Properties of purely reactive Foster and non-Foster passive networks

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The mathematical concept of strongly real functions of positive and negative types is introduced to network theory for the first time. The driving-point reactance/susceptance of a pure Foster network, made up of only ideal positive inductance and capacitance elements, is a strongly real function of real frequency of positive type. As a corollary, for a pure non-Foster network made up of only ideal negative inductance and capacitance elements, the driving-point reactance/susceptance is a strongly real function of real frequency of negative type. It is shown that a condition for a purely reactive passive network to exhibit a positive or negative reactance/susceptance-frequency gradient is that the driving-point immittance should have alternating poles and zeroes lying on the real frequency axis. Finally, it is shown that either purely Foster or non-Foster networks can be constructed by combining ideal Foster and non-Foster reactive elements.

Introduction: The driving-point immittance (either impedance Z or admittance Y) of a passive network realised using lumped elements has only a positive real function [1, 2] of complex frequency $s = j\omega$, where $\omega = (\omega' + j\omega'')$; ω' is the damped angular resonance frequency and ω'' is the Napier frequency. As a result, immittance is represented by a rational function of complex frequency, having polynomials with real coefficients. Also, the right half of the *s-plane* is mapped into the right half of the driving point *Z-* or *Y-plane*, with the immittance function being real for real ω. In the special case of a passive network that contains frequency-invariant reactances [2], the right half of the *s-plane* is also mapped into the right half of the driving point *Z-* or *Y-plane*, but the immittance function is no longer real for real ω. Foster's theorem also states that the driving-point immittance and associated reactive elements are an odd rational function with the Laplace transform [1, 3, 4] for LC circuits satisfying both of the following conditions [5]

$$\frac{\mathrm{d}X_F}{\mathrm{d}\omega'} > 0 \quad \text{and} \quad \frac{\mathrm{d}B_F}{\mathrm{d}\omega'} > 0 \tag{1}$$

where $X_F = \Im\{Z_F(\omega)\}$ is reactance and $B_F = \Im\{Z_F(\omega)\}$ is susceptance; subscript F represents the Foster condition, such that the networks have positive reactance- and susceptance-frequency gradients.

As a corollary to Foster networks, the driving-point immittance and associated reactive elements in non-Foster networks have the following conditions [5]

$$\frac{\mathrm{d}X_{NF}}{\mathrm{d}\omega'} < 0 \quad \text{and} \quad \frac{\mathrm{d}B_{NF}}{\mathrm{d}\omega'} < 0 \tag{2}$$

where subscript NF represents the non-Foster condition, such that the networks have negative reactance- and susceptance-frequency gradients [4, 5]. In practice, lossless non-Foster networks are traditionally implemented by active circuits that can synthesise negative inductance and negative capacitance elements [4, 5]. However, this has also been achieved through the use of negative differential-phase group delay networks [5], originally introduced in [6, 7].

In this Letter, only purely reactive passive networks are considered. Therefore, lossy Foster networks [8, 9] that exhibit driving-point immittances with a positive reactance-frequency gradient, but without exhibiting a positive susceptance-frequency gradient [9], or *vice versa*, thus violating (1), are not considered further.

The mathematical concept of strongly real functions of positive and negative types is introduced here to network theory for the first time. From this, an additional property of ideal lossless Foster networks is identified and then a new condition for networks to obey either (1) or (2) is proven. Finally, examples are given.

Strongly real rational functions: This concept has four associated theorems [10, 11].

Theorem 1: Let $X(\omega)$ be a real rational function (having real coefficients) of degree n

$$X(\omega) = \frac{P(\omega)}{Q(\omega)} \tag{3}$$

Then the following conditions are equivalent:

(i) Poles and zeroes of $X(\omega)$ are interspersed on the ω'-axis.
(ii) $X(\omega)$ is strongly real [i.e. $X(\omega)$ is real only if ω is real].
(iii) $X(\omega)$ can be written as follows, where x_k and b are real

$$X(\omega) = -a\omega + b + \sum_{k=1}^{m} \frac{c_k}{\omega - x_k} \tag{4}$$

with either ($a \le 0$ and $c_k < 0$) or ($a \ge 0$ and $c_k > 0$).

Strongly real rational functions of positive and negative types: It is important to note that any rational function obeying (i) will also obey (ii) and (iii), and *vice versa*. This means that any rational function with interspersed poles and zeroes lying on the ω'-axis will exhibit a positive or a negative gradient along the entire ω'-axis between its poles. However, this depending on the sign of a and c_k in (iii). Owing to (iii), the following will be either positive or negative for any ω' and has no real critical points:

$$\frac{\mathrm{d}X}{\mathrm{d}\omega'} = -\left[a + \sum_{k=1}^{m} \frac{c_k}{(\omega' - x_k)^2}\right] \tag{5}$$

Theorem 2: A strongly real rational function can only be one of two types: positive type if $\omega'' > 0$, implying that $X > 0$ [i.e. mapping the upper half of the *ω-plane* in the upper half of the *Z(ω)-plane*]; or negative type if $\omega'' > 0$, implying that $X < 0$ [i.e. mapping the upper half of the *ω-plane* in the lower half of the *Z(ω)-plane*].

Theorem 3: A strongly real rational function of positive type is strictly increasing in value between its poles, since $a \le 0$ and $c_k < 0$, ensuring (5) is always positive. A strongly real function of negative type has $a \ge 0$ and $c_k > 0$ and must be decreasing in value between its poles.

Theorem 4: If $X(\omega)$ is a strongly real function of positive(negative) type then $-1/X(\omega)$ will also be a strongly real function of positive(negative) type [11].

By inspection of Theorems 1, 3 and 4, since $B(\omega) = -1/X(\omega)$ for any purely reactive network, two lemmata become evident

Lemma 1 (Foster/non-Foster behaviour theorem): Any purely reactive network with poles and zeroes of its driving-point reactance/susceptance alternate on the ω'-axis exhibits either purely Foster behaviour (1) or purely non-Foster behaviour (2) along the entire ω'-axis.

Lemma 2 (combining Theorem 2 with Lemma 1): Since any driving-point immittance for a purely reactive network can be obtained by multiplying its reactance/susceptance by the complex operator j (rotating the *ω-plane* by +90° to give the *s-plane*). Any driving-point immittance with alternating poles and zeroes lying on the imaginary s-axis will exhibit either purely Foster behaviour (1) or purely non-Foster behaviour (2).

If the network exhibits purely Foster behaviour, the right half of the *s-plane* maps in the right-half plane of the driving-point immittance. With purely non-Foster behaviour, the right half of the *s-plane* maps in the left half of the immittance plane. To determine the kind of behaviour, immittance can be calculated for a single value of complex frequency in the right half of the *s-plane* to observe its mapping.

Ideal lossless Foster networks: As stated in [1, 3, 9], for LC networks (with always positive values for inductance L and capacitance C), the driving-port reactance/susceptance has alternating poles and zeroes along the ω'-axis and additionally obey (1). For example, consider the network shown in Fig. 1.

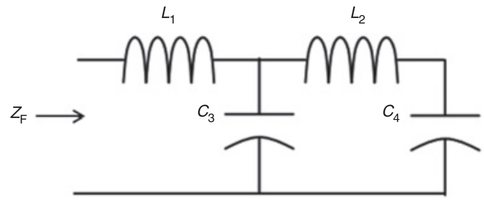

Fig. 1 *Typical LC network topology. Note that random values for L and C would generally produce a network that does not obey either (1) or (2)*

With values of inductances $L_1 = 1\ H$, $L_2 = 2\ H$ and capacitances $C_3 = 3\ F$, $C_4 = 4\ F$, the driving-point impedance at Port 1, having an open circuit at Port 2, is given by:

$$Z_F(s) = \frac{1 + 15s^2 + 24s^4}{7s + 24s^3} \tag{6}$$

$$Z_F(j\omega) = jX_F(\omega) = -j\left(\frac{1 - 15\omega^2 + 24\omega^4}{7\omega - 24\omega^3}\right) \tag{7}$$

The poles for reactance $X_F(\omega)$ in (7) have purely real values of ω at 0 and ±0.54. The zeroes are also found for purely real values of ω at ±0.741 and ±0.275. As predicted by Foster's theorem, there are four solutions for zeroes alternating with the poles on the ω'-axis. Also, as expected, since inductive reactance $X_L(\omega) = \omega L$ is a rational function of positive type, its reactance-frequency gradient obeys (1).

Ideal lossless non-Foster networks: For *LC* networks (with always negative values for inductance *L* and capacitance *C*), the poles and zeroes of the driving-point reactance/susceptance also alternate along the ω'-axis. In addition, the reactance/susceptance is represented by a rational function of negative type, exhibiting non-Foster behaviour as defined by Theorems 2 and 3.

Consider the network topology given in Fig. 1, with values of inductances $L_1 = -1\ H$, $L_2 = -2\ H$ and capacitances $C_3 = -3\ F$, $C_4 = -4\ F$. The resulting driving point impedance at Port 1, having an open circuit at Port 2, is given by

$$Z_{NF}(s) = -\frac{1 + 15s^2 + 24s^4}{7s + 24s^3} \tag{8}$$

$$Z_{NF}(j\omega) = jX_{NF}(\omega) = j\left(\frac{1 - 15\omega^2 + 24\omega^4}{7\omega - 24\omega^3}\right) \tag{9}$$

Since the location of poles and zeroes of $X_{NF}(\omega)$ alternate along the ω'-axis, it is a strongly real function of negative type. Therefore, a single point in the right half of the *s-plane* can be calculated and, using Lemma 2, checked to see that it maps into the left half of $X_{NF}(\omega)$*-plane*, confirming $X_{NF}(\omega)$ obeys (2). Alternatively, the gradient can be determined by taking the derivative of (9); this will always be negative.

Networks employing ideal Foster and non-Foster reactive elements exhibiting Foster behaviour: Considering again the network topology given in Fig. 1, with values of inductances $L_1 = 1\ H$, $L_2 = -2\ H$ and capacitances $C_3 = 3\ F$, $C_4 = 4\ F$ (representing a combination of ideal Foster and non-Foster elements), the driving-point reactance/susceptance still obeys (1). The resulting driving-point impedance at Port 1, having an open circuit at Port 2, is given by

$$Z_F(j\omega) = jX_F(\omega) = -j\left[\frac{1 + \omega^2 - 24\omega^4}{\omega(7 + 24\omega^2)}\right] \tag{10}$$

The poles for $X_F(\omega)$ have values of ω at 0 and $\pm j0.54$; while zeroes have values of ω at ±0.475 and $\pm j0.429$. The poles and zeroes are now either purely imaginary or purely real. Therefore, $X_F(\omega)$ cannot be a strongly real function, since it contains non-real roots. This implies that $X_F(\omega)$ can take real values for non-real ω and that no conclusion can be drawn on the reactance-frequency gradient with the previous theorems. However, the gradient can be determined by taking the derivative of (10) with respect to ω'; this will still always be positive. Note that $B_F(\omega)$ will have the same behaviour since only purely reactive networks are considered here.

Networks employing ideal Foster and non-Foster reactive elements exhibiting non-Foster behaviour: Consider again the network topology given in Fig. 1, with values of inductances $L_1 = -0.2\ H$, $L_2 = -1\ H$ and capacitances $C_3 = -1\ F$, $C_4 = 0.2\ F$. The result is a non-Foster behaviour with the poles and zeroes not lying on the ω'-axis and so the associated driving-point reactance/susceptance will not be a strongly real function of ω. As a result, the behaviour of the reactance/susceptance-frequency gradient must be checked first.

Conclusion: For a purely reactive network, with the use of Lemma 2, a new condition for the driving-point immittance has been found in order for it to exhibit either a purely Foster (1) or non-Foster behaviour (2). This is by means of the mathematical concept of strongly real functions, which to the best of our knowledge is new to network theory. Exceptions to the conditions were also given by examples; a network containing negative(positive) value inductances and capacitances that also exhibit Foster(non-Foster) behaviour. These new properties may give further insight into the behaviour of positive(negative) differential-phase group delay networks that employ Foster(non-Foster) networks; for example with the use of 3D Smith charts [12].

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