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by Yuvraj Anand

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Imperial College London

Department of Mathematics

Algorithmic Market-Making for Options

Author: Yuvraj Anand (CID: 01498584)
Supervisor: Eyal Neumann

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Declaration

The work contained in this thesis is my own work unless otherwise stated.

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To my loving parents,
Mr. and Mrs. Anand.

Abstract

Market-making has become a recognised field of study and since the 1980s we have made much progress in formalising it and considering more complex real-life phenomenon. This paper serves as a theoretical review on results in electronic trading specifically focusing on 2 leading papers in the field, 'High-Frequency Trading in a Limit Order Book' by Avellaneda and Stoikov (2008) [AS08] and 'Algorithmic market-making for Options' by Baldacci *et al.* (2019) [BBG19]. While the first paper formalises for the reader the concepts of the dynamic programming principle and the approach of a market-maker trading through limit orders, the second paper considers a highly complex stochastic volatility model to solve the options market-maker's problem of determining optimal bid and ask quotes on a large book of options. Baldacci *et al.* (2019) [BBG19] approximate the portfolio in terms of its vega, allowing them to turn a high-dimensional stochastic optimal control problem into a low-dimensional functional equation that may be solved using interpolation techniques and a simple Euler scheme. Moreover, this theoretical review also extends 'Algorithmic market-making for Options' by Baldacci *et al.* (2019) [BBG19] by incorporating the appendices of Baldacci *et al.* which aim to relax both the continuous Δ-hedging assumption and constant-Vega approximation. The paper recommends numerically simulating the optimal strategy on a large book options to confirm tractability of the analytical solutions determined.

The above-mentioned papers are monumental in market-making today, yet requiring an understanding of these is reserved mostly to people with high quantitative calibre. This paper consolidates the recent work done in market-making into a self-sufficient theoretical review in order to allow other prospects to gain an introduction, insight and future potential on the subject.

Executive Summary

This section summarises the contents of the paper titled "Algorithmic Market-Making for Options". Since this is primarily a summary for the reader who need not read the paper, it presents no new analysis or findings.

Market-making has become a recognised field of study and since the 1980s we have made much progress in formalising it and considering more complex real-life phenomenon. A market-maker is fundamentally what is known as a liquidity provider. They do this by quoting both bid and ask prices to either directly their clients or via the Exchange to all other market participants for any chosen underlying assets. Since competition in financial markets has risen to the extent that trades must be executed at millisecond intervals, market-makers today are often replaced by market-making algorithms. These algorithms need to be mathematically clever to make money. This is to say they continuously face the risk that the mid-price of the underlying asset held moves adversely without the algorithm being able to liquidate this position.

Ho and Stoll (1981) [HS81] first formalised the concepts of algorithmic trading, followed by, Grossman and Miller's market-making model (1988) [MG88] which extended this Walrasian auctioneer approach on execution-based and arbitrage trading. Since then, we have made much progress in extending these concepts and modelling more complex real-life phenomenon.

This theoretical review focuses specifically on the recent results in electronic trading provided by two leading papers in the field, 'High-Frequency Trading in a Limit Order Book' by Avellaneda and Stoikov (2008) [AS08] and 'Algorithmic market-making for Options' by Baldacci *et al.* (2019) [BBG19].

1. **Purpose:** The two aforementioned papers are monumental in market-making today, yet they require an understanding that is reserved chiefly for people with high quantitative calibre. Thus my work serves to provide a systematic literature review, present two celebrated market-making models in literature, discuss elaborately different problem settings and then produce analytical results on these considered problem settings. To my knowledge and research, this is the first such systematic review that extends and explains in-depth the market-making problem and options market-making problem and then extends the setting of the latter paper to solve for an assumption-less version of the original problem considered by Baldacci *et al.* (2019) [BBG19].

2. **Methods:** The first paper, [AS08], formalises for the reader the concepts of the dynamic programming principle and the approach of a market-maker trading through limit orders. The problem setting introduces the incorporation of limit order trading and the utility function used is the usual Cara utility function, as it is an exponential utility function. In Chapter 2.1, the main foundations of the model by Avellaneda and Stoikov (2008) [AS08] are described: the dynamics of the market mid-price, the arrival rate of market orders subject to their respective distances from the market mid-price and the dealer's utility objective functional.

   Additionally, in chapter 3.1, the second paper, the main foundations of the model by Baldacci *et al.* (2019) [BBG19] are described: the dynamics of the market mid-price using a one-factor stochastic volatility model, the optimisation problem of the options market-maker and lastly the assumptions needed to address the problem from a theoretical point of view and the approximations needed to simulate these optimal quotes.

   The methods used in this paper include secondary research on statistics for intensities in econophysics literature, devising problem settings for two fundamental market-making problems and lastly assessing the need for certain assumptions in order to attain analytical solutions.

3. **Results/ Contributions:** This paper has produced theoretical results to both the problems considered and thus the contributions made shall be individually discussed below.

   Chapter 2.2 solves for the optimal bid and ask quotes and compares them with the dealer's indifference price accounting for their personal risk considerations and given their current inventory. We suggest that the optimal bid and ask quotes may be determined using a straightforward two-step procedure. This intuitive procedure first requires one to solve the non-linear PDE (2.2.4) and determine $i^b(s, q, t)$ and $i^a(s, q, t)$. Next, they need to solve the implicit dependencies (2.2.7) and (2.2.8) in order to calculate the optimal distances from the market mid-price, $\delta^b(s, q, t)$ and $\delta^a(s, q, t)$, and thus solve the market-maker's problem of finding optimal bid and ask quotes. In addition to this, the paper also extends its contributions, namely in Chapter 2.3, by considering a Geometric Brownian Motion for the market mid-price which was earlier governed by a standard Brownian Motion.

   Following this, Chapter 3.1 introduces the Constant-Vega assumption that is essential in simplifying this high-dimensional stochastic optimal control problem of the market-maker into a low-dimensional functional equation. This mathematical simplification allowed me to show in Chapter 3.2 that optimal bid and ask quotes for a book of options with several strikes and maturities may be computed straightforwardly using interpolation techniques and an explicit Euler scheme. The results showcase reducing the dimensionality of the problem using the constant-Vega approximation. According to Baldacci *et al.* (2019) [BBG19], this first step is known as 'Beating the curse of dimensionality'. Once the value function has been reduced to a low-dimensional optimal control problem, we formulate the Hamilton-Jacobi-Bellman equation and thus solve for the optimal controls δ_t^{i,j^*}.

   Even though during the whole course of this chapter, 3.2, we have assumed that the market-maker is continuously Δ-hedged, the last chapter, 3.3, considers the problem without this assumption, in essence generalising the problem for wider applications. This is a significant contribution of the paper as previously, the solution was discussed in Baldacci *et al.*'s appendix yet this paper extends this and verifies that the options market-making problem remains tractable on a large book of options even without the initial assumptions.

4. **Conclusion and Recommendations:** While the first paper formalises for the reader the concepts of the dynamic programming principle and the approach of a market-maker trading through limit orders, the second paper considers a highly complex stochastic volatility model to solve the options market-maker's problem of determining optimal bid and ask quotes on a large book of options. We believe we have achieved methods (discussed above) that scale linearly in the number of options and thus may be computed on a relatively large book of options.

   However, the theoretical review has its limitations. A possible extension to this paper could be the simulation of the optimal strategies suggested by both Avellaneda and Stoikov (2008) [AS08] and Baldacci *et al.* (2019) [BBG19]. This will validate the tractability of the methods and their efficiency in practical applications, i.e. computing the optimal bid and ask quotes for the $N \geq 1$ options in Baldacci *et al.*, confirming that the computations scale linearly in the number of options and therefore then computing the quotes on a relatively large book of options.

   Alternatively, the options market-making problem can also be studied using purely reinforcement-based machine learning techniques as these policies produce efficient results in practice as well. According to latest research by Selser *et al.* (2021) [SKM21], Reinforcement Learning algorithms can be used to solve the classic quantitative finance Market Making problem. This extension pairs well with the practical simulation of the optimal strategy as this will allow us to evaluate both strategies simultaneously according to desired execution qualities.

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Chapter 1

Introduction and Background

This introductory section aims to equip the reader with the required financial knowledge and mathematical tools to approach the more advanced algorithmic trading problems that shall be discussed extensively in Chapters 2 and 3. This chapter is split into three subsections; the first subsection is helpful to gain an insight into the way modern electronic markets operate, the second subsection provides an extensive theoretical overview of stochastic calculus and optimal control theory, whilst the final subsection uses sophisticated mathematical models to initiate discussions on market-making.

Section 1 introduces the reader to electronic markets and looks at what assets are mainly traded on these electronic markets. Furthermore, it identifies who the main types of market participants are, why they trade and how their interactions occur within this market. Next, the fundamentals of an electronic exchange are discussed; this includes limit orders, market orders and how a transaction is carried out. This section ends with a brief explanation of the limit order book.

Section 2 develops a mathematical background to study trading algorithms. This section explains the fundamental portfolio optimisation problem proposed by R. Merton (1971) [RM71], and then explains the dynamic programming principle. This is at the essence of stochastic optimal control theory and underpins a mechanism by which one can attain a solution to the portfolio optimisation problem. Next, the Hamilton-Jacobi-Bellman equation is explained and the framework leading to the solution of the portfolio optimisation problem is shown. After this the Verification Theorem is stated and proved by which one can assure their optimal solution is indeed optimal. Lastly, the section concludes with solving Merton's portfolio optimisation problem and finding the optimal control.

Section 3 serves as a mathematical introduction to market-making. It discusses the preliminary attempts made by Grossman and Miller in modelling the market-maker's problem and briefly looks upon market-making using limit orders, as this shall later be studied in Chapter 2, Avellaneda and Stoikov's solution.

Following the introduction, the reader will be exposed to two papers that are monumental in market-making today; 'High-Frequency Trading in a Limit Order Book' by Avellaneda and Stoikov (2008) [AS08] and 'Algorithmic market-making for Options' by Baldacci *et al.* (2019) [BBG19]. While the first paper formalises for the reader the concepts of the dynamic programming principle and the approach of a market-maker trading through limit orders, the second paper considers a highly complex stochastic volatility model to solve the options market-maker's problem of determining optimal bid and ask quotes on a large book of options.

Baldacci *et al.* (2019) [BBG19] approximate the portfolio in terms of its vega, allowing them to turn a high-dimensional stochastic optimal control problem into a low-dimensional functional equation that may be solved using interpolation techniques and a simple Euler scheme. Last but not the least, this theoretical review also extends 'Algorithmic market-making for Options' by Baldacci *et al.* (2019) [BBG19] by incorporating the appendices of Baldacci *et al.*.

1.1 Electronic Markets

This section introduces the reader to electronic markets and looks at the types of assets mainly traded on these electronic markets. Furthermore, it identifies the main types of market participants, why they trade and how their interactions occur within this market. Next, the fundamentals of an electronic exchange are discussed; this includes limit orders, market orders and how a transaction is carried out. This section ends with a brief explanation of the limit order book.

The references utilised for this section include R. Almgren and N. Chriss (2000) [AC00], Á. Cartea *et al.* (2015) [CJP15], O'Hara *et al.* (1997) [OM97], H. Pham (2009) [HP09], S. Shreve (2013) [SS13], R. Donnelly (2022) [RD22] and R. Donnelly (2022) [RD222].

1.1.1 What are electronic markets and how do they operate?

Electronic markets, as defined by Security and Exchange Commission (SEC) in the U.S.A, are "professional traders acting in a proprietary capacity who generate a large number of trades on a daily basis". Due to the dramatic growth of electronic exchanges over the last decade, the observed market microstructure has been forced to change. For example, it is noticeable that at higher trading volume, the bid-ask spread for large-cap stocks has become tightened.

Furthermore, electronic markets share some common characteristics, including:

1. Since orders need to be executed with 'high-frequency', programs and algorithms must be used in place of humans to execute at the required rate
2. Numerous orders may be submitted and cancelled even after submission
3. Market participants seek to liquidate and close their positions at the end of a trading day.

Therefore, in today's world, there is a multitude variety of financial contracts being traded on electronic markets; here we broadly discuss the main ones.

Shares, equities or corporate stock are claims of ownership on corporations. As per recent studies, Cartea *et al.* [CJP15] suggest that shares are the most common type of asset traded on electronic markets. When these shares are sold to the public, the corporation is simply trying to raise money. They raise money through an Initial Public Offering (IPO) and are listed and traded in an exchange (for example, the New York Stock Exchange (NYSE), the Nasdaq, the London Stock Exchange (LSE) and the Tokyo Stock Exchange (TSE)). Simply put, the share is a right to receive an equal share of the corporation's profits (thus, it gets the name 'share') and to contribute to the firm's decision process via voting rights in the annual shareholder's meetings. They may also be referred to as ordinary shares or common stock in literature.

Alternatively, large corporations may use Bonds, which is another primary instrument, to raise money. Bonds are contracts that necessitate a corporation to pay the holder of its bond a regular income, also known as an interest but give them no decision rights.

Since there is a vast range of financial contracts, they have been categorised into different asset classes according to their underlying characteristics. Bonds are an asset class of their unique type and is often differentiated from cash which may be in the form of savings deposits or treasury bills, essentially investments with a short-term horizon and heavy guarantees and lower returns. More exotic asset classes are in the form of real estate, foreign exchange (FX) or commodities.

An investor can see these asset classes listed on electronic exchanges, usually in mutual funds and exchange-traded funds (more commonly known by its abbreviation, ETFs). ETFs ensure greater liquidity to an investor and a more equity-like setting to provide ease in diversification. A mutual fund is an investment fund that tracks an index and collects money from different investors. Alternatively, when an investor purchases an ETF, they delegate also their money to a portfolio manager. However, there are two main differences between a mutual fund and an ETF. The first is that an exchange-traded fund generates the same return as a specific index (e.g. S&P500). The second is that an investor who purchases an ETF can close their participation in the fund, and in return, the issuer could give the investor a basket of securities which has had the same performance as the ETF.

In recent times beating the benchmark index has become a desired quality risky investors look for in investment firms. However, since mutual funds have to fulfil plenty of fiduciary duties, they cannot always chase these abnormal profits in the midst of regulatory responsibilities. Thus, a new type of investment firm known as hedge funds has become popularised in this field. Because of their bullish, aggressive trading strategies and transparent regulatory requirements, they can to take on more significant risks and use strategies that exploit the leveraging concept.

As a final note to this section, a mathematician needs to have a background in the financial market before studying the dynamics of the stochastic processes involved in the asset state variables. This is because when it comes to designing strategies and algorithms, one must be aware of the limitations of the asset(s) and the particular exchange(s) in consideration. In addition, the analyst involved in studying and making financial decisions must be aligned with the trading objectives of their investors. Thus it is vital to understand the types of market participants that exist in an electronic exchange. This is done extensively in Section 1.1.2.

1.1.2 Classification of market participants

An electronic market revolves around and moves according to its participants and their respective interactions. Despite the fact that each participant looks solely for their own benefit, they may harness different methods to accomplish this shared goal of generating personal profits. Since there are various different ways to make money in the electronic market, algorithms need to be designed meticulously to achieve the desired objectives of the participants. Below we list the most common types of participants that trade in an electronic market:

1. **Market-Maker:** A market-maker, also known as a liquidity provider, aims to create a smooth flow of transactions, as this participant is willing to buy and sell assets at most times. A market-maker will seek to make profits on their quoted bid-ask spreads; this problem shall be further discussed and is the paper's primary focus. We shall take on the role of this participant and assess the optimal quotes they must produce whilst considering different market situations and models.

2. **Fundamental Trader:** This participant uses information known as the 'Fundamentals' of an asset to determine whether to buy or sell it. This information may be in the form of a news release and thus its implications and effects will be measured and then acted on upon by these participants. They use secondary sources of information, including political affairs, economic reports, rumours and the performance of similar assets.

3. **Corporate Issuer:** These corporations generate profits using IPOs, and these transactions generally occur due to a need or prevailing situation faced by the corporation. Moreover, these participants have to option to increase or decrease their supply of shares of the underlying asset by employing a Secondary Share Offering (SSO). These can be share buybacks or converted bonds.

4. **Financial Management Corporation:** These corporations manage funds such as mutual funds or ETFs. They can be distinguished into the two following categories based on the duration of their investment:

    - Short-term Investors: these tend to replicate an index, and a prime example is the ETF.

    - Long-term Investors: these participants base their analysis solely on the 'fundamental value'.

5. **Technical Trader:** These participants work under a critical assumption that the past price history of an asset can determine future movements with some probability. They collect information, analyse, predict and then make market decisions to pursue profits. They use information such as stock charts and stylised facts to assess assets or portfolios. The most common tools Technical traders employ include stock patterns, momentum, moving averages and price points of particular resistance.

6. **High-Frequency Trader:** These traders work to accomplish and execute relatively big orders where execution speed is of utmost priority. They may be grouped into the two following categories below:

   - Execution Traders: This participant is usually an algorithm that executes an order by breaking it into smaller chunks and executing these in a timely fashion such as to minimise the temporary price impact caused, otherwise, by trading the entire order volume at once. These algorithms work under a limited time horizon and a set objective of maximising profits, usually requiring a framework known as the Dynamic Programming Principle (more details on this in Section 2).

   - Arbitrageurs: As the name suggests, these participants seek to find an arbitrage in the market by trading on price inefficiencies to make profits.

1.1.3 Trading in an electronic market and the L.O.B.

There are two fundamental categories of orders on electronic markets; the market orders (MOs) and the limit orders (LOs). Each one is defined as the following:

- Market order: this is used when a dealer must execute their buy/sell trade immediately at the best price available, i.e. the market mid-price. This is trivially considered to be the most aggressive order as the dealer prioritises the execution of the full order volume over the price they will receive for their transaction.
- Limit order: this is used when a dealer seeks to execute an order with a specific price, and up to a specified volume of shares. A limit order gives its dealer control over the execution price, however the limit order is not guaranteed to be executed. Moreover, a limit order will also not execute immediately, unlike market orders, as the dealer will have to wait until their order is matched with incoming orders, or is itself cancelled.

The major differences between them are the urgency and the flexibility with which a dealer must execute their trade.

Lastly, this short primer on electronic markets ends with yet another fundamental concept known as the 'Limit Order Book'. The limit order book is an illustration of the liquidity available at the current time t in the market. Market orders need to be matched with limit orders and there are two main considerations taken into account during this 'matching' procedure.

If a market order arrives at time t, it will be matched with a limit order if:

- the quantity demanded by the market order is **less** than what is offered at the best price available and in this case the 'matching' procedure will match the market orders starting from the earliest processed one until all market orders are complete. This is known as price-time priority.
- the quantity demanded by the market order is **more** than what is offered at the best price available but in this case all available quantity at the first best price is matched, then all the available quantity in the second best price is matched and so on, so forth until the market order is complete. Note that in this case the dealer will be transacting the same underlying asset at different prices by a phenomenon referred to as 'walking the book' or in more formal terminology the 'Temporary Price Impact'.

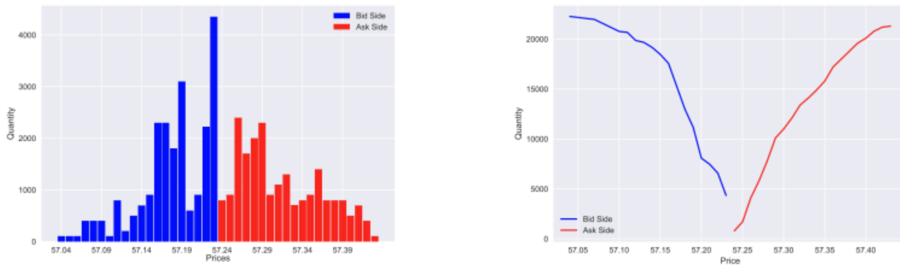

Figure 1.1: An example of a limit order book with its corresponding demand-supply graph.

1.2 Stochastic Optimal Control Theory

This section develops a mathematical background to study trading algorithms. Since financial assets tend to evolve subject to random perturbations, this sub-field of control theory known as *Stochastic Control* aims to study the dynamics of these random systems. This study of random systems has found its applications in economics, management, finance, sports betting, biology and many other subjects. Thus, over the past few decades, much of the work of applied mathematics has been targeted toward this field.

This section first explains the fundamental portfolio optimisation problem, as proposed by R. Merton (1971) [RM71], and then explains the dynamic programming principle. This is at the essence of stochastic optimal control theory and underpins a mechanism by which one can attain a solution to the portfolio optimisation problem. Next, the Hamilton-Jacobi-Bellman equation is explained and the framework leading to the solution of the portfolio optimisation problem is shown. After this, the Verification Theorem is stated and proved by which one can assure their optimal solution is indeed optimal. Lastly, the section concludes with solving Merton's portfolio optimisation problem and finding the optimal control.

The references utilised for this section include R. Merton (1971) [RM71], Á. Cartea *et al.* (2015) [CJP15], T. Bjork (2009) [TB09], O'Hara *et al.* (1997) [OM97], H. Pham (2009) [HP09], S. Shreve (2013) [SS13], E. Neumann (2021) [EN21], and R. Donnelly (2022) [RD22].

1.2.1 Merton's portfolio optimisation problem

Consider a dealer at time t who wishes to maximise their portfolio's expected utility. They can do this by allocating their wealth in a risk-free bank account or a risky asset or any combination of the two. Let us now define the following processes according to Merton [RM71]:

- $B = (B_t)_{0\leq t\leq T}$ is the risk-free bank account and satisfies:

$$dB_t = rB_t dt$$

- $W = (W_t)_{0\leq t\leq T}$ is a Brownian Motion also known as the Wiener Process, hence the W
- $S = (S_t)_{0\leq t\leq T}$ is the discounted risky mid-price process and thus satisfies the Stochastic Differential Equation, SDE:

$$\begin{aligned} dS_t &= (\mu - r)S_t dt + \sigma S_t dW_t, \\ S_0 &= s \end{aligned}$$

- $\pi = (\pi_t)_{0\leq t\leq T}$ is the dealer's self-financing strategy. This indicates the amount of money allocated in the risky asset at time t
- $X^\pi = (X_t^\pi)_{0\leq t\leq T}$ is the dealer's discounted wealth process given the self-financing strategy π. This satisfies the SDE given by:

$$dX_t^\pi = (\pi_t(\mu - r) + rX_t^\pi)dt + \pi_t \sigma dW_t,$$
$$X_0^\pi = x$$

and therefore, the optimal control problem of portfolio maximisation can be formulated as:

$$H^{\pi,t}(s,x) = \sup_{\pi \in \mathcal{A}_{0,T}} \mathbb{E}_{s,x}\left[u(X_T^\pi)\right]. \tag{1.2.1}$$

Here $U(x)$ is the dealer's Utility Function, $\mathcal{A}_{t,T}$ is the set of all admissible strategies for Merton's portfolio optimisation problem. The set of admissible strategies $\mathcal{A}_{t,T}$ corresponds to all the $\mathbb{F}$-predictable self-financing strategies such that:

1. $\int_t^T \pi_s^2 ds < \infty$
2. $\mathbb{E}_{s,x}[.]$ is the conditional expectation given $S_t = s$ and $X_t = x$.

Merton first formalised this problem in 1971 and it is a highly regarded problem as it forms the base case to study stochastic control, a sub-field of the much broader field Control theory. The following few subsections will involve deriving and proving significant mathematical results in Stochastic control theory. This will then allow us to create a framework by which to solve (1.2.1) and in turn solve for the optimal control π^*.

1.2.2 Dynamic programming principle

Furthermore, it is essential to introduce the reader to the definition of filtered probability spaces. They are important in forming the setting for defining and studying stochastic processes and thus must be formally pertained to.

Definition 1.2.1. A filtered probability space, or stochastic basis, $(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\geq 0}, \mathbb{P})$ consists of a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and a filtration $(\mathbb{F}_t)_{t\geq 0}$ contained in $\mathbb{F}$, (gel, 2013).

In a continuous-time setting like the one considered in this paper, it is convenient to impose further conditions, sometimes referred to as the usual conditions or usual hypotheses.

The filtered probability space $(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\geq 0}, \mathbb{P})$ satisfies the usual conditions if the following are met:

1. The probability space $(\Omega, \mathbb{F}, \mathbb{P})$ is complete
2. The σ-algebras $\mathbb{F}_t$ contains all the sets in $\mathbb{F}$ of zero probability
3. The filtration $\mathbb{F}_t$ is right-continuous. This means that for every non-maximal $t \geq 0$, the σ-algebra $\mathbb{F}_{t^+} \equiv \cap_{s>t}\mathbb{F}_s$ is equal to $\mathbb{F}_t$.

We assume that the filtered probability space $(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\geq 0}, \mathbb{P})$ satisfies the usual conditions or hypotheses. In addition to this, we work with a controlled model or state $X = (X_t)_{t\geq 0}$ defined on $\mathbb{R}^n$ with SDE:

$$dX_s = b(X_s, v_s)ds + \sigma(X_s, v_s)dW_s. \tag{1.2.2}$$

Here $W = (W_t)_{t\geq 0}$ is a d-dimensional Brownian Motion (BM) or Wiener Process (hence the index W). $v = (v_t)_{t\geq 0}$ is the control process that satisfies the following conditions:

- Measurable with respect to $\mathbb{F}$
- $v = (v_t)_{t\geq 0}$ is defined on $\mathcal{A} \subseteq \mathbb{R}^n$.

Equation (1.2.2) formulates the behaviour of the continuous-time stochastic process X_t. In an infinitesimal time interval δ, the stochastic process X_t evolves according to a normal distribution with expectation $b(X_t, v_t)\delta$ and variance $\sigma(X_t, v_t)^2\delta$. Moreover, each increment is independent of the process's past behaviour. This is because a Brownian Motion or Wiener process has increments that are independent and normally distributed.

The function $b(X_t, v_t)$ is the drift coefficient and $\sigma(X_t, v_t)$ is the diffusion coefficient. Both these functions satisfy the Lipschitz condition in the set of admissible strategies $\mathcal{A}$ are measurable such that:

$$b(X_t, v_t) : \mathbb{R}^n \times \mathcal{A} \longrightarrow \mathbb{R}^n$$
$$\sigma(X_t, v_t) : \mathbb{R}^n \times \mathcal{A} \longrightarrow \mathbb{R}^{n \times d}$$

The stochastic process X_t is called a diffusion process and it satisfies the Markov property. It can be helpful to interpret the SDE (1.2.2) in this way.

The set of stopping times in the interval $[t, T]$ is denoted as $\tau_{t,T}$. The set of admissible strategies $\mathcal{A}$ is the set of control processes v such that:

$$\mathbb{E}\left[\int_0^T (|b(0, v_t)|^2 + |\sigma(0, v_t)|^2)dt\right] < \infty.$$

Now, we assign f and g to be measurable functions of the form:

$$f : [0, T] \times \mathbb{R}^n \times \mathcal{A} \longrightarrow \mathbb{R}^n$$
$$\sigma(X_t, v_t) : \mathbb{R}^n \longrightarrow \mathbb{R}.$$

In addition to this, g is assumed to be bounded from below and satisfies a growth condition of quadratic order, i.e. $|g(x)| \leq C(1 + |x|^2)$, for all $x \in \mathbb{R}^n$ and some constant C independent of x.

Consider now for all $(t, x) \in [0, T] \times \mathbb{R}^n$ a non-empty subset of controls $v \in \mathcal{A}(t, x)$ such that:

$$\mathbb{E}\left[\int_t^T |f(s, X_s, v_s)|ds\right] < \infty.$$

Thus the gain function $H^v(t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and $v \in \mathcal{A}(t, x)$ is:

$$H^v(t, x) = \mathbb{E}\left[\int_t^T f(s, X_s, v_s)ds + g(X_T)\right].$$

Since this is a portfolio maximisation problem, the goal here is to maximise the gain function over the control process. This can be represented mathematically as:

$$H(t, x) = \sup_{v \in \mathcal{A}(t,x)} H^v(t, x). \tag{1.2.3}$$

Once this equation is solved, one yields an optimal control v^*. v^* is an optimal control if and only if $H(t, x) = H^{v^*}(t, x)$. Additionally, if the control has the form $v_s = a(s, X_s)$ for some measurable function $a : [0, T] \times \mathbb{R}^n \longrightarrow \mathcal{A}$, then this control is called a Markov Control.

We can now go on to state and prove the Dynamic programming principle (DPP), attributed to Richard Bellman (the 'B' in 'HJB' equation stands for Bellman).

Theorem 1.2.2. *(Dynamic Programming Principle). Let us consider (t,x) in $[0,T]\times\mathbb{R}^n$ under the assumptions above mentioned, then the following equation holds:*

$$\begin{aligned} H(t,x) &= \sup_{v\in\mathcal{A}(t,x)}\sup_{\theta\in\tau_{t,T}}\mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},v_s)ds + H(\theta,X_\theta^{t,x})\right] \\ &= \sup_{v\in\mathcal{A}(t,x)}\inf_{\theta\in\tau_{t,T}}\mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},v_s)ds + H(\theta,X_\theta^{t,x})\right] \end{aligned}$$

Proof. Firstly, using the Markov property of X, we can assume that:

$$\begin{aligned} X_s^{t,x} &= X_s^{\theta,X_\theta^{t,x}}, \\ \theta &\le s. \end{aligned}$$

$X_s^{t,x}$ denotes the process X at time s given that $X_t = x$ with $t \le s$, and θ is a stopping time in the interval $[t,T]$. For an arbitrary control v and utilising the law of iterated conditional expectations:

$$H^v(t,x) = \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},v_s)ds + H^v(\theta,x_\theta^{t,x})\right].$$

Since by construction $H^v(t,x)\le H(t,x)$, then this implies:

$$\begin{aligned} H^v(t,x) &\le \inf_{\theta\in\tau_{t,T}}\mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},v_s)ds + H(\theta,x_\theta^{t,x})\right] \\ &\le \sup_{v\in\mathcal{A}(t,x)}\inf_{\theta\in\tau_{t,T}}\mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},v_s)ds + H(\theta,x_\theta^{t,x})\right], \end{aligned}$$

Evaluating the supremum over all the controls $v\in\mathcal{A}(t,x)$ in the left-hand side, gives:

$$H(t,x) \le \sup_{v\in\mathcal{A}(t,x)}\inf_{\theta\in\tau_{t,T}}\mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},v_s)ds + H(\theta,x_\theta^{t,x})\right]. \tag{1.2.4}$$

Now we make a case for the second equality. An arbitrary control $v\in\mathcal{A}(t,x)$ and a stopping time $\theta\in\tau_{t,T}$ are fixed. Using (1.2.3) and for any $\epsilon>0$ and $\omega\in\Omega$, there exists a control $v^{\epsilon,\omega}\in\mathcal{A}(\theta(\omega),X_{\theta(\omega)}^{t,x}(\omega))$ such that:

$$H(\theta(\omega),X_{\theta(\omega)}^{t,x}(\omega)) - \epsilon \le H^{v^{\epsilon,\omega}}(\theta(\omega),X_{\theta(\omega)}^{t,x}(\omega)). \tag{1.2.5}$$

Let us consider the control process $\overline{v}_0(\omega)$:

$$\overline{v}_0(\omega) = \begin{cases} v_s(\omega)\ s\in\ [0,\theta(\omega)] \\ v_s^{\epsilon,\omega}(\omega)\ s\in\ [\theta(\omega),T] \end{cases}$$

Making use of the law of iterated conditional expectation and (1.2.5), it follows:

$$\begin{aligned} H(t,x) \ge H^{\overline{v}}(t,x) &= \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},v_s)ds + H^{v^{\epsilon,\omega}}(\theta,X_\theta^{t,x})\right] \\ &\ge \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},v_s)ds + H(\theta,X_\theta^{t,x})\right] - \epsilon, \end{aligned}$$

because of the arbitrary choice of v, θ and $\epsilon>0$.

Hence, one can conclude:

$$H(t,x) \ge \sup_{v\in\mathcal{A}(t,x)}\sup_{\theta\in\tau(t,T)}\mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},v_s)ds + H(\theta,X_\theta^{t,x})\right]. \tag{1.2.6}$$

(1.2.4) and (1.2.6) prove both sides; therefore, the dynamic programming principle is proved. □

1.2.3 Hamilton-Jacobi-Bellman equation

The HJB equation explains the local behaviour of the goal gain functional equation (1.2.3) when the stopping time θ tends to t. It is also known as the infinitesimal version of the DPP. Here, we shall derive it using the following approach.

Set $\theta = t + h$, and also set a constant control of the form $v = a$, for some constant a, then using equation (1.2.1):

$$H(t,x) \geq \mathbb{E}\left[\int_t^{t+h} f(s, X_s^{t,x}, a)ds + H(t+h, X_{t+h}^{t,x})\right]. \tag{1.2.7}$$

Assuming that H is a function smooth enough to apply the Ito formula in the time interval characterised as $[t, t+h]$, we have:

$$H(t+h, X_{t+h}^{t,x}) = H(t,x) + \int_t^{t+h}\left(\frac{\partial H}{\partial d} + \mathcal{L}^a H\right)(s, X_s^{t,x})ds + local\, martingale,$$

Here $\mathcal{L}_H^a$ is the infinitesimal generator associated to equation (1.2.2), described as:

$$\mathcal{L}^a H = b(x,a)D_x H + \frac{tr(\sigma(x,a)\sigma^T(x,a)D_{xx}H)}{2}. \tag{1.2.8}$$

Substituting (1.2.8) in (1.2.7) gives:

$$0 \geq \mathbb{E}\left[\int_t^{t+h}\left(\frac{\partial H}{\partial t} + \mathcal{L}^a H\right)(s, X_s^{t,x}) + f(s, X_s^{t,x}, a)ds\right], \tag{1.2.9}$$

Now consider dividing (1.2.9) by h and taking $\lim_{h\to 0}$, which gives:

$$0 \geq \frac{\partial H}{\partial t}(t,x) + \mathcal{L}^a H(t,x) + f(t,x,a).$$

As the choice of $a \in \mathcal{A}$ was arbitrary, we have:

$$-\frac{\partial H}{\partial t}(t,x) - \sup_{a\in\mathcal{A}}[\mathcal{L}^a H(t,x) + f(t,x,a)] \geq 0. \tag{1.2.10}$$

In particular, suppose that v^* is an optimal control, and using a similar argument, we can arrive at the conclusion:

$$0 = -\frac{\partial H}{\partial t}(t,x) - \mathcal{L}^{v^*} H(t,x) - f(t,x,v^*),$$

and,

$$-\frac{\partial H}{\partial t}(t,x) - \sup_{a\in\mathcal{A}}[\mathcal{L}^a H(t,x) + f(t,x,a)] = 0, \tag{1.2.11}$$

for all $(t,x) \in (0,T] \times \mathbb{R}^n$.

(1.2.11) is celebrated Hamilton-Jacobi-Bellman, HJB, equation or the DPP equation with terminal condition given by

$$H(T,x) = g(x),$$

for all $x \in \mathbb{R}^n$.

1.2.4 The verification theorem

The verification theorem is of vital importance to control theory. It is used to confirm that a given smooth solution to the HJB equation is consistent with the solution of equation (1.2.3), i.e. the solution to the maximisation of the gain function over the control process. We state the theorem and then prove it, adapting Ryan Donnelly's [RD22] approach.

Theorem 1.2.3. *(The Verification Theorem). Let q be a function in $\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^n)\cap\mathcal{C}^0([0,T]\times\mathbb{R}^n)$ that satisfies a quadratic growth condition, i.e. there exists a constant C independent of x such that:*

$$|q(t,x)| \leq C(1+|x|^2),$$

for all $(t,x)\in(0,T]\times\mathbb{R}^n$.

1. *Suppose,*

$$-\frac{\partial q}{\partial t}(t,x) - \sup_{a\in\mathcal{A}}[\mathcal{L}^a q(t,x) + f(t,x,a)] \geq 0$$

for all $(t,x)\in(0,T]\times\mathbb{R}^n$, and

$$-\frac{\partial q}{\partial t}(t,x) - \sup_{a\in\mathcal{A}}[\mathcal{L}^a q(t,x) + f(t,x,a)] \geq 0$$

for $x\in\mathbb{R}^n$, then $q\geq H$ on $(0,T]\times\mathbb{R}^n$.

2. *Suppose that $q(T)=g$ and suppose there exists a measurable function $\overline{v}(t,x)\in\mathcal{A}$ such that:*

$$-\frac{\partial q}{\partial t}(t,x) - \sup_{\overline{v}\in\mathcal{A}}[\mathcal{L}^{\overline{v}} q(t,x) + f(t,x,\overline{v})] = 0,$$

Considering the SDE

$$dX_s = (X_s,\overline{v}(s,X_s))ds + \sigma(X_s,\overline{v}(s,X_s))dW_s,$$

admits a unique solution $(\overline{X}_s^{t,x})$ and the process $\overline{v}(t,\overline{X}_s^{t,x})\in\mathcal{A}(t,x)$, then $q=H$ on $[0,T]\times\mathbb{R}^n$ and $\overline{v}$ is an optimal Markov control.

Proof. 1. Since we have: $q\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^n)$, for all controls $v\in\mathcal{A}(t,x)$, and τ a stopping time, then using Ito's formula from $(t,s\wedge\tau)$, we obtain:

$$q(s\wedge\tau, X_{s\wedge\tau}^{t,x}) = w(t,x) + \int_t^{s\wedge\tau}\left(\frac{\partial q}{\partial t}(r,X_r^{t,x} + \mathcal{L}^{\mu_r} q(r,X_r^{t,x})\right)dr$$
$$+ \int_t^{s\wedge\tau} D_x q(r,X_r^{t,x})^T\sigma(X_r^{t,x},r)dW_r.$$

Choose $\tau=\tau_n=\inf\{s\geq t: \int_t^s |D_x q(r,X^{t,x})^T\sigma(X_r^{t,x},r)|^2 dr\geq n\}$ so that when $n\to\infty$ then $\tau_n\to\infty$. Thus, the stopped process given by:

$$\left(\int_t^{s\wedge\tau} D_x q(r,X_r^{t,x})^T\sigma(X_r^{t,x},r)dW_r\right)_{t\leq s\leq T}$$

is a martingale, the full result is provided in R.Donnelly [RD22]. By taking an expectation the following is obtained,

$$\mathbb{E}[q(s\wedge\tau, X_{s\wedge\tau}^{t,x})] = q(t,x) + \mathbb{E}\left[\int_t^{s\wedge\tau}\left(\frac{\partial q}{\partial t}(r,X_r^{t,x}) + \mathcal{L}^{\mu_r} q(r,X_r^{t,x})\right)dr\right],$$

and under the assumptions for q:

$$\mathbb{E}[q(s\wedge\tau, X_{s\wedge\tau}^{t,x})] \leq q(t,x) + \mathbb{E}\left[\int_t^{s\wedge\tau} f(X_r^{t,x})dr\right]$$

for all $v\in\mathcal{A}(t,x)$. Ergo,

$$\left|\int|_t^{s\wedge\tau} f(X_r^{t,x},u_r)dr\right| \leq \int_t^T |f(X_r^{t,x},u_r)|dr,$$

as q is of quadratic growth and by the Dominated Convergence Theorem (See E. Neumann [EN21]). As $n\to\infty$,

$$\mathbb{E}[g(X_T^{t,x})] \leq q(t,x) + \mathbb{E}\left[\int_t^T f(X_r^{t,x},u_r)dr\right],$$

for all $v \in \mathcal{A}(t,x)$. Thus, it can be concluded that $q(t,x) \le H(t,x)$ for all $(t,x) \in [0,T] \times \mathbb{R}^n$ due to $v \in \mathcal{A}(t,x)$ being an arbitrary control.

2. Applying Ito's formula to $q(r, (\overline{X}_r^{t,x})$ for $t \in [0,T)$ and $s \in [t,T)$, in order to obtain:

$$\mathbb{E}[q(s, \overline{X}_r^{t,x})] = q(t,x) + \mathbb{E}\left[\int_t^s \left(\frac{\partial q}{\partial t}(r, \overline{X}_r^{t,x}) + \mathcal{L}^{\overline{v}(r,\overline{X}_r^{t,x})} q(r, \overline{X}_r^{t,x})\right) dr\right].$$

From the definition of control process $\overline{v}(t,x)$, it follows:

$$-\frac{\partial q}{\partial t}(t,x) - \sup_{\overline{v} \in \mathcal{A}}[\mathcal{L}^{\overline{v}} q(t,x) + f(t,x,\overline{v})] = 0,$$

implies that,

$$\mathbb{E}[q(s, \overline{X}_s^{t,x})] = q(t,x) + \mathbb{E}\left[\int_t^s f(\overline{X}_r^{t,x}, \overline{v}(r, \overline{X}_r^{t,x}))dr\right].$$

Consider $s \to t$, then

$$q(t,x) = \mathbb{E}\left[\int_t^T f(\overline{X}_r^{t,x}, \overline{v}(r, \overline{X}_r^{t,x}))dr + g(\overline{X}_T^{t,x})\right] = H^{\overline{v}}(t,x),$$

this implies $H^{\overline{v}(t,x)} \ge H(t,x)$. Lastly, we have $q = H$ and $\overline{v}$ is an optimal Markov control.

□

1.2.5 Strategy to solve a stochastic control problem

utilising the Dynamic Programming Principle 1.2.2 and the Verification Theorem 1.2.3, one can devise a strategy to solve a stochastic optimal control problem.

Consider a non-linear HJB equation:

$$-\frac{\partial q}{\partial t}(t,x) - \sup_{a \in \mathcal{A}}[\mathcal{L}^a q(t,x) + f(t,x,a)] = 0,$$

for all $(t,x) \in (0,T] \times \mathbb{R}^n$ and terminal condition $q(T,x) = g(x)$.

1. Fix $(t,x) \in (0,T] \times \mathbb{R}^n$ and solve $\sup_{a \in \mathcal{A}}[\mathcal{L}^a q(t,x) + f(t,x,a)]$ as a maximisation problem with respect to a.

2. Represent with $a^*(t,x)$ the value that attains this maximum in the maximisation problem with respect to a.

3. Using the Verification Theorem, if this non-linear partial differential equation with terminal condition admits a smooth solution q, then q is the solution to the stochastic optimal control problem stated in (1.2.3), and $a^*(t,x)$ is the optimal Markov control.

1.2.6 Solution to the portfolio optimisation problem

Finally, this subsection aims to consolidate the DPP/HJB framework to solve Merton's portfolio optimisation problem stated in equation (1.2.1). Using the HJB equation (1.2.11) with terminal conditions given by $H(T,x) = g(x)$ for all $x \in \mathbb{R}^n$, one can obtain the following:

$$0 = \left(\partial_t + rx\partial_x + \frac{\sigma^2 s^2 \partial_{ss}}{2}\right) H + \sup_{\pi}\left[\pi((\mu - r)\partial_x + \sigma\partial_{xs})H + \frac{\sigma^2\pi^2\partial_{xx}H}{2}\right], \tag{1.2.12}$$

and the terminal condition given by $H(T,x,s) = U(x)$. By observing from the supremum equation and differentiating with respect to π implies that the maximum exists if and only if $\partial_{xx}H(t,x,s) < 0$. Solving for π^* by evaluating the supremum in π, allows us to obtain:

$$\pi^* = \frac{(\mu - r)\partial_x H + \sigma\partial_{xs}H}{\sigma^2 \partial xxH}, \tag{1.2.13}$$

Substituting the obtained π^* back into equation (1.2.12) gives:

$$0 = \left(\partial_t + rx\partial_x + \frac{\sigma^2 s^2 \partial_{ss}}{2}\right) H - \frac{((\mu - r)\partial_x H + \sigma\partial_{xs}H)^2}{2\sigma^2 \partial xxH}. \tag{1.2.14}$$

Since the terminal condition $H(T,x,s) = U(x)$ is independent of s, $H(t,x,s) = h(t,x)$ will be used as the ansatz. Thus, substituting the ansatz into equation (1.2.14) gives:

$$0 = (\partial_t + rx\partial_x)h(t,x) - \frac{\lambda(\partial_x h(t,x))^2}{2\sigma\partial_{xx}h(t,x)}.$$

Since $\lambda = \frac{\mu - r}{\sigma}$, the expression for π^* can be simplified as,

$$\pi^* = -\frac{\lambda}{\sigma}\left(\frac{\partial_x h}{\partial_{xx}h}\right).$$

This implies that π^* solely depends on $U(x)$.

1.3 Market Microstructure

This section serves as a mathematical introduction to market-making. It discusses the preliminary attempts made by Grossman and Miller in modelling the market-maker's problem and briefly looks upon market-making using limit orders, as this shall later be studied in Chapter 2, Avellaneda and Stoikov's solution.

The references utilised for this section include Miller *et al.* (1988) [MG88], Á. Cartea *et al.* (2015) [CJP15], O'Hara *et al.* (1997) [OM97], S. E. Aoud *et al.* (2015) [SF15], C. Lehalle *et al.* (2018) [CS18] and R. Donnelly (2022) [RD222].

1.3.1 What is market-making?

In Section 1, the reader was introduced to a type of market participant known as the 'Passive' market-maker (MM). This individual facilitates trades and makes a profit from the spread of their execution strategy. They must be able to adapt quickly to the ever-changing and volatile market conditions. Additionally, an 'Active' trader is an individual who exploits their ability to predict movements in the mid-price and thus intervene optimally to benefit.

In the words of Á. Cartea *et al.* (2015) [CJP15], *"A key dimension of liquidity as provided by MMs is immediacy: the ability of investors to buy (or sell) an asset at a particular point in time without having to wait to find a counter-party with an offsetting position to sell (or buy)."* Since we are dealing with trading on an active exchange, there will exist various market-makers under competition. Suppose a few MMs dominate a market. In that case, they will have most of the market power and will effectively be responsible for providing liquidity in the form of quoting bid and ask prices to other market participants who look to trade. market-makers are involved in various financial markets where liquidity provision may play an important role, including financial derivatives, equities, currencies, commodities and many others.

As a fictional example, consider a MM who is willing to buy shares of Company Neumann at £49 per share and willing to sell at £51 per share. Let us determine what the quoted spread of this MM is. Since a trader looking to buy from the MM will have to buy at £51 and a trader looking to sell to the MM will have to sell at £49, this difference of £2 is known as the quoted spread. Thus, the MM has a quoted spread of £2. Notice that by posting both the limit orders, the MM ensures liquidity to other market participants who may want to execute either side of the trade at the given time. For the MM's business model to stay sustainable and make profits themselves, they must quote a buy price lower to their sell price, which is intuitive (costs must be lower than gains).

1.3.2 Grossman-Miller market-making model

This model was first studied by Grossman and Miller (1988) [MG88] in attempt to capture the behaviour of market-makers. In this introduction to market-making, we shall look at a slightly rephrased version of it given by Á. Cartea *et al.* (2015) [CJP15].

Consider a scenario with n identical market-makers for a single asset and a set of three days $t \in \{1, 2, 3\}$. On day 1, a liquidity trader, denoted by LT1, sells i units of the asset at time 1, and another liquidity trader, denoted this time by LT2, buys i units of the asset at time 2. At time 3, the price of the asset is given by

$$S_3 = \mu + \epsilon_2 + \epsilon_3,$$

where μ is a constant and ϵ_2, ϵ_3 are two independent normally distributed variables with mean 0 and variance σ^2. S_3 reflects the price movement between time 1 to time 2 and the price movement between time 2 to time 3. Furthermore, it is assumed that all market participants have a risk-averse nature and the same utility function given as:

$$U(X) = -\exp(-\gamma X).$$

At the final time, namely time 3, everything must be settled in cash and all participants should close their accounts with 0 units of the underlying asset, i.e. they must liquidate all of their inventory at the end of the investment horizon. However, we are rather interested in the amount of assets held by the market participants at an intermediate time, time 2. We assume that at time 2, the n market-makers and LT1 respectively hold q_1^{MM} and q_1^{LT1} units of underlying asset.

The problem can now be formulated in an optimisation framework to proceed further. Agent j chooses q_2^j in order to maximise their expected utility given ϵ_2 is realised and made available to all the market participants at time 2:

$$\max_{q_2^j} \mathbb{E}[U(X_3^j)|\epsilon_2],$$

subject to: $X_3^j = X_2^j + q_2^j S_3$ and $X_2^j + q_2^j S_2 = X_1^j + q_1^j S_2$.

This is a concave maximisation problem with solution given by:

$$q_2^j = \frac{\mathbb{E}[S_3|\epsilon_2] - S_2}{\gamma\sigma^2},$$

for all agents.

From the assumptions on S_3 a priori, it follows that,

$$\mathbb{E}[U(X_3^j)|\epsilon_2] = -\exp(-\gamma(X_2^j + q_2^j\mathbb{E}[S_3|\epsilon_2]) + \frac{1}{2}\gamma^2(q_2^j)^2\sigma^2).$$

Ergo, the solution, i.e. units of the underlying asst held by the participant, is represented as,

$$q_2^j = \frac{\mathbb{E}[S_3|\epsilon_2] - S_2}{\gamma\sigma^2}.$$

S_2 can be determined as shown,

$$0 = i + q_1^{LT2} = nq_1^{MM} + q_1^{LT1} + q_1^{LT2} = nq_2^{MM} + q_2^{LT1} + q_2^{LT2} = (n+2)q_2.$$

Since $q_2 = 0$, it implies $S_2 = \mathbb{E}[S_3|\epsilon_2]$. Moreover, from the above expression for $S_3 = \mu + \epsilon_2 + \epsilon_3$, we have that $\mathbb{E}[S_3|\epsilon_2] = \mu + \epsilon_2$. Finally, this means that $S_2 = \mu + \epsilon_2$.

Now we see what happens at time 1 using a similar approach to the previous optimisation problem. Here we consider the maximisation problem as stated below,

$$\max_q \mathbb{E}[U(X_2^j)]$$

subject to: $X_2^j = X_1^j + q_1^j S_1$ and $X_1^j + q_1^j S_1 = X_0^j + q_0^j S_1$.

As done earlier, we repeat a similar approach to determine the solution,

$$q_1^j = \frac{\mathbb{E}[S_2] - S_1}{\gamma\sigma^2}.$$

and S_1,

$$i = nq_0^{MM} + q_0^{LT1} = nq_1^{MM} + q_1^{LT1} = (n+1)\frac{\mu - S_1}{\gamma\sigma^2} \iff S_1 = \mu - \gamma\sigma^2\frac{i}{n+1}$$

Observe that the expectation of future prices is $\mathbb{E}[S_3] = \mu$, yet the price market-makers provide to liquidity traders is $S_1 = \mu - \gamma\sigma^2\frac{i}{n+1}$. The existence of this difference that involves various parameters, expressed as $\gamma\sigma^2\frac{i}{n+1}$, has a few implications;

- the larger the value of n, meaning the more the market-makers, the lower the trading cost
- the larger the order size denoted by i, the higher the trading cost
- the higher the risk aversion nature of the market-maker γ, the higher the trading cost
- the higher the volatility σ in the market,the higher the trading cost.

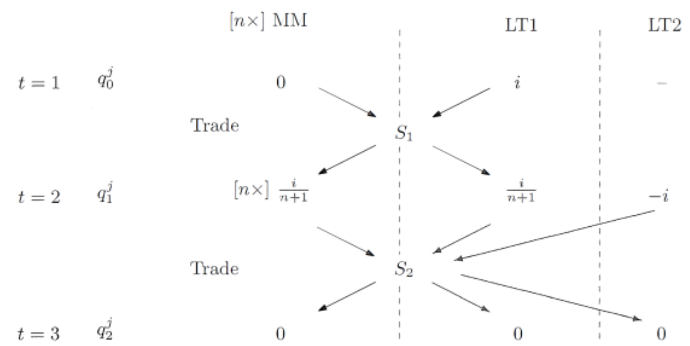

Figure 1.2: An illustrative representation provided by Á. Cartea *et al.* (2015) [CJP15]. It utilises a backwards programming approach to model trading and price setting in the Grossman-Miller market-making model.

1.3.3 Brief on market-making using limit orders

In this subsection, we will prepare to transition from the Walrasian auctioneer in the Grossman-Miller model to a more general setting that captures the measurement of price impact. Thanks to Chapter 1.1.3, we now have the background to study market-making using limit orders. A seminal reference for the theory of market-makers trading using limit orders is the model by Ho & Stoll (1981) [HS81].

The Market-maker's problem is to optimally choose the distance δ from the mid-price S_t, also known as the depths δ^+, δ^- for ask and bid quotes respectively. Once the optimal depths have been selected then the MM will post their sell limit order at $S_t + \delta^+$ and the buy limit order at $S_t - \delta^-$. The frequency of market order arrival has a probability p_+, p_- respectively. Thus, the probability a sell order will be filled is $p_+ P_+(\delta^+)$, where P_+ is the cumulative distribution function representing the probability of a market order walking up the book to where the MM's sell limit order is resting (similarly, P_- is the cumulative distribution function representing the probability of a market order walking up the book to where the MM's buy limit order is resting).

In the next chapter, we shall see Avellaneda and Stoikov's (2008) [AS08] formalisation of the market-making model using limit orders in greater detail. The tools developed in Chapter 1.2 Stochastic Optimal Control Theory, will be particularly helpful when considering δ^+ and δ^- (distance from market mid-price to quoted ask/bid respectively) in a full-fledged dynamic inventory model.

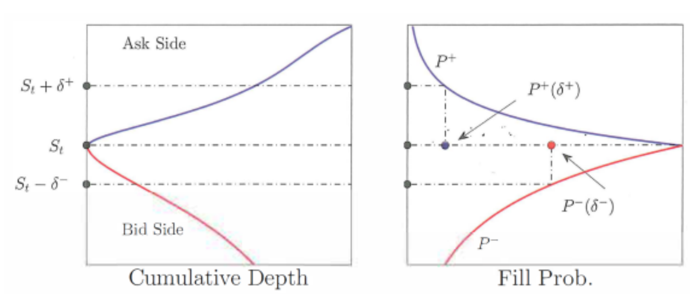

Figure 1.3: The MMs limit order book and probability of execution.

Chapter 2

A Market-Making Model using Limit Orders

This chapter is organised as follows. In section 1, the main foundations of the model by Avellaneda and Stoikov (2008) [AS08] are described: the dynamics of the market mid-price, the arrival rate of market orders subject to their respective distances from the market mid-price and the dealer's utility objective functional. Section 2 solves for the optimal bid and ask quotes and compares them with the dealer's indifference price accounting for their personal risk considerations and given their current inventory.

2.1 Foundations of the Model

2.1.1 Market mid-price of the underlying stock

For mathematical simplicity, the money market is assumed to pay no interest. The evolution of the market mid-price of the stock is described as follows

$$dS_u = \sigma dW_u \tag{2.1.1}$$

with boundary condition at initial value given by $S_t = s$.

The terms in the equation are: S_u market mid-price of the underlying stock, σ is a constant and W_u is a standard one-dimensional Brownian motion or Wiener Process (hence indexed by W). The reason for a zero-drift consideration in this continuous-time model is due to the implicit assumption that the dealer has no opinion on the drift or any auto-correlation structure for the underlying stock.

The use of this market mid-price will be to value the dealer's assets at the terminal point of the investment period. They may not always trade without cost at this price, yet this stochastic nature of the market mid-price enables one to measure the risk of the dealer's current inventory. In a later subsection, the paper introduces and explores the prospects of trading through the use of limit orders.

2.1.2 The problem of an optimising dealer with finite horizon

The dealer aims to maximise the expected exponential utility function of their profit and loss portfolio at the terminal time T. A convex risk measure such as the expected exponential utility function is indeed an important choice as it permits one to define the dealer's indifference price independent of the dealer's wealth.

For mathematical convenience in discussing a more complex situation, the paper first models an inactive trader with a 'frozen inventory' strategy. This essentially means that the dealer merely holds their inventory position at some volume q and does not post any limit orders in the market for the entirety of the investment period $[t, T]$. This case will prove to be helpful when we consider a dealer who posts limit orders and is allowed a variable inventory position during the finite horizon.

The dealer's objective value function is described as

$$v(x,s,q,t) = \mathbb{E}_t\left[-exp\left(-\gamma\left(x+qS_T\right)\right)\right]$$

where x is the initial wealth of the dealer in pounds and the other terms are the same as defined earlier. This value function can alternatively be presented as the following

$$\begin{aligned} v(x,s,q,t) &= -exp\left(-\gamma\left(x+q\mathbb{E}_t\left[S_T\right]\right)\right) \\ v(x,s,q,t) &= -exp\left(-\gamma x\right)exp\left(-\gamma q\mathbb{E}_t\left[S_T\right]\right) \\ v(x,s,q,t) &= -exp\left(-\gamma x\right)exp\left(-\gamma qs\right)exp\left(\frac{\gamma^2 q^2\sigma^2\left(T-t\right)}{2}\right) \end{aligned} \tag{2.1.2}$$

in order to explicit the dependence of the objective value of function on the observed market parameters. Here s is the market mid-price of the underlying stock at the initial time t i.e. $S_t = s$. The last exponential term is indicative of the expected value of the drift-less Brownian motion determined using Ito's lemma.

Furthermore, the idea of indifference bid and ask prices also known as the reservation bid and ask prices may now be formally defined. The indifference bid price is defined as the price that would lead the dealer to becoming indifferent between their current portfolio and their current portfolio plus one stock. The indifference ask is defined as the price that would lead the dealer to become indifferent between their current portfolio and their current portfolio minus one stock.

Remark 2.1.1. It may be worth noting that this is a subjective valuation from the dealer's perspective and may not reflect the price at which trading occurs.

Definition 2.1.2. Firstly, define v as the 4-dimensional value function of the dealer. Their indifference bid price i^b as posed above is represented in the relation given

$$v(x-i^b(s,q,t),s,q+1,t) = v(x,s,q,t). \tag{2.1.3}$$

Next, one can simplify the expression to yield The indifference ask price i^a is similarly represented in the relation given

$$v(x+i^a(s,q,t),s,q-1,t) = v(x,s,q,t). \tag{2.1.4}$$

The careful algebraic manipulation of equations (2.1.2),(2.1.3) and (2.1.4) allows one to compute a closed-form analytical expression for the indifference bid and ask prices. The calculations below are given for the indifference bid price and the same follows for the indifference ask price, yet the author avoids the tediousness of the calculation.

By putting together the results of (2.1.2) and (2.1.3), one can derive:

$$\begin{aligned} v(x-i^b(s,q,t),s,q+1,t) &= -exp\left(-\gamma\left(x-i^b(s,q,t)\right)\right)exp\left(-\gamma(q+1)s\right)exp\left(\frac{\gamma^2(q+1)^2\sigma^2\left(T-t\right)}{2}\right) \\ &= -exp\left(-\gamma x\right)exp\left(-\gamma qs\right)exp\left(\frac{\gamma^2 q^2\sigma^2\left(T-t\right)}{2}\right) \\ &= v(x,s,q,t). \end{aligned}$$

This implies that

$$exp\left(\gamma i^b(s,q,t)\right)exp\left(-\gamma s\right)exp\left(\frac{\gamma^2(2q+1)\sigma^2\left(T-t\right)}{2}\right) = 1,$$

and thus the indifference bid price is given as

$$i^b(s,q,t) = s + \frac{\left(-1-2q\right)\gamma\sigma^2\left(T-t\right)}{2}. \tag{2.1.5}$$

Similarly, using equations (2.1.2) and (2.1.4) the indifference ask price is given as

$$i^a(s,q,t) = s + \frac{(1-2q)\,\gamma\sigma^2\,(T-t)}{2}. \tag{2.1.6}$$

These indifference bid and ask prices apply only when no trading occurs during the entire finite time horizon $[t,T]$. The indifference price given that the dealer holds a volume of q of the underlying stock is an average of the indifference bid price and indifference ask price given q, and is thus denoted as $i(s,q,t)$:

$$i(s,q,t) = s - q\gamma\sigma^2\,(T-t)\,. \tag{2.1.7}$$

Elaborating further on this closed-form expression for the indifference price, we notice that this price is simply an adjustment to the market mid-price with respect to the inventory held by the dealer of the underlying stock.

Consider the following two scenarios.

Let the dealer hold a position of 'Long Stock', this means they hold a positive amount of the underlying stock, i.e. $(q > 0)$. Noticing the negative sign in the coefficient for q in the second term of equation (2.1.7), allows one to deduce that the indifference price must be below the market mid-price (subtracting from the market mid-price). Having an indifference price below that of the market mid-price implies a dealer's want to sell stock.

Alternatively, if the dealer holds a position of 'Short Stock', $(q < 0)$ and the coefficient for q in the second term of (2.1.7) is now positive. Thus, the dealer is willing to buy the underlying stock at a higher price.

2.1.3 The problem of an optimising dealer with infinite horizon

To consider this problem with infinite horizon, one must first evaluate the use of a different objective functional. Since one is given the choice of the terminal time T at which the dealer's performance is measured, the indifference price must depend on the time interval $(T-t)$. This means that the closer the dealer is to the terminal time T, the less risky their volume held of the underlying stock is, since it may be immediately liquidated at the market mid-price S_T.

The need for a stationary version of the indifference price requires the utilisation of an infinite horizon objective given by $\overline{v}(x,s,q)$:

$$\overline{v}(x,s,q) = \mathbb{E}\left[\int_0^\infty -exp\,(-\omega t)\,exp\,(-\gamma\,(x+qS_t))\;dt\right].$$

In accordance with Definition (2.1.1), the stationary indifference prices are similarly defined and determined as:

$$\overline{i^b}(s,q) = s + \frac{1}{\gamma}\log\left(1 + \frac{(-1-2q)\,\gamma^2\sigma^2}{2\omega - \gamma^2\sigma^2 q^2}\right), \tag{2.1.8}$$

and

$$\overline{i^a}(s,q) = s + \frac{1}{\gamma}\log\left(1 + \frac{(1-2q)\,\gamma^2\sigma^2}{2\omega - \gamma^2\sigma^2 q^2}\right). \tag{2.1.9}$$

Here $\omega > \left(\frac{\gamma^2\sigma^2}{2}q^2\right)$, thus it may be thought of as an upper bound on the volume q the dealer can hold of the underlying stock. Deducing from this upper bound, a value such as $\omega = \left(\frac{\gamma^2\sigma^2}{2}(q_{max}+1)^2\right)$ would be a sufficient condition to ensure the indifference prices stay bounded.

2.1.4 The problem of an optimising dealer with limit orders

Consider now a dealer who trades in the underlying through limit orders that they set around the market mid-price given by (2.1.1).

The dealer quotes a bid price p^b and the ask price p^a, and thus is obliged to respectively buy and sell one volume of the underlying stock at these prices, should he be 'hit' or 'lifted' by a market order. These limit orders p^b and p^a may be updated continuously without incurring any costs to the dealer.

The differences that indicate the distances of the limit orders from the market mid-price are denoted as:

$$\delta^b = s - p^b$$

and

$$\delta^a = p^a - s.$$

These distances along with the current shape of the limit order book allow one to deduce a prioritising strategy of executing large market orders. To elaborate on this point further, the need for an accompanying scenario is essential.

Consider a large market order to buy Q units of the underlying stock arrives in the market, the Q unit of limit orders with the lowest ask prices will automatically be executed. However, this causes a temporary market price impact, a phenomenon that describes the occurrence of transactions at prices above the market mid-price. The temporary price impact, Δp, of a trade of volume Q units is algebraically defined as:

$$\Delta p = p^Q - s,$$

where p^Q is the price of the highest (furthest from market mid-price) limit order executed in this trade. The dealer's limit order will only be executed if it lies within the range of the highest market order, i.e. $\delta^a < \Delta p$.

Assumption 2.1.3. Both trading intensities are assumed to be modelled as follows:

1. Market sell orders 'hit' the dealer's buy limit order at Poisson rate $\lambda^b(\delta^b)$, a function decreasing in δ^b.
2. Market buy orders 'lift' the dealer's sell limit order at Poisson rate $\lambda^a(\delta^a)$, a function decreasing in δ^a.

This is intuitive as limit orders further away from the market mid-price should be of lower priority of being executed and thus the dealer receives fewer buy and sell quotes.

Remark 2.1.4. 2.1.5 compiles recent studies in econophysics to present some realistic functional forms for the Poisson trading intensities $\lambda(\delta^b)$ and $\lambda(\delta^a)$.

Moreover, the wealth and inventory are now both stochastic processes that depend on the arrival of market buy and sell orders. The wealth in the dealer's cash account jumps each time there is an executed buy or sell order and these jumps are modelled using a jump Poisson process. The wealth X_t is written as:

$$dX_t = p^a dN_t^a - p^b dN_t^b$$

where p^b, p^a are the dealer's quoted bid and ask price respectively. N_t^b is a Poisson process with intensity λ^b modelling the amount of stocks bought by the dealer. N_t^a is a Poisson process with intensity λ^a modelling the amount of stocks sold by the dealer. The total volume of the underlying stock held at time t, q_t, can be represented by the difference between these Poisson processes:

$$q_t = N_t^b - N_t^a.$$

Lastly, the dealer's objective functional in the setting of trading through limit orders is given by:

$$u(x, s, q, t) = \max_{\delta^b, \delta^a} \mathbb{E}_t \left[-exp \left(-\gamma \left(X_T + q_T S_T \right) \right) \right].$$

In contrast with the previous setting of the problem, the dealer who trades through limit orders indirectly influences the order flow they receive as they are in control of the bid and ask prices. Before delving into the solution to the optimal bid and ask quotes, the paper presents recent results to justify the choice for the Poisson trading intensity λ.

2.1.5 Justification for the Poisson Trading Intensity λ

The microstructure of financial markets has long been a key subject of study within the econophysics community and this paper focuses yet on just a small section of results concerned with the Poisson trading intensity λ. In simple terms, the trading intensity λ is the intensity at which a limit order is executed with respect to its distance δ from the mid-price. There are three main factors needed to quantify this intensity and form a function for this intensity:

1. Overall frequency of market orders
2. Distribution of the size of market orders
3. Temporary impact to mid-price as a result of a relatively large market order.

Combining these three factors indicates that the trading intensity λ decays exponentially or as a power law function.

For mathematical simplicity, a constant frequency Λ for incoming market buy or sell orders is assumed. As suggested by Avellaneda and Stoikov (2008) [AS08],

$$\Lambda \approx \frac{\text{total volume traded over day } x}{\text{average size of market orders on day } x}$$

Secondly, the distribution of the size of market orders tends to obey a power law, according to multiple studies cited below with their approximations for α. Hence, the density of market order size which essentially is a proportionality relation describing the distribution of the size of the market orders is given by:

$$f^Q(x) \propto x^{-1-\alpha} \tag{2.1.10}$$

for large x we have $\alpha = 1.53$ as measured by Gopikrishnan *et al.* (2000) [GPG00], for American stocks $\alpha = 1.4$ cited in Maslow and Mills (2001) [MM01], for shares on the NASDAQ $\alpha = 1.5$ calculated by Gabaix *et al.* (2006) [GGP06].

The temporary impact to the market is generally hard to define or measure and hence there seems to be less agreement on functional relations or statistics on this matter in the econophysics literature. Despite these limitations, a few attempts have found that the change in price Δp resulting from a market order of size Q is obtained by:

$$\Delta p \propto Q^{\beta}. \tag{2.1.11}$$

Here $\beta = 0.5$ according to Gabaix *et al.* (2006) [GGP06] and $\beta = 0.76$ according to Weber and Rosenow (2005) [WR05].

Alternatively, a better fit to the function has been found by Potters and Bouchaud (2003) [PB03] given by

$$\Delta p \propto \log(Q). \tag{2.1.12}$$

Amalgamating the results and statistics captured by these studies, it is possible to derive the Poisson trading intensity λ with which the dealer's orders are executed. As proposed earlier, λ should depend solely on the distance δ^b, δ^a of the dealer's respective buy, ask quote to the mid-price. Thus, the function for the Poisson trading Intensity that describes the arrival of buy or ask orders, should be respectively represented as $\lambda^b(\delta^b), \lambda^a(\delta^a)$.

As an example, consider by the manipulation of equations (2.1.10) and (2.1.12):

$$\begin{cases} \lambda(\delta) & = \Lambda P(\Delta p > \delta) \\ & = \Lambda P(\log(Q) > K\delta) \\ & = \Lambda P(Q > exp(K\delta)) \\ & = \Lambda \int_{exp(K\delta)}^{\infty} x^{-1-\alpha} dx \\ & = Aexp(-k\delta). \end{cases} \tag{2.1.13}$$

Here $A = \frac{\Lambda}{\alpha}$ and $k = \alpha K$. Instead, if a power law is considered for the market price impact as described by equation (2.1.11), then the intensity $\lambda(\delta)$ is given by:

$$\lambda(\delta) = B\delta^{\frac{-\alpha}{\beta}}.$$

On the other hand, Avellaneda and Stoikov's (2008) [AS08] approach rather requires an understanding of the short term liquidity. They suggest that the market impact function could be derived by directly integrating density of the limit order book. Their suggested procedure has in fact been studied previously by Smith *et al.* (2003) [SED03] and Weber and Rosenow (2005) [WR05]. It is often described as the 'Virtual' price impact.

2.2 Results on the Market-Making Model using Limit Orders

After having extensively discussed the problem, the paper now aims to admit a solution to the model and offer a method to test the performance of the optimal quotes. According to the standard approach in stochastic and optimal control theory, this problem too requires setting up the Hamilton-Jacobi-Bellman framework to transform it into a set of equations that may be analytically or numerically computed.

2.2.1 Hamilton-Jacobi-Bellman equation

From Section 2.1, the dealer's objective is summarised by the value function:

$$u(x,s,q,t) = \max_{\delta^b,\delta^a} \mathbb{E}_t\left[-exp\left(-\gamma\left(X_T + q_T S_T\right)\right)\right]. \tag{2.2.1}$$

Here δ^b and δ^a, as first studied by Ho and Stoll (1981) [HS81], are in fact time and state dependent. In their study of the optimal agent problem, their primary approach for analysis was utilising the dynamic programming principle. This allowed them to confirm that the value function u (2.1.1) solves the Hamilton-Jacobi-Bellman equation presented below:

$$\begin{cases} u_t + \frac{\sigma^2 u_{ss}}{2} + \max_{\delta^b} \lambda^b\left(\delta^b\right)\left[u\left(x-s+\delta^b,s,q+1,t\right) - u\left(x,s,q,t\right)\right] + \\ \max_{\delta^a} \lambda^a\left(\delta^a\right)\left[u\left(x+s+\delta^a,s,q-1,t\right) - u\left(x,s,q,t\right)\right] = 0, \\ u(x,s,q,T) = -exp\left(-\gamma\left(x+qs\right)\right). \end{cases} \tag{2.2.2}$$

This non-linear PDE has a solution that is continuous in the variables s, x and t and is dependent on the discrete values of the volume held q.

Since the assumption of expected exponential utility was made in Section 2.1, the non-linear problem may be simplified using the following ansatz:

$$u(x,s,q,t) = -exp\left(-\gamma x\right) exp\left(-\gamma\theta\left(s,q,t\right)\right). \tag{2.2.3}$$

Substituting the ansatz (2.2.3) into (2.2.2) yields the following equation in θ:

$$\begin{cases} \theta_t + \frac{\sigma^2\theta_{ss}}{2} - \frac{\sigma^2\gamma\theta_s^2}{2} + \max_{\delta^b}\left[\frac{\lambda^b(\delta^b)}{\gamma}\left[1 - exp\left(\gamma\left(s-\delta^b-i^b\right)\right)\right]\right] + \\ \max_{\delta^a}\left[\frac{\lambda^a(\delta^a)}{\gamma}\left[1 - exp\left(-\gamma\left(s+\delta^a-i^a\right)\right)\right]\right] = 0, \\ \theta(s,q,T) = qs. \end{cases} \tag{2.2.4}$$

2.2.2 Indifference prices as a function of θ

The indifference bid and ask prices, i^b and i^a, depend directly on the function $\theta(s,q,t)$ and these relations, when the inventory is q, are:

$$i^b(s,q,t) = \theta(s,q+1,t) - \theta(s,q,t). \tag{2.2.5}$$

$$i^a(s,q,t) = \theta(s,q,t) - \theta(s,q-1,t). \tag{2.2.6}$$

2.2.3 Optimal distances δ^b and δ^a

Using the first-order optimality conditions in (2.2.4), the optimal distances δ^b and δ^a are given by the implicit relations:

$$s - i^b(s,q,t) = \delta^b - \frac{1}{\gamma}\log\left(1 - \gamma\frac{\lambda^b\left(\delta^b\right)}{\left(\frac{\partial\lambda^b}{\partial\delta}\right)\left(\delta^b\right)}\right). \tag{2.2.7}$$

$$i^a(s,q,t) - s = \delta^a - \frac{1}{\gamma}\log\left(1 - \gamma\frac{\lambda^a\left(\delta^a\right)}{\left(\frac{\partial\lambda^a}{\partial\delta}\right)\left(\delta^a\right)}\right). \tag{2.2.8}$$

All in all, the optimal bid and ask quotes may be determined using a straightforward two-step procedure. This intuitive procedure first requires one to solve the non-linear PDE (2.2.4) and determine $i^b(s,q,t)$ and $i^a(s,q,t)$. Next, they need to solve the implicit dependencies (2.2.7) and (2.2.8) in order to calculate the optimal distances from the market mid-price, $\delta^b(s,q,t)$ and $\delta^a(s,q,t)$, and thus solve the market-maker's problem of finding optimal bid and ask quotes.

In addition to this, the paper now considers instead a geometric Brownian motion to describe the dynamics of the market mid-price in Avellaneda and Stoikov's model.

2.3 Underlying Stock driven by a Geometric Brownian Motion

Instead of equation (2.1.1):

$$dS_u = \sigma dW_u,$$

This geometric Brownian motion is given by:

$$\frac{dS_u}{S_u} = \sigma dW_u$$

with boundary condition given at initial value by $S_t = s$.

The mean/variance objective functional is given by the usual CARA utility function:

$$V(x,s,q,t) = \mathbb{E}_t\left[(x+qS_T) - \frac{\gamma}{2}(qS_T - qs)^2\right],$$

Here x denotes the initial wealth in pounds.

Following along, the expectation may be taken inside the brackets and computed in order to simplify the value function:

$$V(x,s,q,t) = x + qs - \frac{\gamma q^2 s^2}{2}\left(exp\left(\sigma^2 (T-t)\right) - 1\right).$$

This value function yields indifference prices that are analogous to the ones previously obtained in section 2.2 and are given by:

$$\begin{cases} I^b(s,q,t) = s + \dfrac{(-1-2q)\,\gamma s^2\left(exp\left(\sigma^2(T-t)\right)-1\right)}{2}. \\ \\ I^a(s,q,t) = s + \dfrac{(1-2q)\,\gamma s^2\left(exp\left(\sigma^2(T-t)\right)-1\right)}{2}. \end{cases}$$

Chapter 3

An Options Market-Making Model

This chapter is organised as follows. In section 1, the main foundations of the model by Baldacci *et al.* (2019) [BBG19] are described: the dynamics of the market mid-price using a one-factor stochastic volatility model, the optimisation problem of the options market-maker and lastly the assumptions needed to address the problem from a theoretical point of view and the approximations needed to simulate these optimal quotes. Following this, section 2 introduces the Constant-Vega assumption that is essential in simplifying this high-dimensional stochastic optimal control problem of the market-maker into a low-dimensional functional equation. This mathematical simplification in section 2 allows one to use interpolation techniques and an explicit Euler scheme in order to generate optimal bid and ask quotes for a book of options with several strikes and maturities.

3.1 Foundations of the Model

In order to further our study on the options market-making problem, it is essential to remind the reader of the definition of filtered probability spaces.

Definition 3.1.1. A filtered probability space, or stochastic basis, $(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\in T}, \mathbb{P})$ consists of a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and a filtration $(\mathbb{F}_t)_{t\in T}$ contained in $\mathbb{F}$, according to Gel (2013) [gel13].

T here denotes the ordered time index set such that $\mathbb{F}_s \subseteq \mathbb{F}_t$ for all $s < t$ in T. In a continuous-time setting like the one considered in this paper, it is convenient to impose further conditions sometimes referred to as the usual conditions or usual hypotheses as seen earlier in Section 1.2.2.

The filtered probability space $(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$ satisfies the usual conditions if the following are met:

1. The probability space $(\Omega, \mathbb{F}, \mathbb{P})$ is complete
2. The σ-algebras $\mathbb{F}_t$ contains all the sets in $\mathbb{F}$ of zero probability
3. The filtration $\mathbb{F}_t$ is right-continuous. This means for every non-maximal $t \in \mathbb{R}_+$, the σ-algebra $\mathbb{F}_{t^+} \equiv \cap_{s>t}\mathbb{F}_s$ is equal to $\mathbb{F}_t$.

Remark 3.1.2. For the entirety of this chapter, it is assumed that all the stochastic processes are defined on $(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$.

3.1.1 Price dynamics of the underlying asset

Consider an asset with price dynamics following the one-factor stochastic volatility model given by:

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{\nu_t} S_t dW_t^S \\ d\nu_t = a_{\mathbb{P}}(t, \nu_t)dt + \xi\sqrt{\nu_t} dW_t^\nu. \end{cases}$$

Here $\mu \in \mathbb{R}$, $\xi \in \mathbb{R}_+^*$ and $(W_t^S, W_t^\nu)_{t\in\mathbb{R}_+}$ which are two Brownian Motions that have quadratic covariation $\rho = \frac{d\langle W^S, W^\nu\rangle}{dt} \in (-1, 1)$. Furthermore,$a_{\mathbb{P}}$ is a set of well-defined process such that the process $(\nu_t)_{t\in\mathbb{R}_+}$ remains positive almost surely (a.s.).

Adding on, Heston's model (1993) [SLH93], namely $a_{\mathbb{P}} : (t,\nu) \longrightarrow \kappa_{\mathbb{P}}(\theta_{\mathbb{P}} - \nu)$ where $\kappa_{\mathbb{P}}, \theta_{\mathbb{P}} \in \mathbb{R}_+^*$ satisfy the Feller condition $2\kappa_{\mathbb{P}}\theta_{\mathbb{P}} > \xi^2$, is a primordial example for the classification of function $a_{\mathbb{P}}$. For mathematical simplicity, a one-factor model with the instantaneous variance is chosen, however similar results can be obtained using L. Bergomi's one-factor model (refer to [LB05], [LB15]) with forward variances and also L. Bergomi's two-factor stochastic volatility model (again refer to [LB05], [LB15]).

Under the assumption that interest rate are fixed to 0, J. Gatheral (2011) [JG11] introduces an equivalent risk-neutral/pricing probability measure indexed by $\mathbb{Q}$. The market mid-price and volatility processes are now transformed in the following way:

$$
\begin{aligned}
dS_t &= \sqrt{\nu_t} S_t d\overline{W}_t^S \\
d\nu_t &= a_{\mathbb{Q}}(t,\nu_t)dt + \xi\sqrt{\nu_t} d\overline{W}_t^\nu.
\end{aligned}
$$

Here $\xi \in \mathbb{R}_+^*$ and $(\overline{W}_t^S, \overline{W}_t^\nu)_{t\in\mathbb{R}_+}$ which are another couple of Brownian Motions under the probability measure $\mathbb{Q}$ that have quadratic covariation $\rho = \frac{d\langle \overline{W}^S, \overline{W}^\nu\rangle}{dt} \in (-1,1)$. Furthermore,$a_{\mathbb{Q}}$ is a set of well-defined process such that the process $(\nu_t)_{t\in\mathbb{R}_+}$ remains positive almost surely (a.s.).

Now, a volume of $N \geq 1$ European options is written on this above asset, under the measure $\mathbb{Q}$. For each option $i \in \{1,...,N\}$, the maturity date of the i-th option is denoted by T^i and the price process of the i-the option is denoted by $(\mathcal{O}_t^i)_{t\in[0,T^i]}$. Even though most options in applications are call and/or put options, this stochastic volatility model setting considers any type of European options.

In the one-factor model under measure Q, for all $i \in \{1,...,N\}$ and all $t \in [o,T^i]$, the price process associated with the i-th option $\mathcal{O}_t^i = \mathcal{O}^i(t,S_t,\nu_t)$, where $\mathcal{O}^i$ is the solution to the following partial differential equation on the domain $[0,T^i) \times \mathbb{R}_+^2$:

$$
\begin{aligned}
0 =& \partial_t \mathcal{O}^i(t,S,\nu) + a_{\mathbb{Q}}(t,\nu)\partial_\nu \mathcal{O}^i(t,S,\nu) + \frac{1}{2}\nu S^2 \partial_{SS}^2 \mathcal{O}^i(t,S,\nu) \\
&+ \rho\xi\nu S \partial_{\nu S}^2 \mathcal{O}^i(t,S,\nu) + \frac{1}{2}\xi^2\nu\partial_{\nu\nu}^2 \mathcal{O}^i(t,S,\nu).
\end{aligned}
\tag{3.1.1}
$$

Remark 3.1.3. Option prices depend also on a terminal condition, also called the Maturity, corresponding to the payoff. Since our problem's time horizon is before the maturity of all the $N \geq 1$ options under consideration, this short-term optimisation problem does not require a final condition associated with equation (3.1.1).

Before we can discuss and explain the problem of the market-maker, it is helpful to provide mathematical background on the construction of the jump processes used to describe the dynamics of the inventory process.

3.1.2 Construction of the jump processes

Analogous to the filtered probability space considered earlier, we now work with a slightly different filtered probability space given by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \overline{\mathbb{P}})$. As the generalisation to $N > 1$ options is just a mathematical exercise, we consider for to keep things simple that $N = 1$ and we have only one option.

Consider $\overline{N}^b$ and $\overline{N}^a$ to be two independent compound Poisson processes with intensity 1 and increments following distributions characterised respectively as $\mu^b(dz)$ and $\mu^a(dz)$ on the interval $\mathbb{R}_+^*$.

Consider N^b and N^a to be two processes starting at the origin (i.e. 0), to be the solutions to the coupled SDEs given by:

$$
\begin{aligned}
dN_t^b &= \int_{\mathbb{R}_+^*} \mathbf{1}_{\{N_{t-}^b - N_{t-}^a + z \in \mathcal{Q}\}} \overline{N}^b(dt,dz), \\
dN_t^a &= \int_{\mathbb{R}_+^*} \mathbf{1}_{\{N_{t-}^b - N_{t-}^a - z \in \mathcal{Q}\}} \overline{N}^a(dt,dz).
\end{aligned}
$$

Their respective intensity kernels by Girsanov's theorem under the measure $\overline{\mathbb{P}}^\delta$ are:

$$\lambda_t^b(dz) = \mathbb{1}_{\{q_{t-}+z\in\mathcal{Q}\}}\mu^b(dz),$$
$$\lambda_t^a(dz) = \mathbb{1}_{\{q_{t-}-z\in\mathcal{Q}\}}\mu^a(dz),$$

Here the difference has been replaced by $q_t := N_t^b - N_t^a$. Using the Radon-Nikodym derivative, $\forall\delta \in \mathcal{A}$ the probability measure characterised as $\overline{\mathbb{P}}^\delta$ is shown in the expression below:

$$\left.\frac{d\overline{\mathbb{P}}^\delta}{d\overline{\mathbb{P}}}\right|_{\mathcal{F}_t} = L_t^\delta. \tag{3.1.2}$$

Here $(L_t^\delta)_{t\geq 0}$ is the unique solution to the SDE:

$$dL_t^\delta = L_{t-}^\delta\left(\int_{\mathbb{R}_+^*}(\Lambda^b(\delta^b(t,z)) - 1)\overline{N}^b(dt,dz) + \int_{\mathbb{R}_+^*}(\Lambda^a(\delta^a(t,z)) - 1)\overline{N}^a(dt,dz)\right),$$

where $L_0^\delta = 1$. Using Girsanov's theorem, the jump processes N^b and N^a have intensity kernels under the measure $\overline{\mathbb{P}}^\delta$,

$$\lambda_t^{\delta,b}(dz) = \Lambda^b(\delta^b(t,z))\mathbb{1}_{\{q_{t-}+z\in\mathcal{Q}\}}\mu^b(dz),$$
$$\lambda_t^{\delta,a}(dz) = \Lambda^a(\delta^a(t,z))\mathbb{1}_{\{q_{t-}-z\in\mathcal{Q}\}}\mu^a(dz),$$

This construction has been particularly adapted from Baldacci *et al.* (2019) [BBG19] *(Appendix A.3)* and is essential to ensure that $N^{i,b}(dt,dz)$ and $N^{i,a}(dt,dz)$ are two right-continuous $\mathbb{R}_+^*$-marked point processes, with no simultaneous jumps almost surely. These jump processes describe the transactions of the i-th option on both the bid and ask sides. This will be further discussed in the next subsection, 3.1.3.

3.1.3 Options market-maker's optimisation problem

This problem concerns an options market-maker in charge of providing bid and ask quotes for $N \geq 1$ distinct options over an interval $[0,T]$, where $T < \min_{i\in\{1,...,N\}} T^i$ (refer to Remark 3.1.3). For all $i \in \{1,...,N\}$, $\mathcal{O}_t^i - \delta_t^{i,b}(z)$ and $\mathcal{O}_t^i - \delta_t^{i,a}(z)$ respectively denote the bid and ask prices (per contract) quoted by the options market-maker for z contracts of the i-th option. Moreover, the function $(\delta_t^i(.))_{t\in[0,T]} := (\delta_t^{i,b}(.), \delta_t^{i,a}(.))_{t\in[0,T]}$ is $\mathbb{F}$-predictable and bounded from below by a given constant δ_∞. In practical applications, δ_∞ must be chosen negative enough so that the lower bound is never binding to the problem. From now on, we will use $\mathcal{A}$ to represent the set of $\mathbb{F}$-predictable, $\mathbb{R}^{2N}$-valued functions that are bounded from below by δ_∞. Such processes shall be from hereafter considered as admissible control processes. Inventory process $(q_t)_{t\in[0,T]} := (q_t^1, ..., q_t^N)'_{t\in[0,T]}$ of the market-maker has dynamics described by:

$$dq_t^i := \int_{\mathbb{R}_+^*} z(N^{i,b}(dt,dz) - N^{i,a}(dt,dz)),$$

for all $i \in \{1,...,N\}$. Here $N^{i,b}(dt,dz)$ and $N^{i,a}(dt,dz)$ are two right-continuous $\mathbb{R}_+^*$-marked point processes, with no simultaneous jumps almost surely (as constructed earlier in Subsection 3.1.2). These jump processes describe the transactions of the i-th option on both the bid and ask ends, and their respective intensity processes $(\lambda_t^{i,b}(dz))_{t\in\mathbb{R}_+}$ and $(\lambda_t^{i,a}(dz))_{t\in\mathbb{R}_+}$ are represented as:

$$\lambda_t^{i,b}(dz) := \Lambda^{i,b}(\delta_t^{i,b}(z))\mathbb{1}_{\{q_{t-}+ze^i\in\mathcal{Q}\}}\mu^{i,b}(dz)$$
$$\lambda_t^{i,a}(dz) := \Lambda^{i,a}(\delta_t^{i,a}(z))\mathbb{1}_{\{q_{t-}-ze^i\in\mathcal{Q}\}}\mu^{i,a}(dz).$$

where $(e^i)_{i\in\{1,...,N\}}$ is the canonical or elementary basis of $\mathbb{R}^N$. Morever, the set of authorised inventories which defines the risk limits of the options market-maker are indexed by $\mathcal{Q}$ and $(\mu^{i,b}, \mu^{i,a})$ be two probability measures on $\mathbb{R}_+^*$ that represent the distributions of the transaction volumes.

Consider and refer to O. Guéant (2017) [OG17] for further details, the positive functions for $i \in \{1,...,N\}$ indexed as $\Lambda^{i,b}$ and $\Lambda^{i,a}$ satisfy the so-called classical hypotheses listed below.

1. $\lim_{\delta\to+\infty} \Lambda^{i,b}(\delta) = \lim_{\delta\to+\infty} \Lambda^{i,a}(\delta) = 0$.
2. $\Lambda^{i,b}$ and $\Lambda^{i,a}$ are twice continuously differentiable.
3. $\Lambda^{i,b}$ and $\Lambda^{i,a}$ are both strictly decreasing, namely $\Lambda^{i,b^{'}} < 0$ and $\Lambda^{i,a^{'}} < 0$.
4. $\sup_{\delta\in\mathbb{R}} \dfrac{\Lambda^{i,b}(\delta)\Lambda^{i,b^{''}}(\delta)}{\left(\Lambda^{i,b^{'}}(\delta)\right)^2} < 2$ and $\sup_{\delta\in\mathbb{R}} \dfrac{\Lambda^{i,a}(\delta)\Lambda^{i,a^{''}}(\delta)}{\left(\Lambda^{i,a^{'}}(\delta)\right)^2} < 2$.

Remark 3.1.4. The above classical hypotheses are sufficient to allow for a multitude of relevant, in the practical sense, forms of intensities. The most frequented in literature is the earlier introduced Exponential intensity by M. Avellaneda and S. Stoikov (2008) [AS08] in Chapter 2.

The options market-maker can also buy and sell the underlying asset in addition to quoting prices for the $N \geq 1$ options. Another important assumption considered here is that the asset must be liquid enough to sustain perfect Δ-hedging. Even though in practice, a portfolio that is not vega hedged cannot be optimally Δ-hedged. However for now, we assume that the Δ-hedging is done in continuous-time in order to keep things simple. $(\Delta_t)_{t\in[0,T]}$ will now be used to represent the continuous-time Δ of this portfolio.

$$\Delta_t := \sum_{i=1}^{N} \partial_S \mathcal{O}^i(t, S_t, \nu_t) q_t^i,$$

for all $t \in [0,T]$. Thus, using Ito's formula the resulting dynamics for the cash process of the options market-maker indexed by $(X_t)_{t\in[0,T]}$ is given as:

$$dX_t := \sum_{i=1}^{N} \left(\int_{\mathbb{R}_+^*} z \left(\delta_t^{i,b}(z) N^{i,b}(dt,dz) + \delta_t^{i,a}(z) N^{i,a}(dt,dz) \right) - \mathcal{O}_t^i dq_t^i \right) + S_t d\Delta_t + d\langle \Delta, S\rangle_t.$$

Consider now the Mark-to-Market (MtM) process denoted by $(V_t)_{t\in[0,T]}$ which includes the options market-makers (cash, shares and options) is obtained by the following expression:

$$V_t := X_t - \Delta_t S_t + \sum_{i=1}^{N} q_t^i \mathcal{O}_t^i.$$

Its dynamics are represented as,

$$\begin{aligned}
dV_t &= dX_t - S_t d\Delta_t - \Delta_t dS_t - d\langle \Delta, S\rangle_t + \sum_{i=1}^{N} \mathcal{O}_t^i dq_t^i + \sum_{i=1}^{N} q_t^i d\mathcal{O}_t^i \\
&= \sum_{i=1}^{N} \left(\int_{\mathbb{R}_+^*} z \left(\delta_t^{i,b}(z) N^{i,b}(dt,dz) + \delta_t^{i,a}(z) N^{i,a}(dt,dz) \right) + q_t^i d\mathcal{O}_t^i \right) - \Delta dS_t \\
&= \left(\int_{\mathbb{R}_+^*} z \left(\delta_t^{i,b}(z) N^{i,b}(dt,dz) + \delta_t^{i,a}(z) N^{i,a}(dt,dz) \right) + q_t^i \partial_\nu \mathcal{O}^i(t,S_t,\nu_t)(a_{\mathbb{P}}(t,\nu_t) - a_{\mathbb{Q}}(t,\nu_t))dt \right. \\
&\left. + \sqrt{\nu_t} \xi q_t^i \partial_\nu \mathcal{O}^i(t,S_t,\nu_t) dW_t^\nu \right).
\end{aligned}$$

Following this, $\forall i \in \{1,...,N\}$ the definition of the vega of the i-th option is,

$$\mathcal{V}_t^i := \partial_{\sqrt{\nu}} \mathcal{O}^i(t,S_t,\nu_t) = 2\sqrt{\nu_t}\partial_\nu \mathcal{O}^i(t,S_t,\nu_t)$$

for all $t \in [0,T]$, thus allowing one to re-write the dynamics of the options market-maker's portfolio as:

$$dV_t = \sum_{i=1}^{N} \left(\int_{\mathbb{R}_+^*} z \left(\delta_t^{i,b}(z) N^{i,b}(dt,dz) + \delta_t^{i,a}(z) N^{i,a}(dt,dz) \right) + q_t^i \mathcal{V}_t^i \frac{a_{\mathbb{P}}(t,\nu_t) - a_{\mathbb{Q}}(t,\nu_t)}{2\sqrt{\nu_t}} dt + \frac{\xi}{2} q_t^i \mathcal{V}_t^i dW_t^\nu \right).$$

Recent results in the field of market-making, suggest two plausible objective functions to consider for the options market-maker. As in Chapter 2, M. Avellaneda and S. Stoikov (2008) [AS08] use an expected exponential utility objective functional given by:

$$\sup_{\delta \in \mathcal{A}} \mathbb{E}\left[-\exp -\gamma V_T\right].$$

Here $\gamma > 0$ represents the risk-aversion parameter of the options market-maker. Alternatively, according to Á. Cartea *et al.* (2015) [CJP15] and O. Guéant (2017) [OG17] suggest a risk-adjusted expectation for the objective functional of the form:

$$\sup_{\delta \in \mathcal{A}} \mathbb{E}\left[V_T - \frac{\gamma}{2}\int_0^T \left(\sum_{i=1}^N \frac{\xi}{2} q_t^i \mathcal{V}_t^i\right)^2 dt\right].$$

Even though, the two objective functions considered above are close to one another in application, O. Guéant (2017) [OG17] shows that they both produce optimal quotes in practice. In Baldacci *et al.* (2019) [BBG19], the second objective function is considered and we use the same framework to approach the problem ahead, thus in this case the second objective function is more appropriately presented as:

$$\begin{aligned}\sup_{\delta \in \mathcal{A}} \mathbb{E}\Bigg[\int_0^T \Bigg(\sum_{i=1}^N \Bigg(\Bigg(\sum_{j=a,b}\int_{\mathbb{R}_+^*} z\delta_t^{i,j}(z)\Lambda^{i,j}(\delta_t^{i,j}(z))\mathbb{1}_{\{q_{t-}-\psi(j)ze^i \in \mathcal{Q}\}}\mu^{i,j}(dz)\Bigg) \\ + q_t^i \mathcal{V}_t^i \frac{a_{\mathbb{P}}(t,\nu_t) - a_{\mathbb{Q}}(t,\nu_t)}{2\sqrt{\nu_t}}\Bigg)\Bigg) dt - \frac{\gamma\xi^2}{8}\int_0^T \left(\sum_{i=1}^N q_t^i \mathcal{V}_t^i\right)^2 dt\Bigg],\end{aligned} \tag{3.1.3}$$

and where $\psi(j)$ is:

$$\psi(j) := \begin{cases} +1 \text{ if } j = a \\ -1 \text{ if } j = b. \end{cases}$$

Hence, the value function denoted as $u(t, S, \nu, q)$, is constructed as the following:

$$u : (t, S, \nu, q) \in [0, T] \times \mathbb{R}_+^2 \times \mathcal{Q} \longrightarrow u(t, S, \nu, q).$$

In accordance with equation (3.1.3), the value function for maximisation with respect to $(\delta_s)_{s\in[t,T]} \in \mathcal{A}_t$, where $\mathcal{A}_t$ is the set of admissible controls (earlier defined in Section 3.1.3) on the interval $[t, T]$, is written as:

$$\begin{aligned}u(t,S,\nu,q) = \sup_{(\delta_s)_{s\in[t,T]}\in\mathcal{A}_t} \mathbb{E}_{(t,S,\nu,q)}\Bigg[\int_t^T \Bigg(\sum_{i=1}^N \Bigg(\Bigg(\sum_{j=a,b}\int_{\mathbb{R}_+^*} z\delta_s^{i,j}(z)\Lambda^{i,j}(\delta_s^{i,j}(z))\mathbb{1}_{\{q_{s-}-\psi(j)ze^i\in\mathcal{Q}\}}\mu^{i,j}(dz)\Bigg) \\ + q_s^i\mathcal{V}_s^i \frac{a_{\mathbb{P}}(s,\nu_s) - a_{\mathbb{Q}}(s,\nu_s)}{2\sqrt{\nu_s}}\Bigg)\Bigg) ds - \frac{\gamma\xi^2}{8}\int_t^T \left(\sum_{i=1}^N q_s^i\mathcal{V}_s^i\right)^2 ds\Bigg].\end{aligned}$$

3.1.4 Further assumptions to approximate a solution

O. Guéant (2017) [OG17] addresses this stochastic optimal control problem theoretically, however, in a more application based setting, this approach is ineffective in providing optimal bid and ask quotes. Thus, in order to beat the curse of dimensionality as there are $N \geq 1$ options for the market-maker and so the value function has $N + 2$ variables (not considering the time variable), the following assumptions are helpful in approximating a solution to the options market-maker's problem.

Assumption 3.1.5. The vega of each option over the interval $[0, T]$ is approximated by its value at the initial time (i.e. $t = 0$),

$$\mathcal{V}_t^i = \mathcal{V}_0^i =: \mathcal{V}^i \in \mathbb{R},$$

for all $i \in \{1, ..., N\}$.

This is a valid assumption as long as T is not too large. This essentially means that T has to be sufficiently large enough to ensure several transactions occur across many options, yet it must be small enough so that the constant-vega approximation remains justified, thus smaller than the maturities of the options at the very least. It might be worth noting that this is time-inconsistent, however the output of the model under the constant-vega approximation over a short period of time can be used and then re-run over the next interval in a batching sense. This is a common practice method used in applied optimal control when the parameters need to be estimated in near real-time.

Assumption 3.1.6. The set of allowed inventories has imposed vega risk limits, i.e.

$$\mathcal{Q} = \left\{ q \in \mathbb{R}^N \,\middle|\, \sum_{i=1}^{N} q^i \mathcal{V}^i \in [-\overline{\mathcal{V}}, \overline{\mathcal{V}}] \right\},$$

Here $\overline{\mathcal{V}} \in \mathbb{R}_+^*$ is the vega risk limit of the options market-maker.

This is a natural assumption as a Δ-hedged portfolio must describe its source of risk by only its risk limits. The limitation due to this assumption is that no individual risk limit may be set to each of the separate $N \geq 1$ options.

3.2 Results on the Options Market-Making Model

Now that we have described the dynamics of the market mid price, elaborately explained the options market-maker's problem and discussed the assumptions made and their importance in approximating a numerical solution to the problem.

In these following two subsections the paper will first focus on reducing the dimensionality using the constant-vega approximation. According to Baldacci *et al.* (2019) [BBG19], this first step is known as 'Beating the curse of dimensionality'. Once the value function has been reduced to a low-dimensional optimal control problem, the paper will then formulate the Hamilton-Jacobi-Bellman equation and thus solve for the optimal controls δ_t^{i,j^*}.

3.2.1 Change of variables: reducing dimensions

Consider the following approach under the assumptions from Section 3.1.4. The $N \geq 1$ options create a total of $N+2$ state variables but they may be replaced with just two variables. These two variables are the vega of the portfolio and the instantaneous variance of the portfolio. The dynamics of the vega of the portfolio, which mathematically defined is $\mathcal{V}_t^\pi = \sum_{i=1}^N q_t^i \mathcal{V}^i$, is represented by:

$$d\mathcal{V}_t^\pi = \sum_{i=1}^N \int_{\mathbb{R}_+^*} z\mathcal{V}^i(N^{i,b}(dt,dz) - N^{i,a}(dt,dz)).$$

It then follows that the value function denoted by u must verify:

$$\forall (t,S,\nu,q) \in [0,T] \times \mathbb{R}_+^2 \times \mathcal{Q},$$
$$u(t,S,\nu,q) = v\left(t, \nu, \sum_{i=1}^N q^i \mathcal{V}^i\right),$$

with $v(t,\nu,\mathcal{V}^\pi)$ defined as:

$$v(t,\nu,\mathcal{V}^\pi) = \sup_{(\delta_s)_{s\in[t,T]}\in\mathcal{A}_t} \mathbb{E}_{(t,\nu,\mathcal{V}^\pi}\left[\int_t^T \left(\left(\sum_{i=1}^N \sum_{j=a,b} \int_{\mathbb{R}_+^*} z\delta_s^{i,j}(z)\Lambda^{i,j}(\delta_s^{i,j}(z))\mathbb{1}_{\{|\mathcal{V}_s^\pi - \psi(j)z\mathcal{V}^i| \leq \overline{\mathcal{V}}\}}\mu^{i,j}(dz)\right) + \mathcal{V}_s^\pi \frac{a_{\mathbb{P}}(s,\nu_s) - a_{\mathbb{Q}}(s,\nu_s)}{2\sqrt{\nu_s}} - \frac{\gamma\xi^2}{8}\mathcal{V}_s^{\pi\,2}\right) ds\right]. \tag{3.2.1}$$

Since $(\nu_t, \mathcal{V}_t^\pi)_{t\in[0,T]}$ is a Markov process, the two assumptions from Section 3.1.4 reduce the problem to a low-dimensional optimal control problem of just two state variables. In fact, these two state variables follow $2N$ controlled processes and a standard Brownian Motion (BM).

3.2.2 Hamilton-Jacobi-Bellman equation and solution to optimal controls

In a paper by B. Øksendal and A. Sulem (2007) [OS07], the Hamilton-Jacobi-Bellman equation in accordance with equation (3.2.1) is obtained as the following:

$$0 = \partial_t v(t,\nu,\mathcal{V}^\pi) + a_{\mathbb{P}}(t,\nu)\partial_\nu v(t,\nu,\mathcal{V}^\pi) + \frac{1}{2}\nu\xi^2\partial_{\nu\nu}^2 v(t,\nu,\mathcal{V}^\pi) + \mathcal{V}^\pi \frac{a_{\mathbb{P}}(t,\nu) - a_{\mathbb{Q}}(t,\nu)}{2\sqrt{\nu}} - \frac{\gamma\xi^2}{8}\mathcal{V}^{\pi\,2} + \sum_{i=1}^N \sum_{j=a,b} \int_{\mathbb{R}_+^*} z\mathbb{1}_{\{\mathcal{V}^\pi - \psi(j)z\mathcal{V}^i \leq \overline{\mathcal{V}}\}} H^{i,j}\left(\frac{v(t,\nu,\mathcal{V}^\pi) - v(t,\nu,\mathcal{V}^\pi - \psi(j)z\mathcal{V}^i)}{z}\right)\mu^{i,j}(dz), \tag{3.2.2}$$

with terminal condition specified as $v(T,\nu,\mathcal{V}^\pi) = 0$. Moreover,

$$H^{i,j}(p) := \sup_{\delta^{i,j} \geq \delta_\infty} \Lambda^{i,j}(\delta^{i,j})(\delta^{i,j} - p), i \in \{1,...,N\}, j = a,b.$$

Ergo, we have now a low-dimensional optimal control problem of the HJB type and once we know the formulation of the value function, the optimal bid and ask quotes for the $N \geq 1$ options can be calculated as a function ${\delta_t^{i,j}}^*(z)$. In fact, this ${\delta_t^{i,j}}^*(z)$ represents the distance of the optimal mid-to-bid or optimal ask-to-mid quotes.

$$ {\delta_t^{i,j}}^*(z) = \max\left(\delta_\infty, (\Lambda^{i,j})^{-1}\left(-{H^{i,j}}'\left(\frac{v(t,\nu_t,\mathcal{V}_{t-}^\pi) - v(t,\nu_t,\mathcal{V}_{t-}^\pi - \psi(j)z\mathcal{V}^i)}{z}\right)\right)\right), i \in \{1,...,N\}, j = a,b. $$

Remark 3.2.1. Consider the case where $a_\mathbb{P} = a_\mathbb{Q}$, this implies that v does not depend on ν. Thus, $v(T,\nu,\mathcal{V}^\pi) = w(t,\mathcal{V}^\pi)$ where w solves the simpler HJB equation presented below.

$$ 0 = \partial_t w(t,\mathcal{V}^\pi) - \frac{\gamma\xi^2}{8}{\mathcal{V}^\pi}^2 + \sum_{i=1}^N \sum_{j=a,b} \int_{\mathbb{R}_+^*} z\mathbb{1}_{\{\mathcal{V}^\pi - \psi(j)z\mathcal{V}^i \leq \overline{\mathcal{V}}\}} H^{i,j}\left(\frac{w(t,\nu,\mathcal{V}^\pi) - w(t,\nu,\mathcal{V}^\pi - \psi(j)z\mathcal{V}^i)}{z}\right)\mu^{i,j}(dz) \tag{3.2.3} $$

with final condition given as $w(T,\mathcal{V}^\pi) = 0$.

3.3 Generalising the Options Market-Making Model

3.3.1 Relaxing the Δ-hedging assumption

Even though during the whole course of this chapter we have assumed that the market-maker is continuously Δ-hedged, this subsection considers the problem without this assumption, in essence generalising the problem for wider applications.

We now introduce a new process $(q_t^S)_{t\in[0,T]}$ that tracks the market-maker's inventory of the concerned asset. Using this new process $(q_t^S)_{t\in[0,T]}$, the dynamics of the cash process $(X_t)_{t\in[0,T]}$ of the MM now writes as:

$$dX_t := \sum_{i=1}^{N}\left(\int_{\mathbb{R}_+^*} z\left(\delta_t^{i,b}(z)N^{i,b}(dt,dz)+\delta_t^{i,a}(z)N^{i,a}(dt,dz)\right)-\mathcal{O}_t^i dq_t^i\right)+S_t dq_t^S + d\langle q^s, S\rangle_t.$$

Furthermore, as introduced to earlier but now with new considerations, the Mark-to-Market portfolio value is given by:

$$V_t = X_t + q_t^S S_t + \sum_{i=1}^{N} q_t^i \mathcal{O}_t^i.$$

Its dynamics are given by the expression:

$$dV_t = \sum_{i=1}^{N}\left(\int_{\mathbb{R}_+^*} z\left(\delta_t^{i,b}(z)N^{i,b}(dt,dz)+\delta_t^{i,a}(z)N^{i,a}(dt,dz)\right)+q_t^i\mathcal{V}_t^i\frac{a_{\mathbb{P}}(t,\nu_t)-a_{\mathbb{Q}}(t,\nu_t)}{2\sqrt{\nu_t}}dt+\frac{\xi}{2}q_t^i\mathcal{V}_t^i dW_t^\nu\right)$$
$$+\sqrt{\nu_t}S_t\left(\sum_{i=1}^{N} q_t^i\partial_S O^i(t,S_t,\nu_t)+q_t^S\right)dW_t^S.$$

Moving forward, the mean-variance optimisation problem is formulated as:

$$\sup_{(\delta,q^S)\in\overline{\mathcal{A}}} \mathbb{E}[V_T]-\frac{\gamma}{2}\mathbb{V}\left[\int_0^T \frac{\xi}{2}\mathcal{V}_t^\pi dW_t^\nu + \sqrt{\nu_t}S_t(\Delta_t^\pi+q_t^S)dW_t^S\right],$$

where $\Delta_t^\pi := \sum_{i=1}^N q_t^i\partial_S O^i(t,S_t,\nu_t)$ is the Δ at time t of the market-maker's portfolio. In addition to this, $\mathcal{A}' = \{(\delta_t, q_t^S)_{t\in[0,T]}\}$ such that δ is a $\mathbb{F}$-predictable $\mathbb{R}^{2N}$- valued map bounded from below by δ_∞ and q^S is an $\mathbb{R}$-valued adapted process with $\mathbb{E}\left[\int_0^T \nu_t S_t^2(\Delta_t^\pi+q_t^S)^2 dt\right] < +\infty$.

Since,

$$\mathbb{V}\left[\int_0^T \frac{\xi}{2}\mathcal{V}_t^\pi dW_t^\nu + \sqrt{\nu_t}S_t(\Delta_t^\pi+q_t^S)dW_t^S\right]$$
$$= \mathbb{E}\left[\int_0^T\left(\frac{\xi^2}{4}\mathcal{V}_t^{\pi 2}+\nu_t S_t^2(\Delta_t^\pi+q_t^S)^2+\rho\xi\mathcal{V}_t^\pi\sqrt{\nu_t}S_t(\Delta_t^\pi+q_t^S)\right)dt\right],$$

then it is trivial that the variance term is minimised for $q^S = q^{S^*}$ and,

$$\forall t\in[0,T],\ q_t^{S^*} = -\Delta_t^\pi - \frac{\rho\xi\mathcal{V}_t^\pi}{2\sqrt{\nu_t}S_t}.$$

Additionally, this minimum value is expressed as,

$$(1-\rho^2)\int_0^T \frac{\xi^2}{4}\mathcal{V}_t^{\pi 2}dt.$$

Ergo, the optimisation problem can now be written as,

$$\sup_{\delta\in\mathcal{A}}\mathbb{E}\left[\int_0^T\left(\left(\sum_{i=1}^{N}\sum_{j=a,b}\int_{\mathbb{R}_+^*} z\delta_t^{i,j}(z)\Lambda^{i,j}(\delta_t^{i,j}(z))\mathbb{1}_{\{|\mathcal{V}_t^\pi-\psi(j)z\mathcal{V}^i|\le\overline{\mathcal{V}}\}}\mu^{i,j}(dz)\right)+\mathcal{V}_t^\pi\frac{a_{\mathbb{P}}(t,\nu_t)-a_{\mathbb{Q}}(t,\nu_t)}{2\sqrt{\nu_t}}-\frac{\gamma\xi^2}{8}(1-\rho^2)\mathcal{V}_t^{\pi 2}\right)dt\right].$$

This is the same optimisation problem as we saw in Equation (3.2.1), with the exception that the vol of vol parameter ξ is now multiplied with a term $\sqrt{1-\rho^2}$. This adjustment is made to due to the possibility of reducing risk by trading in alignment with the optimal strategy in the presence of vol-spot correlation.

3.3.2 Relaxing the constant-Vega assumption

Consider the following approach in relaxing the constant-Vega approximation. Let's assume $\forall i \in \{1, ..., N\}$, the process $(\mathcal{V}_t^i)_{t\in[0,T]}$ remains in close proximity to its initial value i.e. $\mathcal{V}^i := \mathcal{V}_0^i$, then using a perturbation method around the constant-Vega assumption we get,

$$\sum_{i=1}^{N} q_t^i \delta_{\sqrt{\nu}} O^i(t, S_t, \nu_t) = \mathcal{V}_t^{\pi} + \epsilon \mathcal{W}(t, S_t, \nu_t, q_t)$$

and similarly for the value function,

$$u(t, S, \nu, q) = v\left(t, \nu, \sum_{i=1}^{N} q^i \delta_{\sqrt{\nu}} O^i(0, S_0, \nu_0)\right) + \epsilon\phi(t, S, \nu, q) = v\left(t, \nu, \sum_{i=1}^{N} q^i \mathcal{V}^i\right) + \epsilon\phi(t, S, \nu, q).$$

Under the assumption that $\mathcal{Q} = \mathbb{R}^N$ and the HJB equation associated with the value function is,

$$\begin{aligned}
0 &= \delta_t u(t, S, \nu, q) + a_{\mathbb{P}}(t, \nu)\delta_\nu u(t, S, \nu, q) + \frac{1}{2}\nu S^2 \delta_{SS}^2 u(t, S, \nu, q) + \frac{1}{2}\nu\xi^2 \delta_{\nu\nu}^2 u(t, S, \nu, q) + \rho\nu S\xi\delta_{\nu S}^2 u(t, S, \nu, q) \\
&+ \frac{a_{\mathbb{P}}(t, \nu) - a_{\mathbb{Q}}(t, \nu)}{2\sqrt{\nu}} \sum_{i=1}^{N} q^i \delta_{\sqrt{\nu}} O^i(t, S, \nu) - \frac{\gamma\xi^2}{8}\left(\sum_{i=1}^{N} q^i \delta_{\sqrt{\nu}} O^i(t, S, \nu)\right)^2 \\
&+ \sum_{i=1}^{N} \sum_{j=a,b} \int_{\mathbb{R}_+^*} H^{i,j'}\left(\frac{u(t, S, \nu, q) - u(t, S, \nu, q - \psi(j)ze^i)}{z}\right)\mu^{i,j}(dz),
\end{aligned}$$

with terminal condition $u(T, S, \nu, q) = 0$.

Evaluating the first-order term in the Taylor expansion with respect to ϵ, we obtain,

$$\begin{aligned}
0 &= \delta_t \phi(t, S, \nu, q) + a_{\mathbb{P}}(t, \nu)\delta_\nu \phi(t, S, \nu, q) + \frac{1}{2}\nu S^2 \delta_{SS}^2 \phi(t, S, \nu, q) + \frac{1}{2}\nu\xi^2 \delta_{\nu\nu}^2 \phi(t, S, \nu, q) + \rho\nu S\xi\delta_{\nu S}^2 \phi(t, S, \nu, q) \\
&+ \frac{a_{\mathbb{P}}(t, \nu) - a_{\mathbb{Q}}(t, \nu)}{2\sqrt{\nu}} \mathcal{W}(t, S, \nu, q) - \frac{\gamma\xi^2}{4}\mathcal{W}(t, S, \nu, q) \sum_{i=1}^{N} q^i \mathcal{V}^i \\
&+ \sum_{i=1}^{N} \sum_{j=a,b} \int_{\mathbb{R}_+^*} H^{i,j'}\left(\frac{v(t, \nu, \sum_{l=1}^{N} q^l \mathcal{V}^l) - v(t, \nu, \sum_{l=1}^{N} q^l \mathcal{V}^l - \psi(j)z\mathcal{V}^i)}{z}\right) \\
&\times ((\phi(t, S, \nu, q) - \phi(t, S, \nu, q - \psi(j)ze^i))\mu^{i,j}(dz),
\end{aligned}$$

again having terminal condition satisfied by $\phi(T, S, \nu, q) = 0$.

Since this is a linear equation, $\phi(t, S, \nu, q)$ attains a Feynman-Kac representation to tame the curse of dimensionality (concerned with computation of quotes/practical applications):

$$\phi(t, S, \nu, q) = \mathbb{E}_{(t,S,\nu,q)}\left[\int_t^T \left(\frac{a_{\mathbb{P}}(s, \nu_s) - a_{\mathbb{Q}}(s, \nu_s)}{2\sqrt{\nu_s}}\mathcal{W}(s, S_s, \nu_s, q_s) - \frac{\gamma\xi^2}{4}\mathcal{W}(s, S_s, \nu_s, q_s)\sum_{i=1}^{N} q_s^i \mathcal{V}^i\right) ds\right]$$

and the jump processes $N^{i,b}$ and $N^{i,a}$ have intensities presented respectively below:

$$\overline{\lambda}_t^{i,b}(dz) = -H^{i,b'}\left(\frac{v(t, \nu_t, \sum_{l=1}^{N} q_{t-}^l \mathcal{V}^l) - v(t, \nu_t, \sum_{l=1}^{N} q_{t-}^l \mathcal{V}^l + z\mathcal{V}^i)}{z}\right)\mu^{i,b}(dz)$$

$$\overline{\lambda}_t^{i,a}(dz) = -H^{i,a'}\left(\frac{v(t, \nu_t, \sum_{l=1}^{N} q_{t-}^l \mathcal{V}^l) - v(t, \nu_t, \sum_{l=1}^{N} q_{t-}^l \mathcal{V}^l - z\mathcal{V}^i)}{z}\right)\mu^{i,a}(dz)$$

$\forall i \in \{1, ..., N\}$.

For practical purposes, one can compute the function ϕ by utilising a Monte-Carlo simulation algorithm. Then, optimal bid and ask quotes accounting for the variation of the Vegas can be determined to the first order in ϵ, a constraint of the perturbation method around the constant-Vega approximation.

This concludes our analysis on the options market-maker's problem. This rigorous theoretical analysis proves that even without the constant-Vega assumption initially posed, the market-making problem is tractable and retains similar scope in practical applications. Therefore, Baldacci *et al.* (2019) [BBG19] have achieved a method that scales linearly in the number of options and thus may be computed on a relatively large book of options.

Chapter 4

Conclusion

Financial mathematics in the past few decades has transformed trading in financial markets. Formal mathematical theories such as asset pricing and replication theory, along with arbitrage theory have allowed new generation technology to benefit from the intricacies present in trading financial assets. High-frequency trading is one such instance of an emerging trend in financial markets. Various papers have cited the uprising of high-frequency trading and have delved into devising models to practice this form of market participation. Statistics from econophysics have also revealed that almost half the volume traded currently on financial markets occurs in black-boxes where a high-number of trade executions are made all based on algorithms.

Since the 1980s, when Ho and Stoll (1981) [HS81] first formalised the concepts of algorithmic trading, we have made much progress in extending these concepts and modelling more complex real-life phenomenon. This theoretical review paper focuses specifically on the recent results in electronic trading provided by two leading papers in the field, 'High-Frequency Trading in a Limit Order Book' by Avellaneda and Stoikov (2008) [AS08] and 'Algorithmic market-making for Options' by Baldacci *et al.* (2019) [BBG19].

In conclusion, these two problems are monumental in market-making today, yet they require an understanding that is reserved chiefly for people with high quantitative calibre. This paper has consolidated this recent work done in market-making into a self-sufficient theoretical review allowing other individuals an introduction, insight and future potential on the subject. While the first paper formalises for the reader the concepts of the dynamic programming principle and the approach of a market-maker trading through limit orders, the second paper considers a highly complex stochastic volatility model to solve the options market-maker's problem of determining optimal bid and ask quotes on a large book of options. Moreover, this theoretical review also extends 'Algorithmic market-making for Options' by Baldacci *et al.* (2019) [BBG19] by incorporating the appendices of Baldacci *et al.*. The appendices consider relaxing the assumptions of continuous Δ-hedging and constant-Vega approximation, which this theoretical review extends and reviews in the main body of the paper itself.

4.1 Key Findings and Contributions

This paper has produced theoretical results to both the problems considered and thus the contributions made shall be individually discussed below.

The first paper, [AS08], formalises for the reader the concepts of the dynamic programming principle and the approach of a market-maker trading through limit orders. The problem setting introduces the incorporation of limit order trading and the utility function used is the usual Cara utility function, as it is an exponential utility function. In Chapter 2.1, the main foundations of the model by Avellaneda and Stoikov (2008) [AS08] are described: the dynamics of the market mid-price, the arrival rate of market orders subject to their respective distances from the market mid-price and the dealer's utility objective functional. Chapter 2.2 solves for the optimal bid and ask quotes and compares them with the dealer's indifference price accounting for their personal risk considerations and given their current inventory. We suggest that the optimal bid and ask quotes may be determined using a straightforward two-step procedure. This intuitive procedure first requires one to solve the non-linear PDE (2.2.4) and determine $i^b(s,q,t)$ and $i^a(s,q,t)$. Next, they need to solve the implicit dependencies (2.2.7) and (2.2.8) in order to calculate the optimal distances from the market mid-price, $\delta^b(s,q,t)$ and $\delta^a(s,q,t)$, and thus solve the market-maker's problem of finding optimal bid and ask quotes. In addition to this, the paper also extends its contributions, namely in Chapter 2.3, by considering a Geometric Brownian Motion for the market mid-price which was earlier governed by a standard Brownian Motion.

Additionally, in chapter 3.1, the second paper, the main foundations of the model by Baldacci *et al.* (2019) [BBG19] are described: the dynamics of the market mid-price using a one-factor stochastic volatility model, the optimisation problem of the options market-maker and lastly the assumptions needed to address the problem from a theoretical point of view and the approximations needed to simulate these optimal quotes. Following this, Chapter 3.1 introduces the Constant-Vega assumption that is essential in simplifying this high-dimensional stochastic optimal control problem of the market-maker into a low-dimensional functional equation. This mathematical simplification allowed me to show in Chapter 3.2 that optimal bid and ask quotes for a book of options with several strikes and maturities may be computed straightforwardly using interpolation techniques and an explicit Euler scheme. The results showcase reducing the dimensionality of the problem using the constant-Vega approximation. According to Baldacci *et al.* (2019) [BBG19], this first step is known as 'Beating the curse of dimensionality'. Once the value function has been reduced to a low-dimensional optimal control problem, we formulate the Hamilton-Jacobi-Bellman equation and thus solve for the optimal controls δ_t^{i,j^*}. Even though during the whole course of this chapter, 3.2, we have assumed that the market-maker is continuously Δ-hedged, the last chapter, 3.3, considers the problem without this assumption, in essence generalising the problem for wider applications. This is a significant contribution of the paper as previously, the solution was discussed in Baldacci *et al.*'s appendix yet this paper extends this and verifies that the options market-making problem remains tractable on a large book of options even without the initial assumptions.

The methods used in this paper include secondary research on statistics for intensities in econophysics literature, devising problem settings for two fundamental market-making problems and lastly assessing the need for certain assumptions in order to attain analytical solutions.

4.2 Limitations and Future Research

It is crucial to understand that the theoretical review has its limitations due to time constraints. A possible extension to this paper could be the simulation of the optimal strategies suggested by both Avellaneda and Stoikov (2008) [AS08] and Baldacci *et al.* (2019) [BBG19]. This will validate the tractability of the methods and their efficiency in practical applications, i.e. computing the optimal bid and ask quotes for the $N \geq 1$ options in Baldacci *et al.*, confirming that the computations scale linearly in the number of options and therefore then computing the quotes on a relatively large book of options.

Last but not least, the options market-making problem can also be studied using purely reinforcement-based machine learning techniques as these policies produce efficient results in practice as well. According to latest research by Selser *et al.* (2021) [SKM21], Reinforcement Learning algorithms can be used to solve the classic quantitative finance Market Making problem. Their optimal agent must find a delicate balance between the price risk of her inventory and the profits obtained by capturing the bid-ask spread. They design an environment with a reward function that determines an order relation between policies equivalent to the original utility function. They compared the performance of their agent with the benchmark symmetric agent and found that their agent, the Deep Q-Learning algorithm, manages to recover the optimal solution. This extension pairs well with the practical simulation of the optimal strategy as this will allow us to evaluate both strategies simultaneously according to desired execution qualities.

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