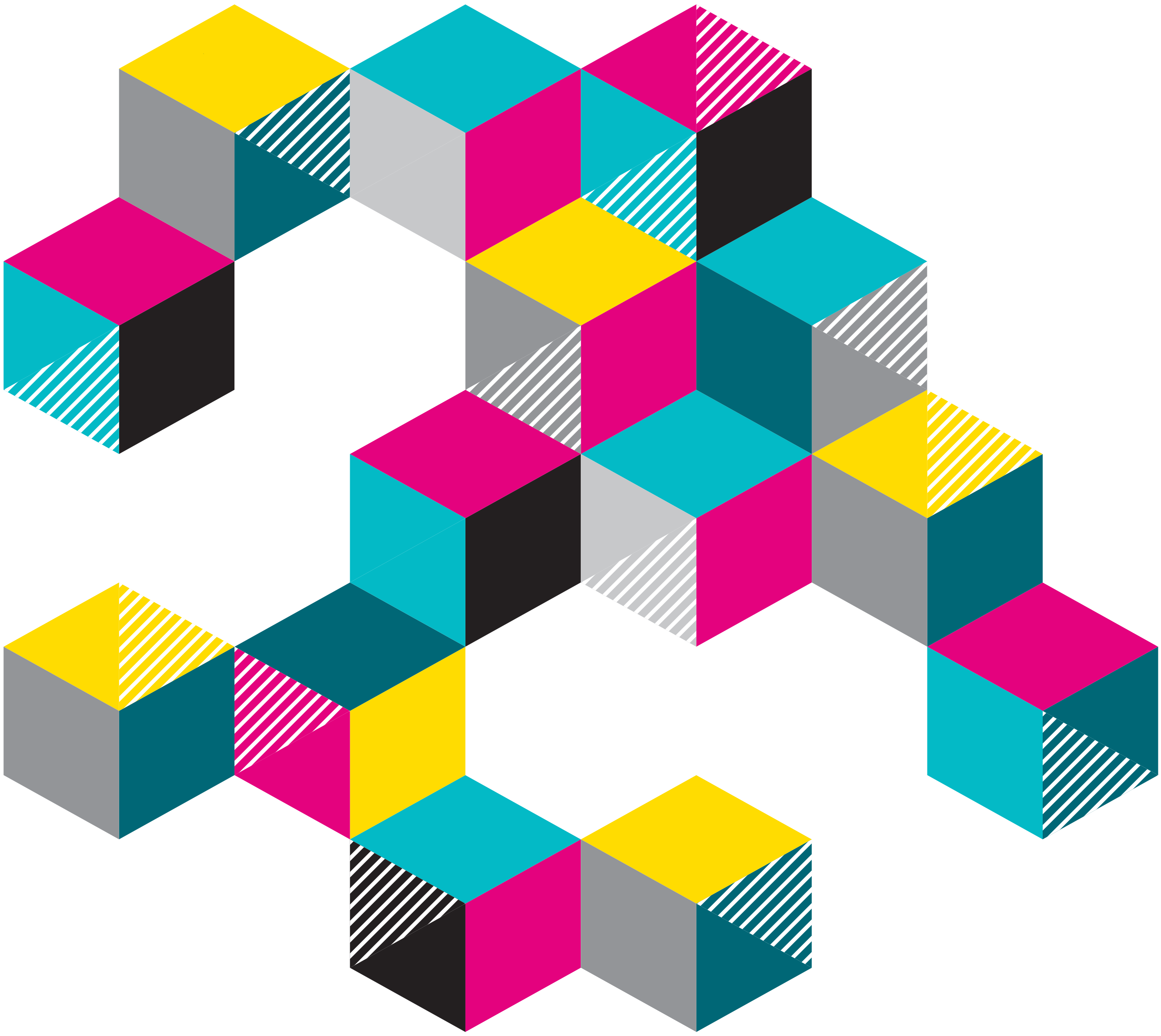
Imperial College London

MATHS TEACHING RESOURCES

For teachers of high-achieving students in KS2

3 Cubic Equations



Welcome

These resources have been put together with you, the primary teacher, at the forefront of our thinking. At Imperial College London we recognise the importance of keeping high-attaining students engaged with maths and want to do everything we can to help teachers provide for all of their students. Maths is critical for us as a facilitating subject for further eventual study in engineering or science, as well as being fascinating in its own right.

The content in this booklet was developed by teachers at Colchester Royal Grammar School in conjunction with Imperial. Based on initial feedback from primary teachers who trialled it, we have included specific guidance and examples to help teachers develop their own understanding of the material but we would welcome any further suggestions for improvement.

The material for the first three booklets (Percentages, Linear Equations and Cubic Equations) has been loosely based on the previous 'Level 6' curriculum content and we are currently looking into developing some further assessment materials to sit alongside these in case schools wish to offer an informal certification.

I hope that you and your students find these resources useful and most of all enjoyable.

George Constantinides

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Cubic equations

Cubic equations are equations in which the highest power of x is three.

SOLVING CUBIC EQUATIONS

Every cubic equation has at least one solution. To solve a cubic equation, the roots of the equation must be found.

There are various methods for solving cubic equations but we will focus on the trial and improvement method.

TRIAL AND IMPROVEMENT

This method requires you to “try” to solve the equation by substituting in different values for the unknown x until a value is found that works. With each substitution you “improve” on your previous attempt and try to get closer to the real value.

This method is sometimes used to solve an equation where there is no exact answer.

These questions often ask you to give the solution to a certain number of decimal places or significant figures.

Trial and improvement

Using a calculator

EXAMPLE 1

$x^3 = 10$ **Solve to one decimal place (1 d.p.)**

★ **TEACHER'S NOTE**
The underlined sentences in these examples can be used as good open questions to ask pupils when working the examples.

How do we go about choosing a value of 'x' to start?

x^3 is a cube number
$2^3 = 2 \times 2 \times 2 = 8$
$3^3 = 3 \times 3 \times 3 = 27$

10 is between 8 and 27 so x must be between 2 and 3

Let $x = 2.5$ (using a calculator)
$x^3 = 2.5^3 = 15.625$

This is greater than 10 (too big) so x must be between 2 and 2.5

Let $x = 2.1$
$x^3 = 2.1^3 = 9.261$

This is less than 10 (too small) so x must be greater than 2.1

Let $x = 2.2$
$x^3 = 2.2^3 = 10.648$

This is greater than 10 (too big) so x must be between 2.1 and 2.2

How do we determine the correct answer for x to one decimal place?

To determine which value is the correct answer to one decimal place we need to look at the x value that is half way between 2.1 and 2.2

Let $x = 2.15$
$x^3 = 2.15^3 = 9.938375$

This is less than 10 (too small) so x must be greater than 2.15. This means to get an answer to one decimal place we will round up. Our answer rounding to 1 decimal place is $x = 2.2$ (1 d.p.)

EXAMPLE 2

$x^3 + x = 51$ **Solve to one decimal place (1 d.p.)**

How do we go about choosing a value of 'x' to start?

When $x = 3$ $\quad x^3 + x = 3^3 + 3 = 30$
When $x = 4$ $\quad x^3 + x = 4^3 + 4 = 68$

51 is between 30 and 68 so x must be between 3 and 4

Let $x = 3.5$ (using a calculator)
$x^3 + x = 3.5^3 + 3.5 = 46.375$

This is less than 51 (too small)so x must be greater than 3.5

Let $x = 3.6$
$x^3 + x = 3.6^3 + 3.6 = 50.256$

This is less than 51 (too small) so x must be greater than 3.6

Let $x = 3.7$
$x^3 + x = 3.7^3 + 3.7 = 54.353$

This is greater than 51 (too big) so x must be between 3.6 and 3.7

How do we determine the correct answer for x to one decimal place?

Next, choose the value that is half way between 3.6 and 3.7

Let $x = 3.65$
$x^3 + x = 3.65^3 + 3.65 = 2.277125$

This is greater than 51 (too big) so x is less than 3.65. For an answer to one decimal place we round down. Our answer rounding to one decimal place is $x = 3.6$ (1 d.p.)

Probing question:

What if we wanted to find the value of 'x' to two decimal places?

EXCERCISES / USING A CALCULATOR

Solve to one decimal place:

1. $x^3 - x = 50$

2. $y^3 + 2y = 18$

3. A number plus its cube is 5. What is the number?

HINT ▸ Write your equation first.

4. The length of a cuboid is 2cm longer than the width. The height is the same as the width. What is the width of the rectangle to the nearest one decimal place if the volume of the cuboid is 200cm^3?

SOLUTIONS / USING A CALCULATOR

★ **TEACHER'S NOTE**
It will be helpful to review rounding to 1 and 2 decimal places prior to this work.

1 $x = 3.8$ (1 d.p.)

Try $x = 3$
$3^3 - 3 = 24$ *(too small)*
$4^3 - 4 = 60$ *(too big)*

50 is between 24 and 60 and so x must between 3 and 4.

Try $x = 3.5$
$3.5^3 - 3.5 = 39.375$
(less than 50, $x = 3.5$ is too small)

Try $x = 3.6$
$3.6^3 - 3.6 = 43.056$
(less than 50, $x = 3.6$ is too small)

Try $x = 3.7$
$3.7^3 - 3.7 = 46.953$
(less than 50, $x = 3.7$ is too small)

Try $x = 3.8$
$3.8^3 - 3.8 = 51.072$
(this is larger than 50, x must be between 3.7 and 3.8)

Try $x = 3.75$
$3.75^3 - 3.75 = 48.984$
(this is smaller than 50, x must be between 3.75 and 3.8)

Therefore, the answer to one decimal place is $x = 3.8$

★ **TEACHER'S NOTE**
To facilitate students working systematically students can show their workings in a table like this

x	$x^3 - x = 50$	
3	$3^3 - 3 = 24$	too small
4	$4^3 - 4 = 60$	too big
3.6	$3.6^3 - 3.6 = 43.056$	too small
3.7	$3.7^3 - 3.7 = 46.953$	too small
3.8	$3.8^3 - 3.8 = 51.072$	too big
3.75	$3.75^3 - 3.75 = 48.984$	too small

2 $y = 2.4$ (1 d.p.)

3 Equation is $x^3 + x = 5$
OR
$x + x^3 = 5$
$x = 1.5$ (1 d.p.)

4 l = w + 2, so the equation for the volume is
$V = w^2(w + 2)$
$200 = w^3 + 2w^2$
w = 5.3 cm (1 d.p.)

Trial and improvement

Using a spreadsheet

This method requires the use of a formula that describes the cubic equation. The formula and the trial values for the unknown are inputted into a spreadsheet such as that on Microsoft Excel.

At the top of the columns we label the values. In the examples below, column A has the input values for the unknown "a", whilst column B generates trial solutions based on the formula and inputted values for "a".

EXAMPLE

Solve the cubic equation $a^3 + a = 20$

	A	B	C
1	a	a^3 + a	
2			
3			
4			
5			
6			
7			
8			

1. Label column A by putting the unknown value "a" into cell A1 and column B with the formula ($a^3 + a$) in cell B1.

NOTE ▸ Spreadsheet show powers by using the ^. E.g. a^3 = a^3, a^5 = a^5

	A	B	C
1	a	a^3 + a	
2	1	=A2*A2*A2+A2	
3	2		
4	3		
5	4		
6	5		
7	6		
8			

OR enter "=A2^3+A2"

2. Enter trial values for "a" into column A.

   Define column B in terms of cell A2, by writing the formula in B2 in terms of A2.

	A	B	C
1	a	a^3 + a	
2	1	2	
3	2	**10**	
4	3	**30**	
5	4	**68**	
6	5	**130**	
7	6	**222**	
8			

3 Copy the formula down for all the values of "a". This will generate the values of $a^3 + a$ for values of $a = 2, 3, 4, 5$, and 6.

We have been ask to solve the cubic equation $a^3 + a = 20$, so we want to find the value in column B that is the closest to 20.

Looking at the values in column B, we see that the solution must be between 2 and 3.

Copy the formula down by placing the cursor at the bottom right corner of cell B2 so it changes into a black cross, left click and drag down the cells.

	A	B	C
1	a	a^3 + a	
2	1	2	
3	2	10	
4	2.1	11.361	
5	2.2	12.848	
6	2.3	14.467	
7	2.4	16.224	
8	2.5	18.125	
9	2.6	20.176	

4 We simply change the values in column A and the formulae will recalculate the values in column B.

Looking at the values in column B, we see that the solution must be between 2.5 and 2.6.

	A	B	C
1	a	a^3 + a	
2	1	2	
3	2	10	
4	2.1	11.361	
5	2.2	12.848	
6	2.3	14.467	
7	2.4	16.224	
8	2.5	18.125	
8	2.55	19.131375	
9	2.56	19.337216	
10	2.57	19.544593	
11	2.58	19.753512	
12	2.59	19.963979	
13	2.60	20.176	

5 Continue changing the values in column A until the required accuracy of 'a'

Looking at the values in column B, we see that the solution must be between 2.59 and 2.60
Therefore, the solution must be 2.6 to 1.d.p.

EXCERCISES / USING A SPREADSHEET

Set up spreadsheets and solve the following to one decimal place:

1. $x^3 + x = 81$

> **HINT** ▸ This image is how to start question 1.

	A	B	C
1	x	x^3 + x	
2	1	=A2^3 + A2	
3	2		
4	3		
5	4		
6	5		
7			
8			

2. $x^3 - x = 123$

3. $x^3 + 2x = 55$

4. $x^3 + x - 10 = 8$

5. $2x^3 - x = 160$

Extension: $2x^3 - x = 160$

Can you solve to 2 decimal places, 3 decimal places and 4 decimal places?

How accurate can you get?

SOLUTIONS / USING A SPREADSHEET

1 $x^3 + x = 81$ ▸ **Answer:** 4.2 (to 1 d.p.)

To find a solution we are looking for a value of x that gives the value of $x^3 + x$ as 81. Therefore, we are looking for a value of x in column A that gives a value of 81 in column B.

Formula in cell B2 should be: = A2^3 + A2. We find that the solution must be greater than 4.24 but less than 4.25. Since we've been told to give the answer correct to 1 decimal place, the answer we are looking for is 4.2.

	A	B	C
1	x	x^3 + x	
2	1	2	
3	2	10	
4	3	30	
5	4	68	
6	5	130	

Solution is between $x = 4$ and $x = 5$

	A	B	C
1	x	x^3 + x	
2	1	2	
3	2	10	
4	3	30	
5	4	68	
6	4.1	73.021	
7	4.2	78.288	
8	4.3	83.807	

Solution is between $x = 4.2$ and $x = 4.3$

	A	B	C
1	x	x^3 + x	
2	1	2	
3	2	10	
4	3	30	
5	4	68	
6	4.1	73.021	
7	4.2	78.288	
8	4.24	80.465024	
9	4.25	81.015625	
10	4.26	81.568776	

Solution is between $x = 4.24$ and $x = 4.25$

2 $x^3 - x = 123$ ▸ **Answer:** 5.0 (to 1 d.p.)

Formula in cell B2 should be: = A2^3 − A2

We find that the solution must be greater than 5.04 but less than 5.05. Since we've been told to give the answer correct to 1 decimal place, the answer we are looking for is 5.0.

3 $x^3 + 2x = 55$ ▸ **Answer:** 3.6 (to 1 d.p.)

Formula in cell B2 should be: = A2^3 + 2*A2

We find that the solution must be greater than 3.62 but less than 3.63. Since we've been told to give the answer correct to 1 decimal place, the answer we are looking for is 3.6.

4 $x^3 + x - 10 = 8$ ▸ **Answer:** 2.5 (to 1 d.p.)

Formula in cell B2 should be: = A2^3 + A2 − 10

We find that the solution must be greater than 2.49 but less than 2.50. Since we've been told to give the answer correct to 1 decimal place, the answer we are looking for is 2.5.

5 $2x^3 - x = 160$ ▸ **Answer:** 4.3 (to 1 d.p.)

Extension ▸ **Answer:**
4.35 (2 d.p.)
4.348 (3 d.p.)
4.3475 (4 d.p.)

With thanks to Susan Gill and Mark Walsh at Colchester Royal Grammar School, as well as those primary teachers who trialled these resources and gave valuable feedback.

Should you have any thoughts as to how we might improve future versions, or if there are other topics you would like us to cover, please email:
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www.imperial.ac.uk/schools