MSc Applied Mathematics Project 2021-22

UNCERTAINTY QUANTIFICATION IN TOPOLOGICAL DATA ANALYSIS

Supervisors: Barbara Bravi and Anthea Monod

Topological data analysis (TDA) is a 21st century branch of data science that leverages ideas from pure mathematics for feature extraction in data. *Persistent homology*[1,5] is a tool from TDA that adapts the classical concept of homology from algebraic topology to data settings as point clouds or finite metric spaces. Persistent homology has been widely applied with great success in many applications including viral evolution[8], medical imaging[3], sensor networks[6], neuroscience[7], finance[2], and many more. Essentially, persistent homology captures the "shape" and "size" of simplicial complexes as skeletal representations of data and summarizes this information in a persistence diagram. Persistence diagrams are algebraic objects by construction that live in a well-defined metric space. They are also random objects, by nature of the underlying point cloud from which they arise.

Despite their widespread use in data analysis, there remains much to be understood about the interplay between the algebraicity and randomness of persistence diagrams. This project will explore this intersection and seek to better understand the random nature of points on a persistence diagram as well as the random nature within persistence diagram space using theory from uncertainty quantification—such as polynomial chaos[9] and spline-based methods[4]—to perform moment and density estimation on persistence diagrams. Time permitting, there may be the opportunity to explore an application to biochemical reaction networks.

PREREQUISITES

The following skills are required for this project.

- Interest in geometry, algebra and topology: Prior experience with algebraic topology is not required, but undergraduate experience (with good results) in both algebra and point-set topology is strongly recommended.
- Experience with probability, statistics, and stochasticity/randomness via appropriate choice of modules offered by the programme.
- Independence and creativity: The intersection of the fields of topological data analysis and uncertainty quantification is novel and though there is some relevant work, the area is wide open for exploration and potential for several different aspects of study. The ideal candidate will play an active role in directing the project in a direction commensurate with their skills and interests.
- Strong programming skills: Ability to write code in your choice of language. Note that the most widely used TDA libraries are written in Python and C++.

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Statistical learning of the generation probability of immune receptors

Supervisor: Barbara Bravi (Department of Mathematics, Imperial College London)
Co-supervisor: Becca Asquith (Department of Infectious Disease, Imperial College London)
Collaborators: Ben Willcox (Birmingham), Carrie Willcox (Birmingham) and Derek Macallan (SGUL)

T cells are cells of the immune system which play a major role in protecting the organism against infections and malignancies. Their action is based on the molecular recognition of infected or cancer cells, which relies on the binding between the T cell membrane receptors and protein targets displayed on the surface of infected or cancer cells. Every individual is endowed with an extremely diverse set of T-cell receptors; such diversity is *generated* through a stochastic process, the V(D)J recombination [1], whereby gene segments are stochastically chosen, combined and modified (through random insertion or removal of nucleotides) to form the gene from which the receptor is translated. After this generation step, the receptors undergo a process of selection based on their binding properties, which essentially discards the receptors that are likely to be non-functional.

The so-called $\gamma\delta$ T cells [2] are a subset of T cells whose functional characterization is far from complete, particularly in comparison to a much better known type of T cells called $\alpha\beta$ T cells. For instance, the statistics of receptor generation and selection have been characterized quantitatively for $\alpha\beta$ T cells, using techniques of statistical learning that allow one to build probabilistic models of generation [3,4] and selection [5] from large datasets of receptor sequences (*i.e.*, the receptors' constitutive chains of nucleotides or amino acids). On the other hand, the same characterization for $\gamma\delta$ T cells is lacking, and even whether or not $\gamma\delta$ T cells are subject to selection during their development is still an open question.

We aim at tackling these questions in a quantitative way by detecting the statistical signatures of generation and selection mechanisms in newly produced datasets of $\gamma\delta$ T-cell receptor sequences. Starting from this dataset, the main objective of this project is to construct a probabilistic model of receptor generation for $\gamma\delta$ T cells, using the statistical learning approach proposed for $\alpha\beta$ T cells [3]. Such an approach is based on the Expectation-Maximization algorithm to infer the parameters of a probabilistic model describing step by step the process underlying generation, *i.e.* V(D)J recombination.

The inferred model will inform us of the diversity and the statistical properties of the $\gamma\delta$ receptor repertoires resulting from generation, setting a baseline expectation in terms of receptor statistical distribution; modelling the differences with respect to this expectation will eventually allow us to detect and quantify signatures of selection. Time permitting, we will start to explore also this second question by learning, using different information in the same dataset, a probabilistic model of $\gamma\delta$ receptor selection.

This work will be performed in collaboration with our experimental colleagues, Ben Willcox, Carrie Willcox and Derek Macallan who have generated the experimental data.

References

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A probabilistic modelling approach to the detection of immune escape in cancer evolution

Supervisor: Barbara Bravi (Department of Mathematics, Imperial College London)
Co-supervisor: Benjamin D. Greenbaum (Department of Epidemiology and Biostatistics, Memorial Sloan Kettering Cancer Center, New York)

Cancer progression is characterized by the appearance of genetic mutations, and what factors shape the observed distribution of mutations is an outstanding open question. In particular, it is not clear what is relative interplay between immune escape, *i.e.*, the occurrence of mutations disrupting immune recognition of cancer cells, and the 'mutational signatures' of different cancer types [1], *i.e.*, the characteristic patterns in the distribution of mutations reflecting the intrinsic or extrinsic factors (such as environmental carcinogens, UV radiation, smoking) from which they might originate.

The objective of this project is to design a quantitative framework to address this question, based on fitness models and probabilistic machine learning tools that are able to measure the cost of mutations in terms of their potential to be recognized by the immune system [2–4]. The design of such framework requires two main steps: *(i)* sampling by Monte Carlo the background distribution purely induced by mutational signatures, starting from genes that are known to harbour oncogenic mutations (such as p53 and KRAS) and using mutational rates that reflect mutational signatures (from the COSMIC database cancer.sanger.ac.uk/signatures); *(ii)* learning the probabilistic models that describe the potential of mutations to be recognized constrained to different genetic make-ups (since differences in the genetic make-up lead to differences in the immune response across individuals). The models' predictions will be then deployed to map the distribution of mutational costs in terms of immune recognition. The MSc project will focus on step *(ii)*, starting from the background distribution for p53 obtained in the work [5].

The result will be a reference model for cancer mutational landscapes providing a baseline expectation of the immune escape potential of typical mutations in cancer, which can be used as term of comparison to detect deviations with respect to this baseline indicating immune escape.

Possible directions that could be developed as a PhD project include: *(i)* a refinement of the background distribution simulation step, by taking into account higher-order dependencies among nucleotides of mutational signatures; *(ii)* comparisons to cancer evolution datasets to define a quantitative measure of immune escape based on the deviation with respect to the reference model's baseline expectation; *(iii)* a systematic application to mouse datasets, where it is possible to obtain a profile of early-stage *vs* late-stage mutations and hence to assess, through the designed framework, the temporal progression of immune escape.

References

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Professor Colin Cotter

Modified CR1-DG0 discretisation for atmosphere and ocean models have worked a lot on compatible finite element methods for geophysical fluid dynamics (see this book for example: https://link.springer.com/book/10.1007/978-3-030-23957-2) which have a number of special properties that make them great for this application area. One disadvantage is that they are not very efficient when using explicit time integration methods because you still have to solve a globally-coupled matrix-vector system at each Runge-Kutta stage. On the other hand, another finite element discretisation, CR1-DG0, only requires the solution of a diagonal matrix, meaning that it is good
for explicit time-integration methods. However, CR1-DG0 does not share the special properties of compatible finite elements. I have
discovered a modification of CR1-DG0 that recovers these properties. I would like to publish this work but it requires the design and implementation of numerical experiments to demonstrate and benchmark this in practice, hence this project. The project will develop code
using the Firedrake library: https://www.firedrakeproject.org/

Efficient solution of energy-conserving methods for geophysical flows:

My recently graduated PhD student Golo Wimmer developed space discretisations for geophysical flows that conserve energy whilst controlling numerical wiggles through upwind discretisations. These have wonderful properties but are yet to impact practical use because
of the lack of efficient algorithms to solve the systems of nonlinear equations that arise from them (Golo demonstrated them using methods that are 10-40 times slower than the algorithms that can be used on standard discretisations). I have designed a new approach to
efficiently solving these equations based on an "augmented Lagrangian" technique, and this project would implement and explore this approach. The project will develop code using the Firedrake library: https://www.firedrakeproject.org/

Finite element neural operators:

Neural operators are neural net architectures that are designed to
learn operators mapping between infinite dimensional function spaces. They can be used to e.g. learn the time-T solution map for
time-dependent PDEs, solve PDE constrained inverse problems, etc. This project will investigate a new type of neural operator that is implementable on finite element spaces defined on unstructured meshes, which can be applied to e.g. fluid dynamics problems in complicated domains such as flow past objects etc. The method is based upon composition of fast approximate solvers for elliptic PDEs with coefficients that play the role of the neural network parameters. In
this project we will explore this finite element neural operator
approach by developing an implementation using the Firedrake library: https://www.firedrakeproject.org/

Michele Coti Zelati: Enhanced diffusion for radial flows

The goal of this project is to study the enhanced diffusion properties of a passive scalar f that satisfies the advection diffusion equation

$$\partial_t f + \boldsymbol{u} \cdot \nabla f = \nu \Delta f, \tag{1}$$

where $\nu > 0$ is a small diffusion parameter and $\boldsymbol{u}$ is a given, time-independent and divergence-free velocity vector field. In fairly general settings, it is not hard to show that the energy of the solution (namely, the L^2-norm) decays exponentially as $\mathrm{e}^{-\nu t}$. However, it is expected that the presence of the flow $\boldsymbol{u}$ speeds up the rate to $\mathrm{e}^{-\nu^q t}$, for some $q \in (0, 1)$ depending on $\boldsymbol{u}$. This is a manifestation of a phenomenon called *enhanced diffusion*.

In this project, we will take a look at the case in which $\boldsymbol{u}$ describes a circular flow, which in polar coordinates look like

$$\boldsymbol{u}(r, \theta) = v(r) \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}, \qquad r \geq 0, \ \theta \in [0, 2\pi), \tag{2}$$

for some smooth decreasing function $v : [0, \infty) \to \mathbb{R}$, with $v'(0) = 0$. The goal is to describe in mathematically rigorous terms the features of Figure 1.
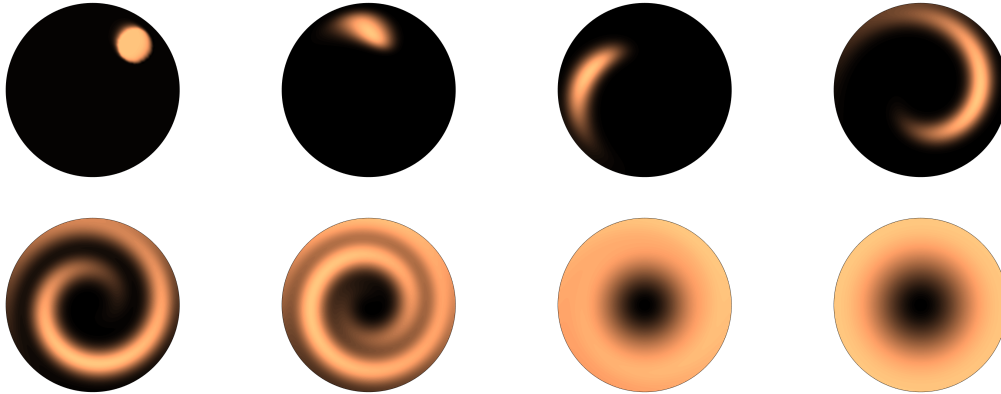

FIGURE 1. The evolution of a drop of slightly diffusive "cream" radial stirred into a "cup of coffee" with impermeable walls. Initially, pure advection is the dominant effect. As time progresses, the solution becomes radially symmetric. After this time, the cream simply diffuses across the (circular) streamlines.

The problem can be approached in various ways: it can be rephrased into studying the spectral properties of the operator $\boldsymbol{u} \cdot \nabla - \nu \Delta$, or it can be studied using an energy method called *hypocoercivity*, see for instance [1].

Prerequisites. Students taking on this project are required to have some basic understanding of differential equations, multivariable calculus, Fourier series and Hilbert spaces.

References

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Michele Coti Zelati: Landau damping for active Brownian particles

Active matter systems consisting of many interacting self-propelled particles occur in many applications ranging from synthetic self-propelled colloids, microtubules, and bacterial suspensions, to large-scale systems such as fish schools, bird flocks and collective robotics.

If one considers identical Brownian particles moving in a periodic box, a so-called mean-field limit procedure allows to derive a partial differential equation for the macroscopic density $f = f(t, \boldsymbol{x}, \theta)$ of particles with direction angle θ, at time t and position $\boldsymbol{x}$, of the form

$$\partial_t f + \nabla \cdot (v(\rho) f \boldsymbol{e}(\theta)) = D_e \Delta f + D_R \partial_{\theta\theta} f. \tag{1}$$

Here, D_e is an effective diffusion depending on how crowded the system is, D_R is a free rotational diffusion with diffusion coefficient, $\boldsymbol{e}(\theta) = (\cos\theta, \sin\theta)$ and v is a nonlinear effective speed, which depends on the the spatial macroscopic density

$$\rho(t, \boldsymbol{x}) = \int_0^{2\pi} f(t, \boldsymbol{x}, \theta) \mathrm{d}\theta. \tag{2}$$

The goal of this project is to study the stability of the homogeneous solution $f_\star = \phi/2\pi$, of mass $\phi \in [0, 1]$, for various models of the form (1). Even when no diffusion is present (i.e. $D_e = D_R = 0$), the problem can be understood by means of a mysterious stability mechanism in plasma physics, known as *Landau damping*. It originates by an oscillatory behaviour known as *phase mixing*, and was only recently understood in an important article by C. Mouhot and fields medalist C. Villani [1]. The project will require the understanding of a powerful theory of oscillatory integrals, know as *stationary phase* method.

Prerequisites. Students taking on this project are required to have some basic understanding of differential equations, multivariable calculus, Fourier series and Hilbert spaces.

References

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Wave localization in random media
Dr Bryn Davies and Prof Richard Craster
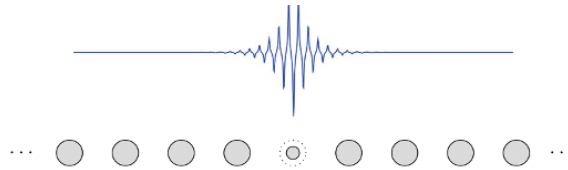Key words: differential equations, wave physics, numerical simulations

Figure 1: When a defect is introduced to a periodic structure, there can exist eigenmodes that are localized in a region of the defect.

This project will be an opportunity to become familiar with cutting-edge ideas in wave physics and contribute to active research. A general principle in wave physics is that if a perturbation is made to a periodic structure then waves with specific frequencies will be localized (that is, guided or trapped) in a region of that perturbation (see Figure 1). This has been studied in many different settings and exploited for many different applications. One perspective on this, which has been widely studied, is that we can assign topological indices to periodic structures and use these to predict the occurrence of localized modes (which are often known as *topologically protected modes* to describe their special properties) [3]. Conversely, the behaviour when random perturbations are made to the structure is less well understood. In this case, localized modes have been shown to occur and *landscape functions* have been used to predict where they may occur [2].

In this project, we are interested in studying the occurrence of localization in randomly perturbed media and investigating the extent to which this can be understood using the established theory of topological edge modes. Initially, we will explore a one-dimensional discrete lattice, modelled using difference equations. A similar system was studied in [4] for the case of deterministic, localized perturbations. The student will develop numerical codes to model the problem and perform experiments to understand the behaviour. They will also need to familiarise themselves with the background literature on Anderson and weak localization [2] and topological insulators [3]. One-dimensional systems based on the Su-Schrieffer–Heeger (SSH) model are of particular interest [1]. The fundamental question is how this theory can be translated to our setting and used to predict the observed behaviour.

If good progress is made, these results can be extended to mass-string systems (modelled by ordinary differential equations) and, subsequently, to multidimensional partial differential systems, similar to those studied in [1].

Prerequisites:
- basic understanding of differential equations,
- some working knowledge of simulations in MATLAB or Python,
- interest in wave physics.

References

[1] H. Ammari, B. Davies, E. O. Hiltunen, and S. Yu. Topologically protected edge modes in one-dimensional chains of subwavelength resonators. *J. Math. Pure. Appl.*, 144:17–49, 2020.

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[4] M. Makwana and R. Craster. Localised point defect states in asymptotic models of discrete lattices. *Q. J. Mech. Appl. Math.*, 66(3):289–316, 2013.

Benchmarking implementations of quantum stochastic simulators

Dr Thomas Elliott

The field of quantum information theory combines information theory with quantum physics [1], opening a new paradigm of information processing. Quantum computers – devices that can perform computations with quantum information – promise to provide significant advantages for fields that rely on immense computational power, such as machine learning, financial modelling, and computational biology. While the realisation of quantum computers poses an extreme engineering challenge, in recent years the threshold where quantum computers can outperform our best classical computers (at specific tasks) has been surpassed [2], heralding the beginning of practical quantum computational advantages.

One such area of application is the simulation of complex stochastic processes. To replicate the behaviour of a stochastic process, it is in general necessary to store information about the past of the process, requiring the simulator to have a memory. It has been shown that simulators capable of storing and processing quantum information can do this with a lower memory overhead than possible with any classical counterpart [3]. In some circumstances, this quantum advantage can scale without bound [4]. Given the ubiquity of stochastic simulation, these results are expected to find application across the quantitative sciences.

This project will explore the proof-of-principle implementation of such quantum stochastic simulators on cloud-accessible quantum processors, such as the *IBM Quantum Experience* [5]. Focussing on toy processes that require only a few quantum bits (qubits) to realise, the student will first trial the implementation on classical simulators of noisy quantum processors, and then later on real quantum processors. We will then investigate how the performance of these implementations can be quantified and evaluated. Subsequently, we will explore methods for improving this performance, such as incorporating quantum error correction techniques and recompiling computations to require fewer quantum operations.

The project will be at least in part numerical, with scope for a significant analytical component. By the end of the project, the student will have gained experience in state-of-the-art research in quantum computation and information theory. Some prior knowledge of the field would be useful, but not essential. Familiarity with basic quantum mechanics is required.

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[2] Arute, F. et al. *Quantum supremacy using a programmable superconducting processor* Nature 574, 505 (2019).
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DIFFUSE INTERFACE METHODS FOR TWO-PHASE COMPLEX FLUIDS

Dr Andrea Giorgini

The motion of two-phase fluids and their interaction at the interface is a fascinating and challenging problem in fluid mechanics. Complex phenomena already occur in simple experiments when the spatial regions occupied by a single flow is deformed by moving fluid structures, and the interface area decreases its characteristic length scale, develops singularities and changes its topology. These phenomena are ubiquitous in materials science, engineering and biology: from coalescence of drops to phase separation, from topological defects of liquid crystals to electrowetting, and from non-isothermal flows to tumor growth dynamics. The Diffuse Interface (DI) theory, also called Phase Field theory, represents nowadays a successful method to simulate fluid mixtures, being able to capture the main features of the mutual interplay in the motion of two fluids and large interface deformations. The key concept of DI theory is to represent the interface (or phase boundary) as region with finite thickness among which physical quantities have a rapid but smooth variation. The interfaces are described as the level set of a order parameter, which is usually the fluid concentration or rescaled volume fraction. The dynamics of the order parameter is deduced from the principles of fluid mechanics, thermodynamics and statistical mechanics. In contrast to other methods in which the interface is a time-dependent surface, the main advantage of the DI formulation is the Eulerian description of the interface. In addition, DI models can be regarded as a regularization of Sharp Interface models (free boundary problems) for simple fluid mixtures driven by surface tension or surface evolution problems driven by mean curvature. On the other hand, DI models can be used to represent the dynamics of complex fluids (polymers, liquid crystals and biofluids), which are those mixtures whose molecular interaction at the microscopic scale affects the macroscopic dynamics.
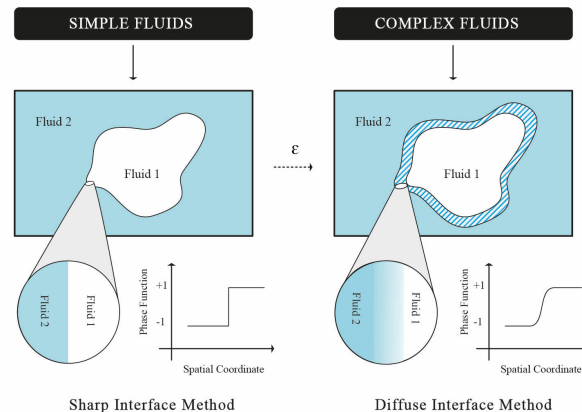
Project Description. This project is devoted to the mathematical study of systems of partial differential equations originating from the DI theory for fluid mixtures. These models consist of the Navier-Stokes equations for the velocity of the mixture coupled with the phase field equations for the order parameter. The student's task will consist in studying the well-posedness theory and longtime behavior of a DI system for complex fluids.

CAHN-HILLIARD DYNAMICS IN MATERIALS SCIENCE

Dr Andrea Giorgini

The Cahn-Hilliard equation is a famous model in Materials Science. It was proposed in 1958 by J.W. Cahn and J.E. Hilliard to provide a mesoscale description of the evolution of microstructures during the phase separation in a binary alloys system. Such phenomenon is characterized by an early stage where the so-called spatial spinodal decomposition takes place, followed by the coarsening process. In the latter, the average size of these spatial domains with same phase increases over time at the expense of the smaller ones (see the figure below). This occurs when a homogeneous mixture undergoes a rapid cooling below a certain critical temperature. The Cahn-Hilliard equation and its variants are particular Diffuse Interface (Phase Field) models and have been employed for different phenomena which are characterized by pattern formation, segregation-like processes and interface motion, such as grain boundary, nucleation, liquid-liquid phase transition and crystallization. In a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, we consider the Gindzburg-Landau free energy

$$E(\phi) = \int_\Omega \frac{\varepsilon}{2}|\nabla\phi|^2 + \frac{1}{\varepsilon}\Psi(\phi)\,\mathrm{d}x,$$

where ϕ is the phase function, ε is a parameter related to the thickness of the interface, and the free energy density Ψ is the Flory-Huggins potential

$$\Psi(s) = \frac{\theta}{2}\left[(1+s)\log(1+s) + (1-s)\log(1-s)\right] - \frac{\theta_0}{2}s^2, \quad s \in [-1, 1],$$

where the constant parameters θ and θ_0 fulfill the conditions $0 < \theta < \theta_0$. The Cahn-Hilliard system is the gradient flow with respect to the $(H^1(\Omega))'$ metric of the total free energy $E(\phi)$, namely

$$\partial_t\phi = \Delta\big(-\varepsilon\Delta\phi + \frac{1}{\varepsilon}\Psi'(\phi)\big), \quad \text{in } \Omega \times (0, \infty),$$

subject to physical boundary conditions.

Project Description. This project is devoted to the study of the recent theoretical results concerning local, nonlocal and fractional Cahn-Hilliard equations with logarithmic (singular) potentials. More precisely, the student's task will consists in understanding the techniques for the existence and uniqueness of weak solutions, regularity theory and validity of the separation property for Cahn-Hilliard type equations. The study of numerical approximations can be also discussed.
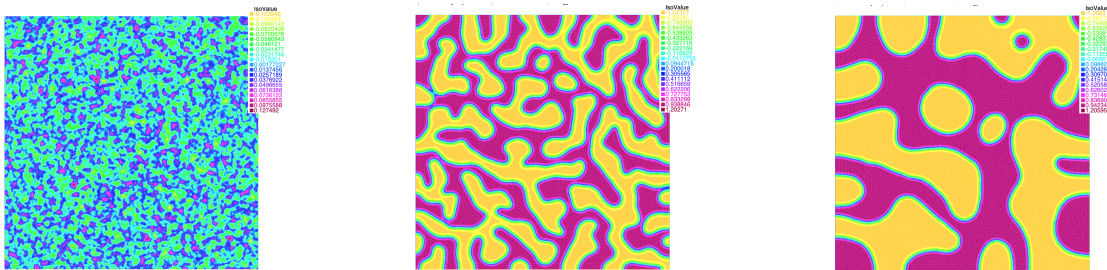

FIGURE 1. Different stages of the phase separation in the Cahn-Hilliard equation

Optimal vibration control of slender structures

Supervisor: Dante Kalise

DESCRIPTION

In this project we will study active vibration control strategies for slender structures such as bridges, trusses, and aircraft wings. We will study this problem in the framework of PDE-constrained optimization, where the control is designed by optimizing a performance index (vibration mitigation plus control effort) constrained to a PDE governing the structural vibration dynamics. Since standard structural vibration models are linear, after a suitable discretization the active vibration control problem can be cast as a Linear-quadratic Regulator (LQR) problem, and can be solved trough a large-scale Algebraic Riccati Equation. As an extension, we will study a mathematical formulation for the optimal location and design of sensors and actuators in the framework of bi-level optimization.

PLAN

The project is split into three stages:

(1) Modelling and approximation of structural vibration phenomena: beam and plate models, time/frequency-dependent formulations, FEM and spectral approximation.
(2) PDE-constrained optimization. The LQR problem for structural dynamics, optimality conditions for the stationary problem. Numerical approximation of large-scale control problems.
(3) Actuator/sensor aspects: piezoelectric actuators, optimal actuator/sensor location and design.

A background/interest on computational mathematics and mathematical modelling with PDEs is essential.

REFERENCES

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Deep Neural Networks for Real-time Trajectory Planning

Supervisor: Dante Kalise

DESCRIPTION

Different problems in robotic locomotion can be studied in the framework of optimal control theory. For example, we can look for a set of actions (controls) which take a drone from point A to B by minimizing the amount of time or energy that is required to complete the task. These control signals are often expressed as feedback laws, that is, real-time actions which can be implemented solely based on the current state (position, velocity) of the robot. In this project, we will develop a deep learning formulation of the optimal trajectory planning problem outlined above. y combining methods from dynamical systems, control theory, and polynomial systems, we will generate a synthetic dataset of optimal input-output pairs. This synthetic dataset will be used to cast a supervised learning problem to approximate an optimal feedback law to be applied in real time after a measurement of the current state of the system.

PLAN

The project is split into three stages:

(1) A brief introduction to optimal control: minimum time formulation, bang-bang controls, switching structures.
(2) Parametrizing optimal control problems as nonlinear and/or polynomial systems. Solution strategies via iterative methods or computer algebra.
(3) Synthetic data generation and supervised learning approaches for synthesizing feedback laws. Applications in robotic locomotion.

Prerequisites: basic knowledge of dynamical systems, computational simulation tools, and machine learning (DNNs, supervised learning). Programming skills in Matlab and/or Python. Willingness to learn about polynomial systems and/or computer algebra is desirable.

REFERENCES

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(3) B. Azmi, D. Kalise and K. Kunisch. *Optimal Feedback Law Recovery by Gradient-Augmented Sparse Polynomial Regression,* in Journal of Machine Learning Research, 22(48):1–32, 2021.

Time-optimal control of agent-based dynamics

Supervisors: Dante Kalise and Grigorios A. Pavliotis

Description

In this project we will study the synthesis of control laws for agent-based models arising in large animal populations (bird flocks, fish shoal, and sheep) and swarm robotics. The objective is to determine optimal control signals which are able to steer the population towards a desired state, e.g. a certain spatial configuration or flocking state. For this, we will follow a dynamic optimization approach, deriving optimality conditions which characterize an optimal intervention. In particular, we will focus on the synthesis of optimal controls which are able to achieve the objective in a minimum amount of time. The derivation of optimality conditions guides the construction of numerical methods for an effective control synthesis. Moreover, we will explore the control synthesis as the number of agents grows, paving the way for a mean-field modelling of the time-optimal control problem.

A background on dynamical systems, numerical analysis, and optimisation/control is desirable.

Plan

The project is split into three stages:

(1) Mathematical modelling of collective behaviour phenomena: animal behaviour, agent-based models, and swarm robotics.
(2) Time-optimal control, optimality conditions for agent-based dynamics.
(3) Computational synthesis of control laws through dynamic optimization. Mean-field scaling of the control problem.

References

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(3) M. Bongini, M. Fornasier, M. Hansen, and M. Maggioni . *Inferring interaction rules from observations of evolutive systems I: The variational approach,* Math. Models Methods Appl. Sci. 27(5):909–951 (2017).
(4) R. Bailo, M. Bongini, J.A. Carrillo and D. Kalise. *Optimal consensus control of the Cucker-Smale model,* IFAC-PapersOnLine 51(13)(2018):1–6.

Self-sustaining processes and coherent structures in two-dimensional flows

Supervisor: Andrew Walton

Description

The topic of transition to turbulence in wall-bounded flows is a classical problem. Reynolds pioneering experiment (1) demonstrates how pipe flow undergoes this transition from an orderly or laminar state to a disordered or turbulent one. One well-established route from laminar to turbulent flow occurs via a sequence of steps, beginning with the linear instability of infinitesimal disturbances (e.g. see (2)). This description, however, is problematical in view of the fact that many wall-bounded flows such as the pipe flow mentioned above can be shown to be stable to small disturbances. Evidently, any theory which claims to explain the route to transition in such flows must therefore be inherently nonlinear.

Within a turbulent flow it is often possible to discern the existence of coherent structures despite the apparent randomness of the overall motion. For example, streaky structures possessing cross-stream variation can be seen in many experimental visualizations of shear flows (e.g. (3)). In addition these streaky structures often appear to be accompanied by a longitudinal vortex structure in the cross-stream plane.

Numerical simulations of turbulent flow (e.g. (4)) have made it possible to observe an interaction between three distinct structures in the flow: a flow in the cross-stream plane with no downstream dependence, known as a roll flow, a streamwise component possessing some cross-stream variation (a streak), and finally a three-dimensional travelling wave propagating in the streamwise direction.

The aim of this project is to explore mathematically the interaction between these three components in a three-dimensional flow and to examine whether a similar mechanism can operate in two dimensions.

Plan

There are possibilities for the project to be more or less computationally-oriented depending on the interests of the student but there will be some numerical computation (in Python or Matlab) required.

The project will commence with some reading on Self-Sustaining Processes and Coherent Structures which will be made available by me. The student will be encouraged to fill in some of the details within these notes and to reproduce some of the computations.

Next, a self-sustaining process for two-dimensional flows involving wave-mean flow interaction will be studied theoretically for a general parallel (or nearly parallel) base flow.

Following this, the effect of this mechanism upon specific flows such as pipe flow and plane and annular Poiseuille and Couette flows will be studied.

If time allows the instability structures derived will be compared with exact solutions of the two-dimensional Navier-Stokes equations.

Prerequisites: Some prior exposure to fluid mechanics is desirable together with a knowledge of Matlab or Python.

References

1. An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels. Reynolds, O. (1883) *Phil. Trans. R. Soc.*, **174**, 935-982.

2. Introduction to hydrodynamic stability, Drazin, P. G. (2002) Cambridge Texts in Applied Mathematics.

3. The structure of turbulent boundary layers, Kline S. J., Reynolds, W. C., Schraub, F. A.& Runstadler, P. W. (1967) *J. Fluid Mech.*, **30**, 741–773.

4. Regeneration mechanisms of near-wall turbulence structures. Hamilton, J. M., Kim, J. & Waleffe, F. (1995) *J. Fluid Mech.*, **287**, 317–348.

The nonlinear stability of plane Couette-Poiseuille flow with zero mean advection velocity

Supervisor: Andrew Walton

Description

The topic of transition to turbulence in wall-bounded flows is a classical problem. Reynolds pioneering experiment (1) demonstrates how pipe flow undergoes this transition from an orderly or laminar state to a disordered or turbulent one. One well-established route from laminar to turbulent flow occurs via a sequence of steps, beginning with the linear instability of infinitesimal disturbances (e.g. see (2)). This description, however, is problematical in view of the fact that many wall-bounded flows such as the pipe flow mentioned above can be shown to be stable to small disturbances. Evidently, any theory which claims to explain the route to transition in such flows must therefore be inherently nonlinear.

A particular flow which can be shown to be linearly stable is that between parallel moving walls subject to a constant streamwise pressure gradient such that the resulting velocity profile, when averaged across the channel, is zero. This flow can be set up relatively easily in a laboratory and, despite the property of linear stability referred to above, exhibits transition to turbulence as the relative wall speed is increased. An interesting feature of this process is that before the flow becomes fully turbulent we observe the formation of a self-sustained turbulent spot in an otherwise laminar flow (3).

Although the linear stability of this flow has been well-studied there are very few theoretical studies of the nonlinear stability properties of this flow. This project aims to use high Reynolds number asymptotic methods including the use of nonlinear critical layer theory to understand the interactions occurring between the boundary layers and the core flow that self-sustain this sophisticated nonlinear structure.

Plan

The project will commence with a derivation of the basic flow and an investigation of its linear stability properties via the solution of an Orr-Sommerfeld eigenvalue problem using Chebyshev collocation (4).

Once the linear stability is established we can take a nonlinear asymptotic approach at high Reynolds number where the disturbance field is practically free from viscosity except within thin layers adjoining each wall and a region where the basic flow velocity is close to the phase speed of the propagating perturbation: this region is known as a critical layer. By linking the properties of the various layers we can establish a self-sustaining interaction in which the amplitude of the disturbance can be related to its wavelength and phase speed. Although a substantial part of this analysis is analytical, some numerical calculations are necessary.

With the self-sustaining mechanism established, we can calculate the distortion to the mean flow induced by the disturbance and compare it to that found in the experiments referred to earlier.

Prerequisites: Some prior exposure to fluid mechanics is desirable together with a knowledge of Matlab or Python.

References

1. An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels. Reynolds, O. (1883) *Phil. Trans. R. Soc.*, **174**, 935-982.

2. Introduction to hydrodynamic stability. Drazin, P. G. (2002) Cambridge Texts in Applied Mathematics.

3. Couette-Poiseuille flow experiment with zero mean advection velocity: subcritical transition to turbulence. Klotz, L., Lemoult, G., Frontczak, I., Tuckerman, L. S. & Wesfreid, J. E. (2017) *Phys. Rev. Fluids*, **2**, 043904.

4. Stability and transition in shear flows. Schmid, P. J. & Henningson, D. S. (2001) Applied Mathematical Sciences, Springer.

Investigation of the stability properties of shear flows over compliant surfaces

Supervisor: Andrew Walton

Description

The behaviour of fluid flows over flexible surfaces is a challenging one for theoretical fluid dynamicists. The traditional instabilities one encounters over a rigid wall are supplemented by elastic modes arising from the response of the boundary. In this project we will consider flows through channels with either one or both walls possessing compliant properties. The flow can be generated by applying a pressure gradient, moving the walls or a combination of both effects. A simple spring-backed plate model will be adopted to describe the motion of the boundary and we will assume that any perturbations of the surface are small, so that their effect on the basic flow will be a linear one. In the case of plane Couette flow, where the flow over rigid walls is linearly stable, it is possible to show that the surface flexibility destabilizes the flow. The aim of the project is to investigate the linear stability properties of the ensuing flow both at asymptotically large Reynolds number and at finite Reynolds number. The former analysis requires the use of matched asymptotic expansions, while the latter involves computation and requires the student to write some of their own code.

Plan

There are possibilities for the project to be more or less computationally-oriented depending on the interests of the student but there will be some numerical computation (in Python or Matlab) required.

The project will start by considering plane Couette flow subject to rigid boundaries. The linear stability eigenvalue problem will be posed and solved numerically (using either existing code or that developed by the student) and the flow will be visualised.

Following this, there will be some modelling of the flexible boundaries resulting in a modified stability problem that incorporates the wall compliance via a boundary condition. This new problem will be solved for a range of parameters, and the effects of wall stiffness upon stability will be documented quantitatively.

Next, the instabilities induced will be analyzed asymptotically at large Reynolds number, leading to multi-zone interactive structures that elucidate the key physical mechanisms underlying the instability process. The solutions of the corresponding high Reynolds number eigenrelations will be compared with the computations.

If time allows, the same ideas will be applied to plane Poiseuille flow, in which the rigid state is already unstable and also to a linear combination of Poiseuille and Couette flows in which the wall sliding speed provides an extra parameter.

Prerequisites: Some prior exposure to fluid mechanics is desirable together with a knowledge of Matlab or Python. Some experience of perturbation methods and matched asymptotic expansions would also be helpful, but not essential.

References

1. Introduction to hydrodynamic stability, Drazin, P. G. (2002) Cambridge Texts in Applied Mathematics.

2. On the stability of plane Poiseuille flow between compliant boundaries. Nagata, M. & Cole, T. R. (1999) *Transactions on Modelling and Simulation*, **vol. 21**
   https://www.witpress.com/Secure/elibrary/papers/CMEM99/CMEM99023FU.pdf

3. Instabilities in a plane channel flow between compliant walls, Davies, C. & Carpenter, P. W. (1997) *J. Fluid Mech.*, **352**, 205–243.

Pavel Berloff - Estimating Eulerian Transport Tensors from the Lagrangian Trajectories

Stirring of ocean water by mesoscale currents (``eddies'') leads to large-scale transport of many important
oceanic properties (``tracers'').
These eddy-induced transports can be related to the large-scale tracer gradients, using the concept of turbulent
diffusion.
The concept is widely used to describe these transports in the real ocean and to represent them in climate models.
This study focuses on the inherent complexity of the corresponding coefficient tensor (``K-tensor'') and its
components, defined here in all its spatio-temporal complexity.
Results so far demonstrate that this comprehensive K-tensor is space-, time-, direction- and even tracer-dependent.
Using numerical simulations with idealized intermediate-complexity models of the North Atlantic circulation,
it was shown that these properties lead to many discoveries about the involved eddy effects.

This Project will address important question: to what degree components of the K-tensor can be approximated from
trajectories of infinitesimal passive (i.e., Lagrangian) particles released in the flow?
This is motivated by the fact that most of the relevant oceanic observations are made in terms of the Lagrangian
particle approach.
The leading hypothesis is that the real differences between the actual K-tensor and its Lagrangian approximation
will be huge, and this knowledge will shake up the whole research field.
Knowing these differences quantitatively, as well as qualitatively, will be practically important new knowledge.
The student will be expected to work with model output data and master theory of turbulent transport.

Prerequisites: Interest in mathematical modelling, coding skills, ODEs and PDEs, numerical methods.

Collective Behavior of interacting particle systems
MSc project $2021 - 2022$
Professor G.A. Pavliotis

Interacting particle systems arise in many applications raging from plasma physics and stellar dynamics to biology and even to algorithms for sampling and optimization and to mathematical models in the social sciences.

Quite often, the interactions between particles (agents) at the microscale lead to the emergence of collective behavior at the macroscale. This collective behavior can manifest itself in terms of the formation of clusters (in stellar dynamics models), the emergence of consensus (in models for opinion formation), of synchronization (in systems of interacting nonlinear oscillators), the swarming of animal populations etc.

The goal of this project is the numerical and analytical study of systems of interacting particles that exhibit collective behavior and, in particular, the formation of clusters. in particular, we will study the Langevin dynamics

$$\ddot{\mathbf{q}}_i = -\nabla \Phi_N(\mathbf{q}^N) - \gamma \dot{\mathbf{p}}_i + \sqrt{2\gamma\beta^{-1}}\, \dot{\mathbf{B}}_i, \tag{1}$$

where $\mathbf{q}^N = (\mathbf{q}_1, \ldots \mathbf{q}_N)$ denotes the position of N particles in $\mathbb{R}^2$, $\mathbf{p}^N = (\mathbf{p}_1, \ldots \mathbf{p}_N)$ the momentum vector, γ the friction coefficient, β the inverse temperature and $\mathbf{B}_1, \ldots \mathbf{B}_N$ a collection of N independent two-dimensional standard Brownian motions. The potential Φ_N consists of a confining potential and of pairwise interactions,

$$\Phi_N(\mathbf{q}^N) = \sum_{j=1}^{N} V(\mathbf{x}_j) - \frac{\kappa}{2} \sum_{i,j=1}^{N} W(\mathbf{x}_i, \mathbf{x}_j), \tag{2}$$

where W is a symmetric function. We will consider, in particular the case of Gaussian interactions

$$W(\mathbf{x}, \mathbf{y}) = e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{2\sigma^2}} \tag{3}$$

Other choices, such as the Morse potential $W(x) = \lambda^2(e^{-2(x-x_e)-e^{-(x-x_e)}}$, will also be considered. The parameter (interaction strength) κ can be either positive or negative and it can depend on the number of particles, e.g. $\kappa \sim \frac{1}{N}$. We will consider the dynamics (1) both in $\mathbb{R}^{2N} \times \mathbb{R}^{2N}$ as well as in $\Lambda^N \times \mathbb{R}^{2N}$, where Λ denotes a box of size L in $\mathbb{R}^2$ with periodic boundary conditions. The goal will be to study the problem by means of extensive numerical simulations as well as by studying analytically the mean field/thermodynamic limit $N \to +\infty$.

Prerequisites for this project are statistical mechanics, stochastic differential equations and numerical methods for ODEs, SDEs, and PDEs. Useful references are [4, 1, 5, 2, 3].

References

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[2] A. Campa, T. Dauxois, D. Fanelli, and S. Ruffo. *Physics of long-range interacting systems*. Oxford University Press, Oxford, 2014. With a foreword by David Mukamel.

[3] N. Martzel and C. Aslangul. Mean-field treatment of the many-body Fokker-Planck equation. *J. Phys. A*, 34(50):11225–11240, 2001.

[4] J. Messer and H. Spohn. Statistical mechanics of the isothermal lane-emden equation. *Journal of Statistical Physics*, 29(3):561–578, 1982.

[5] H. A. Posch, H. Narnhofer, and W. Thirring. Dynamics of unstable systems. *Phys. Rev. A (3)*, 42(4):1880–1890, 1990.

Stochastic Gradient Descent in Continuous Time

Professor G.A. Pavliotis

The methods and models of machine learning are rapidly becoming de facto tools for the analysis and interpretation of large data sets. The ability to synthesise and simplify high-dimensional data raises the possibility that neural networks may also find applications as efficient representations of known high-dimensional functions. Training a given neural network remains one of the central challenges in applications due to the slow dynamics of training and the complexity of the objective function. Parameter optimisation in machine learning typically relies on the stochastic gradient descent algorithm (SGD), which makes an empirical estimate of the gradient of the objective function over a small number of sample points. In order to study the properties of stochastic gradient descent for neural network optimisation, it is possible to recast the standard training procedure in terms of a system of interacting particles. It is then possible to use tools from stochastic differential equations and statistical mechanics to study the improve methodologies for the training of neural networks. The goal of this project will be to study recent works on the dynamical study of the training of neural networks and to implement some of the proposed methodologies for improving the performance of the SGD algorithm.

Sirignano, J.; Spiliopoulos, K. Stochastic gradient descent in continuous time. SIAM J. Financial Math. 8 (2017), no. 1, 933–961.

Sirignano, J.; Spiliopoulos, K. Stochastic gradient descent in continuous time: a central limit theorem. Stoch. Syst. 10 (2020), no. 2, 124–151.

Parameters as interacting particles: long time convergence and asymptotic error scaling of neural networks
G Rotskoff, E Vanden-Eijnden
Advances in neural information processing systems, 7146-7155 (2018)

Neural networks as interacting particle systems: Asymptotic convexity of the loss landscape and universal scaling of the approximation error
GM Rotskoff, E Vanden-Eijnden stat 1050, 22 (2018)

Parameter estimation for the McKean-Vlasov stochastic differential equation
L Sharrock, N Kantas, P Parpas, GA Pavliotis
arXiv preprint arXiv:2106.13751

MSc Applied project proposals – Pruessner, Gunnar

Project 1 - Interacting particles far from non-equilibrium

The study of non-equilibrium systems is often concerned with entropy and entropy production, that is the rate with which entropy is being produced by a process. While this is normally studied as an ensemble average, some authors are using methods based on single, stochastic trajectories.
We are interested in studying the entropy production for a non-trivial Markov process. We consider the motion of a particle that is constrained to move on a ring of states. The dynamics of the particle are governed by a Poisson process, the particle can hop to the nearest right or left site independently after an exponentially distributed waiting time, that depends on the position of another particle that is moving randomly as well.
The entropy production for this process can be calculated for different levels of description: The two particles may be considered distinguishable or indistinguishable and we may consider an ensemble of trajectories or a single one. We want to use traditional as well as field-theoretic methods. The aim of the project is to survey the different methods and to devise a procedure to efficiently change from one description to another.

Project 2 - Entropy production of random walk on evolving graphs

Similar to the project above, we want to study the entropy production for a particular type of process. In the present project we consider the entropy production when the very Markov matrix of the underlying process changes.
For many systems the state space changes with time. If we model the dynamics on the state space as a random walk on a graph, there will be a competition of timescales between the "mixing time" of the process on the graph and the characteristic time of the change of the graph. What features does the entropy have as a consequence? A related problem is the characterisation of the entropy production of the evolving graph itself (e.g. comparing different remodelling dynamics). This project will require serious initial numerical input.

Project 3 - Field theory of defect pair creation and annihilation under coarse-graining

Active nematics can be characterised by the creation and annihilation of defect pairs with opposite topological charge. Unlike passive nematics, these defects behave like "active" quasi-particles. Can we use Doi-Peliti field theories to characterise these effective dynamics and study the entropy production of the active vs passive case?

Project 4 - Entropy production determines the material properties of logic active matter

This is in the spirit of Marchetti's recent papers on logic active matter, where active particles have some internal state that is modified by interaction/measurement. The setup is a 1d box with active particles where each particle is either self-propelling to the left or to the right (the internal state is 1 bit), maybe with a bit of diffusion. With some rate m each particle "measures" it's surrounding area and detects whether it's nearest neighbour is to its left or its right (1 bit of information). It then adjusts the direction of its self-propulsion so that it moves away from its nearest neighbour. If the

measurement rate is sufficiently fast (compared to the self-propulsion), the particles will organise in what resembles a 1d lattice (they "crystallise"), while for low measurement rates this should looks like a gas of run-and-tumble particles. The idea is to explore these two phases and the transition. For example, what happens if I compress the box in the "active solid" phase? Do I get pressure waves and how do they propagate? What kind of material is this? Also, the entropy production of the measurement itself is an interesting question, which can also be explored at the level of a single particle, if we fix the position of its neighbours to simplify things.

Transition to chaos in model spatially-developing flows

Supervisors: Dr. P. K. Ray and Prof. D. T. Papageorgiou

Project Description

Consider the figure below. The image on the left is a bifurcation diagram for the logistic map, $x_{n+1} = rx_n(1 - x_n)$; this map was analyzed by Feigenbaum 40 years ago in his foundational study on the period-doubling route to chaos [1]. The image on the right illustrates the spatial development of a fluid boundary layer as it *transitions* from a steady, laminar flow to fully-developed turbulence. The basic question motivating this project is, can the bifurcation diagram on the left at all describe the dynamics depicted in the boundary layer on the right? Boundary layers have great practical significance – consider water flowing through pipes in your home, air flow over an aircraft wing, or hurricanes after landfall. While these flows are undoubtedly important, they are also enormously complicated and require simulation and analysis of the 3-D Navier-Stokes equations. In this project, we make a pragmatic compromise and focus on the spatially-developing 2-D Kuramoto-Sivashinsky (K-S) equation, $u_t + (u + c)u_x + \nabla^2 u + \nabla^4 u = 0$, which retains many important features of the Navier-Stokes equations but is simpler to analyze and simulate. Transition to chaos in the K-S system has already been investigated for confined dynamics on a periodic domain, $x \in [0, L)$ [2], and in this project, we will analyze open flows developing along the half line, $x \in [0, \infty)$. Numerical simulations will be used as a 'laboratory' for investigating transition scenarios. Statistical methods and nonlinear time series analysis will be applied to simulation results and connections to insights gained from both chaos theory and linear stability analysis will be explored and explained.
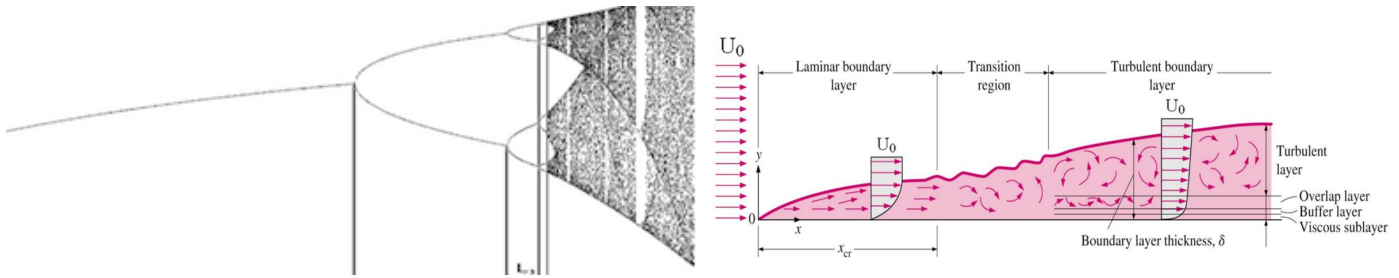
Learning Outcomes

There will be several learning outcomes emerging from this project:
- You will learn about numerical methods for nonlinear PDEs and acquire proficiency in scientific computing
- You will also learn about nonlinear time series analysis and statistical analysis of complex spatio-temporal data
- You will learn about linear stability analysis of spatially developing flows and asymptotic methods used to connect linear theory to observed nonlinear dynamics.

Background: The following courses (or equivalent) from the Applied Mathematics program could prove useful, however not all are essential: Fluid Dynamics I/II, Hydrodynamic Stability, Asymptotic Analysis, Numerical Solution of ODEs, Computational PDEs. Some programming experience is essential.

References

[1] S.H. Strogatz, Nonlinear dynamics and chaos, 2000.

[2] Y.-S. Smyrlis and D.T. Papageorgiou. Predicting chaos for infinite-dimensional systems: The Kuramoto-Sivashinsky equation, a case study. Proc. Natl. Acad. Sci. USA, 88:11129-11132, 1991.

Waves and turbulence in pedestrian flows

Supervisor: Dr. Prasun Ray

Project Description

Analysis and optimization of pedestrian flows is fundamentally important for both urban design and public health and safety. These flows can exhibit a range of complex phenomena, and the aim of this project is to use simulations of pedestrian flows to investigate these dynamics. You will simulate pedestrian flows in canonical configurations such as uni- and bi-directional traffic in a hallway using the descision-based model developed in [1], and analyze the waves and turbulence that are known to develop under certain conditions. You will also carry out comparisons with experimental studies of pedestrian behavior and critically assess the strengths and weaknesses of the model and develop improvements as appropriate.

Illustration of a pedstrian (p_1) analyzing her path (taken from [1])

Learning Outcomes

There will be several learning outcomes emerging from this project:
· You will learn about numerical methods for large systems of ODEs and optimzation; you will also acquire proficiency in scientific computing
· You will learn about mathematical modeling of pedestrian behavior
· You will also learn about the dynamics of linear and nonlinear waves in the context of pedestrian traffic

Background: The following courses (or equivalent) may be useful, however not all are essential: Hydrodynamic Stability, Numerical Solution of ODEs, Computational PDEs, Scientific Computation. Python programming experience will be helpful.

Reference

[1] M. Moussaid, D. Helbing, & G. Theraulaz. How simple rules determine pedestrian behavior and crowd disasters. Proc. Natl. Acad. Sci. USA, 108::6884-6888 2011.

The projects involve mathematical modelling (fluids, waves,...), asymptotic analysis and for some projects also numerical calculations.

A. Vertical oscillations of levitated "Leidenfrost" drops. Drops are able to levitate for relatively long times above a sufficiently hot substrate, supported by a cushion of their own vapour. This is the "Leidenfrost effect," which you can observe in your kitchen by drizzling water on a very hot pan. There is currently a lot of interest in studying the rich dynamics of Leidenfrost drops, including their high mobility, symmetry breaking and spontaneous motion, self-induced oscillations and more; experiments are far ahead of theory. In this project, we will try to rationalise recent observations of Leidenfrost drops exhibiting vertical oscillations which can grow into limit-cycle-like bouncing or trampolining. We will start by modelling the vertical dynamics of a levitated Leidenfrost solid (a chunk of "dry ice").

B. Trapped modes. Waveguides are devices that are designed to convey propagating waves (acoustic, electromagnetic, etc.). Nonetheless, waves can also be perfectly trapped within a waveguide by suitably designed obstacles, namely without interacting with waves propagating towards or outwards from the obstacle. Such waves are described by solutions to the Helmholtz equation known as trapped modes, or bound states in the continuum. In this project, we will combine asymptotic methods and conformal mappings to identify and calculate approximations for trapped modes in slit channels. This approach will allow us to identify and describe new trapped modes as well as simplify existing results in the literature.

C. Active particles interacting with a boundary. Chemically active particles self-propel in a viscous fluid by generating around themselves an asymmetric solute distribution, which in turn animates a flow around the particle. This project is concerned with chemically active particles that are isotropic in their geometry and surface properties — they self-propel, in a random direction, as a consequence of a symmetry breaking instability associated with a positive feedback loop between advective-diffusive solute transport and Stokes flow. We will use asymptotic methods and Stokes-flow theory to study the trajectories of isotropically active particles interacting with a distant boundary.

D. Vortex shedding in strong-field electrophoresis. Electrophoresis refers to the field-driven motion of a charged particle that is immersed in an electrolyte. Direct numerical simulations in the strong-field regime have recently revealed the formation of vortices on one side of the particle; above a critical field strength, the vortices periodically detach resulting in an unsteady flow reminiscent of the Karman vortex street (only at zero Reynolds number!). We will use asymptotic tools to study the Poisson–Nernst–Planck–Stokes set of equations governing this phenomenon, and investigate the origin of these vortices and their stability, building on existing strong-field theories in the literature.

MSc projects with Dr Igor Shevchenko

- **Large scale low-frequecny variability of the midlatitude ocean circulation**
  Understanding origins of the large-scale low-frequency variability (LFV) of the ocean is not only one of the central questions in the Earth system modelling and geophysical fluid dynamics, but also one of the serious challenges in predictive understanding of climate change. The midlatitude atmosphere and ocean possess significant interannual variability and several large-scale variability modes on decadal and interdecadal timescales. Physical origins of the LFV modes remain unclear, and it is not even known to what extent these origins are intrinsic atmospheric, intrinsic oceanic, or coupled oceanic-atmospheric. This project focuses on studying the intrinsic oceanic LFV.

- **Absorbing boundary conditions for nonlinear wave equations**
  Many problems in science and engineering are naturally formulated in unbounded domains; typical examples originate from fluid dynamics, solid mechanics, aerodynamics, electrodynamics, acoustics, etc. However, numerical simulations of such problems require a finite computational region. This project is aimed to design absorbing boundary conditions for efficient and robust numerical simulations of nonlinear wave equations in unbounded domains.

- **Stochastic parameterisations for ocean models**
  Stochastic parameterisations of oceanic eddies play an important role in geophysical fluid dynamics because of their ability to represent complex physical processes with relatively simple models. In this project we develop parameterisations for the quasi-geostrophic model of wind-driven ocean gyres and analyse their efficiency in modelling unresolved scales.

- **Multiscale oceanic energetics**
  The goal of this project is to study inter-scale energy transfers in the ocean, examine the multi-scale nature of the forward and backward energy cascade, and how the energy transfers depend on viscosity.

- **Modelling the ocean with primitive equations**
  Modelling the ocean with primitive equations is a vast and active area of research in geophysics. The goal of this project is to simulate and study ocean currents in the North Atlantic with using the Regional Ocean Modelling System (ROMS).

- **Bifurcation analysis of dynamical systems with degenerative solutions**
  In this project we consider convection in a porous material saturated with fluid and heated from below. This problem belongs to the class of dynamical systems with nontrivial cosymmetry, which gives rise to a hidden parameter in the system and continuous families of infinitely many equilibria, and leads to non-trivial bifurcations. It is planned to study nonlinear phenomena resulting from the existence of cosymmetry, describe different non-classical bifurcations, and the selection scenarios (namely, which of infinitely many equilibria can be realized in physical experiments).

Dr Philipp Thomas - Stochastic processes in biology

Web: https://www.ma.imperial.ac.uk/~pthomas

PROJECT 1: **Hidden Markov models on trees to understand cancer cell proliferation**

Supervisors: Fern Hughes (f.hughes19@imperial.ac.uk), Dr Alexis Barr (a.barr@lms.mrc.ac.uk), Dr Philipp Thomas (p.thomas@imperial.ac.uk)

Modern microscopy experiments allow us to observe dividing cells and their offspring over time. Such technologies promise to reveal crucial insights into diseases like cancer - a disease of abnormal cell growth. To understand the biological mechanisms of cell proliferation and division, we require stochastic models that can account for cell-to-cell variation observed in the data.

A simple model for this stochastic process is a Markov chain where each cell divides based on a stochastic internal state. In this project, you will extract the hidden dynamics of these states from single-cell experimental data of healthy and cancer cells acquired by our experimental collaborators (Barr Lab @ Hammersmith Hospital campus) using Hidden Markov Models (HMM). To this end, we will develop the common sequential HMM to a tree-structured HMM, which accounts for the branching process nature of cell division. In particular, we aim to extend the expectation maximisation algorithm to fit tree-structured data and the Viterbi algorithm to extract the hidden dynamics. The developed methodology is expected to advance our understanding of cancer cell proliferation and heterogeneity.

Prior knowledge of biology is not required. Experience with HMMs is beneficial but not essential.

Literature:
Murphy - Machine Learning: A Probabilistic Perspective (2012) - Book
(http://noiselab.ucsd.edu/ECE228/Murphy_Machine_Learning.pdf)

Bishop - Pattern Recognition and Machine Learning (2006) - Book
(https://www.microsoft.com/en-us/research/uploads/prod/2006/01/Bishop-Pattern-Recognition-and-Machine-Learning-2006.pdf)

Nakashima et al. 2020 - Lineage EM algorithm for inferring latent states from cellular lineage trees (https://doi.org/10.1093/bioinformatics/btaa040)

Mohammadi et al. 2021 (preprint) - A lineage tree-based hidden Markov model to quantify cellular heterogeneity and plasticity (https://doi.org/10.1101/2021.06.25.449922)

PROJECT 2: **Age-structured models of cancer cell populations**

Supervisors: Fern Hughes (f.hughes19@imperial.ac.uk), Dr Alexis Barr (a.barr@lms.mrc.ac.uk), Dr Philipp Thomas (p.thomas@imperial.ac.uk)

Cells can enter a resting state called 'quiescence', where they temporarily stop cycling and dividing. In cancer, we believe quiescence can allow cells to escape drug treatment, resulting in tumour recurrence once they re-enter the cell cycle. In this project, you will combine stochastic modelling with data from Dr. Alexis Barr's lab at the LMS to understand how quiescence gives cancer cells a proliferative advantage over healthy cells.

You will combine agent-based stochastic models with data from a variety of lung cancer cell lines. For each cell line, you will use the growth curve data along with measured quiescent fraction to inform an age structured model of a cell population. The aim is to understand how the emergence of a quiescent subpopulation alters population growth and measured phenotypic distributions and how they are altered in cancer. The theoretical predictions will be fitted to the data to provide a mechanistic understanding of cancer cell proliferation and guide future experiments.

Literature:
Thomas, Philipp - Making sense of snapshot data: ergodic principle for clonal cell populations. (https://doi.org/10.1098/rsif.2017.0467)

Barr, Alexis R., et al. - DNA damage during S-phase mediates the proliferation-quiescence decision in the subsequent G1 via p21 expression. (https://doi.org/10.1038/ncomms14728)

PROJECT 3: **Stochastic population dynamics in time-dependent environments**

Supervisor: Dr Paul Piho (p.piho@imperial.ac.uk), Dr Philipp Thomas (p.thomas@imperial.ac.uk)

Controlling the growth of cell populations is an important problem in biomedical applications. In this project, we will investigate solutions to age-structured population dynamics with noisy time-dependent inputs corresponding to effects of the environment.

We study and compare the effects of noise in age-dependent and age-independent population growth dynamics both via analytical and simulation-based methods. In particular, we develop methods for simple populations to quantify their asymptotic growth. Of particular interest will be to distinguish features of individuals in growing from dying populations (super-/subcritical processes).

Some basic knowledge of stochastic processes is advantageous but not necessary.

Literature:
Inaba, H (2017). Age-structured population dynamics in demography and epidemiology (Chapter 1), Springer Book.

PROJECT 4: **Approximate inference methods for live cell microscopy**

Supervisor: Dr Barbara Bravi (b.bravi@imperial.ac.uk), Dr Philipp Thomas (p.thomas@imperial.ac.uk)

Cellular reaction networks are inherently noisy. Understanding the sources of this variation is important since it affects many essential cell functions. Modern time-lapse microscopy allows us to answer this question but the wealth data it generates requires sophisticated statistical inference tools to connect data with mechanistic models.

The chemical master equation models intracellular reactions as a continuous-time Markov chain. While inference tools for Markov chains are well developed, they do not immediately apply to stochastic reaction networks because the number of accessible states in these networks is, at least in principle, unbounded. The idea of this project is to combine the Baum-Welch algorithm (expectation maximization) for inference of Markov chains from time series data with the Finite State Projection Algorithm (FSP) to find the unknown parameters of the network. The FSP algorithm is a finite-dimensional approximation of the master equation which also provides error estimates and convergence guarantees. You will probe the efficacy of the methodology using cutting-edge live cell microscopy data of bacteria.

Literature:
Wikipedia - Baum-Welch Algorithm
(https://en.wikipedia.org/wiki/Baum%E2%80%93Welch_algorithm)

Munsky et al (2006) *The finite state projection algorithm for the solution of the chemical master equation* J. Chem. Phys. 124, 04410 (https://aip.scitation.org/doi/10.1063/1.2145882)

PROJECT 5: **Bayesian inference of stochastic reaction networks**

Supervisor: Dr Philipp Thomas (p.thomas@imperial.ac.uk)

Biochemical reactions occur at random times in living cells. To reliably estimate the rates of these reactions, we need to take into account this stochasticity observed in vivo. Bayesian inference is often the preferred choice for this purpose, but it is usually expensive because it

requires sampling from the posterior distribution of parameters via MCMC. In this project, we will explore an alternative route to the inverse problem. The goal will be to use an asymptotic expansion of the stochastic process to approximate the posterior distribution directly. The results are expected to provide analytical insights into how informative stochasticity is about the parameters underlying the system under study.

Literature:

Fröhlich, F., Thomas, P., Kazeroonian, A., Theis, F. J., Grima, R., & Hasenauer, J. (2016). Inference for stochastic chemical kinetics using moment equations and system size expansion. *PLoS Computational Biology*, *12*(7), e1005030.

Thomas, P., & Grima, R. (2015). Approximate probability distributions of the master equation. *Physical Review E*, *92*(1), 012120.

Phonons in an expanding or contracting BEC – Dr Ryan Barnett

In quantum mechanics, any particle can be classified either as a fermion or a boson. When a collection of bosons is cooled to a sufficiently low temperature, an exotic state of matter called a Bose-Einstein condensate (BEC) is formed. Since its realisation in 1995 (for a good pedagogical review see [1]), BEC has proven to be a fertile ground for applications in quantum simulations. For example, they can provide clean realisations of toy models used to understand disparate physical systems.

The low-energy excitations in BECs are often called phonons and are similar (though sometimes superficially) to vibrational modes of a crystal or sound waves in air. In this project, we will investigate such collective excitations of a BEC that is *expanding* or *contracting*. Experiments along these lines have recently been carried in [2] and have strong analogies with certain cosmological models. We will attempt to understand the puzzles found in these experiments using quantum field theory methods of many-particle systems.

[1] A. J. Leggett, *Bose-Einstein condensation in the alkali gases: Some fundamental concepts*, Rev. Mod. Phys. 73, 307

[2] S. Banik, M. Gutierrez Galan, H. Sosa-Martinez, M. Anderson, S. Eckel, I. B. Spielman, G. K. Campbell, *Hubble Attenuation and Amplification in Expanding and Contracting Cold-Atom Universes* arXiv:2107.08097

Exotic states in rotating condensates – Dr Ryan Barnett

Diagonalising the Hamiltonian corresponding to a charged particle confined to a plane with a constant perpendicular magnetic field is one of the canonical problems in quantum mechanics. The resulting spectrum has a fascinating feature: there are (typically) large degeneracies associated with each eigenenergy. Such collections of degenerate states are called Landau levels, and the one with lowest energy is called the lowest Landau level (LLL). These large degeneracies play an essential role in (fractional) quantum Hall systems [3].

Systems of bosons in the LLL have been experimentally explored much less, partially due to the fact that most bosons used the lab are neutral (i.e. they carry no charge). For such systems, a magnetic field (as experienced by charged particles) can be mimicked by mechanically rotating the system. Recent experiments have reached the LLL regime using such rotation with the additional technical trick of geometrical squeezing [4]. In these experiments a bosonic crystalline phase was observed.

This project will aim to understand this experiment using the method of exact diagonalisation.

[3] R. B. Laughlin, *Fractional quantization*, Rev. Mod. Phys. 71, 863 (1999)

[4] Biswaroop Mukherjee, Airlia Shaffer, Parth B. Patel, Zhenjie Yan, Cedric C. Wilson, Valentin Crépel, Richard J. Fletcher, Martin Zwierlein, *Crystallization of Bosonic Quantum Hall States*, arXiv:2106.11300

Dynamics of Active Fluids: an exploration of the Toner-Tu equation

Supervisor: Thibault Bertrand

Description

Active systems take energy from their environment to transform it into motion. These systems are driven far from equilibrium [1] and display a wealth of new phenomena forbidden by equilibrium thermodynamics, including the emergence of novel collective properties including large scale collective motion [2], clustering [3], and self-jamming [4]. Studying active matter offers hope to uncover new physics, shine light on complex biological processes and perspectives to develop functional materials and smart devices. Complex and robust collective behaviors can be the result of interactions between very simple constituent agents; finding a general framework to understand how active particles synergistically interact to perform a task is appealing and has many applications.

While the dynamics of conventional fluids is governed by the famous Navier-Stokes equation [5], the dynamics of active fluids is well-described by the Toner-Tu equation [6,7,8,9,10]. This equation was originally derived on the basis of symmetry considerations [6]. For the past two decades, several studies have rederived hydrodynamic equations by systematically coarse-graining microscopic models of active particles to finally end up with a Toner-Tu equation [10].

Plan

Using a combination of analytics and numerical simulations, we will study the emergent phases stemming from the activity in the Toner-Tu equation. On the numerical side, we will develop various methods to numerically solve the Toner-Tu equation and compare them [11]. These methods may include: finite differences, finite elements and/or classical pseudo-spectral and spectral methods which were hugely successful in classical hydrodynamics.

Prerequisites: Good coding skills are essential. Familiarity with partial differential equations and/or fluid dynamics would is desirable.

References

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[11] J. Dunkel, S. Heidenreich, K. Drescher, H. H. Wensink, M. Br, R. E. Goldstein. Phys. Rev. Lett., 110, 228102 (2013)

Inertial Active Matter

Supervisor: Thibault Bertrand

Description

In recent years, a growing effort has been devoted to research in the field of *Active Matter* [1]. Active matter generically describes the dynamics of large systems of agents that consume energy from their surroundings to perform a function. Interactions between agents and with their environment may have striking consequences, e.g. it can generate novel behaviour including self-organisation and collective motion that are not observed in passive systems. These systems occur across the natural world and understanding their structure is a fundamentally interdisciplinary field of research. A classical example of collective motion is the flocking behaviour observed in birds, fish and bacteria [2,3].

Agent-based modelling in which one stipulates the microscopic interactions between agents allows us to investigate the link between agent-agent interactions and the emergence of particular macroscopic behaviour, both computationally and analytically [4]. Of particular interest are *minimal* models which reduce the description of agents to a few key features, providing the foundation for the development of more specific and complex models. In the *Active Brownian Particle* (ABP) model, particles are self-propelled with force F_p in a direction that diffuses in time with diffusivity D_R. A system of ABPs is commonly described by the following equations of motion for individual positions, $\mathbf{r}_i$, and orientations, θ_i:

$$\zeta \dot{\mathbf{r}}_i(t) = \sum_{j \neq i} \mathbf{F}(\mathbf{r}_i, \mathbf{r}_j) + F_p(\cos \theta_i, \sin \theta_i), \quad \dot{\theta}_i(t) = \sqrt{2 D_R} \eta_i(t)$$

where $\mathbf{F}$ is the agent-agent interaction and $\eta_i(t)$ is a noise term. In deriving these equations one assumes that the friction ζ is large, a fair assumption when modelling microscopic organisms such as cells or bacteria. This is called the *overdamped* limit. In models where particles interact only via purely-repulsive interactions $\mathbf{F}$, agents can crowd and form dense clusters leading to the so-called *Motility Induced Phase Separation* [5].

While a lot of effort has been devoted to understanding active matter in the overdamped limit, the underdamped or inertial limit remains elusive [6]. Mathematically, this implies that inertia is not negligible and may have striking consequences. For instance, it is thought that the presence of inertia can hinder the mechanism that generates the MIPS phase separation. We often work in the overdamped limit because it makes analytical progress more tractable. From a modelling standpoint, understanding the role of inertia is necessary to relate these minimal models to larger scale physical systems like animal flocking or robotic swarms.

Many questions remain unanswered in these underdamped systems: how can we quantify the effect of inertia on the presence of MIPS? What are the differences in the physical properties of the clusters formed with and without inertia? Is there a new collective behaviour that is present only when we have inertia present in the system?

Plan

The student will develop interacting active particle simulations in the underdamped limit including that of systems exhibiting MIPS that includes the effect of inertia. This will involve solving a 2nd order stochastic differential equation. From here, the candidate would be expected to investigate the role that inertia plays in perturbing the well-established results for the overdamped systems. A starting

test for the code would be for the candidate to recreate the results seen in the overdamped systems before expanding.

While the project would be mainly computational, the candidate would also be expected to develop an understanding of the analytical study undertaken on the overdamped versions of the model. We believe that these questions are currently the subject of much attention and that this project provides the candidate with a great opportunity to make a relevant contribution to the work being done in the field.

Prerequisites: Solid coding skills in C/C++, matlab or python. A strong interest in mathematical physics is essential.

References

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[6] H. Löwen, J. Chem. Phys. 152, 040901, 2020

Worm blobs and the dynamics of entangled active filaments

Supervisor: Thibault Bertrand

Description

Living systems at all scales aggregate in large numbers for a variety of functions including mating, predation, and survival. The majority of such systems consist of unconnected individuals that collectively flock, school, or swarm [1,2]. Already striking collective behaviors can emerge from simple interactions between biological agents, but in some cases, the individuals can even be are physically connected to each other, forming an additional class of entangled active matter systems with emergent collective properties. In particular, recent experiments showed that aquatic worms such as the California blackworm (Lumbriculus variegatus) can entangle their bodies into dense blobs to protect themselves against environmental stresses [2].

From a theoretical point of view, these macroscopic worms can be thought of as active Brownian polymers [3], whose length and the stiffness is dependent on the species under consideration. In recent years, there has been huge research interest in the study of active polymers; existing models usually describe these active polymers a beadstrings in which each bead (or monomer) is subject to both thermal fluctuations but also an active force (or self-propulsion). Two strategies have been considered to model this active forcing: (i) the active force on each monomer can be applied in a random direction subject to rotational diffusion *or* (ii) the active force can be applied tangentially to chain. These two different conventions have been shown to lead to very different structural properties. The first kind of chain undergoes shrinkage followed by swelling of the structure with active forces [4], whereas the tangentially driven chain has been shown to form spirals and undergo snake-like motion transiently [5].

While the behavior of active polymer in porous media has been the focus of some recent work with a recent study showing that stiffness of active filaments promote their efficient transport in two-dimensional porous media [6,7], studies of dense assemblies of active polymers are crucially lacking in the literature. We know that while flexible chains curl up and tend to get trapped when they form spirals, stiff active polymers do not. In this project, we will investigate the collective dynamics of worm blobs.

Plan

The project will try to answer a number of interesting questions. Our main aim is to understand the dynamics of active polymers and focus on the competition between knotting/entanglement of the filaments and the possible emergence of collective behavior in dense assemblies. To do so, we will extend and analyse computer simulations by extending existing models [8].

After an initial review of the literature on the subject, you will develop numerical simulations which you will compare to existing results. We will then extend the existing literature by exploring numerically a variety of conditions. Our numerical results will be interpreted in the framework of existing theories for active matter and transport in crowded environments.

Prerequisites: Good coding skills in the programming language of your choice is essential (recommended programming languages: C/C++, matlab or python).

References

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3D Modelling of Cells and Tissues

Supervisor: Thibault Bertrand

Description

Understanding the mechanics and dynamics of cells and tissues is of vital importance to our understanding of a diverse array of biological processes, from morphogenesis to cancer progression. Recent experimental advances have allowed for tissues to be imaged with cellular resolution and so motivate cell-resolution modelling to describe and analyse the behaviour seen [1].

One branch of such modelling are vertex models. Vertex models are a popular, recently developed class of 2D cell-resolution models that have successfully replicated several experimental findings [2,3]. However, these models are limited in the types of tissues they can describe as most biological tissues are inherently three dimensional. Recent work has sought to rectify this by extending a vertex model to 3D for an inert tissue of non-motile cells [4]. However, cells are inherently motile and capturing this activity is crucial for properly describing biological tissues. The aim of this project is to develop a 3D model of a motile tissue and explore the diverse array of behaviour such a tissue displays.

Plan

The student will first seek to implement a vertex based model in a similar manner to [4]. Following this, we will develop a method for introducing activity to model a tissue of motile cells. This is more complicated than in the 2D case as we now have to consider momentum conservation between cells, as opposed to cells moving on a rigid substrate.

Beyond the implementation of an active vertex model in 3D, this project has a broad scope and the model developed can be used to explore a variety of questions depending on the interests of the student. Possible avenues include studying the effects cell motility has on the rigidity transition to understand how cell aggregates fluidize, a process of vital importance in morphogenesis, or examining how single motile cells migrate through tissues, which is central to our understanding of cancer metastasis and the immune response.

Prerequisites: Good coding skills in the programming language of your choice is essential (recommended programming languages: C/C++, matlab or python). An interest in mathematical physics and biomathematics is essential.

References

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Swelling and shrinking dynamics in a thermo-responsive hydrogel

Supervisor: Thibault Bertrand

Description

Swelling is a fundamental process in biology, engineering, and the earth sciences: tissues swell after injury, wooden structures swell with humidity, and dry soils swell after rainfall. Macroscopically, swelling is the volumetric growth of a porous material due to the spontaneous imbibition of additional pore fluid. Swelling is distinct from other growth processes because of the fundamental role of hydrodynamics: local expansion of the pore structure is coupled to the evolving fluid distribution, making swelling inherently dynamic and poromechanical.

The mechanics of polymeric gels has attracted great interest over the years [1-5] and more recently in the context of hydrogels [6,7]. A hydrogel is a cross-linked network of hydrophilic polymers saturated with water. Hydrogels can experience extremely large and reversible changes in volume during swelling, which can result in complex changes in shape and the development of surface patterns [3,8,9,10]. Hydrogels have found a wide variety of practical applications; for example, they are widely used for moisture absorption and in soft contact lenses. In biomedical engineering, they are used for drug delivery, wound dressing, and as a scaffold for tissue engineering [11,12]. They have also shown promise for use as sensors, actuators, and flow controllers, and as a model system in soft granular matter [13].

We previously studied the swelling and drying of a sphere of hydrogel [14]. For that, we developed a dynamic model based on large-deformation poromechanics and the theory of ideal elastomeric gels. We used our model to study the complex internal dynamics of swelling and drying, and to highlight the fundamentally transient nature of these strikingly different processes.

Thermo-responsive hydrogels are gels whose degree of swelling depends on the ambient temperature. They can become hydrophobic when heated above a certain temperature, which results in the expulsion of much of the interstitial fluid and significant shrinking of the gel. These offer a route to programmable material in which the shape may be tuned by control parameters.

Plan

The dynamics of swelling and shrinking due to temperature changes in thermo-responsive hydrogels remains poorly understood. In this project, we will extend our poromechanical model for the swelling of a spherical gel to include temperature dependent behavior [14]. First, we will consider the case of a material in which temperature equilibration is very fast (i.e. the case where swelling and shrinking is done at constant temperature). We will then explore the case which includes proper thermal transport. To do so, we will use both analytics and numerics (including finite difference methods). You will be responsible for developing new codes and extending existing codes.

Prerequisites: Good coding skills in the programming language of your choice is essential (recommended programming languages: C/C++, matlab or python). An interest in fluid mechanics is essential.

References

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MSc Projects 2021/22

Dr Martin Rasmussen (m.rasmussen@imperial.ac.uk)

Project 1. Resilience in dynamical systems. Resilience describes how much a dynamical system is away from a bifurcation point where the behaviour of the system changes fundamentally. Several concepts of resilience have been described in the literature. This project builds upon the recently developed notion of intensity, which has been explored for continuous time in [1] and for discrete time in [2]. The approach studies pertubations of the dynamics using a set-valued or control system. Intensity has the advantage that it is a purely dynamical quantity and for this reason more meaningful than other approaches. This project aims first at understanding how the continuous-time theory is related to the discrete-time approach. This is of particular importance when one uses discretisations. Moreover, the project aims at extending the theory from determistic dynamical systems to random dynamical systems. While for random systems with bounded noise, such an extension seems straightforward (at least conceptually), it needs to be explored how these ideas extend to systems with unbounded noise and also to certain questions for bounded noise systems.

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Project 2. Rate-induced tipping in discrete-time dynamical systems. Tipping points describe bifurcations where the output of a dynamical system changes disproportionately compared to the change in the parameter. In [1], several mathematical mechanisms for tipping have been proposed, including rate-induced tipping, where a system is pushed across a bifurcation point through a nonautonomous change in the parameter that is non-adiabatic, i.e. fast enough, so that it cannot be regarded as a constant parameter in the dynamics. This project aims at understanding rate-induced tipping in the discrete-time context and builds on the results obtained by

PhD student Michael Hartl [2]. In contrast to the continuous-time case, the discrete-time case is almost unexplored and more interesting, since one can observed non-intuitive behaviour already in one dimension.

[1] P. Ashwin, S. Wieczorek, R. Vitolo, and P. Cox, *Tipping points in open systems: bifurcation, noise-induced and rate-dependent examples in the climate system*, Philosophical Transactions of the Royal Society A – Mathematical, Physical, and Engineering Sciences **370**(1962), 2012.

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