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Effective permeability of a metamaterial: Against conventional wisdom

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A method for finding the effective permeability of metamaterials is presented, based on the interaction between electromagnetic and magnetoinductive waves. Assuming a coupled circuit model for the interaction, a dispersion equation is derived that exhibits two types of bandgaps, one leading to complex solutions and the other to purely evanescent waves. Although losses are disregarded, the effective permeability (in contrast to established theories) is shown to have an imaginary part in part of the stop band, while its real part remains finite in both the pass band and the stop band. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.3696075]

It was recognised a century and a half ago that there is a simple way describing the electromagnetic properties of materials. Instead of considering the intricacies of the microscopic structure, it is possible to describe most of those properties by introducing the macroscopic parameters of permittivity, permeability, and conductivity. Clausius[1] and Mossotti[2] were pioneers of this art; their original concepts are still valid today, and the Clausius-Mossotti equations can be found in any textbook on Solid State Physics.

Effective medium theories[3–11] (and how to derive the material parameters from them) have enjoyed a new lease of life since the subject of Metamaterials became of general interest. It is difficult to choose between those theories, because of the scarcity of experimental data. There is however broad agreement on a number of issues. Restricting the discussion to lossless magnetic metamaterials containing resonant elements it is agreed that as the frequency increases towards the resonant frequency the real part of the permeability tends to plus infinity, then switches suddenly to minus infinity, and rises monotonically from there towards its value at infinite frequency. In the absence of losses, the imaginary part of the permeability (the part that causes attenuation) is necessarily zero. In the presence of losses, the real part of the permeability remains finite, but there is then an imaginary part as well that may reach a high value at the resonant frequency. An increase in losses leads to lower real and higher imaginary permeability. There are some variations on this basic picture (as many as five are noted in Ref. 12) but no major departures.

A different approach was offered by Syms *et al.*[13] based on a coupled circuit model for the interaction between electromagnetic (EM) waves and magneto-inductive (MI) waves propagating in an array of split-ring resonators (SRRs). Here, SRRs were described as L-C circuits and the EM wave as a transmission line. Rods were also modelled but are unnecessary here. A dispersion relation for coupled EM-MI waves was derived, with a forbidden gap arising as a direct consequence of the interaction. The paper investigated the stop bands and pass bands for various combinations of SRRs and rods, with and without coupling between the elements. The aim of the present paper is to extend that analysis by deriving the effective permeability directly from the dispersion equation. We shall first quote the dispersion equation for a lossless system, define the complex permeability, and show how attenuation may occur in a lossless structure.

We assume the model of Fig. 1. The physical structure is a linear array of SRRs to which a transverse EM wave is coupled via its magnetic field (Fig. 1(a)). Both waves are represented using circuit models (Fig. 1(b)). The lower transmission line is that of an EM wave with inductance L' and capacitance C', where $L' = \mu_0 a$ and $C' = \epsilon_0 a$, a is the period and μ_0 and ϵ_0 are the free space permeability and permittivity. The MI wave is also represented by lumped element resonators with inductance L and capacitance C, but in addition, the elements are coupled to each other by a mutual inductance M. Coupling between the waves is represented by a mutual inductance M'. Assuming propagation in the form $\exp\{j(\omega t - nka)\}$, where ω is the angular frequency, k is the propagation constant, and n is the element number, the dispersion equation is obtained[13] as

$$\{w^2 - 2\rho^2 + 2\rho^2\cos(ka)\}\{w^2[1 + \kappa\cos(ka)] - 1\} = w^4 q^2. \quad (1)$$

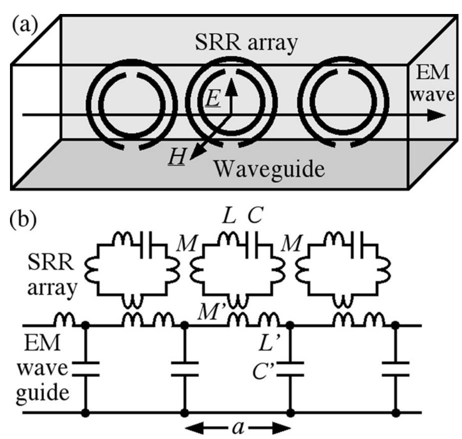

FIG. 1. (a) Physical model and (b) circuit model for the interaction between electromagnetic and magnetoinductive waves.

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Here, $w = \omega/\omega_0$ is the normalised frequency, $k = k' - jk''$ is a complex propagation constant with real part k' and imaginary part k'', $\omega_0 = (LC)^{-1/2}$ is the resonant frequency of the elements, $\rho = (c/a)(LC)^{1/2}$, c is the velocity of light, $\kappa = 2M/L$ is the coupling coefficient between the elements and $q^2 = M'^2/(LL')$ is the coupling coefficient between the EM and MI waves. q^2 is a dimensionless parameter, playing the same role as the filling factor in other theories.

Fig. 2(a) shows the dispersion characteristic for the typical parameter values $q^2 = 0.02$, $\rho = 20$, and $\kappa = -0.2$. Fig. 2(b) shows the details of the gap region, and Fig. 2(c) shows the variation of $k''a$ with frequency over the same range. The gap lies between the normalised frequencies $w_1 = 1.114$ and $w_3 = 1.132$, but notably, it is made up by two different regions, one below and one above $w_2 = 1.122$. Above w_3, the dispersion curves look the same as those in Refs. 3 and 14. However, they are different below w_3 since the unperturbed MI wave, a backward wave, is omitted from these theories. Inclusion of the backward wave means that the lower branch has a maximum, allowing the presence of a pair of complex waves (real parts the same, imaginary parts of opposite sign) between the points P and Q as shown in Fig. 2(b). It needs to be emphasised that there is an imaginary part although are absent. This result is against conventional wisdom, basically different from other treatments aimed at determining the effective permeability. The more general result that there can be lossless attenuation in a periodic system has been known for a long time.[15] For more recent studies of similar anomalous behaviour in generalised negative index media and media containing arrays of holes, see Refs. 16 and 17.

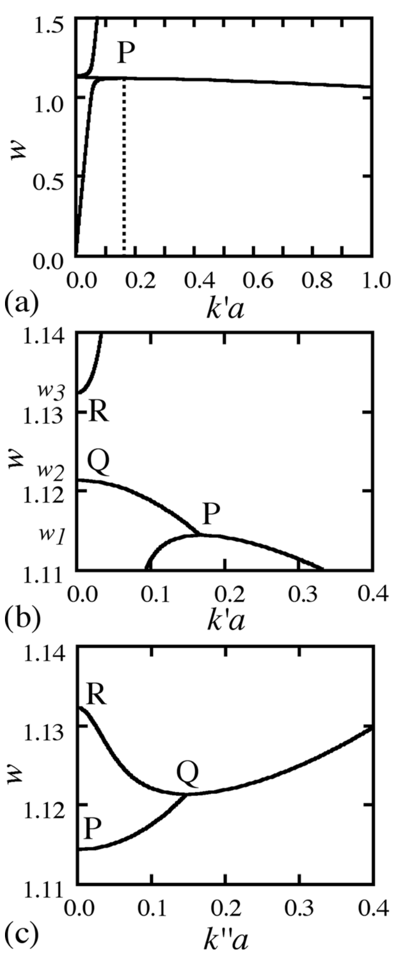

FIG. 2. (a) and (b) Dispersion characteristics and (c) frequency dependence of $k''a$.

The dispersion curve of Fig. 2(a) offers immediately a qualitative explanation of the phenomena due to the coupling between the EM and MI waves. At low frequency, the curve is characteristic of an unperturbed EM wave. As the frequency increases, the interaction with the MI wave bends the curve towards higher values of k, until it reaches the point P. Beyond that, we have a backward wave that retains the properties of the MI wave. However, up to P, we may regard the wave as a modified EM wave, whose speed drops below the velocity of light. Above w_3, we have a fast wave that gradually reduces its velocity to that of an unperturbed EM wave once again.

This qualitative explanation may be made more rigorous by defining an effective permeability. The unperturbed wave vector is equal to $k_{EM} = \omega(\mu_0\epsilon_0)^{1/2}$ while the actual wave vector is k. Hence, the relative permeability $\mu_r = \mu_r' - j\mu_r''$ is equal to

$$\mu_r = (k/k_{EM})^2 = (k'^2 - k''^2 - 2jk'k'')/k_{EM}^2 \qquad (2)$$

We shall first discuss the variation of μ_r' qualitatively, on the basis of the dispersion curve of Fig. 2(a). At low frequencies $\mu_r' = 1$. It rises monotonically with w until it reaches w_1, corresponding to the point P. The lower branch has its maximum value here. As the frequency increases further, k becomes complex. Its real part decreases and its imaginary part grows, as shown in Figs. 2(b) and 2(c). Eq. (2) implies that μ_r' then declines monotonically, reaching zero when $k' = k''$. As k' reduces further, μ_r' becomes negative, reaching its maximum negative value when $k' = 0$ at $w = w_2$. Between w_2 and w_3, in the upper gap, k' remains zero but k'' is non-zero. Hence, μ_r' is still negative at that point and starts growing because k'' declines. At w_3, both k' and k'' are zero; hence, $\mu_r' = 0$.

The actual variation of μ_r' is shown in Fig. 3(a), for the same parameters as before. Note that, because Eq. (1) has

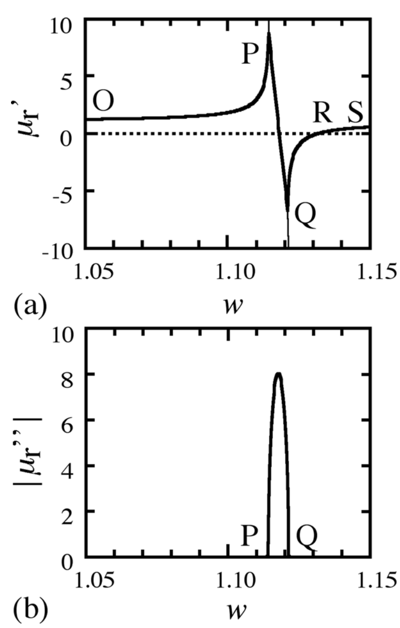

FIG. 3. Frequency variations of (a) μ_r' and (b) $|\mu_r''|$.

multiple solutions, there are two possible variations and these are shown superimposed. The thick line (the curve O-P-Q-R-S) can be associated with the EM wave. The thin line (which shares the region P-Q, but which diverges elsewhere) can be associated with the MI wave, and is therefore devoid of meaning as permeability. The former variation is similar to curves available from other theories, but it should be emphasized that those results were derived for the lossy case. There are two major differences here. First, the maxima and minima of μ_r' are now finite, not infinite. This result can be attributed to the fact that the maximum deviation of the EM branch from the light-line is also finite, and limited to the range P-Q in Fig. 2(b). Second, the derivative of μ_r' is discontinuous at P and Q due to the sudden appearance and disappearance of k''.

The corresponding variation of $|\mu_r''|$ is shown in Fig. 3(b). Again, two variations are shown, but these are identical. μ_r'' is of course zero at P and Q. However, in contrast to all other theories, μ_r'' is not zero but finite in between. We can now have two distinct physical interpretations. Relying on the dispersion equation, we can argue that an electromagnetic wave will decline between P and Q (w_1 and w_2) because k'' is finite. Alternatively, we can argue that the wave declines because the effective permeability has an imaginary part between these frequencies. Between Q and R (w_2 and w_3) the effective permeability has no imaginary part but, nevertheless, the wave declines because μ_r' is negative.

For 1D systems, the approach above offers a very simple route to homogenization. Other effects, such as non-nearest neighbour coupling and retardation could clearly be added. The permeability may be used for normal calculations, for example, finding the reflection and transmission coefficients at the interface between free space and a slab of the metamaterial. Despite the appearance of complex k-values, we have verified numerically that power is conserved. In conclusion, the effective permeability of a magnetic metamaterial has been determined based on the single assumption that the EM wave is coupled to a MI wave. It has been shown that the real part remains finite in the whole frequency band and may become negative in a band that starts above the resonant frequency. An imaginary part may also be present although losses are entirely absent.

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