appeared in the convertible market, which was driven by rising interest rates, higher volatility and massive issuance of new convertible bonds in 2012, where the global issuance increased after 2007 for the first time. After that, there was positive net issuance each year from 2013 to 2020, and 2020 saw the largest global new issuance since 2001, at around \$159 billion[2].

The outbreak and pandemic of COVID-19 in 2020 caused high volatility in the financial mar-



Source: BofA Global Research (2022)[1]
Figure 1.1: Global convertible bond market value.

ket, which essentially activated the convertible markets. At the same time, companies hit by the pandemic started to recapitalize and aimed to seek a cheaper source of financing, which makes convertibles bonds an ideal approach to raising rescue capital. Also, central banks set interest rates at very low levels to simulate consumption, so the financing terms for companies were very attractive. Low interest rates also led to strong demand from major market participants, such as hedge funds. As a result, global convertible issuance boomed and has even climbed to its historical high water mark since 2007, with ten new issuances per week at some point post-COVID over 2020. [2]

This upward trend continued into 2021 but did not persist any longer in the following year, instead an unusually long quiet issuance period has shown in 2022. Commodity futures prices experienced a dramatic increase as a result of the Russia-Ukraine war that broke out in late February of 2022, the whole world was experiencing unusually high inflation and many economies even showed signs of recession. The rising interest rates, widening credit spreads and falling stock prices led to negative performance in the convertible bond market. The poor performance has started to cause outflows from convertible-focused funds, leading to a large decrease in the cumulative NAV of funds. The uncertain risk environment is another factor that caused the quiet issuance of convertible bonds.

Generally speaking, when markets are very volatile, the issuance of new securities, such as IPOs, new convertible bonds, and new straight bonds, becomes fairly quiet. Investors tend to be very cautious about committing their capital to a new issue when the market is going through turbulent price moves because it is very difficult to price a new deal as the daily prices are quite unstable. The quiet issuance period in 2022 has been longer than usual. The western sanctions on Russia for its invasion of Ukraine led to a surge in the oil price level, which in turn gave rise to persistent inflation. To ease the inflationary problem, central banks implemented contractionary monetary policy by raising base interest rates and shrinking the money supply in the market. This monetary correction also discouraged people from consumption and investors from lending out money, resulting in a low ebb for corporate fundraisings, with a 25% slide in the first half of 2022. [5] At the same time, the financial market suffered from turbulence due to the rising price level, which made companies delay planned IPO and equity financing undergo a fall in price. The slowdown also happened in the high-yield bond market, the global issuance slipped compared to the previous year. However, markets move in cycles, although the convertibles market is going through such a long quiet period, it is more likely that there would be a lot of new issuance in the near future.

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[2]Data source: M. Youngworth. Global convertibles primer. Global convertibles, BofA global research, 2021

Chapter 2

Capital Structure Models

In this chapter, the capital structure of a company is predicted by structural models, and the probability of default will be modelled as an endogenous variable using option pricing theory. For simplicity, the simplest capital structure is used, assuming that the company has only a zero coupon bond in liability and the residual claim, equity.

2.1 The Merton Model

One of the most widely used structural models that use a company's capital structure to model its debt value is the Merton model [6] introduced by Robert Merton in 1974. Many assumptions are required for this Black-Scholes-type pricing model, such as those on a firm's value dynamics.

2.1.1 Geometric Brownian Motion

As one of the critical assumptions, Merton assumes that firm value follows a continuous stochastic diffusion process described by the following stochastic differential equation.

$$\mathrm{d}V_t = (\alpha V_t - c)\mathrm{d}t + \sigma V_t \mathrm{d}W_t, \tag{2.1.1}$$

where:

- α is the instantaneous expected rate of return on firm value V per unit of time
- σ is the instantaneous volatility of return on firm value per unit of time
- $W = (W_t)_{t\geq 0}$ is the standard Brownian Motion process
- c represents the total net cash flow the firm pays to its investors (for example, dividends paid to shareholders and interest payments to bondholders) and cash received from new financing. A positive c indicates cash outflow whereas a negative c refers to cash inflow.

Merton applied the simplest possible capital structure, assuming that the company has only equity and a risk-free zero coupon bond on its liability, so in its balance sheet, the value of assets equals the value of equity and the value of the zero coupon bond. Under these assumptions, Merton modelled underlying assets value A_t follows Geometric Brownian motion, assuming a constant interest rate and the only source for uncertainty is from volatility σ of assets' return.

$$\log A_t = \log A_0 + (r - \frac{\sigma^2}{2})t + \sigma W_t \tag{2.1.2}$$

Since $W = (W_t)_{t\geq 0}$ is a Brownian motion process, W_t is distributed as a Gaussian with mean zero and variance t, so we can write

$$W_t = \sqrt{t} \cdot N(0,1)$$

and hence

$$\ln A_T = \ln A_t + \left(r - \frac{1}{2}\sigma^2\right) \cdot (T - t) + \sigma\sqrt{T-t} \cdot N(0,1)$$

Equivalently,

$$A_T = A_t \cdot e^{\left(r-\frac{1}{2}\sigma^2\right)\cdot(T-t)+\sigma\sqrt{T-t}\cdot N(0,1)} \tag{2.1.3}$$

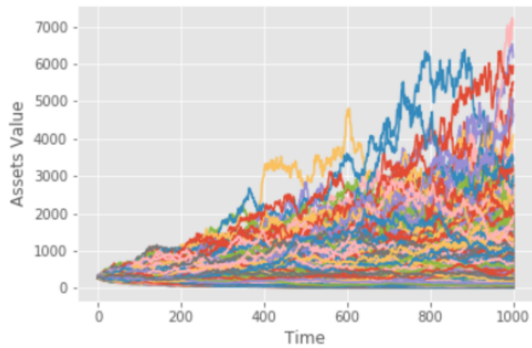


Figure 2.1: Firm value dynamics under Geometric Brownian Motion

At bond maturity T, bondholders either get the principal value (FV) of the bond or the asset value of the firm, whichever is smaller. With the residual claim, shareholders will get the remainder. If asset value ends up less than the liability level, all assets are first liquidated to pay out the company's debt obligations because bondholders have higher seniority in claims than shareholders. Although all assets are used to repay debt obligations, not 100% of the bondholders are able to get full repayment, so the company fails to meet its financial obligation and defaults on its debt. So in this case, the value of equity is zero. If asset value ends up greater than the liability level, the company will be able to make the full amount of debt repayment and shareholders will get whatever is remaining. Therefore, the value of the company's equity and debt at maturity are:

$$D_T = \min(FV, A_T), \qquad E_T = \max(A_T - FV, 0) \tag{2.1.4}$$

The value of equity at maturity of the bond is quite similar to the payoff of a standard European Call option, where the Call option is in the money when the underlying asset's value rises above the face value of the bond. This shows that equity can be regarded as an analogue of a Call option on asset value with the face value of the bond as the strike level. Using the Black Scholes option pricing formula, the value of equity is:

$$E_t = N(d_1)\, A_t - N(d_2)\, Xe^{-r(T-t)} \tag{2.1.5}$$

$$\text{where} \qquad d_1 = \frac{\ln\frac{A_t}{X} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \qquad \text{and} \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

Parameters in the functions are:

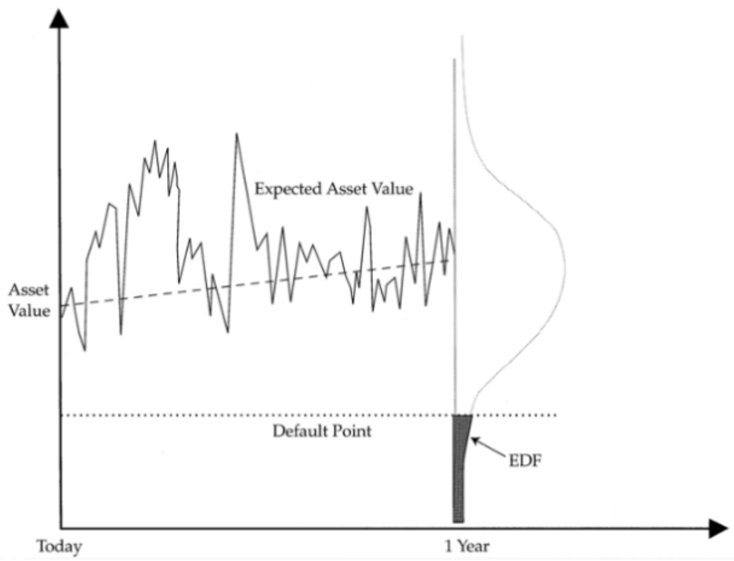
- E_t is the market value of equity at time t
- A_t is the asset value of a firm at time t
- X is the face value of a zero coupon bond
- r is the risk-free interest rate
- σ is the volatility of return on assets
- N denotes the cumulative distribution function of the Gaussian random variable

With a constant volatility level σ and constant risk-free rate r, equity value at time t can be derived using equation (2.1.5). Then the value of debt is derived using the accounting equation:

$$D_t = A_t - E_t \tag{2.1.6}$$

Combining equations (2.1.2), (2.1.5) and (2.1.6) above, Merton further concluded a closed-form solution for evaluating the value of debt.

In the structural model, default is treated as an endogenous variable, which happens when asset value is lying below the liability level.



Source: Stephen Kealhofer, Quantifying Credit Risk I: Default Prediction (2013)
Figure 2.2: Frequency distribution of assets value and probability of default.
[7, Page 31]

Figure 2.2 illustrates the asset value dynamics and probability of default, assuming the company has a single debt liability, equity, and no other obligations. The x-axis represents the investment horizon of a bond which matures one year from now, the y-axis represents the market value of asset value. Asset value follows a lognormal diffusion process, the more volatile the process is, the more likely that the assets will end up with an extreme value. The dotted line represents the face value of the bond that needs to be paid back at maturity, when asset value falls below this value, the company is incapable of making the whole amount of payment and will default. The curve at one year shows the probability of asset value ending up with different values, where the area under the dotted line and colour shadowed by black refers to the likelihood that default happens in one year. Therefore, the probability of default is associated with the market price of assets, the face value of the liability and the volatility of asset return. The lower the asset value or the higher the liability or the higher the volatility, the more likely that default will happen.

The coupon rate of a bond compensates the bondholders for the credit risk they have exposure to, namely the possibility that a company will not be able to fulfil its debt obligations. Therefore, the credit spread should be the effective coupon rate such that the market value of bonds is equal to the present value of all future cash flows made by issuing companies, incorporating the likelihood of default and possible recovery, discounted at the risk-free interest rate.

$$\exp(r_{risky} * T) = D_T \qquad \Longrightarrow \qquad CS = -\left[\frac{1}{T}\log\left(\frac{D_T}{D_0}\right) - r\right] \tag{2.1.7}$$

Suppose we have a zero coupon bond with \$1 million notional. If we are given the initial bond value, initial asset value, risk-free interest rate and volatility of asset value as inputs, it is possible to derive a functional relationship between equity value and credit spread. The initial equity value E_0 can be derived by subtracting the initial bond value from the initial asset value. We assumed that the asset value follows Geometric Brownian Motion, so following equation (2.1.2) with Monte Carlo Simulation, we are able to predict its value at maturity as A_T. Then we can apply equation (2.1.4) to compute the final debt value D_T, finally, credit spread CS is derived by equation (2.1.7).

2.1.2 Monte Carlo Simulation

Monte Carlo Simulation is a mathematical technique to obtain an estimate for unknown events, by assigning multiple values to a variable to get multiple results and averaging the results.

The basic procedure to simulate the value of a European call option can be summarized as follows:

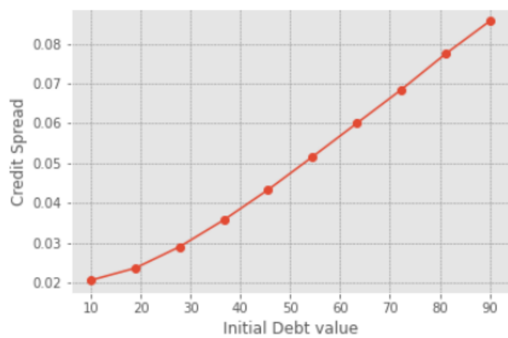
- Generate sample paths of the underlying asset price.
- For each simulated sample path, evaluate the payoff of the option.
- Take the sample average of the payoff over all sample paths and multiply with a discount factor to derive the option price.

According to Merton, equity can be viewed as a European Call option on the asset value of the underlying company, with debt face value as the strike price, so Monte Carlo simulation is also applicable in valuing equity. In detail, we follow the algorithm below to simulate the credit spread under varied initial asset value A_0.

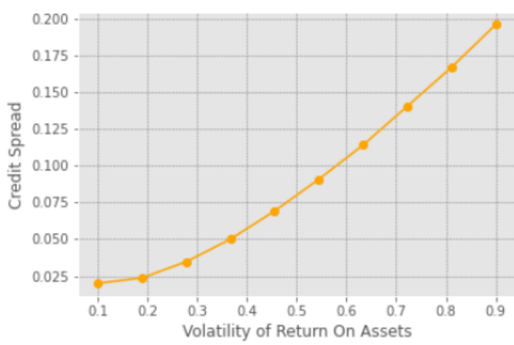
Algorithm: Valuation of Convertible Bond with Monte Carlo Simulation

1. Generate a $N(0, 1)$ variable Z.
2. Set $A_{t+\Delta t} - A_t = A_t(r\Delta t + \sigma_A \sqrt{\Delta t} Z_t)$, where
   - $\Delta t = \frac{T}{N}$ defines the time interval in the discrete model
   - σ_A is the volatility of the asset price
   - r is the constant risk-free interest rate
3. The asset price path $(A_t)_{0 \leq t \leq T}$ can be generated by starting from A_0 and repeating (1) - (2) N times.
4. Set ending equity value by $E_T = \max(A_T - FV, 0)$
5. Repeat (1) - (4) M times independently to get E_{T_1},....,E_{T_M}.
6. Set $\bar{E_T} = \frac{E_{T_1} + ... + E_{T_1}}{M}$,
7. Multiply the average by the discount factor e^{-rT}, the result is the initial equity value E_0
8. Compute initial bond value by $D_0 = A_0 - E_0$
9. Derive credit spread using formula (2.1.7), here we assume $D_T = FV$ and ignore any default at $t = T$

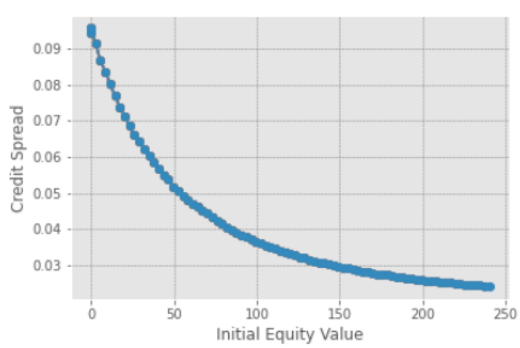
Figure 2.3 shows the relationship between credit spread and three key model inputs - initial debt value D_0, initial equity value E_0 and volatility of return on assets σ_A. Here I assume the company holds a single debt obligation, a zero coupon bond with maturity of five years, initial value A_0 fixed at \$100 in subfigure 2.3(a) and 2.3(b), whereas spanning over interval $[10, 20, ..., 300]$ in 2.3(c). The annual risk-free rate r is static at 2%, and annual volatility on asset price σ_A is fixed at 40%. In subfigure 2.3(a), with a fixed A_0, the credit spread is wider if a greater percentage of the company



(a) Relationship between credit spread and initial debt value.



(b) Relationship between credit spread and volatility level.



(c) Relationship between credit spread and initial equity value, with initial debt value fixed at 60.

Figure 2.3: Model prediction result of credit spread and its relationship with model inputs.

is financed by debt compared to equity, this makes sense as a heavily leveraged company tends to expose its stakeholders to more credit risk. In subfigure 2.3(b) we analyse the impact of σ_A on credit spread with initial debt and equity level fixed at $D_0 = 60$ and $E_0 = 40$ respectively. The upward curve shows that credit spread increases with volatility of asset value σ_A, which makes sense again since higher volatility implies higher default risk for bondholders. In subfigure 2.3(c), with a fixed D_0 at 60, the higher E_0 is, the lower the credit spread is. Intuitively, investors of a less leveraged company face less credit risk, thus requiring less yield compensation so we expect the credit spread to be narrower.

One limitation of the Geometric Brownian Motion process is that the model cannot generate paths including jumps. A Geometric Brownian Motion is a typical diffusion-type process with a continuous path. In reality, the firm value might not be continuous all the time, if default happens, the stock price might experience a sudden drop thus becoming discontinuous:

$$S^+ = S^-(1-\eta) \tag{2.1.8}$$

where S^- refers to the stock price before default, S^+ refers to the stock price after default, and η represents the loss on the stock price due to default with $0 \leq \eta \leq 1$. However, a random variable that follows a Geometric Brownian model is guaranteed to have the feature of continuity, therefore, it fails to capture the extreme movements and price dynamics, so may not be realistic when taking the likelihood of default into consideration.

2.1.3 Jump-Diffusion Process

Merton introduced the Jump-Diffusion model in 1976, as indicated by its name, a jump-diffusion model combines a diffusion process and a jump process to capture the discontinuous behaviour in the price dynamics. In this approach, the diffusion process is modelled by Geometric Brownian motion and the jump process is characterized by the Poisson process.

A jump-diffusion process $(X_t)_{t\geq 0}$ has the form: [8]

$$X_t = X_0 + I_t + R_t + J_t = X_t^c + J_t$$

where

- I_t is the Ito integral part:
$$I_t = \int_0^t \Gamma_s dB_s$$
- R_t is the Riemann intergral part:
$$R_t = \int_0^t \Theta_s ds$$
- J_t is the pure jump part, which is a pure jump process with $J_0 = 0$ (e.g. Poisson process N_t).

Besides, X_t satisfies the PDE:

$$\frac{dX_t}{X_{t-}} = \mu dt + \sigma dW_t + Y dN_t$$

where $Y = e^Z - 1$, $Z \sim N\left(0, \sigma^2\right)$ and N_t is a Poisson process with intensity λ.

To get a more realistic prediction for future credit spread, we assume that the value of the underlying assets follows a jump-diffusion process with additional parameters characterizing the price jump due to default:

- Mean of jump size, $m = 0$
- Standard deviation of jump, $v = 0.1$
- Intensity of jump, i.e. number of jumps per year, $\lambda = 1$

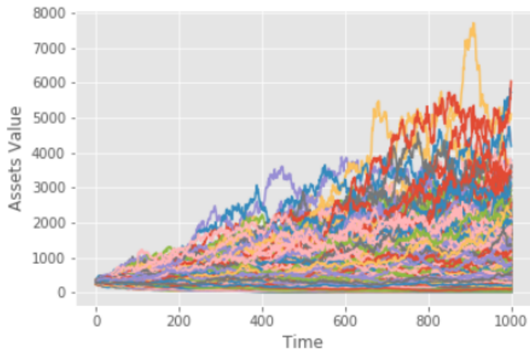
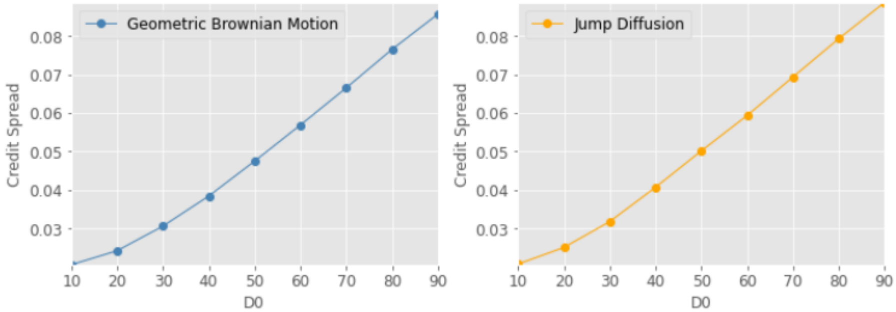


Figure 2.4: Firm value dynamics under Jump-diffusion model

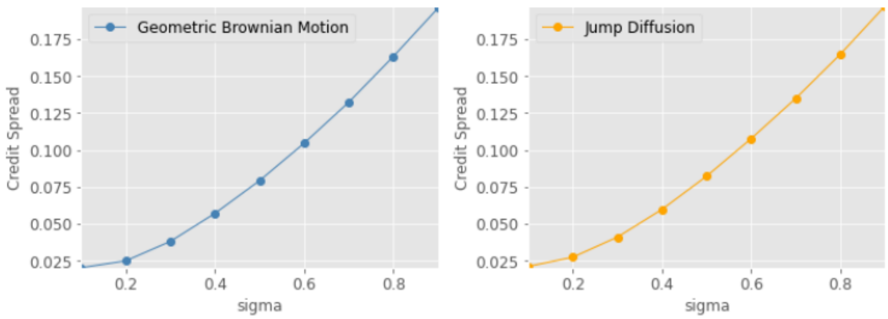
Figure 2.5 shows the predicted results for credit spread and its relationship to initial debt value, volatility level, initial equity value and ending equity value by assuming the asset value follows Geometric Brownian Motion and a Jump-diffusion process respectively. Because the jump-diffusion model is able to capture extreme movement, the asset paths generated by this model are more volatile, leading to a higher credit spread than Geometric Brownian Motion. This has been reconfirmed in these charts. If we set other variables such as D_0, E_0, E_T, σ as fixed, for any arbitrary value that the x-axis takes, the credit spread in the right charts is always higher. The relationship between credit spread and those variables is expected to be less stable as well, this is shown by the wider confidence interval in Table 2.1.

Asset Paths	95% Confidence Interval
Geometric Brownian Motion	$(0.0252, 0.0262)$
Jump-Diffusion	$(0.0263, 0.0276)$

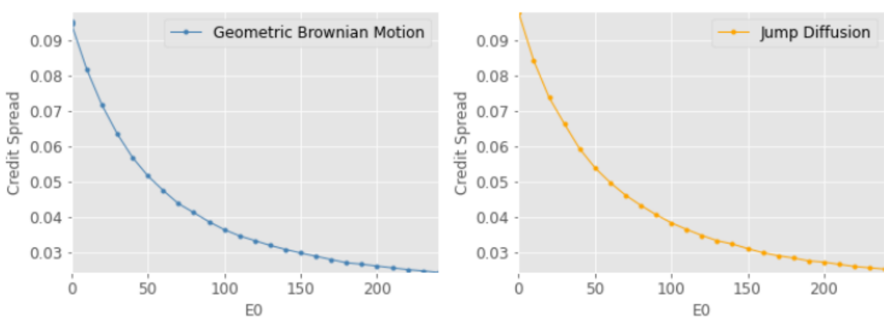
Table 2.1: 95% Confidence Interval of credit spread.



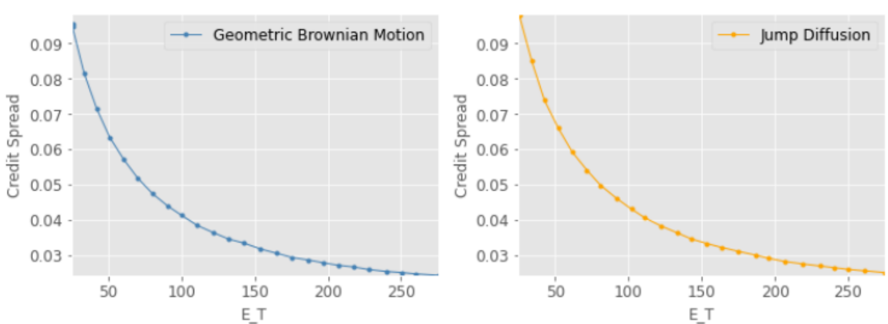
(a) Relationship between credit spread and initial debt value.



(b) Relationship between credit spread and volatility level.



(c) Relationship between credit spread and initial equity value.



(d) Relationship between credit spread and end equity value.

Figure 2.5: Model results for Geometric Brownian Motion and Jump-Diffusion Processes.

2.2 Probability of Default

The structural model regards default as an endogenous variable, it happens when asset value is lower than the bond obligations at maturity, so the probability that companies default at maturity can be defined in a straightforward way as:

$$
\begin{aligned}
\text{Probability of default} &= \mathcal{P}\left(A_T < D_T\right) \\
&= \mathcal{P}\left(\log A_0 + (r - \frac{\sigma_A^2}{2})T + \sigma_A W_T < \log D_T\right) \\
&= \mathcal{P}\left(W_T < \frac{-\log(\frac{A_0}{D_T}) - (r - \frac{\sigma_A^2}{2})T}{\sigma_A}\right) \\
&= \mathcal{P}\left(Z < \frac{-\log(\frac{A_0}{D_T}) - (r - \frac{\sigma_A^2}{2})T}{\sigma_A \sqrt{T}}\right) \\
&= \Phi\left(-\frac{\log(\frac{A_0}{D_T}) + (r - \frac{\sigma_A^2}{2})T}{\sigma_A \sqrt{T}}\right) \\
&= \Phi\left(-d_2\right) \qquad (2.2.1)
\end{aligned}
$$

where Z is a random variable that follows the standard normal distribution, $\Phi(\cdot)$ is the cumulative density function for the standard normal variable Z:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

σ_A refers to the volatility of underlying asset value, V_0 is the company's asset value at the initial time and can be computed using (2.1.6) at $t = 0$:

$$V_0 = A_0 = D_0 + E_0$$

where by (2.1.5):

$$E_0 = N\left(d_1\right) A_0 - N\left(d_2\right) X e^{-r(T)} \qquad (2.2.2)$$

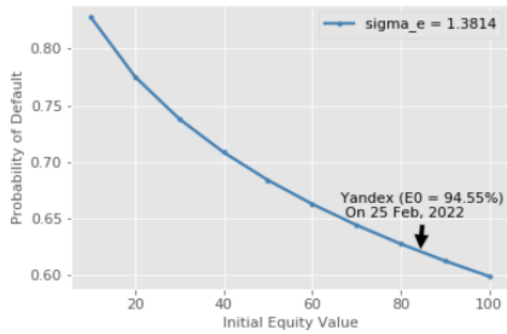
From Itô's lemma,

$$
\begin{aligned}
\sigma_E E_0 &= \frac{\delta E}{\delta V} \sigma_A V_0 \\
&= N(d_1) \sigma_A V_0 \; [9] \qquad (2.2.3)
\end{aligned}
$$

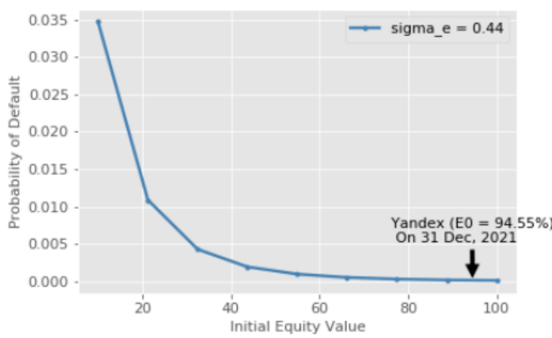
For a company with publicly traded shares, σ_E can be estimated from the standard deviation of the time series of share returns and E_0 can be computed as market capitalization at $t = 0$, that is the number of shares outstanding times the stock price at $t = 0$. σ_A and V_0 are the only two unknown variables on equations (2.2.2) and (2.2.3), so they can be solved by combining two equations with input values of E_0 and σ_E. Further, the probability of default can be computed using (2.2.1).

Let's look at a sample company Yandex, often referred to as "Russia's Google", the largest technology company in Russia with a market share of over 42%, mainly providing Internet-related products and services, such as a search engine. The Russia-Ukraine war and the Western sanctions have led to Yandex's share price plummeting by about 75% from November of 2021, and their American Depositary Receipts (ADRs) being halted from trading on 28^{th} February 2022 because of volatility. [10]

Yandex has 323.2 million shares outstanding and has issued convertible bonds with $1.125 billion in face value, which mature on 3^{rd} March 2025. On 25^{th} February 2022[1] the last business day before the trading of its ADRs were halted on the Nasdaq exchange, its share price decreased to \$18.94 per share, with historical volatility σ_E rising to 1.38 if measured using a 30-day rolling window. $E_0 = 84.48\%$ of A_0, assuming an environment using the 10-year Treasury rate at 2% as a proxy for the risk-free rate. Combining equations (2.2.2) and (2.2.3), the probability of default on this day was approximately 59.90%. Figure 2.6(b) shows the predicted probability of default for companies with the same volatility and interest rate as that of Yandex on 25^{th} February 2022, with initial equity value E_0 spanning over 10% to 100% of initial asset value. The black arrow annotates where Yandex would be on the chart with an equity funding ratio of 84.48%.



(a) Probability of default on 25^{th} February 2022, after the war.



(b) Probability of default on 31^{st} December 2021, before the war.

Figure 2.6: Relationship between the initial equity value and probability of default.

To see the impact of the war on the probability of default, if we look into days before the war, for instance on 31^{st} December 2021[2], where stocks were traded at \$60.50 per share, with σ_E at 0.44 if using a 30-day rolling window and $E_0 = 94.55\%$, measured in percentage of A_0, similarly probability of default is solved as 0.24%. Figure 2.6(a) shows the result for the probability of default on this day and the result for Yandex is annotated with an arrow.

Figure 2.7 shows the relationship between initial equity value E_0, volatility of share returns σ_E and probability of default for a general company with arbitrary E_0 and σ_E value. Yandex is annotated with its equity financing ratio and probability of default on the two days we discussed previously. The chart re-affirms that the probability of default increases with the volatility of share returns σ_E, and the default is less likely to happen if the ratio of equity to assets is high.

[1]Data source: Bloomberg Finance L.P.
[2]Data source: Bloomberg Finance L.P.

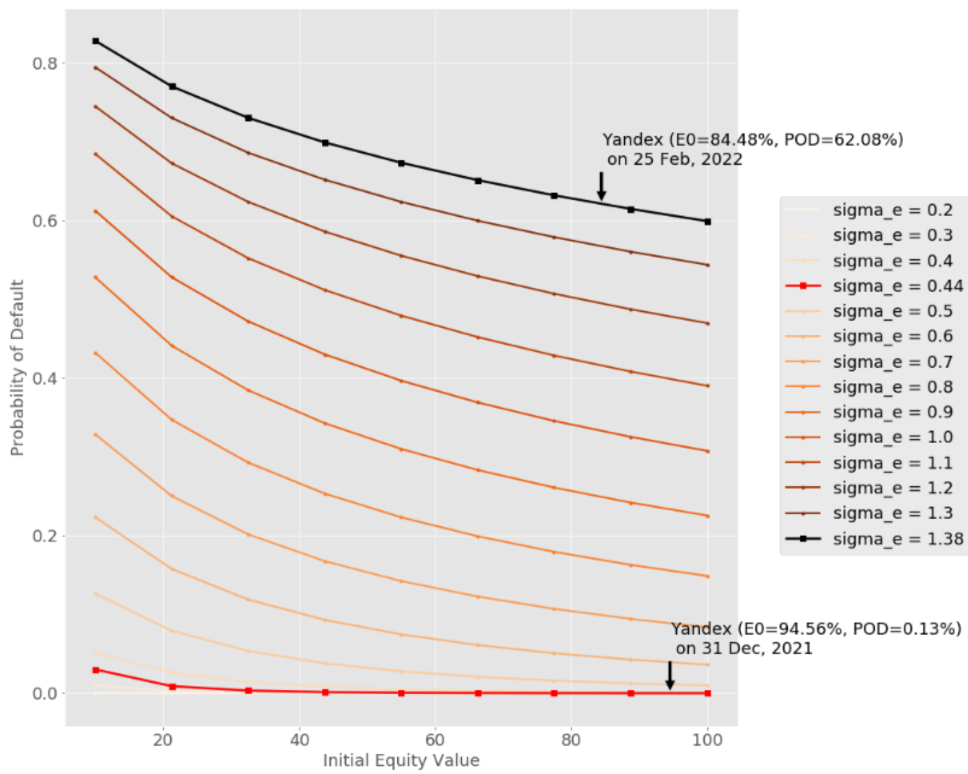


Figure 2.7: Relationship between E_0, σ_e and probability of default.

We have seen how a structural model captures the probability of default as a parameter endogenous to the model. This model highlights the intrinsic link between the value of a company's equity and debt. It is this link, which we can see empirically in the correlation between stock prices and credit spreads, that we go on to further explore in this thesis as a means to hedge credit risk in convertible bonds.

Chapter 3

Valuation Models For Convertible Bonds

Convertible bonds are corporate bonds with an additional embedded equity option, which enable them to be converted into shares at a predetermined number. In this thesis, we assume that investors are rational when making investment decisions, so they always act in a way that maximises their own utility. Because of this embedded option of conversion, the bondholders are entitled to convert the bond into equity at their discretion. More precisely, the conversion happens only when the stock price is greater than the conversion price under the assumption of rationality, which makes the final payoff equal to the greater of conversion value and face value.

$$V_T = \max(\text{FV}, C_r S_T)$$

The payoff is similar to that of a Call option. Because convertible bonds can be converted at the holders' discretion at any time before or at maturity, this makes the valuation of convertible bonds similar to that of American options. Numerical methods are widely used when solving option valuation problems, especially the three well-known classical parametric models: the tree models, the finite difference method, and the Monte Carlo Simulation method. To be specific, the tree models are intuitive methods to value both American and European options. Finite difference methods find the option's price by solving some differential equation that the derivative satisfies. The method can be divided into the explicit scheme, implicit scheme and Crank Nicolson scheme. Compared with the others, Monte Carlo simulation is more flexible. The aim is to determine the option value by taking the average of a sufficiently high number of underlying prices.

In this Chapter, we first focus on pricing convertible bonds using a structure model presented by Goldman Sachs (1994) and solving it numerically using a binomial tree model with the Cox-Ross-Rubinstein parametrization method. Then we move to a reduced-form model proposed by Ayache et al (2003), applying the finite difference method to solve partial differential equations, and comparing the result of varying default assumptions on our models.

3.1 The Structural Model

In the structural model, the probability of default is treated as an endogenous variable, it occurs when the asset value of the underlying company lies below the barrier of total outstanding financial obligations. The structural model focus on the capital structure of a company, since equity and bonds can be viewed as claims on the underlying assets, they can be interpreted in terms of options in the structural model.

Because of the mixed behaviour of convertible bonds, the difficulty in the valuation process is how to treat the inseparable bond and equity components with the appropriate credit spread,[11] as credit spread is a key variable that has an impact on bond value and there is a considerable link between credit spread and underlying stock price.

Goldman Sachs (1994) assumed that convertible bonds would either end up like pure equity or pure corporate bonds and proposed a way to interpret the hybrid behaviour by introducing the probability of conversion. Since the pure bond and pure equity components are subject to different credit risks, as a hybrid security, the discounted rate applied for pricing a convertible at each node of the pricing tree should be between the risk-free rate and risky interest rate, specifically a risk-free rate plus the credit spread adjusted for the probability of conversion. We applied this strategy when using backpropagation in a binomial tree model.

3.1.1 Binomial Tree Model

The Binomial tree model was first proposed by William Sharpe in 1978 and formalized by Cox, Ross and Rubinstein in 1979, with the assumption that the market is frictionless and arbitrage-free, and the underlying stock price follows Geometric Brownian Motion. In addition, we assume that interest rates and the volatility of stock returns are known constants, and any uncertainty brought by default is fully priced in the credit spread, which is a known constant as well, so that the movement of the future stock price is the only uncertainty we need to capture.

In the Binomial tree model with Cox-Ross-Rubinstein (CRR) parametrization method (1979), time grids over the interval $[0, T]$ are defined by

$$\{0, \Delta t, 2\Delta t, ..., (N-1)\Delta t, T\}, \qquad \Delta t = \frac{T}{N}$$

where N is the number of total time nodes in the time interval.
In a similar manner, space grids over space interval $[s_{min}, s_{max}]$ are defined by

$$\{s_{min}, s_{min} + \Delta s, s_{min} + 2\Delta s, ..., s_{min} + 2(N-1)\Delta s, s_{max}\}, \qquad \Delta s = \frac{s_{max} - s_{min}}{2N-1}$$

S_k^n is the stock price at time n and state k, and refers to the k^{th} possible stock price level at time n, and

$$S_k^{n+1} = \begin{cases} S_k^n & \text{with probability } q_u \\ S_{k+1}^n & \text{with probability } q_d \end{cases}$$

where u and d are the up and down factors of stock price movement, with CRR parametrization method, are defined as:

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}$$

The condition $ud = 1$ makes our binomial tree recombine, which reduces the complexity of our numerical scheme as this condition makes the number of nodes increase linearly as the number of time steps increases rather than exponentially. The probability pair (q_u, q_d) measures the risk-neutral probability of moving upward and downward, where

$$q_u = \frac{\mathrm{e}^{r\Delta t} - d}{u - d}, \quad q_d = 1 - q_u$$

Therefore, the value of the underlying share price dynamics at time n and state k are given by

$$S_k^n = S_0 u^{n-k} d^k = S_0 u^{n-2k}, \qquad n = 0, 1, ..., N, \quad k = 0, 1, ..., n$$

For simplicity, we assume that the underlying stock pays no dividends and the convertible bond is a zero coupon bond without any call or put provisions. The value of convertible bond when the stock price is S_k^n is defined as V_k^n.

The valuation process of a convertible bond is similar to that of an American option, which is valued by rolling back the values through the tree and comparing them with what the holder would get upon conversion.

First we define the payoff function $g(\cdot)$ for a convertible bond, under the assumption of no default risk, the conversion happens only if the conversion value is greater than the bond floor, which is the value of an equivalent straight bond:

$$g(S_k^n) := \begin{cases} \max(C_r S_k^N, FV) & n = N \\ \max(C_r S_k^n, B_k^n) & n < N \end{cases} \tag{3.1.1}$$

where B_k^n denotes the value of bond floor when the underlying stock price is S_k^n.

We start from maturity, compute the value of a convertible bond at terminal time $n = N$, which is the greater of its redemption value and its conversion value, simply the payoff function $g(\cdot)$, where

$$V_k^N = g\left(S_k^N\right), \quad \text{where} \quad k = 0, 1, ..., N \tag{3.1.2}$$

Then we loop backward in time for $n = N-1, N-2, ..., 0$. The conversion option is exercised if the conversion value is greater than that of the equivalent straight bond, so the conversion payoff at time n is:

$$V_k^{n,conv} = g\left(S_k^n\right)$$

If investors use their right to convert bonds into stocks, the convertible bonds are equivalent to stocks after conversion, and investors have no credit exposure, so the risk-free interest rate should be used as the discount rate. If investors do not exercise the option and thus cannot share the potential stock price appreciation, then holding a convertible bond is just like holding a straight bond, which is subject to credit risk. So in this case, it should be discounted at the risky rate, which equals the risk-free rate added with credit spread CS:

$$r_{risky} = r + CS$$

Therefore the risky rate should be used when computing the value of the straight bond in (3.1.1), where B_k^n at time n is computed as the average present value of the straight bond at two connected future nodes, discounted at the risky interest rate. [12]

$$B_k^n = \mathrm{e}^{-(r+CS)\Delta t}\left(\frac{1}{2}B_k^{n+1} + \frac{1}{2}B_{k+1}^{n+1}\right)$$

Meanwhile, we should also clarify conversion probability p. At maturity, p is defined to be one at nodes where conversion happens, and zero otherwise. Moving backwards in time, p takes the average probability of two connected future nodes, i.e. [13]

$$p_k^n = \frac{p_k^{n+1} + p_{k+1}^{n+1}}{2}$$

Since the conversion probability p at each node is between 0 and 1, and convertible bonds have hybrid features from both equity and straight bonds, it is reasonable to use a credit-adjusted discount rate y, which is interpolated from the risky and risk-free interest rates with respect to the likelihood that conversion happens. [13]

$$y = rp + (r + c_s)(1-p)$$

Therefore, if the conversion option is not exercised, the value of a convertible bond at time n is the expected value of two connected future nodes with risk-neutral probability, discounted by one-time step by $e^{-y\Delta t}$, where y is the credit-adjusted discounted rate.

$$V_k^{n,hold} = \mathrm{e}^{-y\Delta t}\left(q_u V_k^{n+1} + q_d V_{k+1}^{n+1}\right)$$

Hence, the value of a convertible bond when stock price $s = S_k^n$ is the maximum of conversion payoff and holding value:

$$V_k^n = \max\left(V_k^{n,conv}, V_k^{n,hold}\right) \quad \text{where} \quad \begin{array}{l} n = 0, 1, ..., N-1 \\ k = 0, 1, ..., n \end{array} \tag{3.1.3}$$

where C_r is the conversion ratio defined at issuance of the convertible bonds. Then we use this backward induction algorithm to work back along the binomial tree at all nodes until we reach time 0 and get the required time-zero convertible bond value V_0^0.

The algorithm for pricing the convertible bond using the binomial tree models is given below:

Algorithm: Valuation of Convertible Bond with Binomial Tree Model

1. Set the initial parameters:
   - S_0: Initial underlying stock price
   - T: Maturity time in years
   - r: Annual risk-free rate
   - C_r: Conversion ratio of convertible bond
   - FV: Face value of the convertible bond
   - CS: Credit spread
   - N: Number of time steps
2. Calculate the up and down factors u and d
3. Calculate the risk-neutral probabilities q_u and q_d
4. Work out the underlying stock price at the maturity time
5. Calculate the convertible bond payoff and conversion probability at maturity time T by (3.1.2)
6. Using for loops to work backwards from node N to node 0, calculate convertible bond price and conversion probability at each node by (3.1.3)
7. Repeat until we reach $n = 0$, we return the initial value V_0^0

Numerical Result

Figure 3.1 Shows the relationship between convertible bond value and initial stock price when credit spread is fixed at $CS = 3\%$. Here we assume volatility on share returns is constant at σ_E = 20%, risk-free rate $r = 2\%$, the convertible bond has a conversion ratio $C_r = 1$ and redemption value at maturity equal to bond principal $FV = 100$. With a fixed credit spread, the convertible bond price increases with S_0. When $S_0 \leq 60$, convertible bond is regarded as "busted", where the equity option value is so low that the convertible is essentially a straight bond. Whereas when S_0 is high enough, the slope of the curve is approximately one, which implies a high correlation between convertible bond and equity, it means the conversion is quite likely to happen so that the convertible bond is almost equity-like.

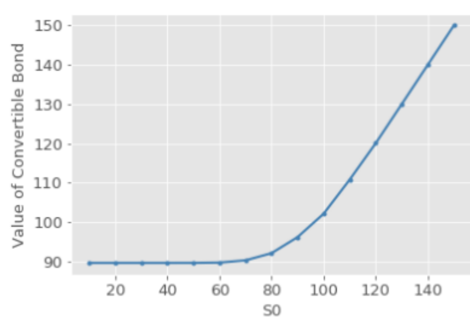


Figure 3.1: Value of convertible bond derived by binomial tree model.

To see the impact of the credit spread as well, we set CS as a variable with value spanning over $[1\%, 2\%, ..., 9\%]$, result is shown in figure 3.2. When S_0 is very low, holders will hold the "busted" convertible bonds so they are equivalent to straight bonds, the wider the credit spread, the lower the bond price, so the curve shifts downward. However, because we assume that there is no credit risk in equity, the payoff of convertible bonds is almost the same as that of stock at a fairly high share price, and the impact of credit spread on its price is negligible. Therefore, we can observe

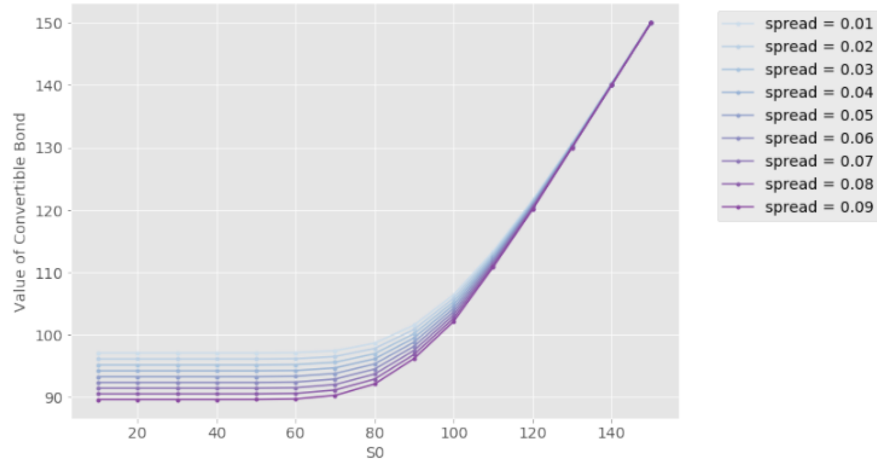


Figure 3.2: Relationship between initial stock price, credit spread and value of the convertible bond.

from the chart that as the stock price rises, the distance between the curves gradually becomes smaller, this is because the price of the convertible bond is less affected by credit risk.

We use real-world data to calibrate the model. The 2025 Convertible bond issued by Yandex has conversion ratio $C_r = 3{,}329.17$ and face value $FV = 200{,}000$. To calibrate the model we use data from this company, on 24^{th} February 2022[1]: stock price $S_0 = \$18.94$ with volatility $\sigma_E = 50\%$, the bond will mature on 3^{rd} March 2025 so time to maturity $T = 3.02$ when expressed in years and credit spread $CS = 20\%$. We assume a 2% risk-free rate r. Using this model, Yandex's convertible bond price is 55.35% of its face value, which is slightly lower than 58.635%, the last price of the day. This is because we assume the simplest convertible bond structure that has no coupon payment or provisions. In fact, the coupon payment of Yandex's convertible bonds is made semi-annually at a coupon rate of 0.75% and it has an embedded call provision which enables early redemption by issuers, both of these features impact the value of convertible bonds.

3.2 The Reduced-Form Model

In the reduced-form model, default is not longer an endogenous variable defined in terms of a company's balance sheet structure, but is modelled by some exogenous hazard rate process.
Ayache et al (2013) argued that the extreme assumptions that stock price will collapse to zero immediately or be totally unaffected upon default are unrealistic, and the splitting of convertible bonds into two distinct components is questionable. Instead, the market must gradually react to defaults, which should be reflected in the falling stock price, with losses on equity price at some value between 0% to 100%. Moreover, they proposed to model credit risk by a stochastic model assuming an explicit correlation between credit and equity, and proposed to model the stock price dynamics by a process:

$$S^+ = S^-(1 - \eta)$$

where S^+ denotes stock price after default, S^- denotes stock price before default, and η is a number between 0 and 1, representing the percentage that the company's stock price drops by when default happens. [14] For simplicity, we assume a convertible bond with no provisions and underlying stock paying no dividends.

[1]Data source: Bloomberg Finance L.P.

3.2.1 The No-Default Model

For convertible bonds with no credit risk, the valuation process is quite similar to solving the PDE for American options. Here we assume the interest rate r is known. Under the assumption that the underlying stock price follows Geometric Brownian Motion with infinitesimal expected return rdt and variance σ^2, its dynamics are described by:

$$\frac{dS}{S} = rdt + \sigma dW_t \tag{3.2.1}$$

where W_t is a stochastic variable following the standard Brownian motion process. Applying Ito's lemma on the value of a Call option $V(S,t)$, we have:

$$dV = \left(rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW \tag{3.2.2}$$

If we construct a delta-hedge portfolio for a short position of one Call option V, then we need to simultaneously buy $\frac{\partial V}{\partial S}$ number of shares to keep this portfolio delta neutral. [9] Payoff of the hedged portfolio over infinitesimal time interval $[t, t + \Delta t]$ is:

$$\Delta \Pi = -\Delta V + \frac{\partial V}{\partial S} \Delta S$$

Plug in discrete versions of equation (3.2.1)and (3.2.2), we have:

$$\Delta \Pi = \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) \Delta t \tag{3.2.3}$$

Notice that the stochastic variable W_t is offset, so there is no uncertainty in the payoff of the portfolio. By the no-arbitrage principle, the return of the hedged portfolio should equal the risk-free interest rate, thus:

$$r\Pi\Delta t = \Delta\Pi \tag{3.2.4}$$

combining (3.2.3) and (3.2.4), we obtain the well-known Black Scholes partial differential equation upon simplification:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Define $\mathcal{L}V$ by

$$\mathcal{L}V := -\frac{\partial V}{\partial t} - \left(\frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \right)$$

Suppose a convertible bond has three provisions under its indenture: a convertible provision entitles the holder to convert it into common shares at a predetermined number C_r, a call provision allows the issuer to call back the bond at B_c and pay the redemption value to holders earlier than scheduled maturity, and a put provision enables the holder to sell the bond to the issuer prior to maturity at B_p. Intuitively, to maximise one's payoff, the holder will either sell back the bond and get prepayment when the current interest rate rises above the coupon rate on the bond or convert the bond into common shares when the stock price increases. On the contrary, the issuer will either make repayment at B_c if they call back the bond when there is a low interest rate in the financial markets or repay the holder at maturity if the conversion provision expires. Hence, the price of a convertible is unbounded, bounded by put constraint, or bounded by call constraint.

Therefore, the valuation of a convertible bond is the solution to the following constrained problem:

$$\mathcal{L}V = 0$$

subject to the constraints:

$$V \geq \max(B_p, C_r S)$$
$$V \leq \max(B_c, C_r S)$$

with terminal constraint:

$$V(S, t = T) = \max(F, C_r S)$$

In detail, the linear complementarity problem can be divided into three cases:

- $B_c > C_r S$:

$$\begin{pmatrix} \mathcal{L}V = 0 \\ (V - \max(B_p, C_r S)) \geq 0 \\ (V - B_c) \leq 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{L}V \geq 0 \\ (V - \max(B_p, C_r S)) = 0 \\ (V - B_c) \leq 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{L}V \leq 0 \\ (V - \max(B_p, C_r S)) \geq 0 \\ (V - B_c) = 0 \end{pmatrix}$$

- $B_c \leq C_r S$:

$$V = C_r S$$

3.2.2 The Default Model

The Partial Default Model

The partial default model can be regarded as an extension of the TF model suggested by Tsiveriotis and Fernandes (1998). Recall that convertible bonds have both equity and bond characteristics, Tsiveriotis and Fernandes (1998) suggest that the valuation of convertible bonds can be split into two parts with different credit risks, the equity part, which has zero default risk and the fixed income part, which is subject to issuers' credit risks. [15] The value of the convertible bond V_t is equal to the sum of the two components:

$$V_t = E_t + B_t$$

where E_t refers to the equity part and B_t refers to the fixed income part. Therefore the value of the convertible bond is the solution of a pair of coupled partial differential equations.

Ayache et al (2003) [14] further incorporated default risk into the valuation of convertible bonds by assuming a recovery rate R on the value of the bond, where $0 \leq R \leq 1$, so redemption value on bond upon default is:

$$B = RX$$

where B is the immediate post-default bond value, and X is the bond value prior to default, for coupon bond $X = FV$. What's more, Ayache et al (2003) assumed default risk is diversifiable and incorporated default risk into the valuation of convertible bonds by assuming the probability of default on an infinitesimal time interval $[t, t + \Delta t)$ conditional no default in [0, t) is given by

$$\mathcal{P}\left(\tau \in [t, t + dt) \mid \tau \notin [0, t)\right) = pdt$$

where $p = p(S, t)$ is a deterministic hazard rate on interval $[t, t + dt)$ Similarly, if we construct a delta hedging portfolio with a long position on one convertible bond V and a short position on β shares, payoff of the hedged portfolio is:

$$\Pi = V - \beta S$$

Applying Ito's lemma with hedge ratio $\beta = B_S$ to eliminate the stock risk from the portfolio, we have:

$$d\Pi = \left[\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}\right] dt + o(dt) \tag{3.2.5}$$

where $o(dt)$ indicates the terms that go to zero faster than dt. Consider the loss on bond upon default, equation (3.2.5) becomes:

$$d\Pi = \left[\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}\right] dt - pdt(V - RX) + o(dt) \tag{3.2.6}$$

The assumption on diversifiable default risk gives:

$$E(d\Pi) = r\Pi dt \tag{3.2.7}$$

Combining equations (3.2.6) and (3.2.7), we have:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - (r + p)V + pRX = 0$$

Define

$$\mathcal{L}V := -\frac{\partial V}{\partial t} - rS\frac{\partial V}{\partial S} - \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r + p)V - pRX$$

The Total Default Model

The Total default model assumes that the stock price collapses to zero immediately upon default, similar to the derivation of the partial default model, we have:

$$d\Pi = \left[\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}\right] dt - pdt(V - RX - \beta S) + o(dt)$$

and by setting hedge ratio $\beta = B_S$ we have:

$$\frac{\partial V}{\partial t} + (r+p)S\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - (r+p)V + pRX = 0 \quad (3.2.8)$$

and the price of the convertible bond is

$$\begin{aligned} V &= FV \exp\left[-\int_t^T (r(u) + p(u)(1-R))du\right] \\ &= FV \exp\left[-\int_t^T (r(u) + CS(u))du\right] \end{aligned}$$

where credit spread $CS = p(1-R)$ can be thought as an analogy of expected loss (EL) expressed in terms of probability of default (POD) and loss given default (LGD) as

$$EL = POD \cdot LGD$$

It is obvious that equations (3.2.8) and (3.2.2) will have different solutions, so the change in assumption on underlying stock price upon default will also change the valuation of convertible bonds.

3.2.3 The Hedge Model

The Hedge Model can be regarded as a general application of the partial and total default model, where Ayache et al (2013) [14] assumed the stock price upon default as a gradual collapse process with coefficient η, where $0 \le \eta \le 1$ and

$$S^+ = S^-(1-\eta)$$

η indicates the percentage of loss on stock if default happens, the partial default model is the case when $\eta = 0$, where the stock price is unaffected by default, and the total default model corresponds to the case when $\eta = 1$, where stock price immediately drops to zero. Investors can choose to convert even upon default with conversion value CV as:

$$CV = C_r S^+ = C_r S^-(1-\eta)$$

By applying a similar approach of constructing a hedged portfolio and assuming diversifiable default risk, we have:

$$d\Pi = \left[V_t + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}\right] dt - pdt\,(V - V_S \eta S) + pdt \max(C_r S(1-\eta), RX)$$

and

$$V_t + (r + p\eta)SV_S + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - (r+p)V + p\max(C_r S(1-\eta), RX) = 0$$

Defining:

$$\mathcal{M}V \equiv -V_t - \left(\frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + (r+p\eta)SV_S - (r+p)V\right)$$

The linear complementarity problem for the hedge model is similar to that of a risk-free bond, with $\mathcal{L}V$ in equation (3.2.1) changed to

$$\mathcal{M}V - p\max(C_r S(1-\eta), RX)$$

3.2.4 Finite Difference Method

The Finite Difference Method provides an approach to solve a PDE numerically by approximating the derivatives in a discrete manner. There are three specific finite difference methods: the explicit scheme, the fully implicit scheme and the θ scheme. The explicit scheme approximates the first time-derivative by the forward difference,

$$\frac{\partial V}{\partial \tau}(S_k, \tau_n) \approx \frac{V_k^{n+1} - V_k^n}{\Delta \tau}$$

whereas the fully implicit scheme applies backward difference,

$$\frac{\partial V}{\partial \tau}(S_k, \tau_n) \approx \frac{V_k^{n} - V_k^{n-1}}{\Delta \tau}$$

Both of them approximate the first and second-order spatial derivatives by central difference.

$$\frac{\partial V}{\partial S}(S_k, \tau_n) = \frac{V_{k+1}^n - V_{k-1}^n}{2\triangle x}$$

$$\frac{\partial^2 V}{\partial S^2}(S_k, \tau_n) = \frac{V_{k+1}^n - 2V_k^n + V_{k-1}^n}{\Delta x^2}$$

The θ scheme can be regarded as a weighted average of the explicit scheme and the fully implicit scheme with weights 1 - θ and θ respectively. When $\theta = 0.5$, we have the Crank-Nicolson scheme.

All of the three schemes are convergent, but the explicit scheme is conditionally stable, which put some constraints on how we can define time and spatial grids point, as the stability requires $\Delta t \leq \alpha \Delta S^2$ for some constant α. [16] The implicit scheme is unconditionally stable but requires solving variational inequality by matrix inversion, which is difficult to implement. The stability of the Crank-Nicolson scheme is also guaranteed but involves a more complicated calculation process. Moreover, it is generally used to handle the non-smooth initial conditions at the coupon reset date in actual practice.

In this section, we provide the discretization of differential equations for convertible bonds under three cases discussed in the previous section, solving them by a fully implicit scheme. [11]

Fully Implicit Scheme

First, we define $\tau = T - t$ to change the PDE into an initial value problem, then the Black Scholes PDE becomes:

$$\frac{\partial V}{\partial \tau}(S,\tau) - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(S,\tau) - rS\frac{\partial V}{\partial S}(S,\tau) + rV(S,\tau) = 0 \tag{3.2.9}$$

Similar to the algorithm of the binomial tree method, we define a set of grid points in the time domain and space domain: $[0,T] \times [s_{min}, s_{max}]$ by:

$$\Delta t := \frac{T}{N}, \quad \triangle x := \frac{S_{\max} - S_{\min}}{M}.$$

with values:

$$\begin{array}{lll} t_n & := n\triangle t, & n = 0, 1, \ldots, N \\ S_k & := S_{\min} + k\triangle x, & k = 0, 1, \ldots, M \end{array}$$

Then we apply the implicit scheme, approximate the first time derivative to V by backward difference and approximate the first and second spatial derivative to V by central difference, the finite difference approximation of equation (3.2.9) becomes: [17, Chapter 2]

$$\frac{V_k^n - V_k^{n-1}}{\Delta \tau} - \frac{\sigma^2 (S_k^n)^2}{2} \frac{V_{k+1}^n - 2V_k^n + V_{k-1}^n}{\Delta x^2} - rS_k^n \frac{V_{k+1}^n - V_{k-1}^n}{2\Delta x} + rV_k^n = 0$$

By rearranging terms, it becomes:

$$\begin{aligned} V_k^{n+1} &= \left(\frac{\triangle\tau}{\triangle x^2}\frac{\sigma^2(S_k^n)^2}{2} - \frac{\Delta\tau}{2\triangle x} r S_k^n\right) V_{k-1}^n + \left(1 - \frac{\Delta\tau}{\Delta x^2}\sigma^2(S_k^n)^2 - r\Delta\tau\right) V_k^n \\ &\quad - \left(\frac{\triangle\tau}{\Delta x^2}\frac{\sigma^2(S_k^n)^2}{2} + \frac{\Delta\tau}{2\triangle\tau} r S_k^n\right) V_{k+1}^n \\ &= -A_k^n V_{k-1}^n + (1 - B_k^n) V_k^n - C_k^n V_{k+1}^n \end{aligned} \tag{3.2.10}$$

where

$$A_k^n := \sigma^2(S_k^n)^2\frac{\Delta t}{2\triangle x^2} - rS_k^n\frac{\Delta t}{2\triangle x}, \quad B_k^n := -\sigma^2(S_k^n)^2\frac{\Delta t}{\Delta x^2} - r\Delta t, \quad C_k^n := \sigma^2(S_k^n)^2\frac{\Delta t}{2\triangle x^2} + rS_k^n\frac{\Delta t}{2\triangle x}$$

The recursive relationship (3.2.10) holds for k = 1, 2,..., M-1, if we define value of V_0^n and V_M^n by boundary conditions (3.2.11), then the iterative matrix is:

$$[\mathbb{I} - L^n] V^n - B^n V^{n-1} = 0$$

where

$$L^n := \begin{bmatrix} 0 & 0 & 0 & \cdots & & & & & 0 & 0 \\ A_1^n & B_1^n & C_1^n & 0 & \cdots & & & & 0 & 0 \\ 0 & A_2^n & B_2^n & C_2^n & 0 & \cdots & & & & \\ & & & & \ddots & & & & & \\ \vdots & & & & & & A_{M-2}^n & B_{M-2}^n & C_{M-2}^n & 0 \\ & & & & & & 0 & A_{M-1}^n & B_{M-1}^n & C_{M-1}^n \\ 0 & 0 & \cdots & & & & 0 & 0 & 0 & 0 \end{bmatrix}$$

If we define payoff of the convertible bond at time T as:

$$g(S_T) = \max(C_r S_T, FV)$$

with boundary conditions:

$$V(s_{min}) = FV, \qquad V(s_{max}) = C_r s_{max} \tag{3.2.11}$$

then for any $\tau \in (0, T)$, the value of the convertible bond at time τ is:

$$V(\tau, S_\tau) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau - t)} g(S_\tau)\right] \tag{3.2.12}$$

where $\mathcal{T}_{t,T}$ is stopping times between t and T. Equation (3.2.12) can be written in a discrete way: [18]

$$V^n = \max[e^{-r\Delta t}\mathbb{E}\left(V^{n+1} \mid \mathcal{F}_n\right), g(S_n)]$$

which is equivalent to

$$\min\left(V^n - e^{-r\Delta t}\mathbb{E}\left(V^{n+1} \mid \mathcal{F}_n\right), V^n - g(S_n)\right) = 0$$

Written in matrix form, it becomes:

$$\min\left(([\mathbb{I} - L^n] V^n) - B^n V^{n-1}, V^n - g^n\right) = 0$$

Here V^n can be solved numerically by applying Jacobi method [19] or Gauss-Seidel method [20] on variational inequality. The PDE for the partial default model, the total default model and the hedge model can be solved numerically by discretizing PDE in a similar manner.

Parameters	Value
Risk-free interest rate r	2%
Volatility on underlying equity σ	20%
Time to maturity T	1
Face value of convertible bond FV	100
Conversion ratio C_r	1
Minimum stock price $s_{\min}$	0
Maximum stock price $s_{\max}$	150
Number of time grid points N	1000
Number of spatial grid points M	150
Probability of default p	30%
Recovery rate R	30%
Loss on equity upon default η	30%

Table 3.1: Model Parameters for the convertible bond.

Numerical Result

We then follow the fully implicit scheme to solve for the value of the convertible bond with the value of market and option parameters as shown in table 3.1. Figure 3.3 shows the relationship between bond value and underlying stock price, which is aligned with the result delivered by the binomial tree model, a higher stock price implies a more valuable convertible bond while a lower stock price might make the convertible bond busted.

Comparison between the risk-free bond and defaultable bond with equity under the total default model and the partial default model is shown in Figure 3.4. We can see from the chart that the red line (no default model) is higher than the other two lines when stock price is lower than 100, which makes sense as bonds subject to no credit risk normally are more valuable. The purple line (partial default model) is higher than the blue line (total default model) because both equity and bond are likely to default under the total default model, whereas only bonds are subject to credit risk in the partial default model, therefore in the former case, investors have less credit risk exposure, which effectively makes the bonds more valuable. When the stock price is very high, the conversion probability is very high so that convertible bonds can be regarded as stock. With greater credit spread, defaultable bonds have a lower price than risk-free bonds.

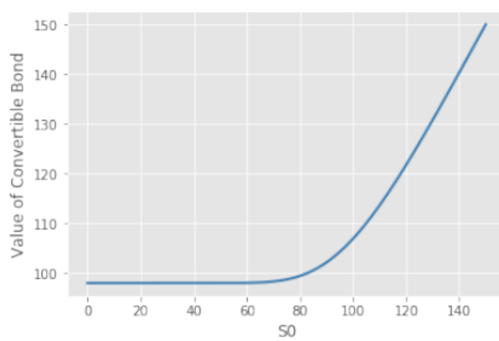


Figure 3.3: Value of convertible bond with no credit risk.

The loss in stock price immediately after default can be thought of as a result of the correlation between equity and credit risk. To see the influence of this correlation coefficient on the convertible bond price, we solved the PDE for the total default model under a fully implicit scheme with respect to different η levels, the result is shown in Figure 3.5(a) and Figure 3.5(b). For η value less than 0.70, higher eta (loss on stock price upon default) corresponds to a lower convertible bond value, whereas for η greater than 0.70, the pattern has reversed. If we look at this threshold, when η=0.7,

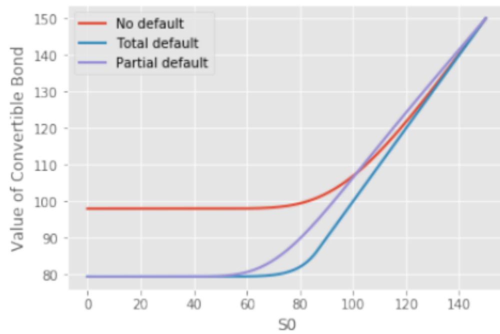
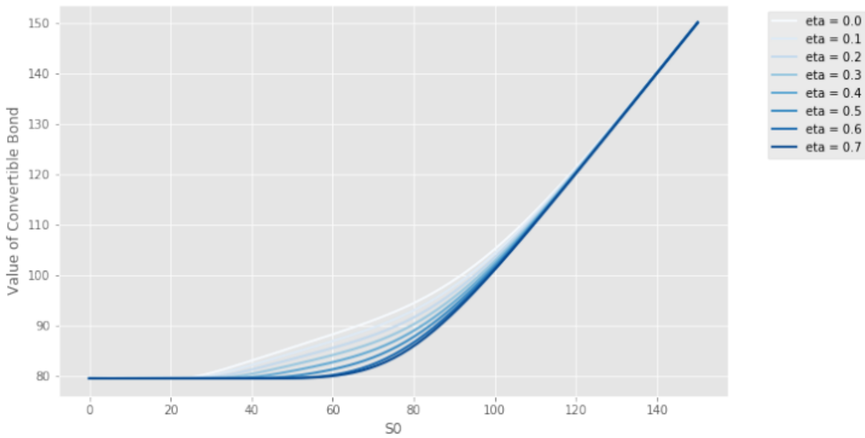
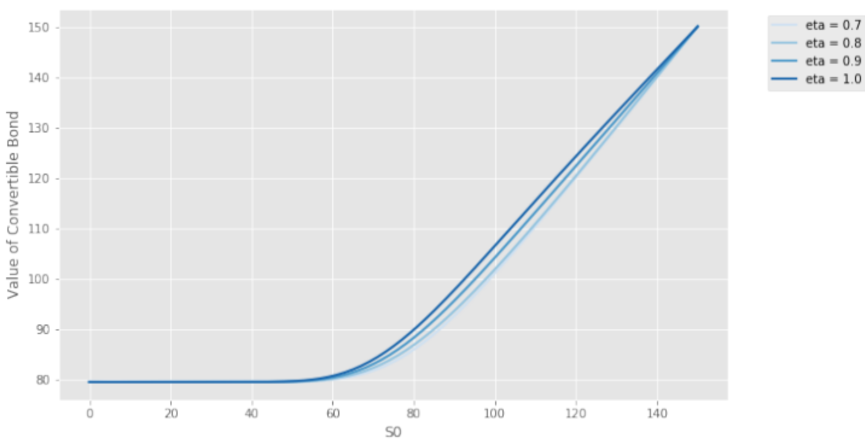


Figure 3.4: Value of convertible bond under different default assumptions.

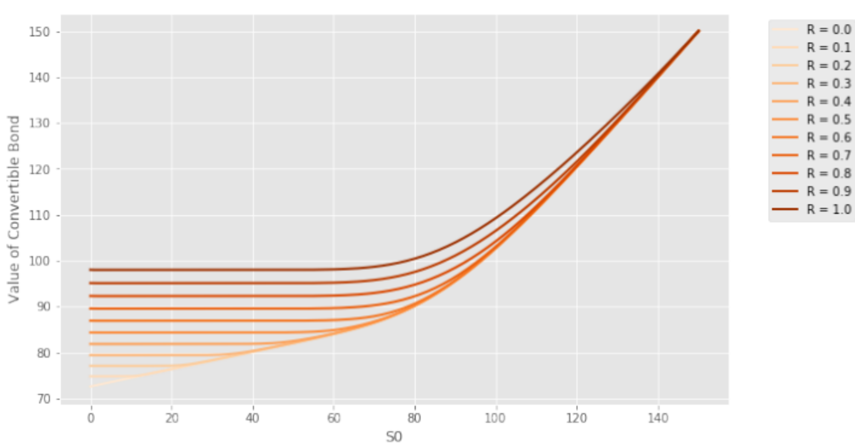
if default happens, convertible bondholders have two options, either continue holding the bond and receive recovery amount $R \cdot FV = 30$, or convert the bonds into common equities with conversion value $CV = C_r S^-(1-\eta) = 30$, then bondholders are indifferent between two options. For $\eta < 0.7$, CV becomes the maximum, so it is optimal to convert the bonds, which implies that the conversion option embedded in bonds is more valuable, thus increasing the value of the convertible bond. On the contrary, when $\eta > 0.7$, the stock price drops too far upon default, so the recovery value of the bond is able to cover more loss. Figure 3.5(c) illustrates the impact of recovery rate R on the convertible bond value. The bond value increases with R because a higher recovery rate indicates lower loss given default.



(a) Value of convertible bond for η less than 0.7.



(b) Value of convertible bond for η greater than 0.7.



(c) Hedge model with different recovery rate.

Figure 3.5: Value of convertible bond under hedge model.

Chapter 4

Evaluation of Methods of Hedging Credit Exposure

In the introduction, we noted that credit is one of the key factors driving the valuation of convertibles, and that in 2022, with the severe equity market pullback following two years of high convertible bond issuance, much of the convertible universe has increased in credit sensitivity. Furthermore, as global interest rates have started to rise in 2022, credit spreads have followed suit. This can be seen broadly in the moves higher in the Markit CDX North America High Yield Index, shown in Figure 4.1.

Source: Bloomberg L.P.(2022)
Figure 4.1: Markit CDX North America High Yield Index.

Under these conditions, it may be increasingly valuable for convertible investors to consider ways of hedging credit exposure in convertible bonds. Below is a discussion of some of the methods investors could use to do this.

4.1 Direct Hedges

4.1.1 Credit Default Swap

A Credit Default Swap (CDS) is a contract between two parties that enables the transfer of credit risk arising from holding the underlying asset, known as the reference entity. Protection buyers make periodical payments to protection sellers until the CDS expires or defaults, whichever happens first. In the event of default, the protection seller is obliged to buy the reference bond at its face value. This means that the transfer of value to the protection buyer will be equal to the difference between the face value of the reference obligation and the recovery rate if a default occurs. An investor holding a long position in the reference obligation combined with a CDS will receive the full principal amount of the bond without any losses whether default happens or not, apart from in the exceptional circumstance of a ‘double default’, i.e. the situation that both the reference entity and the CDS counterparty simultaneously default. In this thesis, we ignore the possibility of double default, as the likelihood is extremely small.

Because of this explicit link between bonds and CDS, CDS spreads and credit spreads – defined as the difference in yield between a given bond and the risk-free asset of the same maturity – with the same reference entity should be very close. Indeed this is the case, as shown by Houweling and Vorst (2001) and Hull et al (2003), who both showed that price discrepancies between bond spreads and CDS premia are within around 10 bps.[21] Therefore, investors can buy CDS protection to hedge the change in bond value caused by the change in the credit risk of the company. For this reason, CDS contracts are widely used to tailor credit exposure or hedge against credit risk. [22]

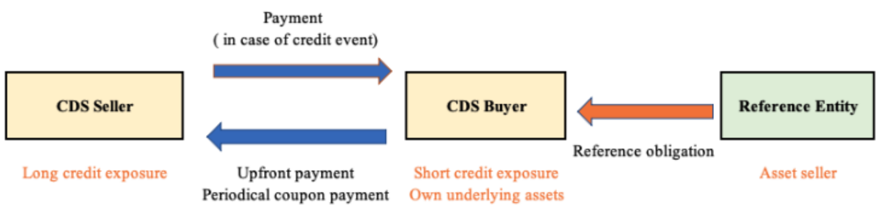


Figure 4.2: Mechanism of CDS.

There are however downsides to trading CDS to hedge credit risk. CDS contracts are not standardised and trade Over The Counter (OTC), so inevitably, there are liquidity issues and counterparty risks. Typically, investors trade CDS with maturities less than 10 years, with 5-year CDS being the most liquid tenor, which can lead to problems such as maturity mismatches. Also, it is often the case that convertible bonds are not listed as reference obligations for CDSS, which introduces basis risk. These risks reduce the hedging effectiveness. [23]

4.1.2 Asset Swapped Convertible Option Transaction

Asset Swapped Convertible Option Transactions (ASCOTs) are effectively an option on convertible bonds. They are used to split a convertible bond into two components: a corporate bond and an option to convert the convertible bond into shares of the issuing companies. An ASCOT consists of an American call option with a floating strike price and an asset swap. Typically, three participants are involved in this structured strategy, an ASCOT buyer, an ASCOT seller and a financial intermediary.

Investors who want to hedge credit risk and retain exposure to equity volatility may consider buying an ASCOT. They would sell the convertible bond to an intermediary and buy an American call option on the convert, which allows them to buy back the bond when they wish to convert it into equity. Investors who want to retain credit exposure to enhance yield will become ASCOT sellers. They will buy the bond and enter an asset swap option as the fixed leg to exchange payment with some financial intermediaries, where they make the fixed coupon payment and receive the floating payment at a floating rate equal to the addition of the relevant interest rate and the Asset Swap spread (ASW). ASW is the spread that makes the initial value of an asset swap zero, where the initial value is computed by adding up all discounted future cash flows, hence the ASW depends on credit risk. This is how the fixed leg maintains credit exposure. The third entity involved is typically a financial intermediary, these are the counterparties between ASCOT buyers and sellers. They write an American call option on the bond and buy the bond from ASCOT buyers at a discount, then enter an asset swap as the floating leg, paying the floating rate and receiving the fixed coupon rate on the convertible bond. [4]

ASCOTs provide investors with the option to hold convertible bonds without credit exposure, it is popular for people who think convertible bonds are undervalued and expect the underlying value to rise. Because of the structure, ASCOTs are sensitive to both the volatility of underlying shares

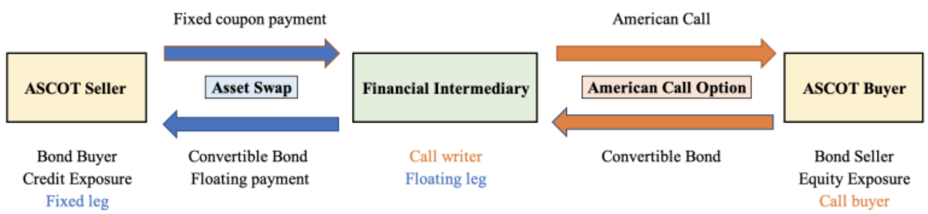


Figure 4.3: Mechanism of ASCOT.

and to credit risk, which makes them a perfect tool to hedge the credit risk of convertible bonds. ASCOTs offer a credit hedge when credit sellers are unable to trade with CDS due to counterparty risk concerns or when there is no CDS market available.

However, illiquidity is a problem for all ASCOT products, as asset swaps are non-standardized products traded OTC and there is heterogeneity in the transaction terms and conditions of financial intermediaries. So it might be costly and time-consuming to match the counterparties in terms of their willingness and capability, and to unwind the positions. These reasons contribute to why ASCOTs exist on only a small proportion of the convertible bond market. Estimates around 3.5% of convertibles globally have tradeable ASCTO markets.

4.1.3 Non-Convertible Bonds

Another hedging instrument that can be used are corporate bonds, without embedded equity options, namely "straight bonds". Straight bonds are bonds that pay regular coupon payments and the principal payment at maturity, without any equity option embedded. For convertible bonds where we want to eliminate credit risk, we can use straight bonds issued by the same underlying company with the same seniority in the claim. The implied spread calculated given the market price of the straight bond can be used as an estimate for the credit spread of the company. We simply enter an opposite position in credit exposure by trading straight bonds to offset the credit risk of the hedged portfolio. To hedge the credit risk of a long position in convertible bonds, investors need to short sell the related straight bonds. Compared to credit derivatives, straight bonds have better liquidity, which makes the transaction easier.

Figure 4.4 shows the distribution of average borrow rate over bonds. The borrow rate on bonds is quite high compared to the expected return from a convertible arbitrage position, with an average value of around 1% [1] for cross-region corporate bonds in a universe of 4725 corporate bonds from 23 regions. We can see from Figure 4.5 below that, for convertible bonds universe with valuation discounts ranging from around 4% cheap to around 3% expensive over the last five years, paying a nearly 1% borrow rate to short sell straight bonds against convertible bonds is often not practical, as with an average convertible life of around 3.5 years, paying 1% borrow rate will offset a large proportion of expected profits from the trade, unless the valuation discount narrows quickly.

[1]Estimated at 5^{th} August 2022 using investment bank borrow desk data.

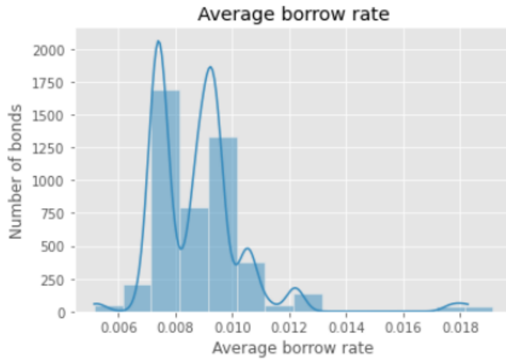


Figure 4.4: Histogram of straight bonds borrowing rate over bonds.

Source: BofA Global Research, ICE Data Indices, LLC (2022)[2]
Figure 4.5: Convertible bond returns over the last five years.

4.2 Correlation Hedges

4.2.1 CDS Index

CDS indices are the most liquid products in the CDS market and the market for CDS indices has grown steadily, accounting for around 50% of the market compared with the single name CDS. [24] In recent years, more and more CDS indices are traded via clearing houses of exchanges.

Investing in the CDS indices is equivalent to holding a portfolio of single-name CDS contracts, where investors can take a long or short position on the credit risk of some specific markets or sectors. This structure provides an easier and less costly way for investors to gain broad credit exposure or hedge credit risk through the correlation between convertible bonds and underlying reference entities in the asset pool. Index components are reconsidered and updated periodically to ensure good liquidity and maintain an approximately constant duration. When a credit event happens, the entity is removed from the index, and the notional is reduced as well. [24]

The good liquidity narrows the bid-ask spread, so CDS indices have lower transaction costs, which makes them more attractive than single-name CDS and bonds for hedging credit exposure. Also, the high liquidity gives potential to a larger trade size, which enables investors to take a large position at their discretion. Since most convertible bonds do not have associated CDS, hence companies are very likely to be not covered directly by CDS indices. To hedge credit exposure, one can take an opposite position on CDS indices covering the same sector or market, and the credit risk can be hedged by the correlation between the convertible bond and the market.

However, hedging credit by CDS indices involves basis risk. Basis risk refers to the probability of loss incurred by an imperfect hedge, i.e. losses are not exactly offset by the hedge. A credit hedge can be effective when the convertible bond is highly correlated to the reference entities of index components, but this is not always the case. High basis risk occurs when the correlation is weak or there are no close matching indices for the underlying bonds. Moreover, a CDS index is not guaranteed to pay out of the convertible bond company defaults. Similar to CDS, duration mismatch is another common problem when hedging credit using CDS indices. [25]

4.2.2 Equity Hedge

Hedging credit risk using equity is the alternative strategy that we have proposed in this thesis, which is structured by offsetting the risk in security price associated with the shift of credit spread, combined with the correlation between credit risk and the stock price of the security issuer.

Using equity to hedge credit risk removes one sort of basis risk: the hedge is specific to the bond issuer. However, it introduces another: the number of shares in the credit-equity hedge is not fixed but is defined through correlation, so we classify it as a correlation hedge. Compared to the direct hedge with credit derivatives, equity has better liquidity as it is exchange-traded. Compared to the direct hedge with straight bonds, equity typically has a far lower borrow rate. Compared to indirect hedge with CDS indices, equity has lower basis risk, as it is working directly on the same underlying. Most importantly, many companies do not have CDS or straight bonds, so these options do not exist.

Credit hedge options	Liquidity	Cost	Basis risk	Likelihood of being paid	Availability[2]
CDS	Low	Low	Low	High	16%
ASCOT	Low	Low	Low	High	4%[1]
Straight bond	Medium	High	Low	High	27%
CDS index	High	Low	High	Low	-
Equity	High	Low	Medium	High	100%

Table 4.1: Summary of different credit hedging options.

Table 4.1 draws a comparison among all the hedging instruments we introduced in this chapter. From the results, it seems that currently there are no perfect strategies on the market for hedging credit risk, and these widely used instruments will more or less bring some other side effects, such as increasing other risks or increasing costs. The problem with direct hedge by credit derivative is caused by their poor liquidity and lack of availability, problems for straight bonds are in their expensive borrow cost, problems for correlation hedge is in high basis risk. Equity outperforms many of these instruments in the above metrics, and all convertible bond issuers have underlying equity. In this regard, using equity as an instrument to hedge credit risk is a strategy worth exploring.

[1]The percentages in the column "Availability" are measured based on our convertible bond universe, which includes 425 convertible bonds across sectors and regions.
[2]Estimated by investment bank analysts.

Chapter 5

Hedging Credit Exposure with Equity

From the capital structure model we discussed in Chapter 1, equity and bonds can both be regarded as claims on a company's assets, therefore the credit risk associated with bonds will impact the value of equity. Because bondholders have priority in the claim, they get paid prior to shareholders upon liquidation. Shareholders have the residual claim on the company's assets, so they will get whatever is left. If the credit risk that a company is facing increases, i.e, the probability of default increases, then equivalently it is more likely for the asset value to end up lying below the debt level, and therefore more likely that no value will be left for shareholders.

The Efficient Markets Hypothesis states that prices fully reflect all information available to the public. If credit markets are pricing an increase in credit risk, this should also be reflected in stock prices. The link between credit and equity has been shown theoretically in Figure 2.3(c) where the implied credit spread increases as the equity value drops. We also demonstrate this empirically below in Figure 5.1, which shows the negative correlations between the overall credit quality of the underlying basket and the broad U.S. equity market.

Source: Bloomberg L.P.(2022)
Figure 5.1: Scatter plot between S&P 500 Index and Markit CDX North America High Yield Index.

5.1 Relationship Between Hazard Rate and Credit Spread

The two main components of credit risk are the probability of default, or the hazard rate, and loss given default (i.e. 1 - recovery rate). Credit spreads and CDS spreads, which we treat equivalently from here given the results from Houweling and Vorst (2001) and Hull et al (2003) referenced in Chapter 4, encompass both of these components of credit risk.

When a company enters into default, the main source of uncertainty for investors is the loss caused by the potential default, which is the expected loss. A risk-neutral investor will require compensation for this potential loss, for example, they are able to purchase bonds at a price with greater discount, or equivalently, there is a wider credit spread. Therefore, expected loss expressed as the percentage of the redemption value of the bond can be thought as an approximation for the credit spread. If we divide time into infinitesimally small time intervals with length dt, in each discrete time step $[t, t + dt]$, the expected loss in this interval is:

$$EL = h \cdot (1 - R) \tag{5.1.1}$$

where h is the hazard rate defined by:

$$h := P\left(\tau \in [t, t + dt) \mid \tau \notin [0, t)\right)$$

and h is assumed to be static on each infinitesimally small time interval. R is the recovery rate under default. So the credit spread is roughly proportional to the hazard rate.

The assumption of a constant hazard rate is unrealistic because the capital structure of a company varies with the value of its underlying assets, which we model as following Geometric Brownian Motion or a jump-diffusion process. It is more appropriate to describe the hazard rate using a stochastic process. Ayache et al (2013) proposed modelling the stochastic hazard rate by a parsimonious model to reflect the negative correlation between credit spread and stock price.[11]

$$h_t = h_0 \left(\frac{S_t}{S_0}\right)^{-p} \tag{5.1.2}$$

where h_0 is the hazard rate estimated at $S = S_0$ and the future hazard rate h_t is decreasing with S_t increasing, and vice versa. p is a positive number representing the ratio of the stock's jump-diffusion volatility to the spread volatility.[26] Combining with equation (5.1.1), we have:

$$\begin{aligned} CS_t &= h_t \cdot (1 - R) \\ &= h_0 \left(\frac{S_t}{S_0}\right)^{-p} (1 - R) \end{aligned}$$

where CS denotes the credit spread. Taking the logarithm of both sides of the equation, we have:

$$\begin{aligned} \log(CS_t) &= -p \log\left(\frac{S_t}{S_0}\right) + \log\left(h_0(1 - R)\right) \\ &= -p \log S_t + \log\left(S_0^p h_0 (1 - R)\right) \\ &= -p \log S_t + c \end{aligned} \tag{5.1.3}$$

for some constant c. This suggests that the relationship between credit and equity can be written as an explicit functional form as equation (5.1.3). Therefore, using historical CDS quotes as a proxy for credit spread and the stock price of underlying companies, fitting a linear regression model enables us to find an estimate for p. [26] Then if we can estimate the base hazard rate h0, we are able to model the future hazard rate.

5.2 Credit-Equity Correlation

On the basis of (5.1.2), we propose two possible models that have different requirements on input data. The first one is called the level model, which requires specific value of credit spread (denoted as CS) and equity value at time t:

$$(CS_t, S_t)$$

The second one is called the return model, which requires returns over a specified period for arbitrary time t. We used 21 trading day returns:

$$\left(\frac{CS_t}{CS_{t-21}}, \frac{S_t}{S_{t-21}}\right)$$

These two models are compared by their prediction results of CDS with real CDS market data. Note that we use CDS data to give an estimate of the market price for the credit spread for a company at a specific tenor.

5.2.1 The Level Model

As illustrated in the previous section, credit spread is roughly proportional to hazard rate, so equation (5.1.2) can be written as

$$\frac{CS_t}{CS_0} = \left(\frac{S_t}{S_0}\right)^{-p} \tag{5.2.1}$$

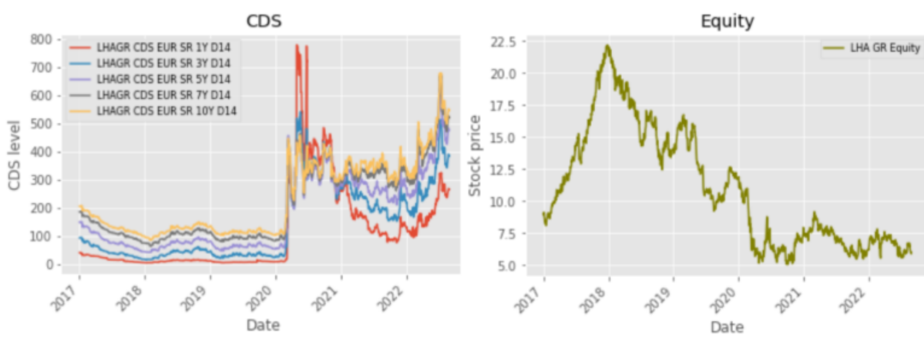
Rearranging the terms in the equation, we have:

$$\begin{aligned} \log CS_t &= -p \log S_t + \log\left(CS_0 \cdot S_0^p\right) \\ &= -p \log S_t + c \end{aligned}$$

where $c = g(S_0, CS_0, p)$ for some function g. Note that it is aligned with equation (5.1.3).

In the level model, we fit a linear regression model on historical data $(\log \text{CDS}_t, \log S_t)$ and estimate parameters (p, c).

Take Deutsche Lufthansa AG as a sample company with available data for both convertible bonds and CDS. Lufthansa is a German airline company, mainly providing air transportation services worldwide, which recorded a 63% drop in full-year revenues in 2021 amid Covid hit to air travel and is expecting recovery on loosening Covid-19 restrictions. Below is a chart showing the time series of Lufthansa's stock price and CDS level for several years from 1^{st} January 2017 to 26^{th} August 2022. Here we selected 5-year CDS for the regression analysis since it has a long available history and the most complete data compared to other tenors.



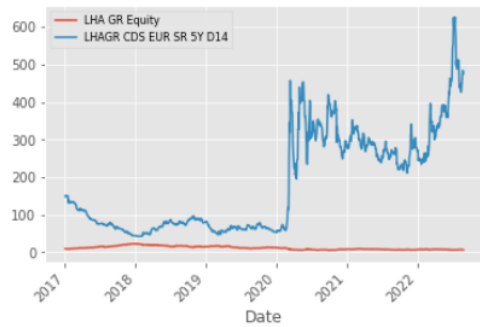
Source: Bloomberg L.P.(2022)
Figure 5.2: Historical stock prices and CDS levels of Lufthansa under several tenors.

An inverse linear relationship can be observed from the scatter plot of log value of equity and CDS (Figure 5.4), which suggests the regression function to take the form:

$$\log CS_t = -p \log S_t + c$$

Data is split into training and testing datasets, where the training dataset consists of data from 1^{st} January 2017 to 31^{st} December 2020, and the testing dataset uses data from 1^{st} January 2021 to 26^{th} August 2022. We fit a linear model on training data and verify the robustness of the model by using testing data. The regression line is:

$$\log CS_t = 9.33 - 1.53 \log S_t$$



Source: Bloomberg L.P.(2022)
Figure 5.3: Historical stock prices and 5-year CDS levels of Lufthansa.

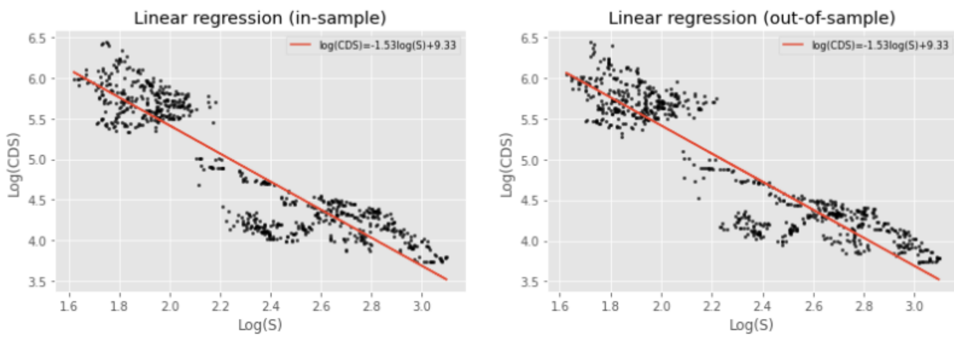


Figure 5.4: Linear regression on stock prices and 5-year CDS levels of Lufthansa, level model.

where p should be the negative slope, so $p = 1.53$. We assess the fitness of the regression model by examining R-squared and Mean Squared Error.
Using CDS with different tenors for the regression analysis will have different slope coefficients in the regression function, which leads to different values of p in the hazard rate model. We estimated the value of p for each tenor of CDS by fitting the linear regression model, the results are shown in table 5.1.

Tenor	Slope	Intercept	R-squared	MSE	RMSE
1	- 3.37	11.40	0.44	2417	0.11
3	- 2.22	9.50	0.53	2605	0.12
5	- 1.53	8.33	0.65	2489	0.12
7	- 1.18	7.78	0.59	2778	0.12
10	- 1.01	7.52	0.52	3179	0.13

Table 5.1: Out-of-sample fitness for all-tenor CDS, level model.

Figure 5.5 shows the result of p for each CDS with tenors 1, 3, 5, 7, and 10 years. The pattern shows that there might be a negative linear relationship between p and tenor, where:

$$p = a + b \times tenor$$

for some constants a and b. So credit spread can be written as a function of stock price and tenor, i.e.

$$CS = f(Equity, tenor)$$

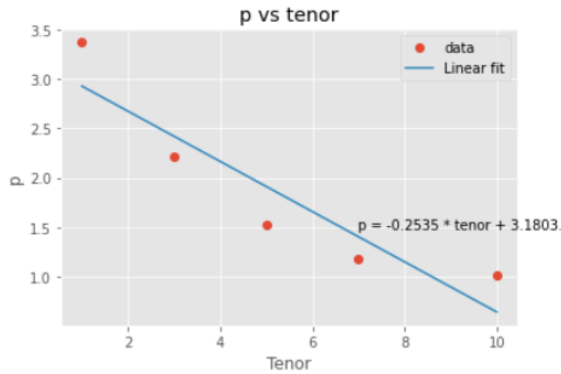


Figure 5.5: Relationship between p and tenor, level model.

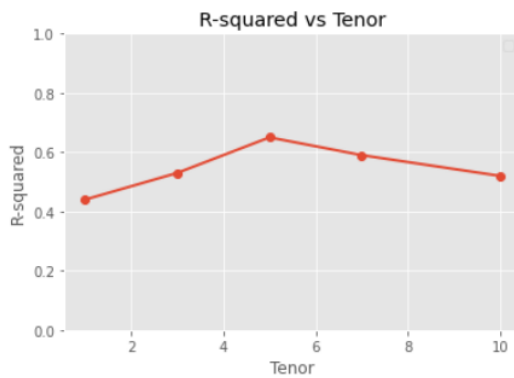


Figure 5.6: Relationship between R-squared and tenor, level model.

Figure 5.6 shows the relationship between R-squared and tenor. When the tenor is 5 years, R-squared has reached its maximum, which makes sense because 5-year CDS has the most data available with a long history and R-squared is increasing with the sample size.

Relationship Between p and Credit Sensitivity

Recall that in (5.1.2), p is defined as the ratio of the stock volatility to the spread volatility. Back to the level model, alternatively we fix (S_0, CDS_0) at the price of 2^{nd} February 2017 and predict CDS using the optimal estimates for parameters (p, c), where c is a function of p, S_0 and CDS_0.

Figure 5.7 demonstrates the change in credit-equity relationship with respect to p. Observing the chart, we found that p has a connection with convexity and companies with a higher p parameter present greater convexity in the credit-equity relationship.

For companies with a higher p, when the stock price is low, the CDS spread increases by a larger amount for a 1% decrease in stock price, which means a company with a higher p is more sensitive to credit risk. According to the capital structure model, all else equal, a lower stock price means a lower equity component of the capital structure, and the theoretical model shows this means higher credit spreads.

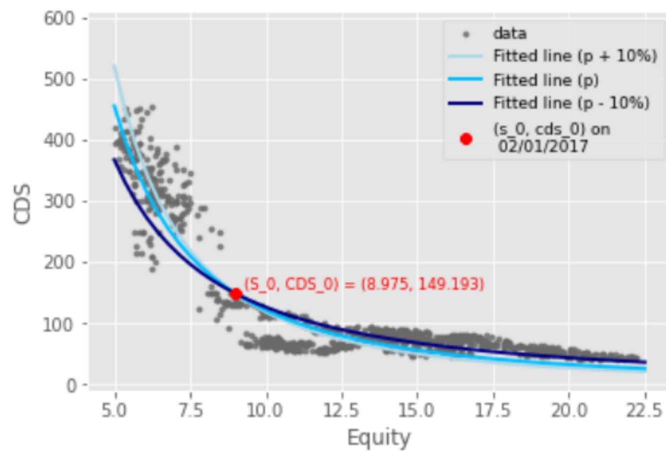


Figure 5.7: Relationship between credit and equity with varied p, level model.

5.2.2 The Return Model

Alternatively, if we evaluate the credit-equity relationship on a return basis using some fixed time period, say 21 trading days, equation (5.1.2) can be written as

$$\frac{CS_{t-21}}{CS_0} = \left(\frac{S_{t-21}}{S_0}\right)^{-q} \tag{5.2.2}$$

for some positive q. Divide equation (5.2.1) by (5.2.2), we have:

$$\frac{CS_t}{CS_{t-21}} = k\left(\frac{S_t}{S_{t-21}}\right)^{-p} \tag{5.2.3}$$

note p is different from the p used in the level model and it should be some function of p in the level model and q, we denote it as p here for coherency. By rearrangement, the equation becomes

$$\log\left(\frac{CS_t}{CS_{t-21}}\right) = -p\log\left(\frac{S_t}{S_{t-21}}\right) + \log k$$

This suggests that we can fit a linear regression line on historical return data $\left(\log\left(\frac{CS_t}{CS_{t-21}}\right), \log\left(\frac{S_t}{S_{t-21}}\right)\right)$ and estimate parameters (p, k) by minimising the squared errors.

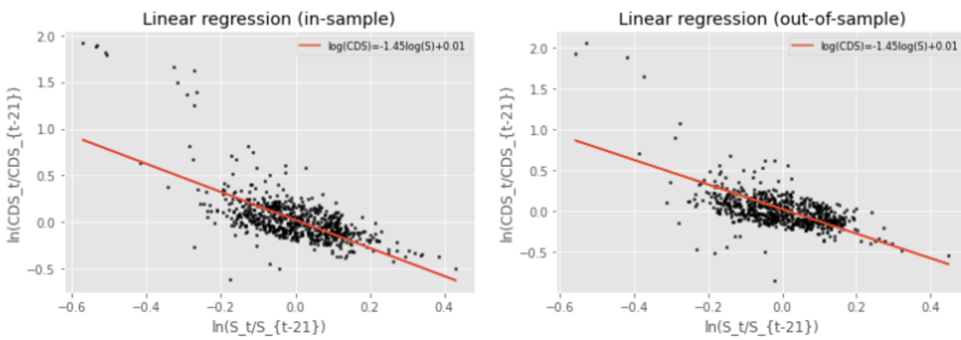


Figure 5.8: Linear regression on stock prices and 5-year CDS returns of Lufthansa, return model.

Tenor	Slope	Intercept	R-squared	MSE	RMSE
1	- 2.52	0.03	0.73	1015	0.08
3	- 1.94	0.01	0.70	1682	0.10
5	- 1.45	0.01	0.70	2127	0.11
7	- 1.14	0.01	0.65	2355	0.11
10	- 0.99	0.00	0.62	2490	0.12

Table 5.2: Out-of-sample fitness for all-tenor CDS, return model.

Figures 5.9 and 5.10 show a comparison between the level and return model in terms of relationship between p, R-squared and tenor. For this particular company, the return model seems to have better model fitness since R-squared values dominate the level model at each tenor.

If we apply a similar regression approach and find the value of p for companies with both convertible bonds and liquid CDS, we can look at some potential factors for p. Figure 5.11(a) shows the relationship between p and D/E ratio, the points are quite scattered and the low R-squared (0.0119 for the level model and 0.0346 for the return model) indicates a lack of relationship. We want to understand this further so look at filtering by sector and find that there is some relationship, as R-squared increases in both models (0.2391 for the level model and 0.4250 for the return model). As shown in figure 5.11(b), we exclude the capital-intensive companies in sectors such as utilities,

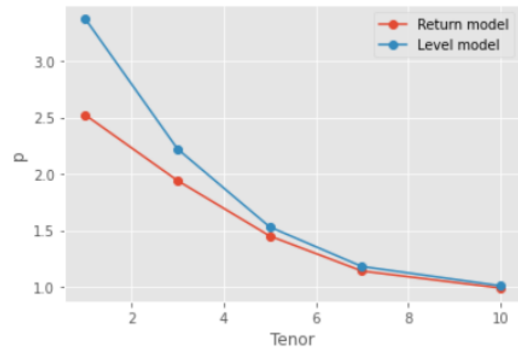


Figure 5.9: Relationship between p and tenor, comparison between level model and return model.

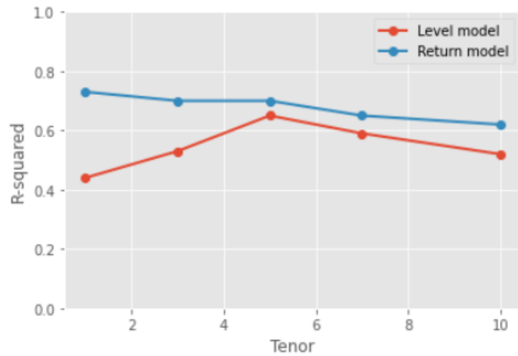


Figure 5.10: Relationship between R-squared and tenor, comparison between level model and return model.

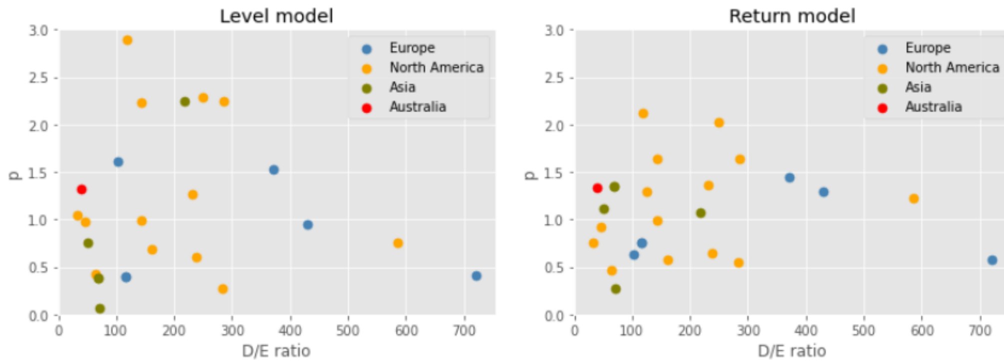
finance and airlines to avoid liquidity and debt market troubles. We can observe a positive trend which indicates a higher p is associated with a higher D/E ratio. To see the relationship between p and credit-worthiness, we plot p against the mean CDS spread as shown by Figure 5.11(c). When CDS spread increases, p increases as well. CDS levels are higher if there is higher credit risk, which means that for the same share price change, if the credit risk is higher, the CDS spread will also move more.

In general, scatter plots between p and other metrics are more scattered under the level model and more concentrated under the return model with a higher R-squared value, which indicates that the return model has better performance in estimating the value of parameter p.

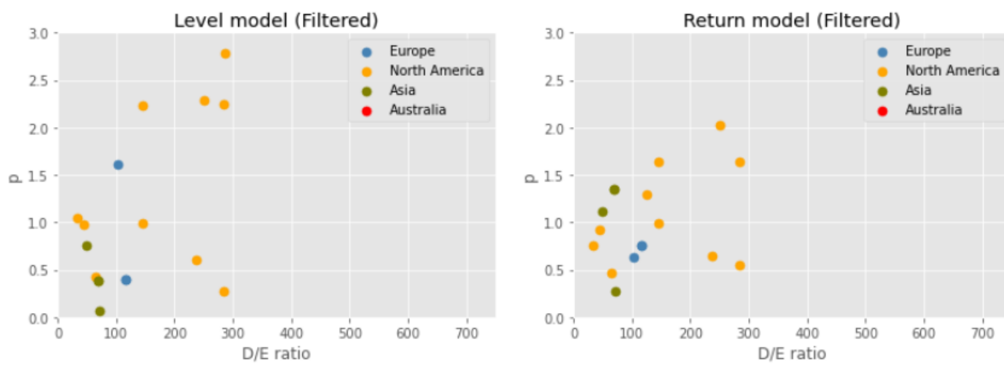
5.2.3 Credit as a Function of Equity and Tenor

Instead of running a linear regression on each tenor to solve for multiple regression lines, we can directly assume a linear function between daily stock price, CDS price and CDS tenor on training data, then simultaneously solve for parameters of the function by minimising the sum of squared errors to get an estimate of the p variable at each tenor. As a result, the optimal estimates of parameters are:

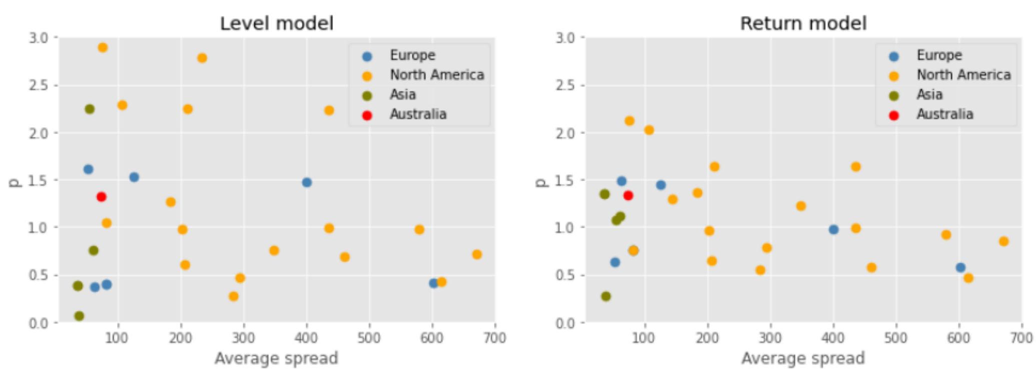
$$[a, b, c] = [2.29986, -0.0887, 8.9827]$$



(a) Relationship between p and D/E before filtration.



(b) Relationship between p and D/E after filtration.



(c) Relationship between p and average CDS spread.

Figure 5.11: Relationship between the value of p and credit risk, comparison between level model and return model.

Hence the functional relationship between CDS, equity and tenor can be approximated using the optimal parameters:

$$p = 2.2998 - 0.0887 \times \text{tenor}$$
$$\log(CDS) = -p\log(S) + 8.9827 \tag{5.2.4}$$

Figure 5.12 and 5.13 show the result, as the tenor of CDS increases, p decreases, which indicates

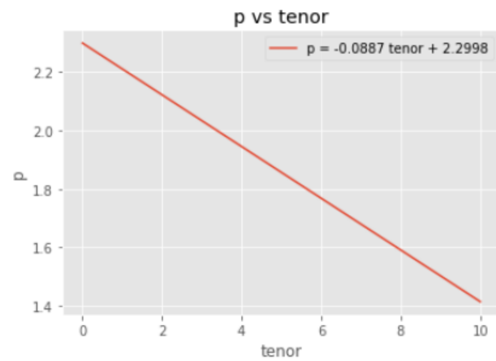


Figure 5.12: Functional relationship between p and tenor.

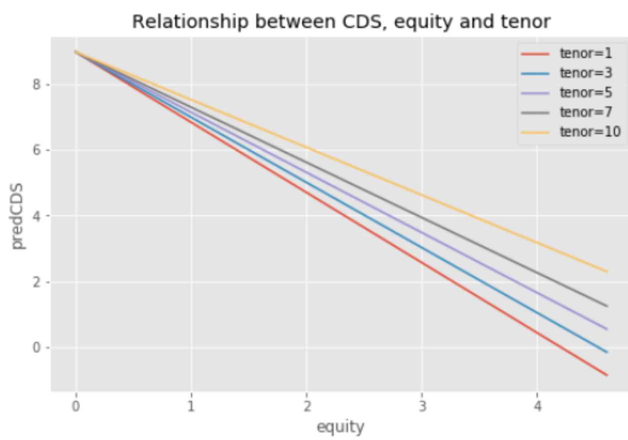


Figure 5.13: Functional relationship between CDS, equity and tenor (CDS and equity are in logarithm scale).

a flatter line, so at the same equity level, the predicted CDS spread will increase. This makes sense, as there will be more uncertainty in the longer holding period, thus CDS with longer tenor normally have a wider spread.

Chapter 6

Systematic Analysis of Credit Hedging with Equity

In this chapter, we perform back-testing on companies with convertibles by forming delta hedging portfolios with different hedging strategies. We then draw a comparison among those strategies by comparing statistics on each portfolio to find the best approach to hedging credit risk and the environment in which that method will produce the best hedge. We formed four portfolios for performance comparison.

Portfolio 1: The Unhedged Portfolio

Portfolio 1 is a portfolio with an initial holding of \$1 million of convertible bonds. We hold this without trading or hedging until the end of the data set. The price of the convertible bond changes every day, and the profit and loss (P&L) of portfolio 1 comes from changes in the price of the convertible bonds:

$$\begin{aligned}\text{P\&L}_1 &= \text{Return from convertible bond}\\ &= \text{Convertible bond price} \times \text{Quantity of convertible bond in portfolio}\end{aligned}$$

Portfolio 2: The Portfolio with Dynamic Black-Scholes Delta Hedge

In portfolio 2, we bought \$1 million of convertible bonds and held with dynamic delta hedging until the end of the data set to make the hedged portfolio delta neutral, according to the Black-Scholes model, which does not incorporate credit risk. Dynamic delta hedging is a strategy that involves creating an offsetting position by trading the underlying shares to reduce the volatility of securities associated with price movement in underlying shares, aimed to keep the hedged portfolio delta-neutral. The P&L comes from the change in price of the convertible bonds and proceeds from dynamic delta hedging.

$$\text{P\&L}_2 = \text{Return from convertible bonds} + \text{Return from delta hedge}$$

The hedge ratio is defined as the comparative value of equity hedge with the aggregate size of the convertible bond, delta is a good measure of it as delta measures the sensitivity between the value of the convertible bond and underlying stock price. The hedge ratio can be computed by multiplying the Black-Scholes delta with the conversion ratio. Short selling the stock requires entering into a position by borrowing stocks first and then selling them, thus borrow cost is inevitable. The borrow cost is calculated by:

$$\text{Borrow cost} = \text{Stock price} \times \text{Borrow rate} \times \frac{\text{Borrowing days}}{365} \times \text{Number of shares}$$

The return from delta hedging consists of the proceeds from selling shares minus borrow cost.

$$\text{Return from delta hedge} = \Delta\text{Stock price} \times \text{Number of shares} - \text{Borrow cost}$$

Portfolio 3: The Portfolio with Delta Hedge and Credit Hedge

In portfolio 3, we bought \$1 million of convertible bonds and held until the end of the period, delta hedging by selling underlying stocks, credit hedging by buying CDS protection under the same name, assuming no credit-equity correlation. The P&L comes from the change in the price of convertible bonds, proceeds from selling stock and CDS protection, i.e. credit hedging. Credit hedging is a strategy to flatten the moves in the valuation of a portfolio due to moves in the credit curve by trading CDS protection. For a specified shift in CDS spread, the value of our un-hedged portfolio will move by the total value of convertible bonds times the dollar credit sensitivity of convertible bonds with respect to the credit curve. We want to offset this amount with a position in the CDS. To match the credit sensitivity of the convertible with CDS, we need the dollar credit sensitivity of CDS, the return from the credit hedge is:

$$\begin{aligned}\text{Return from credit hedge} &= \text{CDS notional required}\\ &\quad \times \Delta\text{CDS level} \times \text{CDS dollar CR01}\end{aligned}$$

And the P&L of portfolio 3 is:

$$\text{P\&L}_3 = \text{Return from convertible bonds} + \text{Return from delta hedge} + \text{Return from credit hedge}$$

Portfolio 4: The Portfolio with Credit-Adjusted Delta Hedge

In Portfolio 4, we bought \$1 million of convertible bonds and held until the end of the period, delta hedging using delta adjusted for credit risk. In Chapter 5, we assumed the credit-equity correlation by:

$$\log(CS) = -p\log(S) + c$$

and p can be written as a function of tenor, where

$$p = a + b \times \text{tenor}$$

Therefore the price of the convertible bonds can be expressed as a function of credit and equity, where credit is a function of equity and tenor.

$$V = f(S, CS) = f(S, g(S, tenor))$$

for some function f and g. Applying the chain rule, the sensitivity of convertible bonds to the underlying stock price is:

$$\begin{aligned}\frac{dV}{dS} &= \frac{\partial V}{\partial S} + \frac{\partial V}{\partial \text{Credit}}\frac{\partial\ \text{Credit}}{\partial S}\\ &= Delta - CR01 \times \frac{p}{S}\exp(-p\log S + c)\\ &= Delta + Delta\ Adjustment\end{aligned}$$

Because delta is conventionally defined as the change in a security price for a 1% change in the price of the underlying assets, the delta adjustment needs to be scaled for an equivalent stock price change. So the delta adjusted for credit is:

$$\begin{aligned}\text{Credit-adjusted Delta} &= Delta + Delta\ Adjustment \times 1\%S\\ &= Delta - CR01 \times \frac{p}{S}\exp(-p\log S + c) \times 1\%S\end{aligned}$$

Multiplying this amount by the conversion ratio gives the credit-adjusted hedge ratio. The P&L of portfolio 4 is the sum of returns from holding convertible bonds and returns from a credit-adjusted delta hedge.

$$\text{P\&L}_4 = \text{Return from convertible bonds} + \text{Return from credit-adjusted delta hedge}$$

The performance of the above four portfolios is compared by some statistical metrics such as annualized return, annualized volatility, the Sharpe ratio and maximum drawdown. We measure the hedging effectiveness of different strategies by measuring how much the variance of the hedged portfolio has been reduced.

6.1 Companies with Convertible Bonds and CDS

We take Deutsche Lufthansa AG as a representative for those companies with CDS data. Lufthansa is a German airline company mainly providing air transportation services. As in Chapter 5, We collected its data and split it into training and testing datasets, then used the training dataset (2^{nd} January 2017 - 31^{st} December 2020) to estimate the functional relationship between credit, p and tenor, and used testing dataset(4^{th} January 2021 – 2^{nd} August 2022) to compare the performance of the four portfolios. Four charts in Figure 6.1 show the performance in each portfolio in terms of cumulative return (R), daily return (r), high water mark (HWM) and drawdown (DD), where:

$$R_t = \frac{\text{NAV}_t}{\text{NAV}_0} - 1$$

$$r_t = R_t - R_{t-1}$$

$$\text{HWM}_t = \max_{0 \le s \le t} R_s$$

$$\text{DD}_t = R_t - \text{HWM}_t$$

Then we annualized return and volatility by:

$$R_{\text{Annualized}} = (1 + R_T)^{\frac{252}{n} - 1}$$

$$\sigma_{\text{Annualized}} = \sqrt{252}\, \sigma_{\text{Daily}} \quad \text{where} \quad \sigma_{\text{Daily}} = \sqrt{\frac{\sum (r_t - \bar{r})^2}{n-1}}$$

We draw a comparison of these four portfolios on their hedging effectiveness and the results are shown in Figure 6.1 and Table 6.1. We can see that the return on the convertible bond is generally negative due to the impact of the ongoing pandemic in 2021, but the performance has become worse since March this year, due to the war between Ukraine and Russia. With a series of negative effects on the economy, such as inflation, recession and volatility, convertible bond returns are further sinking into the negative.

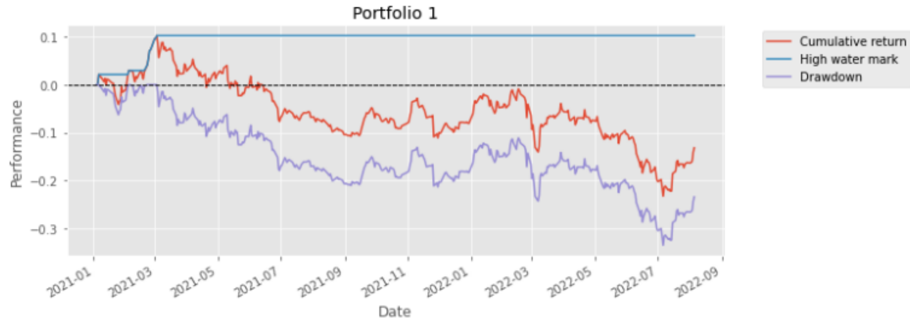
However, the hedging effect of our hedged portfolios did moderate this negative trend to some extent, which is reflected in returns and drawdowns, as they are not as negative as the unhedged portfolio. Besides, volatility has been greatly reduced. Hedging the uncertainty on convertible bonds from stock price movement alone improves results a lot, with credit hedging it improves still further. For Lufthansa, if we ignore the link between credit and equity, and hedge the credit risk by purchasing CDS protection and transferring the credit risk to a third party, we can see from the charts and table below that we will achieve the best performance, as portfolio 3 has the highest annualized return, the largest Sharpe ratio and the smallest maximum drawdown. If we incorporate credit-equity correlation into our hedging strategy and hedge credit risk by selling an additional amount of stock on top of the amount required by the classic delta hedging strategy, then we can achieve the best variance reduction effect because annualized volatility is greatly reduced to its minimum level in portfolio 4. It is worth noting that the return of portfolio 4 is worse than the return of portfolio 3, or even 2. Of this return, -0.03% is from additional borrow cost. This is something we want to explore with a broad universe of bonds.

	Annualized Return	Annualized Volatility	Sharpe Ratio	Max Drawdown
Portfolio 1	- 0.1109	0.1684	- 0.7774	- 0.3357
Portfolio 2	- 0.0590	0.0646	- 1.2233	- 0.1581
Portfolio 3	- 0.0224	0.0581	- 0.7289	- 0.0975
Portfolio 4	- 0.0725	0.0569	- 1.6168	- 0.1607

Table 6.1: Performance comparison for Lufthansa.

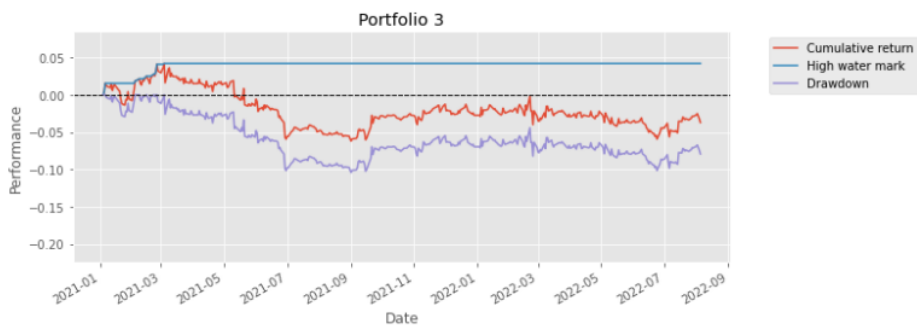
6.1.1 Results From Back-Testing

We start by looking at the companies that have CDS to explore the general relationship between credit and equity which is applicable to all convertible bonds, even to those that do not have CDS.

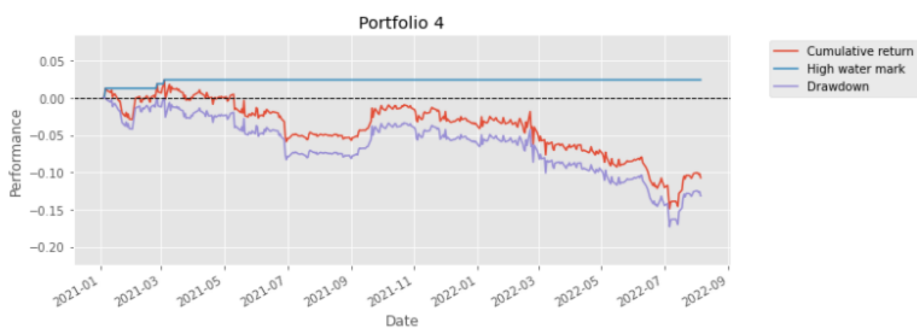


(a) Portfolio 1: The unhedged portfolio.

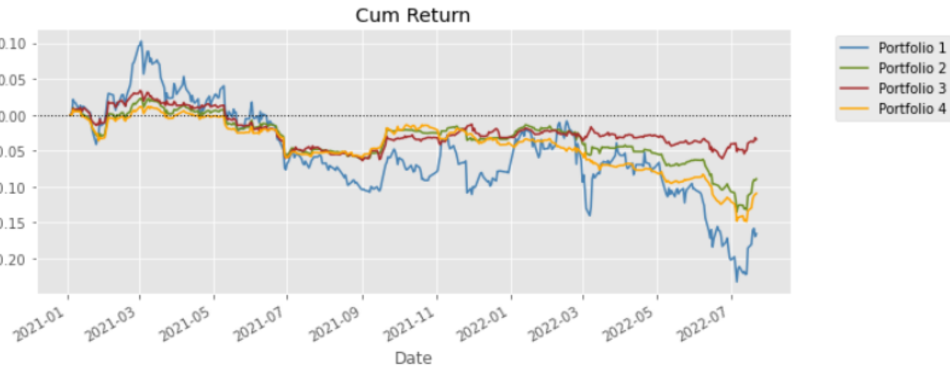
(b) Portfolio 2: The portfolio with dynamic delta hedging.



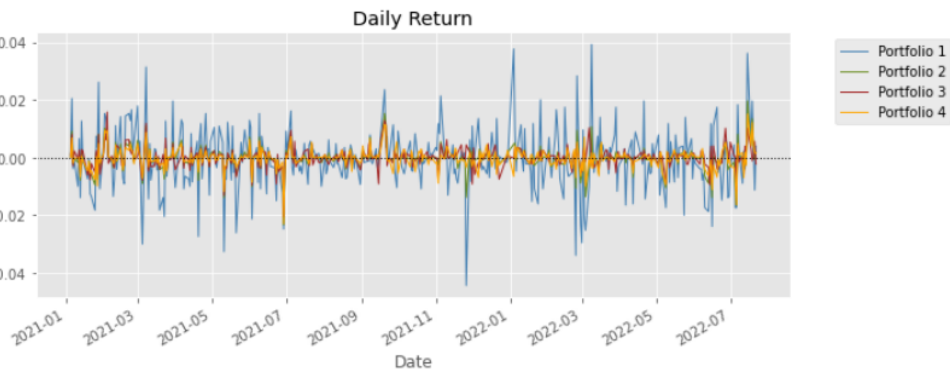
(c) Portfolio 3: The portfolio with delta hedging and credit hedging.



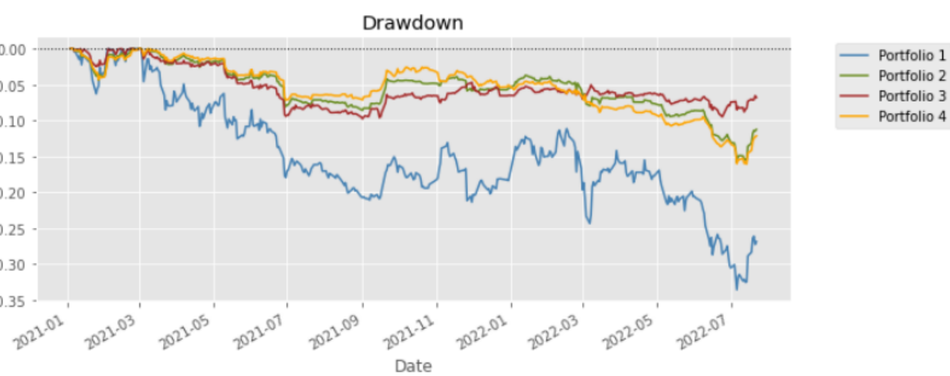
(d) Portfolio 4: The portfolio with credit-adjusted delta hedging.



(e) Cumulative return.



(f) Daily return.



(g) Drawdown.

Figure 6.1: Performance comparison for Lufthansa.

According to the Efficient Market Hypothesis, the liquidity of CDS is very important because its price can reflect all public information under the efficient market. For companies that do not have liquid CDS, we do not classify these companies as 'having CDS', as the trading of CDS is not sufficient to be of good use.

To define what we mean by 'liquid', we calculate the percentage of days in the testing dataset that the change in CDS is less than 1 bp for all companies in our universe that have both convertible bonds and CDS. In that case, we can see from 6.2(a) that most companies have this value greater than 90%. As shown in Figure 6.2(b) and 6.2(c), the percentage of companies with changes in CDS less than 1 bp in the past several days is reduced if we look in a longer window. We decided to use a 21-day return, discarding all CDS that do not move by more than 1bp over a 21 business day period more than 10% of the time. This was our threshold to determine if a company had sufficiently liquid CDS. This left 65 convertible bonds where the company had liquid CDS data available.

To draw a general conclusion, we ran this back-testing for the 65 convertible bonds in our universe where the company had liquid trading CDS. Here we filtered the company universe by excluding those companies where CDS does not change (i.e. Δ CDS < 1 bps) more than 10% of the time when using 21-day returns and removed exchangeable, cross-currency and mandatory convertible bonds to simplify the calculation and keep the formula of the hedge ratio consistent. In this chapter, we denote the annualized volatility in the fours portfolios as σ_1, σ_2, σ_3 and σ_4.

Figure 6.4(a) shows the volatility in all portfolios and assesses the hedging effectiveness by comparing volatility with and without various hedges. We can observe from the charts that all bonds show a variance reduction when Black-Scholes delta hedged, as σ_2 is significantly lower than σ_1. Delta measures the sensitivity of the convertible bond value to the underlying stock price movement. The bonds with the highest delta see most variance reduction, as shown in Figure 6.4(f), where proportional of volatility eliminated by Black-Scholes delta heding is defined by $1-\frac{\sigma_2}{\sigma_1}$. This indicates that the effectiveness of Black-Scholes delta hedging is positively related to delta, as expected. The variance-reducing impact of credit hedging is positively related to the CDS level, as shown in Figure 6.4(g), where proportional of volatility eliminated by credit hedging is defined by $1-\frac{\sigma_3}{\sigma_2}$. This makes sense, wider credit spreads mean greater credit risk, and wider credit spreads tend to be more volatile, this can be seen in Figure 6.3, the volatility of CDS level increases with median CDS. In the cases where the CDS level is tight, adding a credit hedge will not have significant impact as there is hardly any credit risk. Hence we expect an effective variance-reducing impact for low delta convertible bonds with wider CDS spread.

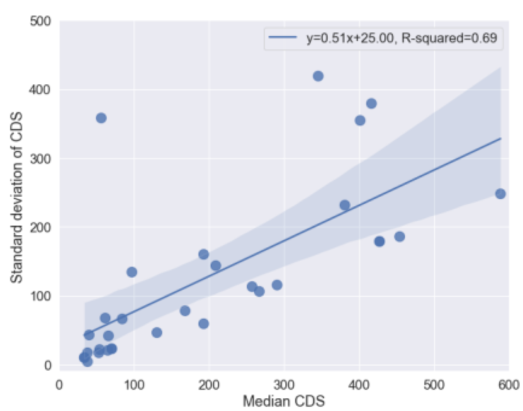
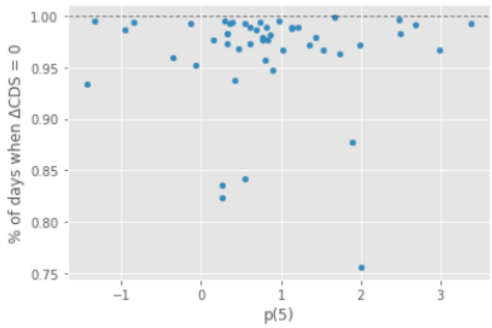
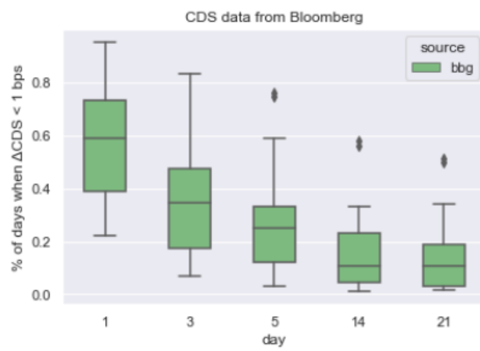


Figure 6.3: Scatter plot of median CDS spread and volatility of CDS spread.

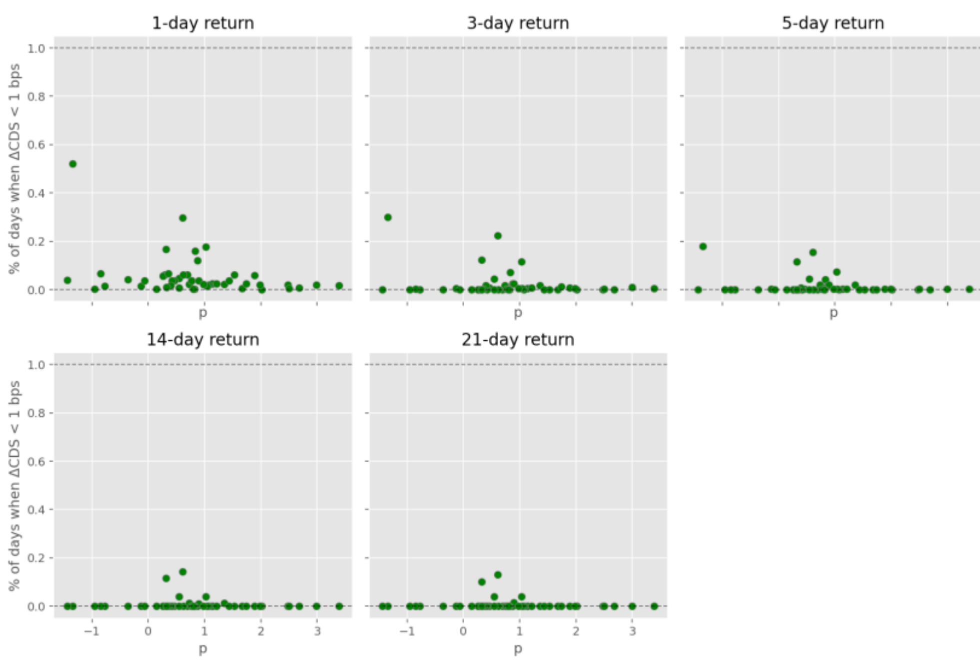
Figure 6.4(c) shows the annualized returns in four portfolios, the returns of the hedged portfolios are in much lower magnitude than the unhedged portfolio 1, indicating that various hedging



(a) Percentage of days when the change in 5-year CDS level is less than 1 bp.



(b) Boxplot for CDS liquidity.



(c) Percentage of days when the change in 5-year CDS level is less than 1 bps.

Figure 6.2: Liquidity of 5-year CDS.

	Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4
Average annualized return	- 4.6%	0.4%	1.1%	0.6%

Table 6.2: Average annualized returns in four portfolios.

	Return from additional borrow cost	Return in portfolio 4
Median	- 0.0070%	- 0.4547%
Min	- 1.2236%	- 16.6085%
Max	- 0.0003%	25.3871%

Table 6.3: Impact from additional borrow cost in portfolio 4.

strategies reduce variance on convertibles effectively as we expected. Figure 6.4(d) displays the annualized return series in the two credit-hedging portfolios and Table 6.2 shows the average performance, where portfolio 3 has an average return higher than portfolio 4 by around 0.5%.

The way we treat transaction costs impacts this difference. In these back-tests we have included borrow cost. This is a frictional cost of hedging that is present for stocks, but not for CDS. Therefore even if we were able to perfectly replicate the exposure of a CDS hedge with equity, the equity strategy would face an additional layer of costs via borrow that the CDS did not face, so we expect returns to be worse for portfolio 4. The impact of borrow for the additional quantity of short shares used in portfolio 4 is displayed in table 6.3.

However, there are an additional set of costs that we did not consider in this thesis, due to lack of available data. That is: the bid-ask spread. For these back-tests, we ignored this factor and assumed all securities are traded at the close price, i.e. assuming zero bid-ask spread. This trading cost for convertible bonds would affect all portfolios equally, so we are not introducing any bias into the comparison by ignoring it. Bid-ask spread on CDS will only affect portfolio 3, and can be a significant factor. Bid-ask spread for stocks is much lower than for CDS, as liquidity is much greater, particularly in the relatively liquid universe of names that we are considering in this thesis. While portfolios 2-4 all face the bid-ask spread in stocks as they all delta hedge, portfolio 4 shorts more shares than portfolio 2 or 3, so this impact should be higher for portfolio 4. Overall, we expect this introduces a bias into our results that unfairly benefits portfolio 3 over portfolio 4. This is an area into which we would like to do more research, though it is outside the scope of this thesis.

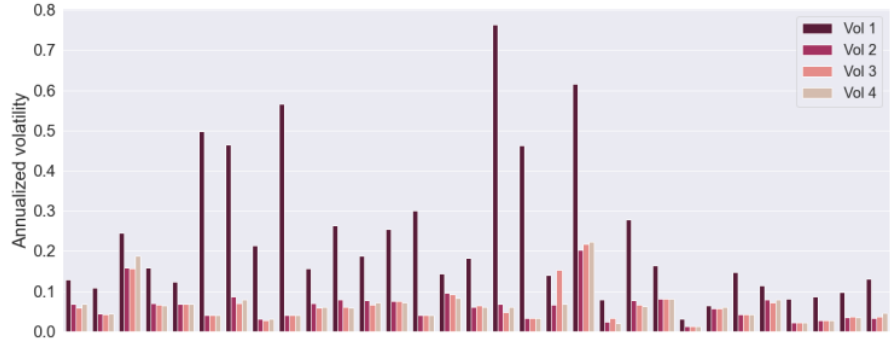
Generally, credit hedging delivers better performance as measured by variance reduction for some bonds but adds more volatility to the hedged portfolio for the other bonds. Therefore, we propose that credit hedging will lead to a greater reducing impact under certain circumstances and split our universe into categories of convertible profiles.

Classification by Delta

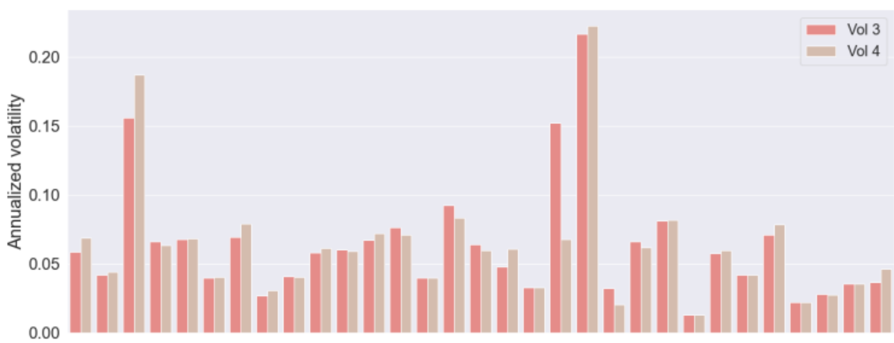
The first criterion is delta. We divided convertible bonds into three categories, convertible bonds with a median delta lower than 30% are in the same group, namely 'low delta'. The 'high delta' group consists of convertible bonds with a median delta greater than 80% and the 'balanced delta' group is for those convertible bonds with median delta between 30% and 80%. The effectiveness of hedging is determined by the level of variance reduction, which is measured by the statistic metric F:

$$F = \frac{\sigma_i^2}{\sigma_2^2} \quad \text{where} \quad i = 3, 4 \tag{6.1.1}$$

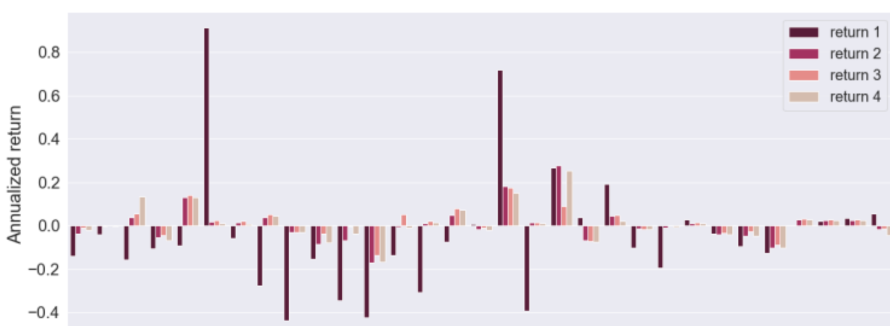
If F is less than 1, it indicates that variance in the credit-hedged portfolio is greater than that of portfolio 2, so we managed to further reduce variance successfully by including a credit hedge using CDS or equity. Conversely, if F is greater than 1, it means the credit hedge does not work well, as there is no improvement observed in variance reduction. F=1 indicates that convertible bonds are not sensitive to credit risk, so adding the credit hedge to our model has no difference in the impact on overall volatility.



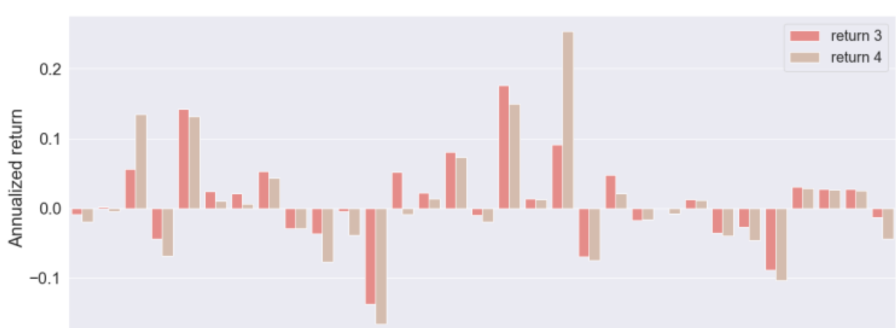
(a) Annualized volatility in four portfolios.



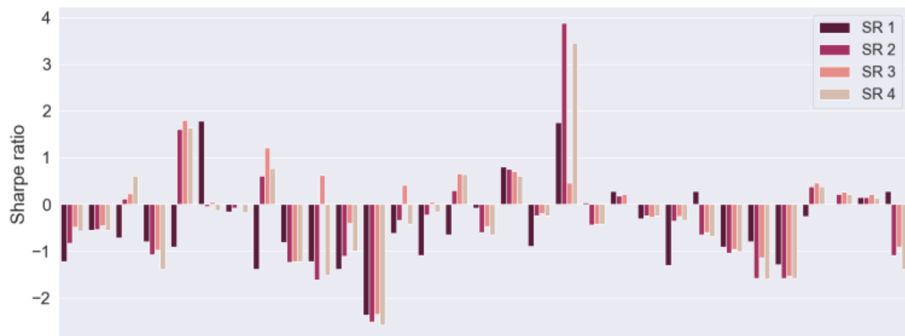
(b) Annualized volatility in portfolio 3 and portfolio 4.



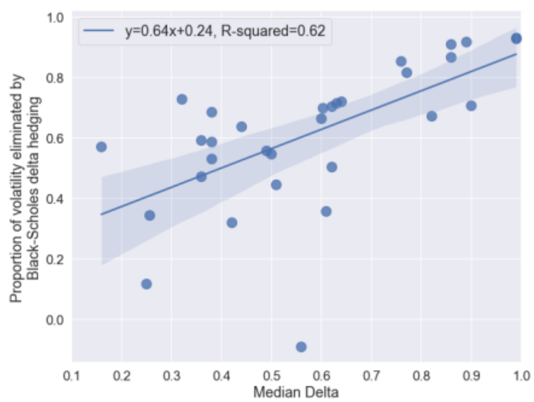
(c) Annualized return in four portfolios.



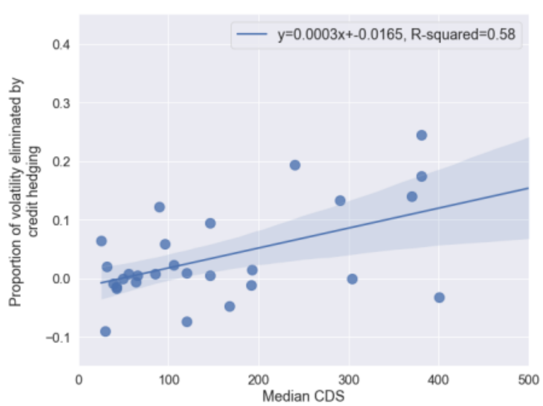
(d) Annualized return in portfolio 3 and portfolio 4.



(e) Sharpe ratio in four portfolios.



(f) Comparison of portfolio 1 and portfolio 2: scatter plot of the proportion of volatility eliminated by Black-Scholes delta hedging and median delta.



(g) Comparison of portfolio 2 and portfolio 4: scatter plot of the proportion of volatility eliminated by credit delta hedging using CDS and median delta.

Figure 6.4: Back-testing results for companies in our universe that have CDS.

		Credit hedge with CDS		Credit hedge with Equity	
Delta	Count	Median	95% CI	Median	95% CI
Low	3	0.960	(0.939,1.011)	1.004	(0.765,1.147)
Balanced	21	0.982	(0.679,1.524)	0.984	(0.869,1.101)
High	7	1.000	(0.794,1.114)	0.994	(0.911,1.098)
Total	31	0.983	(0.769,1.343)	0.994	(0.905,1.069)

Table 6.4: Median variance reduction by credit hedging for convertible bond profile classified by delta.

We compare the performance of two credit hedge methods and their results are shown in Table 6.4. Generally, credit hedging shows more variance reduction when there is considerable uncertainty brought by credit exposure. We can see that when using CDS to credit hedge, F rises as the delta increases and F is close to 1 for companies in the high delta group. This means that the credit hedge works well under a low delta and it has barely any reducing impact under a high delta, which is aligned with our expectation. A high delta indicates a high likelihood that the convertible will be converted to equity, and a low likelihood that it will be redeemed for cash. It means the embedded conversion option is deep-in-the-money. Such convertibles have low sensitivity to credit, due to the low likelihood that a holder will redeem for cash. This explains why the credit hedging strategy does not have a significant influence on reducing portfolio variance. Conversely, for low delta convertibles where the embedded conversion option is deep-out-of-money, the likelihood that the holder will redeem for cash is high, so the credit sensitivity is high. Hence, credit hedging strategies should have more impact on performance for convertible bonds with low delta, all else equal.

The results for equity hedge under low delta (show in Table 6.5) are different from our expectation, we believe this is due to the small sample size as there are only three companies in the 'low delta' group after filtration. Recall in chapter 4 we presented a table (4.1) to compare several widely-used credit hedging instruments in various dimensions, where the number of companies that had liquid CDS only occupies 16% of our universe. This lack of availability posed constraints on our sample size, which thereby introduces noise to the result. This can be seen in the wider confidence intervals for this group, and we note that X of the 3 bonds saw a variance reduction that was within the 95% confidence intervals.

Another reason for this result may be the narrow credit spreads. In general, the magnitude of variance reduction is influenced by the credit risks implied by both CDS levels and credit sensitivity of the convertibles. For the two bonds that we do not show reduced variance when credit hedged using equity, although both have reasonable sensitivity to credit, both companies are of very low credit risk, with median CDS level over the last five years lower than 60 bps. The back-test for the convertible of Takashimaya Co Ltd has a delta hedged volatility of only 1.3%, it is very hard to reduce the variance of such a position. For both Dexus Finance Pty Ltd and Takashimaya Co Ltd, their credit spreads are so low that the impact to delta when adjusting for credit is very small, so the resultant differences in variance were very small. For JetBlue Airways Corporation where there is some credit risk with around 106 bps as 5-year median credit spread, we see a reduction in variance in portfolio 4, as we would expect. For these three convertible bonds in the 'low delta' group with low to moderate median CDS level, while low delta implies high credit sensitivity, CDS hedging is less impactful when CDS spreads are low, as there is not much credit risk. Meanwhile, equity, a more volatile hedging instrument is used to proxy CDS and hedge the credit risk of convertible bonds, which may introduce more uncertainty to our portfolio.

Comparing the last two columns we can see that F is generally smaller under CDS hedge, this means the variance-reducing impact from credit hedging using CDS is better than using equity. Generally, CDS is regarded as the primary choice of credit hedging. As a direct hedging tool, using CDS perfectly hedges the credit risk of convertible bonds with the basis risk as low as

[1]CR01 of a convertible bond measures the credit sensitivity of a convertible bond's value to a 1 bp change in its credit spread

Company	ISIN	Median CDS	CR01[1]	Vol 1	Vol 2	Vol 3	Vol 4
JetBlue Airways Corp	US477143AP66	106.18	-0.03	14.42%	9.47%	9.25%	8.30%
Takashimaya Co Ltd	XS1915588559	29.24	-0.03	3.03%	1.30%	1.28%	1.30%
Dexus Finance Pty Ltd	XS1961891220	63.95	-0.03	6.49%	5.72%	5.75%	5.99%

Table 6.5: Volatility for low delta convertible bonds

possible. Equity is considered a correlation hedge, given that we do not know the exact number of shares that need to be traded at any point during the holding period to offset the credit risk. A prerequisite of a good hedging result is that we are able to estimate this number accurately.

Classification by D/E ratio

The second criterion is the Debt-to-Equity ratio, in this case, we divided convertible bonds into two categories, those with a high D/E ratio (greater than 50% quantile of convertible bonds in the universe, 134) and with a low D/E ratio (less than 50%). The result is shown in table 6.6, where variance is reduced to a larger degree for companies with large D/E ratio. This indicates that an effective variance-reducing impact will be perceived for convertible bonds from a highly leveraged issuer. This makes sense since a highly leveraged firm with a high D/E ratio is more sensitive to credit risk, so adding a credit hedge is able to further reduce the variance.

		Credit hedge with CDS		Credit hedge with Equity	
D/E ratio	Count	Median	95% CI	Median	95% CI
Low	13	0.991	(0.848,1.183)	1.004	(0.904, 1.043)
High	12	0.929	(0.781,0.983)	0.961	(0.799,1.162)
No D/E data available	6	-	-	-	-
Total	31	0.983	(0.850,1.053)	1.002	(0.884,1.069)

Table 6.6: Variance reduction by credit hedging for convertible bond profile classified by D/E ratio.

6.1.2 Should We Hedge Credit Exposure Using CDS or Equity?

Variance reduction results show that although using equity as a credit hedging tool can effectively reduce the variance, the overall hedging effect is not as good as using CDS. The structure of CDS enables it to perfectly match the credit exposure of the bond. This perfect hedge removes a great deal of the uncertainty in the payoff of holding CDS-protected bonds with the same reference identity, leaving exposure to interest rates and equity volatility as the dominant remaining factors. Conversely, an equity hedge is considered a correlation hedge, the hedging performance is influenced by the accuracy in periodical estimation of the unknown hedge ratio adjusted for credit risk, which is a less than perfect hedge. This will introduce basis risk to the risk basket. Although adding CDS to a portfolio will inevitably increase counterparty risk and raise concerns about liquidity issues, CDS remains the primary cost-efficient product for managing credit exposure if it is available for bond creditors.

However, the majority of companies that issue convertible bonds do not have CDS available, or it is not sufficiently liquid to use as a hedging tool. Therefore, it is very important to look for cost-effective alternatives to CDS for one's hedging needs in the turbulent credit market. Regarding its ability to effectively hedge credit risk with cost as low as possible, ample availability and good liquidity, equity can be used as a substitute to hedge credit risk in the absence of CDS.

6.2 Convertible Bonds Without CDS

6.2.1 Determine the Relationship Between Credit and Equity

For companies that do not have CDS, we want to estimate an appropriate credit spread for the convertible bond. The first and simplest method for doing this is to use credit spreads estimated by investment bank analysts. Generally, estimates for credit are more stable than CDS market prices. It remains a question as to whether these more stable estimates provide a better or worse reflection of creditworthiness than CDS pricing. Generally, there is some evidence that market prices move with more volatility than company fundamentals. For stocks, the famous paper Market Volatility and Investor Behavior by Robert Schiller [27] commented on the excess volatility found in stock prices relative to simple present value efficient market models; it's possible that credit markets also exhibit more volatility than company fundamentals suggest they should, in which case more stable credit estimates may actually be a better descriptor of company credit risk. This paper does not intend to make any statements about this, other than that it is unclear whether more stable estimates are more or less useful than credit market pricing in this exercise.

For companies that have CDS, we computed their p value in Section 6.1.1. The histogram below (6.5) shows the distribution of p with companies after filtration with a median value of 1 and a mean value of 1.2. Note that the distribution of p is right-skewed, and we have the mean that is greater than the median, this means the most common values will be overestimated if using mean as an estimate. So we assume p is fixed at 1 and use it to back-test the credit-adjusted delta hedge for the convertible bonds issued by companies that do not have CDS.

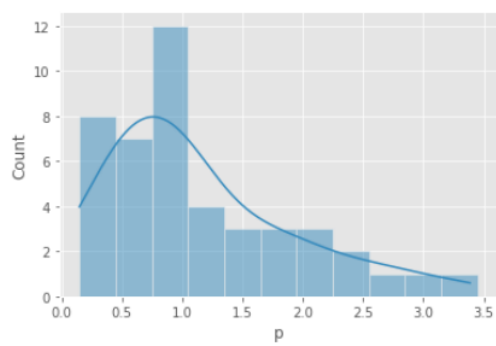


Figure 6.5: Histogram for p values.

Recall that we introduced two models estimating the credit-equity relationship in Chapter 5. In the return model, intercept $\log k$ is very close to zero. For simplicity, we assume it to be zero (so $k = 1$) and effectively have only one parameter p left that needs to be estimated. To reduce the company-specific impact brought by parameter c in the credit-equity relationship, we predict the return of credit spread rather than the level of credit spread. Recall that the relationship between returns of credit spread and stock returns defined in the return model is:

$$\log\left(\frac{CS_t}{CS_0}\right) = -p\log\left(\frac{S_t}{S_0}\right)$$
$$\frac{CS_t}{CS_0} = \left(\frac{S_t}{S_0}\right)^{-p} \implies CS_t = CS_0 \cdot \left(\frac{S_t}{S_0}\right)^{-p}$$

To compute the credit-adjusted delta, we apply the chain rule:

$$\begin{aligned}\frac{dV}{dS} &= \frac{\partial V}{\partial S} + \frac{\partial V}{\partial CS}\frac{\partial CS}{\partial S} \\ &= \text{Delta} + \text{Delta Adjustment}\end{aligned}$$

Where:

$$\frac{\partial CS}{\partial S} = CS_0 \cdot (-p) \left(\frac{S_t}{S_0}\right)^{-p-1} \cdot \frac{1}{S_0}$$

When stock price changes by 1%, the change in credit spread is:

$$\begin{aligned}\Delta CS &= \frac{\partial CS}{\partial S} \times 1\% \times S_t \\ &= CS_0 \cdot (-p) \left(\frac{S_t}{S_0}\right)^{-p} \times 1\% \\ &= CS_t \times (-p) \times 1\%\end{aligned}$$

So alternatively, the credit-adjusted delta can be computed by

$$\text{Credit-adjusted Delta} \;=\; \text{Black Schole's Delta} \;+ \text{CR01} \times CS_t \times (-p) \times 1\% \qquad (6.2.1)$$

6.2.2 Requirement of an Estimate for Credit

We test for the robustness of the $p\,(tenor)$ functional relationship defined in Chapter 5 in the testing data and compare the results of predicting the credit spread using different approaches.

- **Approach 1**: Applying functional relationship defined in (5.2.3) to the testing dataset and compute implied credit spread directly.
- **Approach 2**: Predict credit spread using a rolling window of 21 trading days with the optimal Least Square parameters estimated in the previous rolling window, i.e. 21 trading days prior to the prediction time interval. In this case, parameters are re-parametrised in each window, so $[a, b, c]$ are dynamic for testing data in different windows.

$$\log CS_t = -\tilde{p} \log S_t + \tilde{c}$$

- **Approach 3**: Predict return of credit spread using a rolling window of 21 trading days with the optimal Least Square parameters estimated in the training dataset. Parameters are expected to be static.

$$\log\left(\frac{CS_t}{CS_{t-21}}\right) = -\bar{p} \log\left(\frac{S_t}{S_{t-21}}\right) + \log(\bar{k})$$

- **Approach 4**: Predict return of credit spread by using a rolling window of 21 ttrading days with the optimal Least Square parameters re-parametrised in the previous rolling window, so parameters are dynamic.

$$\log\left(\frac{CS_t}{CS_{t-21}}\right) = -\tilde{p} \log\left(\frac{S_t}{S_{t-21}}\right) + \log(\tilde{k})$$

where $\tilde{p}$ and $\tilde{k}$ denotes the dynamic parameters with value reparameterized in each window, $\bar{p}$ and $\bar{k}$ denotes the static parameters with value estimated in the training dataset.

The charts below show the real CDS data and our predicted result and the fitness result is stored in Table 6.7. Figure 6.6(a) and 6.6(b) show the in-sample and out-of-sample prediction results, where we can see that the out-of-sample prediction result deviates quite a lot from real CDS data with even reversed patterns. This is also shown in Table 6.7, where relative the maximum MSE is observed when applying approach 1 with a negative R-squared. Note that the R-squared is evaluated on the testing dataset rather than the training dataset. In the definition of R-squared, it measures how well is the data is explained by the fitted model, and

$$R^2 = 1 - \frac{RSS}{TSS}$$

$$RSS = \sum_{i=1}^{n} \left(Y_i - \hat{Y}_i\right)$$

$$TSS = \sum_{i=1}^{n} \left(Y_i - \bar{Y}\right)$$

$$ESS = \sum_{i=1}^{n} \left(\hat{Y}_i - \bar{Y}\right)$$

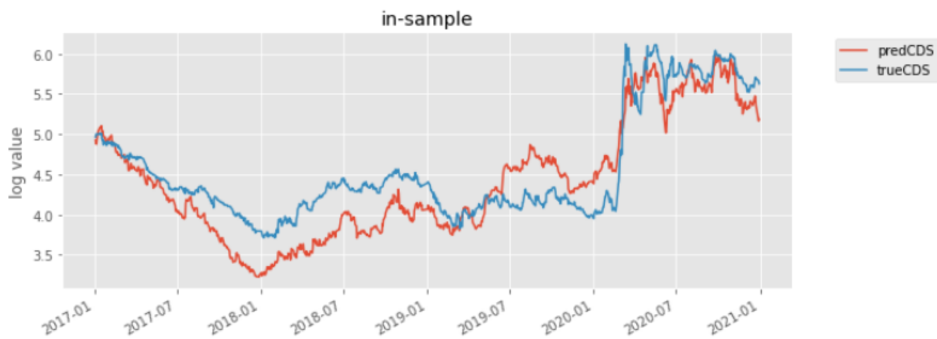
where TSS is defined as the total variation in data, ESS and RSS are defined as the variation in data explained and unexplained by the fitted model, so R-squared is no longer equal to the square of the correlation coefficient, thus is not guaranteed to be bounded by 0 and 1 when the model is evaluated separately on training and testing data. [28]

CDS predicted by different approaches	Relative MSE	R-squared
Approach 1	1.925	-2.148
Approach 2	0.155	0.746
Approach 3	0.277	0.547
Approach 4	0.231	0.622

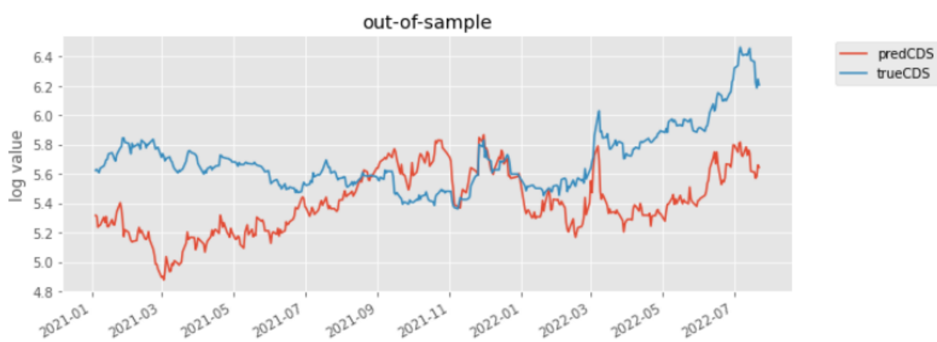
Table 6.7: Out-of-sample prediction result.

Figure 6.6(c) compares the result from approaches 1 and 2 and we find that the prediction outcome is closer to the true CDS, which implies prediction using a rolling window helps us to capture the true pattern. Figure 6.6(d) compares the result from approaches 2 and 4 to see the impact of c parameters, where CDS predicted by level (green line) lies closer to real data, which indicates that approach 2 outperforms approach 4 in terms of model fitness and the c parameter does improve our model performance. This is also shown by model fitness metrics, as approach 2 has higher R-squared and lower MSE than approach 4. Also, reparameterization improves the model fitness as well. Figure 6.6(e) and Table 6.7 draw a general comparison of all approaches by showing the result by visualization or by statistics metrics. Among all approaches, predicting CDS using a rolling window of 21 trading days with the impact of the c parameter dominating the prediction outcome, with the highest R-squared value and the lowest relative MSE.

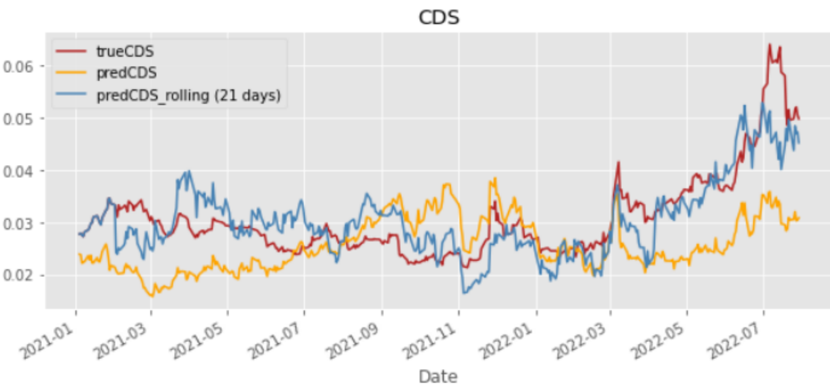
However, the choice of c is quite company-specific, c might vary a lot for different companies, therefore we should be cautious about the value of c when deciding a general functional relationship between credit and equity that works for all companies in the universe. Considering the sensitivity of the prediction result to the c parameter, we use approach 4 (predict CDS returns using a rolling window without the impact of c parameters) as our primary method to predict credit spread for companies without CDS data in the later sections. However this approach also has some limitations, for example, it requires CDS data or at least a reasonable and appropriate estimate for the credit spread 21 days ago to predict the credit spread of today.



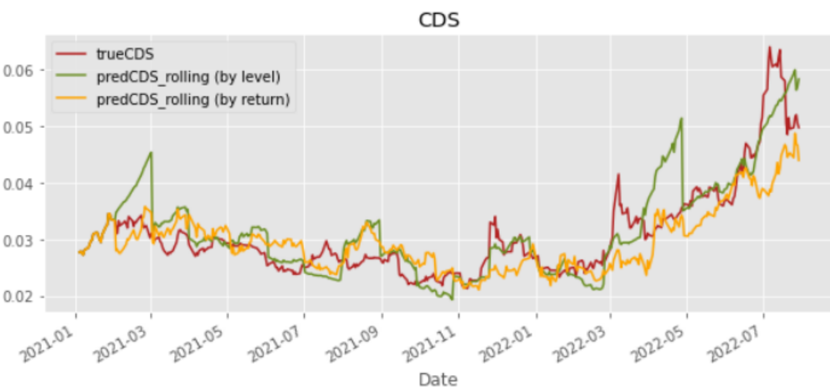
(a) In-sample performance.



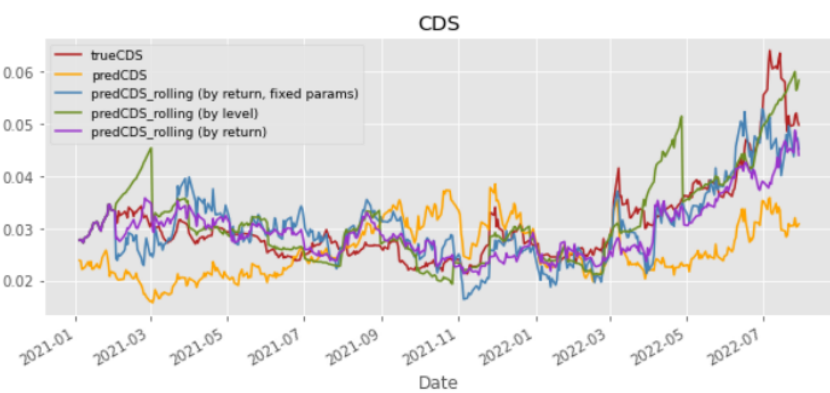
(b) Out-of-sample performance.



(c) Performance comparison: the impact of the rolling window.



(d) Performance comparison: level model and return model.



(e) Performance comparison: all approaches.

Figure 6.6: Performance of credit estimating approaches.

6.2.3 Methods to Estimate Credit

To ensure our prediction method works well, we need to estimate the credit spread 21 days ago and combine it with a rolling window of 21-day stock returns to predict the current credit spread.

Implied Spread by Bloomberg DRSK Function

The first approach is using 5 year CDS spread implied by the Bloomberg Issuer Default Risk model Likelihood of Default (DRSK function). [29]

Implied Spread using a Convertible Bond Pricing Model

Another approach is using a convertible bond pricing model to imply credit spread, which requires estimating volatility. We consider 2 methods for finding this parameter:

- by using volatility estimate from investment bank analysts
- by using historical volatility

Numerical Result

We tested these three methods by selecting companies with and without CDS data and implementing our hedging strategies to compare the degree of variance reduction between portfolio 2 and portfolio 4. We performed an analysis of companies with CDS data here, as it allows us to determine the accuracy of all estimation methods through direct comparison with real CDS market data and helps us to decide which methods are appropriate for companies without CDS data. In this chapter, we pick Lufthansa and Umicore to represent the two cases above.

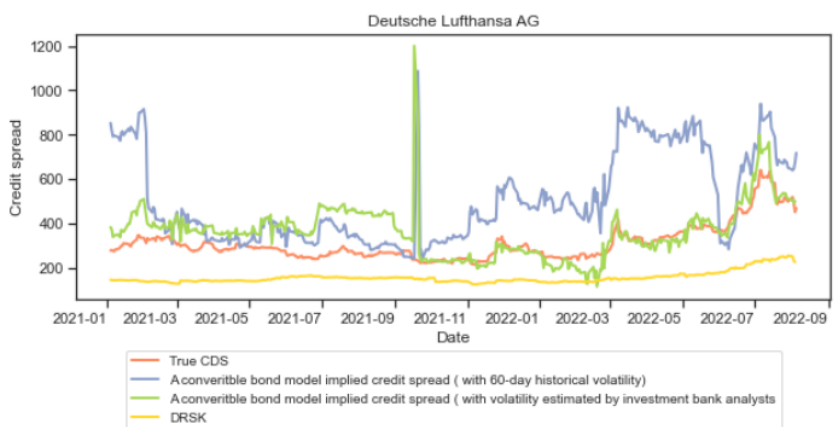


Figure 6.7: Implied spread comparison: Lufthansa (a company with both convertible bond and CDS).

Implied credit spread	Portfolio 1	Portfolio 2	Portfolio 4	Vol reduction
Convertible model(volatility estimated by investment bank)	0.165	0.096	0.093	0.938
Convertible model(60-day historical volatility)	0.165	0.095	0.098	1.068
DRSK	0.165	0.096	0.094	0.965

Table 6.8: Annualized Volatility in hedged portfolios: Lufthansa (a company with both convertible bond and CDS).

Figure 6.7 and Table 6.8 display the results for Lufthansa, a company with CDS spread data.

Among them, the implied spread calculated by a convertible bond model with the volatility estimated by investment bank analysts (the green line) best captures the true pattern and has the best hedging results, where we obtain the smallest F at 0.938 compared with when the credit-equity correlation is not considered. The spread calculated by a convertible bond model with 60-day historical volatility as the volatility input (the blue line) is highly volatile, even pushing up the volatility of the hedged portfolio and may benefit from being stabilized by adding a regularisation parameter. The implied spread approximated by the 5-Year CDS spread from the Bloomberg Issuer Default Risk model (the yellow line) is deviating from the true CDS line, it captures the general trend but smooths out fluctuations thus fails to capture many features that lie within the real data (the red line) and might lead to some bias if applied to dynamic hedging strategies.

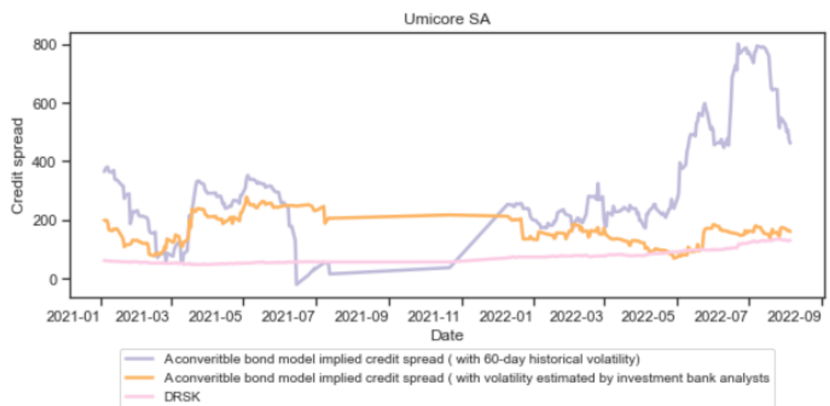


Figure 6.8: Implied spread comparison: Umicore (a company with convertible bonds that does not have CDS).

Implied Spread	Portfolio 1	Portfolio 2	Portfolio 4	Vol reduction
Convertible model(volatility estimated by investment bank)	0.138	0.046	0.046	0.979
Convertible model(60-day historical volatility)	0.138	0.045	0.049	1.268
DRSK	0.138	0.046	0.045	0.966

Table 6.9: Annualized Volatilities in hedged portfolios: Umicore (a company with convertible bond that does not have CDS).

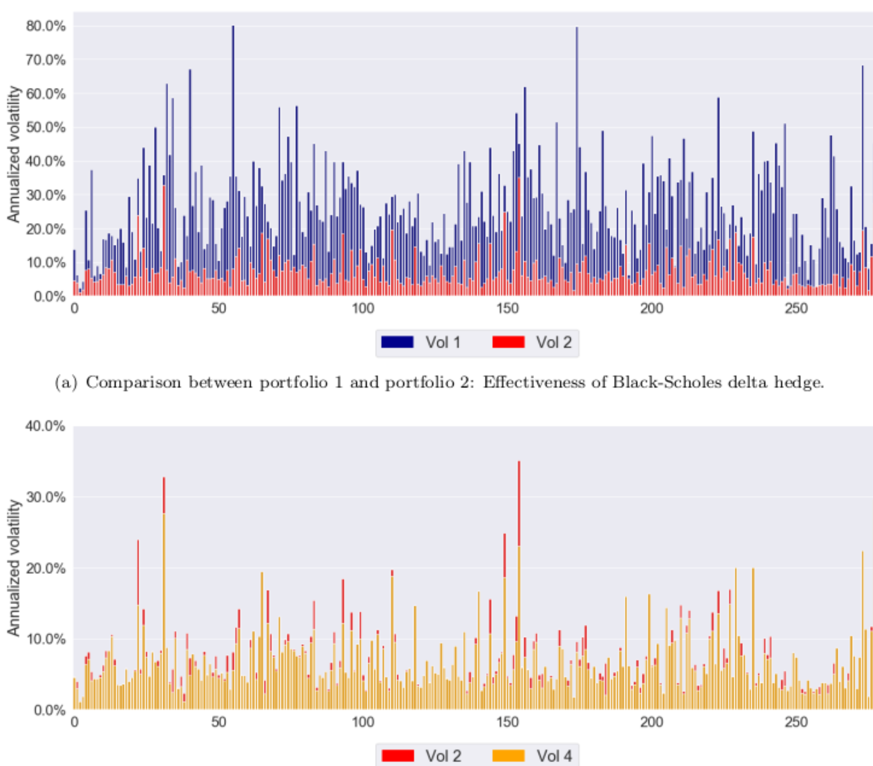
Figure 6.8 and Table 6.9 stores the results for Umicore, a company without CDS spread data. Again, the implied spread calculated by a convertible bond pricing model with the volatility estimated by investment bank analysts (the orange line) has the best hedging results, where F is as low as 0.966 compared with when the credit-equity correlation is not considered, whereas hedging strategies developed by using credit spread estimated by the other two methods do not effectively reduce the variance.

6.2.4 Results Analysis

We take Umicore as a sample company for those companies without CDS data. Umicore is a global materials technology company providing refining, metal and manufacturing services. It issued a convertible bond which will mature on 23^{rd} June 2025. Here, we predicted CDS using implied credit spread computed by a convertible bond given daily convertible bond price, volatility parameter estimated by investment bank analysts, borrow rate and financing rate with a rolling window of 21 trading days. We constructed the hedged portfolio similarly to the previous section, table and charts demonstrating the portfolio performance can be found in A.1.

Again, variance is reduced by applying the Black-Scholes delta hedging strategy to offset the risk on the convertible bond price associated with underlying stock price movement. Modifying delta by taking credit-equity correlation into the calculation, hedging credit by selling an additional number of shares reduces the variance of the hedged portfolio to an even lower level. Our strategy works well for this company.

We applied our strategy to the convertible bonds in our universe, filtering out companies with exchangeable, cross-currency and mandatory convertible bonds. Again, we use F defined in equation (6.1.1) as a measure of the variance-reducing impact.



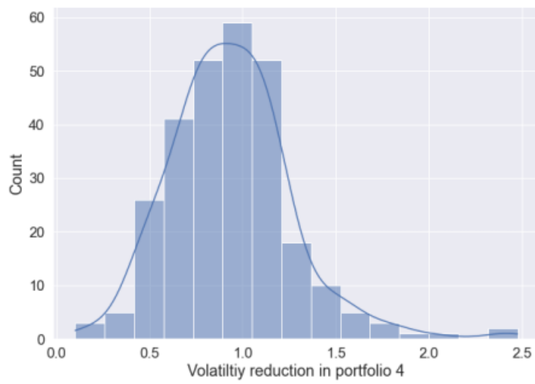
(a) Comparison between portfolio 1 and portfolio 2: Effectiveness of Black-Scholes delta hedge.

(b) Comparison between portfolio 2 and portfolio 4: Effectiveness of credit hedge using equity.

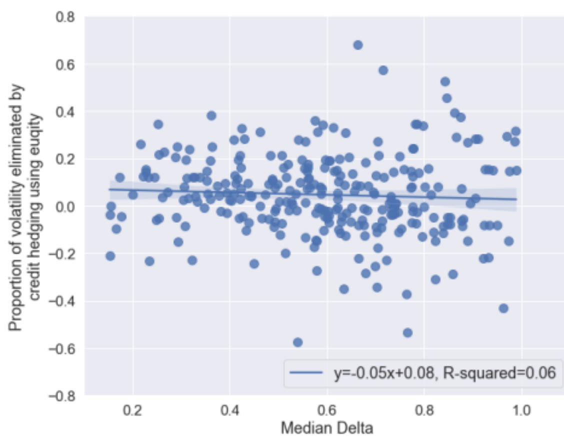
Figure 6.9: Back-testing results for all companies in our universe that do not have CDS I.

Classification by Delta

The charts shown in Figure 6.9 display the results from back-testing. Similarly to the results in Section 6.1.1, Black-Scholes delta hedging works well as σ_2 is greatly reduced compared with σ_1. σ_4 is reduced for some bonds but not for all bonds, indicating that the credit-adjusted delta hedge works under some conditions, such as credit sensitivity and credit risk level. Figure 6.10(a) shows the histogram of F, where the distribution of F is right-skewed, with a median at 0.915, indicating that the majority of convertible bonds show some variance-reducing impact brought by the equity hedge strategy. We find a negative relationship between hedging effectiveness and median delta from figure 6.10(b) that generally the proportion of volatility reduced by equity hedge is positive for convertible bonds with smaller delta, and the number becomes negative under a large delta.
If we classify convertible bonds into different groups by delta, table 6.10 shows the result within each group, where F is less than 1 for low delta and approximately equal to 1 when delta is high. This indicates that our strategy for hedging credit risk works well for low and balanced delta as



(a) Histogram of volatility reduction for companies in our universe that do not have CDS.



(b) Scatter plot of the proportion of volatility eliminated by credit hedging using equity and median delta.

Figure 6.10: Back-testing results for all companies in our universe that do not have CDS II.

the variance is reduced compared to the Black-Scholes delta hedge, there is no improvement in hedging effectiveness when delta is high. For high delta, convertible bonds and equity are almost perfectly correlated, and convertible bond return is very sensitive to equity but insensitive to credit risk, therefore adding a credit hedging strategy will not reduce variance. For the balanced delta bonds, variance reduction effect is lying between that of low delta and high delta, as we expected.

Delta	Count	Median variance reduction	95% CI
Low	25	0.888	(0.793, 1.019)
Balanced	208	0.915	(0.894, 0.981))
High	45	0.978	(0.839, 1.076)
Total	278	0.915	(0.909, 0.987)

Table 6.10: Variance reduction for convertible bonds with no CDS, profile classified by delta.

Classification by D/E ratio

If we classified convertible bonds into different groups by D/E ratio, table 6.11 shows the result within each group. Results are aligned with convertibles bonds that have CDS, a greater variance-reducing impact is seen on convertibles with issuer with high D/E ratio and greater credit risk.

D/E	Count	Median variance reduction	95% CI
Low	143	0.965	(0.885, 1.044)
High	138	0.920	(0.865, 0.974)
No D/E data available	9	-	-
Total	290	0.942	(0.894, 0.991)

Table 6.11: Variance reduction for convertible bond with no CDS, profile classified by D/E ratio.

We have demonstrated that equity can be used to hedge credit exposures in convertible bonds, as we see a significant variance reduction on average when altering the standard Black-Scholes delta hedge to incorporate credit-equity correlation. When selecting a hedge, this ability to reduce variance is important, but the cost of a hedging strategy is also important. In Section 6.1.1 we noted that CDS are relatively low cost. There is no explicit borrow cost for CDS, and although the bid-ask spread is an important factor, it is likely that proxy hedging credit with equity is more expensive than using CDS directly. However, for the majority of the universe of convertible bonds, CDS are not an available instrument, so the question becomes, is the cost of proxy hedging CDS with equity small enough that this can still be a valid option?

	Portfolio 1	Portfolio 2	Portfolio 4
Average annualized return	- 9.0%	0.8%	0.4%

Table 6.12: Average annualized returns in three portfolios.

In Table 6.12 we compare the returns from portfolio 2 and 4, where portfolio 4 has an average return lower than that in portfolio 2, this may be because volatility is reduced at the expense of return when hedging credit risk using equity. To bring these two dimensions together, we compare the Sharpe ratio in portfolio 4 to that of portfolio 2. We can see from the table 6.13 that we can see that Sharpe ratio is reduced for low delta bonds. This is the category that has the greatest credit exposure, and so the area that can benefit most from credit hedging. The Sharpe ratio is significantly worse for high delta bonds, which we expected, as we are not expecting to see reduced volatility and the additional delta hedge will introduce additional cost, worsening returns. For balanced convertibles, the Sharpe ratio is worse for the credit adjusted delta hedge. This area requires further research to understand, and we comment on our ideas around this below. We would also comment that we believe the equity borrow data that we have is quite conservative, and this could be impacting the Sharpe ratio by dampening the return more than is realistic.

Delta	Count	Portfolio 2	Portfolio 4
Low	25	- 0.56	- 0.40
Balanced	208	-0.25	- 0.34
High	45	- 0.06	- 0.51
Total	278	- 0.25	- 0.38

Table 6.13: Median Sharpe Ratio for convertible bond profile classified by delta.

6.2.5 Discussion and Future Research Direction

We have compared the results of hedging credit risk under different instruments and explored the circumstances in which they are most effective for hedging.

In Chapter 4, we compared the strengths and weaknesses of these hedging instruments, focusing on the direct hedging instrument, CDS that directly hedges credit risk through transfers to third parties and equity that utilizes its correlation with credit for their competitive cost-efficiency and tolerant basis risk. In Chapter 5, we developed closed-form formula to demonstrate the credit-equity link explicitly and precisely by calibrating the proposed models with historical data. In Chapter 6, we redefine delta by applying this formula so that the number of shares that need to be traded as part of a hedging strategy reflects the credit-equity correlation and the credit sensitivity of the convertible bond. Finally, we evaluate the variance reduction effect by building various unhedged and hedged portfolios, and back-testing over our universe to compare the remaining volatility levels in each portfolio. We concluded that a credit hedge is effective in an environment of low delta and high credit spread, i.e. high credit sensitivity and great credit risk, and the hedging effectiveness is an overlay of these two factors.

Our choice of hedging tool should be based on the specific circumstance. For instance, when we have liquid CDS then it is the ideal hedging tool. However, CDS hedge, in spite of excellent numerical results, is not applicable to hedge credit exposure for many bonds. The first limitation of this technique is the lack of availability. Indeed, only 16% of convertible bond issuers in our universe have available CDS. In this regard, CDS is not a universal hedging tool for all convertible bonds. Moreover, CDS contracts are not standardised and trade OTC, so inevitably this introduces liquidity problems.

So in practice, in the absence of CDS and an environment of low equity borrow cost, an equity hedge is preferred. However, this may introduce basis risk because of the unknown number of shares that need to be traded to offset the risk exposure. Under this strategy, the effectiveness of the hedge depends on our ability to accurately estimate that amount, which is fundamentally dependent on the effectiveness of our assumptions about the convertible bond, and the stability of the relationship between equity and credit. We know that the value of a convertible bond depends on the underlying equity price and the creditworthiness of the corresponding issuer, so the value of convertible bond can be expressed as a function of two inputs, stock price and credit spread.

The work and results of this thesis prompted the deliberation of another important factor, the sensitivity of volatility to movements in the underlying stock price. As volatility is an parameter in the valuation of a convertible bond, if this parameter is itself related to stock price moves, the valuation equation becomes:

$$V = f\left(S, CS(S), \mathrm{Vol}(S)\right)$$

where CS denotes the credit spread and Vol denotes the volatility. To decide the number of shares traded in the modified delta hedging strategy, we compute the sensitivity of convertible bond price to price change in the underlying equity. By taking the chain rule, we have:

$$\frac{dV}{dS} = \underbrace{\frac{\partial V}{\partial S}}_{\text{Gamma trading (1)}} + \underbrace{\frac{\partial V}{\partial CS}\frac{\partial CS}{\partial S}}_{\text{Credit trading (2)}} + \underbrace{\frac{\partial V}{\partial \mathrm{Vol}}\frac{\partial \mathrm{Vol}}{\partial S}}_{\text{Vega trading (3)}} \tag{6.2.2}$$

Gamma trading is the classic dynamic delta hedging strategy. In this thesis, we only discuss how to reflect credit-equity correlation into delta hedging but ignored the correlation between volatility and equity. In (6.2.2), the bond price is negatively correlated to credit spread, and credit spread is negatively correlated to stock price, so the partial derivative on V to S increases in credit, and term (2) is positive. An additional number of shares need to be sold to hedge the change in bond price associated with change in credit spread. However, because of the embedded option, convertible bond price rises with volatility, volatility and stock price tend to move in opposite directions, so the partial derivative decreases in vega, and term (3) is negative. This implies that we may need to reduce the equity hedge to offset the price change brought by changing volatility. Term (2) and term (3) work against each other and it is possible that the impact of (3) outweighs that of (2), such that we are over-hedging for convertible bonds by selling too many shares. It is potentially those extra shares we sold that led to an increase in the overall variance in the hedged portfolio, and reduced the effectiveness of our hedging strategy. This is an interesting potential area of further research.

Conclusion

In this thesis, we proposed using equity as an alternative credit-hedging instrument to proxy CDS and manage credit risk exposure on a hybrid security, convertible bonds. Our strategy is developed based on the classic dynamic delta hedging using a convertible bond valuation model. We assume that the value of a convertible bond depends on the underlying equity price and the creditworthiness of its issuer, so the value of a convertible bond can be expressed as a function of two inputs, stock price and credit spread. A key improvement presented by our model is the inclusion of the credit-equity correlation and we thereby establish a link between the two original model inputs. Combining the hypothetical relationships in the explicit functional form using the chain rule, we derived a formula for delta adjusted for credit risk. We then back-tested our new hedging strategy on 360 convertible bonds in our universe, comparing results to credit hedging with CDS for the 65 companies that also had CDS available. The hedging effectiveness is assessed by the degree to which the variance of the hedged portfolio is reduced.

In general, our model's performance in reducing variance is consistent with the theoretical results and our expectations. Therefore we conclude that in the absence of CDS, equity is an attractive and cost-effective hedging product with effective risk hedging ability, and we expect a more significant variance-reducing impact in an environment of high credit risk and strong credit sensitivity. We have also identified an area for further research, namely the impact that of the relationship of volatility with equity. Further exploration here may produce even better results in terms of variance reduction, and will very likely reduce the cost of a hedging strategy, as the negative relationship between equity and volatility will reduce the delta hedge quantity, and so reduce borrow costs. This then should have an even greater positive impact on the overall Sharpe ratio.

Appendix A

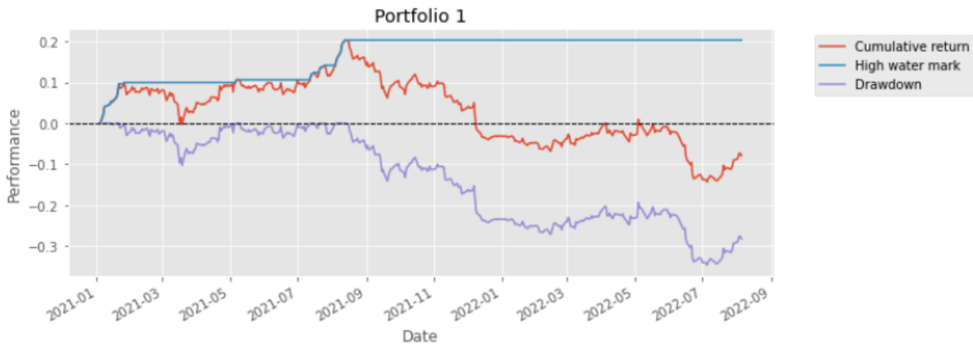
Figures and tables

A.1 Chapter 6 Figures

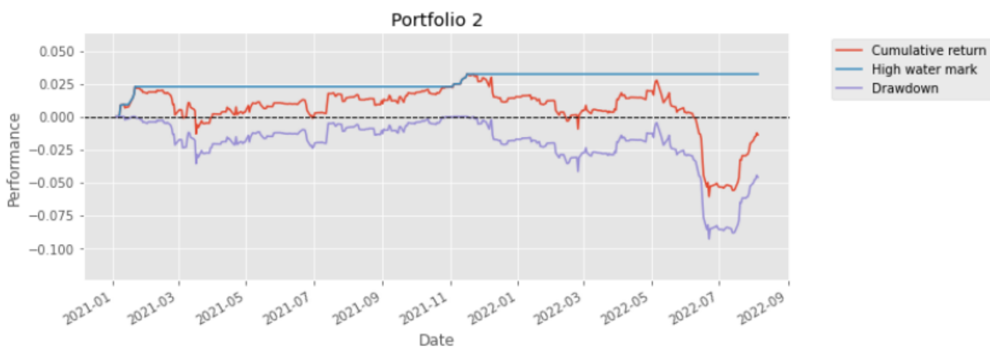
Portfolio Performance and statistical results for Umicore, a company with convertible bond issues in our universe that does not have CDS.

	Annualized Return	Annualized Volatility	Sharpe Ratio	Max Drawdown
Portfolio 1	- 0.0466	0.1378	- 0.4831	- 0.3434
Portfolio 2	- 0.0078	0.0462	- 0.6022	- 0.0941
Portfolio 4	- 0.0160	0.0457	- 0.7894	- 0.0901

Table A.1: Performance comparison for Umicore.

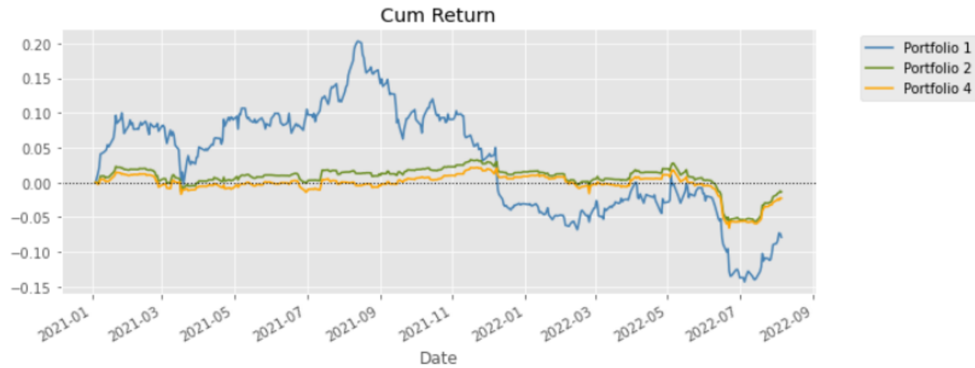


(a) Portfolio 1: the unhedged portfolio.

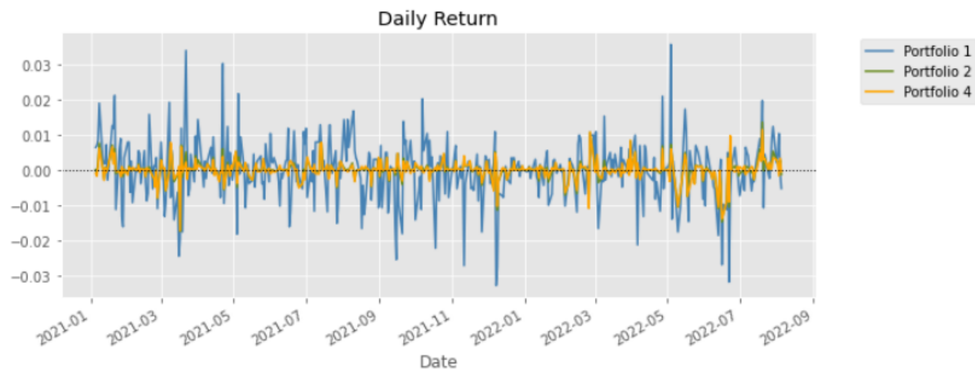


(b) Portfolio 2: The portfolio with dynamic delta hedging.

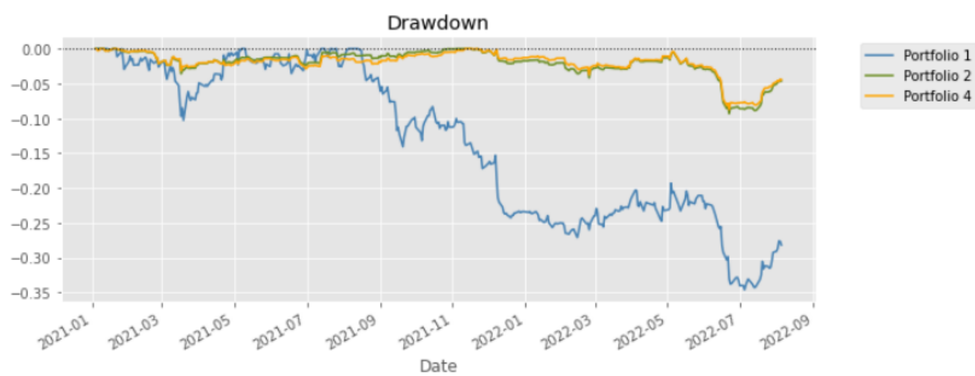
(c) Portfolio 4: The portfolio with credit-adjusted delta hedging.



(d) Cumulative return.



(e) Daily return.



(f) Drawdown.

Figure A.1: Performance comparison for Umicore.

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