MEng Individual Project

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Solutions of the Hitchin Equations from the Nahm Transform

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Abstract

Using the Nahm transform we construct solutions to the Hitchin equations in $\mathbb{R}^2$. These solutions can be interpreted as a limiting case of doubly periodic instantons where the periods of the compact, periodic directions tend to zero. $U(1)$ solutions are constructed where the Higgs field in the Nahm manifold are taken to be zero. $SU(2)$ solutions are approximated and the results are applied to the doubly periodic instantons. We also highlight a correspondence between solutions to the Hitchin equations in $\mathbb{R}^2$ and the periodic monopoles.

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Declarations

This thesis is completed by the author only unless otherwise stated.

Chapter 2, Chapter 3 and Chapter 4 contain a review of previously published work of other academics which I do not claim any credits towards. Chapter 5, excluding Section 5.5 contains my original work unless stated otherwise. The remaining sections of the thesis contains the abstract, introduction, conclusions and appendices.

Contents

Chapter 1

Introduction

1.1 Motivation

Gauge theories are defined via Lagrangians which are invariant under local Lie group transformations. The concept of gauge symmetries is in essence, a redundancy in our description of nature. As a consequence of Noether's theorems, for every local Lie symmetry, a gauge field is produced and coupled to matter fields that have the associated symmetry. The interaction between matter fields and gauge fields gives rise to particle interactions.

The simplest example of a gauge theory is Maxwell's classical electromagnetism [1] where the gauge group is $U(1)$, which is abelian. Yang-Mills theory is a gauge theory first introduced to describe properties of isotopic spin [2]. It is based on non-abelian Lie groups such as $SU(2)$. Non-abelian gauge theory is one of the cornerstones of theoretical physics as it forms the basis of the standard model with gauge group $SU(3) \times SU(2) \times U(1)$, which unifies the description of electromagnetism, weak and strong interactions.

Gauge theory is also rooted in geometry. We can draw direct correspondences between gauge theory with bundle geometry terminologies such as identifying "field strength" with "curvature" and "gauge type" with "principal fibre bundle"[3]. Then it would not be surprising that the study of the topological, as well as the dynamical properties of gauge fields, were extremely fruitful and they gave rise to the field known as topological solitons. Topological solitons are simply smooth solutions to the equations of motion where they admit distinct differences from the trivial vacuum solution. In classical gauge theory, instantons are one class of the topological solitons, along with vortices, monopoles and domain walls (kinks). Existence of these pseudo-particles has implications outside classical field theory as we can infer information in associated theories such as (D-brane) String and other supersymmetric theories [4, 5], which form another branch of theoretical physics.

To construct these solitons explicitly, various methods have been developed by applied and pure mathematicians alike. Solitons that exist most naturally in Yang-Mills and Yang-Mills-Higgs are instantons and monopoles respectively. In the cases of instantons, the Atiyah-Drienfeld-Hitchin-Manin (ADHM) construction [6] is a formalism that uses purely algebraic methods. For monopoles, the available methods are more diverse but they can be realised as a set of equivalences between the objects [7]

1. Monopole solution in $\mathbb{R}^3$ with maximal symmetry breaking at spatial infinity of a classical group G.

2. Certain families of spectral curves (spectral data).

3. Solutions to Nahm's equation satisfying specific boundary conditions (Nahm data).

In this thesis, we will be focusing on the transformation between objects 1 and 3 as above, which is an example of the Nahm transform. Nahm [8, 9] originally introduced this construction as a modification to the ADHM construction where the instanton is invariant under the translation of one coordinate. This lead to the construction of elementary monopoles as they are simply instantons invariant under time translation. In the paper by Braam and Van Baal [10], Nahm's original construction is generalised to a transformation that maps (anti) self-dual instantons on the base four torus to (anti) self-dual instantons on the dual four torus. This paper was also when the name Nahm transform was coined. However, for the construction of instantons on the four torus, the Nahm transform has not yielded new solutions. The Nahm transform maps one hard problem into another equally hard problem as finding instantons in the base four torus is same as finding instantons in the dual four torus.

However, on a generalised four torus, one can take various combinations of periods and let them tend to zero or infinity. In these extreme limits, the Nahm transform offers the benefit of dimensional reduction between the base and dual generalised four torus[1]. For example, in Nahm's original construction of a monopole, the base "torus" is $\mathbb{R}^3$ and the dual "torus" is an interval of $\mathbb{R}$. The dimensional reduction property has been used by numerous authors to study instantons on a generalised torus, classified by the base manifold where the instantons exist on. Some of the well-studied cases are

- Periodic instantons by Kraan and Van Baal [11, 12]. Periodic instantons were also independently investigated by Lee and Lu [13] at same time using the Nahm transform.
- Doubly periodic instantons by Ford and Pawlowski [14, 15].
- Periodic monopoles by Cherkis, Kapustin and Durcan [16, 17].
- Doubly periodic monopoles by Ward [18].

The study of these solutions was very fruitful in revealing properties of these solitons as well as implications outside classical dynamics, such as supersymmetric Yang-Mills in the case of periodic monopoles [16]. However, working with the Nahm transform is very technical. Although the necessary ingredients to perform the transform are known in general (the Nahm equation, Weyl operators)[10], the calculations required to arrive at an explicit solution can be extremely difficult. This motivates us to examine simpler versions of the Nahm transform by varying the base and dual manifold where the Nahm transform maps between, as well as the input Nahm data.

1.2 This Thesis

In this thesis, we are concerned with solutions to the Hitchin equations in $\mathbb{R}^2$. The Hitchin equations can be considered as dimensionally reduced Yang-Mills field equations, thus their solutions correspond to "instantons" existing only in $\mathbb{R}^2$. Application of the Nahm transform to the Hitchin equations is simpler as the Nahm and inverse Nahm transform take the same form, as well as the fact that the base and dual manifold are the same. As the Nahm transform will map solutions to the Hitchin equations between the two $\mathbb{R}^2$ planes, we will attempt to construct complex solutions on the base $\mathbb{R}^2$ from elementary solutions

[1]The detail of dimensional reduction will be discussed in section Chapter 4

on the dual $\mathbb{R}^2$ plane. This problem is worth studying due to its connection to other unsolved problems such as the double flux Aharonov Bohm (AB) effect[2][19] and periodic monopoles. The study of the Hitchin equations has been extensive in the past, but only a few publications attacked the Hitchin equations through the Nahm transform. However, some solutions to the Hitchin equations can naturally be seen as the limiting configurations of doubly periodic instantons, which we will take as the starting point of our analysis.

The thesis is structured as follows. In Chapter 2 and Chapter 3 we will review instantons and monopoles respectively, as well as introduce the Hitchin equations as dimensionally reduced field equations. In Chapter 4 we will review the Nahm transform and give an example constructions of monopole solutions using the Nahm transform. We will also consider the Nahm transform for the doubly periodic instantons to set up the consideration of constructing solutions to the Hitchin equations. The main results of the thesis are presented in Chapter 5 where we consider $U(1)$ and $SU(2)$ solutions to the Hitchin equations. The chosen simple Nahm potentials are abelian solutions to the Hitchin equations which describe the AB fluxes through the $\mathbb{R}^2$ plane.

[2]More of the AB effect will be discussed in Chapter 5.

Chapter 2

Instantons

In this chapter, we will review the general properties of Yang-Mills instantons on $\mathbb{R}^4$ in Section 2.1 as well as some instanton solutions to the pure Yang-Mills self-duality equation in Section 2.3. The content of this chapter follows from the reviews by M.K.Prasad [20], D.Tong [21] as well as N.Manton [22]. For more in-depth discussion please refer to those reviews.

2.1 General Instantons

In this section, we will be following the conventions in [20]. The pure Yang-Mills action is given by

$$S = -\frac{1}{2e^2}\int \mathrm{Tr}(F^{\mu\nu}F_{\mu\nu})d^4x, \tag{2.1.1}$$

We will be using Euclidean metrics[1] with signature $(+,+,+,+)$, thus the covariant and contravariant indices are treated as the same. In the literature, the action might differ by a multiplicative constant. This does not change the form of the dynamical equation (equations of motion) of the field as the constants can be removed by appropriate scaling arguments. The constant e is the gauge coupling constant and $F^{\mu\nu}$ is the field strength tensor defined by

$$F_{\mu\nu} = \partial_\nu A_\mu + \partial_\mu A_\nu + [A_\mu, A_\nu], \tag{2.1.2}$$

where A_μ is the gauge potential and $\mu,\nu \in \{0,1,2,3\}$ represents 4 dimensional space. We will use the standard notation when taking summation over Greek letters e.g. μ,ν, which take values $\mu,\nu \in \{0,1,2,3\}$, when taking summation over Roman letters e.g. i,j,k, which take values $i,j,k \in \{1,2,3\}$ unless otherwise specified. Here $[A,B]$ is the commutator of A and B with the definition $[A,B] = AB - BA$. We denote A_0 as the "time" gauge potential and A_1, A_2, A_3 as the 3 dimensional "space" gauge potential.

We will consider the adjoint representation of the group where the basis (generator) elements $\{T^a\}_{a=0}^N$ of the Lie algebra $\mathfrak{g}$ for our gauge group satisfies the relation $\left[T^a, T^b\right] = f^{abc}T^c$, where f^a_{bc} are the structural constants[2]. Note that $N < \infty$ is the dimension of the group as we are only considering finite dimensional Lie groups and we can always define a pairing such that such that $\left\langle T^a, T^b \right\rangle = \delta^{ab}$. The gauge potential and field strength written in terms of $\{T^a\}$ satisfies the relation $A_\mu = eT^aA^a_\mu$ and $F^a_{\mu\nu} = \partial_\nu A^a_\mu + \partial_\mu A^a_\nu + ef^{abc}A^b_\mu A^c_\nu$ respectively. Instantons are the finite action solutions to the Euler-Lagrange equation

[1] Instantons in the more physical Minkowski space is a tricky issue and it will not be investigated here.

[2] It is always possible to consider the basis of a Lie algebra as set of anti-Hermittian and traceless $N \times N$ matrices by simple Lie theory

obtained from standard calculus of variations on (2.1.1). It is a second order differential equation

$$\partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = [D_\mu, F^{\mu\nu}] = 0, \tag{2.1.3}$$

where D_μ is known as the covariant derivative and it is defined as

$$D_\mu = \partial_\mu + A_\mu. \tag{2.1.4}$$

Gauge transformation of gauge potential and field strength is defined by

$$A_\mu \longrightarrow g^{-1}A_\mu g + g^{-1}\partial_\mu g, \qquad F_{\mu\nu} \longrightarrow g^{-1}F_{\mu\nu}g, \tag{2.1.5}$$

respectively, where $g = e^{\lambda^a(x)T^a} \in G$ is a unitary operator and $\lambda^a(x)$, $a = 1, \dots N$ are N arbitrary functions. The dual electromagnetic tensor ${}^\star F_{\mu\nu}$ is defined as

$${}^\star F_{\alpha\beta} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}, \tag{2.1.6}$$

where its components are the components of the Hodge dual of $\mathcal{F} = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$, which is field strength written in form notation. Following the approach of Belavin *et al.* in [23], we write (2.1.1) as

$$S = -\frac{1}{4e^2}\int \mathrm{Tr}((F^{\mu\nu} \mp {}^\star F_{\mu\nu})^2) \pm 2\mathrm{Tr}(F_{\mu\nu}{}^\star F^{\mu\nu})d^4x. \tag{2.1.7}$$

Since we are concerned with finite action solutions, we demand the field strength to tend to zero as $|x|$ tends to ∞. This is only possible with A_μ satisfying

$$A_\mu \xrightarrow{|x|\to\infty} g^{-1}\partial_\mu g, \tag{2.1.8}$$

for some $g \in G$ defined at spatial infinity boundary of $\mathbb{R}^4$ and it is called a pure gauge. In fact, substitution of (2.1.8) into (2.1.7) yields

$$S_{min} \geq \frac{8\pi^2}{e^2}|k|, \tag{2.1.9}$$

where k is known as the Pontraygin index which is an integer. It is computed with the formula

$$k = -\frac{1}{16\pi^2}\int \mathrm{Tr}(F_{\mu\nu}{}^\star F^{\mu\nu})d^4x, \tag{2.1.10}$$

where the Pontraygin index k measures the degree of the maps from boundaries of space-time $\partial\mathbb{R}^4_\infty \cong \mathbb{S}^3_\infty$ to $SU(N)$.

Restricting the gauge group to $SU(2)$, we have that $SU(2)$ is a manifold isomorphic to $\mathbb{S}^3$. Considering an instanton solution A_μ at spatial infinity, it is a smooth map from $\mathbb{S}^3_\infty$ to $SU(2) \cong \mathbb{S}^3$. Then geometrically speaking, topological charge k is the number of times the solution "wraps" $\mathbb{S}^3_\infty$ around $SU(2)$. Topological charge k is a topological invariant quantity, *i.e.* a solution of topological charge k_0 cannot be continuously deformed to a solution with topological charge $k_1 \neq k_0$. We say that solutions of different topological charge belong to different topological *sectors* to emphasise this fact. By the result $\pi_n(\mathbb{S}^n) = \mathbb{Z}$, we have $k \in \mathbb{Z} = \pi_3(\mathbb{S}^3)$ which is also known as the second Chern class or Pontraygin number [23] in topology. Thus, the minimum action is given by

$$S_{min} = \pm\frac{1}{2e^2}\int \mathrm{Tr}(F_{\mu\nu}{}^\star F^{\mu\nu})d^4x, \tag{2.1.11}$$

where equality of (2.1.9) can be attained by the first order (anti) self duality equation

$$^\star F_{\mu\nu} = \pm F_{\mu\nu}. \tag{2.1.12}$$

The realisation of the (anti) self-duality equation is a dynamical problem rather than a topological problem [20]. The configuration space of solutions to (2.1.12) can be considered as the quotient space $\mathcal{A} \setminus \mathcal{G}$ where $\mathcal{A}$ is the space of gauge potentials and $\mathcal{G}$ is the space of gauge transformation. This space is highly non-trivial due to the existence of topological sectors and topological charge. An interesting interpretation of instantons arises from this observation where instantons are tunnelling paths between the vacuums of gauge equivalent solutions in the same topological charge [24].

2.2 Hitchin Equations

The Hitchin equations [25] are the dimensional reduced self-duality equation (2.1.12) on $\mathbb{R}^2$. To form the Hitchin equations from the self-duality equation in $\mathbb{R}^4$, we "freeze" two of the coordinates of $\mathbb{R}^4$, without loss of generality we can choose them to be x_1 and x_2. The self-duality equation is then reduced to $\mathbb{R}^2$ as solutions are no longer be dependent on x_1 and x_2. We form the following objects

$$x = x_0 + ix_3 \in \mathbb{C}, \qquad \Phi = \frac{1}{2}(A_1 - iA_2), \tag{2.2.1}$$

where x is the complex variable of the $\mathbb{R}^2$ plane and Φ is a complex "Higgs" field. Writing the remaining gauge potential as a complex potential $A_x = \frac{1}{2}(A_0 - iA_3)$ and its complex conjugate, we can write down the field strength as

$$F_{x\bar{x}} = \partial_{\bar{x}} A_x - \partial_x A_{\bar{x}} + [A_x, A_{\bar{x}}]\,. \tag{2.2.2}$$

Substituting into (2.1.12) yields the celebrated Hitchin equations in complex coordinates[14].

$$[D_x, \Phi] = 0, \tag{2.2.3a}$$

$$F_{x\bar{x}} = \left[\Phi, \Phi^\dagger\right] = [D_x, D_{\bar{x}}]\,, \tag{2.2.3b}$$

where the covariant derivative $D_{\bar{x}}\Phi = \partial_{\bar{x}} + [A_{\bar{x}}, \Phi]$ and $\Phi^\dagger$ denoting the complex conjugate transpose of Φ.

2.3 Instanton Solutions

The first general solution to $SU(2)$ Yang-Mills equation is for the case $k = 1$ proposed by Belavin, Polyakov, Schwartz and Tyupkin [23] which is known as the BPST instanton. We will not present the original construction by Belavin *et al.* , instead, we will consider the BPST solution as a special case of the 't Hooft ansatz. The presentation of the solution will be in the same form as [22]. We introduce anti-symmetric tensor $\sigma_{\mu\nu}$ with

$$\sigma_{i0} = \tau_i, \quad \sigma_{ij} = \epsilon_{ijk}\tau_k = -\frac{i}{2}\left[\tau_i, \tau_j\right], \quad i, j \in \{1, 2, 3\}, \tag{2.3.1}$$

where $\{\tau_i\}$ are the Pauli matrices:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3.2}$$

The gauge potential ansatz is

$$A_\mu = \frac{i}{2}\sigma_{\mu\nu}\partial_\nu \log\left(\rho(x)\right). \tag{2.3.3}$$

One can alternatively give the components of gauge potential as [20]

$$A^a_\mu = -\frac{1}{e}\bar{\eta}^a_{\mu\nu}\partial_\nu \log(\rho(x)) \tag{2.3.4}$$

with standard skew-Hermitian, traceless $\mathfrak{su}(2)$ generators $T^a = \frac{\tau^a}{2i}$ to show the dependencies to gauge coupling. The 't Hooft symbols $\bar{\eta}^a_{\mu\nu}$ introduced in [26] satisfies $\sigma_{\mu\nu} = \tau^a\bar{\eta}^a_{\mu\nu}$, $a = 1, 2, 3$. Substitution of the ansatz into the Yang-Mills self-duality equation (2.1.12) gives that ρ satisfies the Laplace equation in $\mathbb{R}^4$

$$\partial_\mu\partial^\mu\rho(x) = 0, \tag{2.3.5}$$

which we have the one pole solution $\rho(x) = 1 + \frac{\lambda^2}{|x-a|^2}$, and the BPST gauge potential defined as

$$A_\mu = \frac{i}{2}\sigma_{\mu\nu}\partial_\nu \log\left(1 + \frac{\lambda^2}{|x-a|^2}\right), \tag{2.3.6}$$

where $a \in \mathbb{R}^4$ is arbitrary. There are a total of five real parameters in the solution with $\lambda \in \mathbb{R}$ corresponding the physical size of instanton, while $a \in \mathbb{R}^4$ corresponding to the location of instanton in 4-dimensional space. We can see at the point $x = a$, is an apparent singularity of the solution. This singularity is an artefact of the gauge and it can be removed with a suitable gauge transformation. Gauge invariant quantities such as the action density (2.1.1) are smooth in all gauges, thus it is non-singular even when computed with (2.3.6).

As $\rho(x)$ is chosen as the solution to the Laplace equation (2.3.5), 't Hooft extended the construction of BPST instantons to charge k instantons with

$$\rho(x) = 1 + \sum_{i=1}^{k}\frac{\lambda_i^2}{|x-a_i|^2}. \tag{2.3.7}$$

We can interpret this as k charge one BPST instantons with size λ_i and at space-time a_i where λ_i and a_i are arbitrary. This solution has $5k$ parameters as each of the k instantons contributes 5 parameters as in the BPST case. In fact, the index theorem by Atiyah *et al.* [27] shows the solutions of $SU(2)$ instantons depends on exactly $8k - 3$ parameters, where k is the Pontraygin index. Hence the 't Hooft solutions have the correct number of parameters for $k = 1$, but it is missing $3k - 3$ parameters for $k > 1$. We will define the gauge "orientation" of instantons as the orientation of instanton embedding in the gauge group $SU(2)$. These 3 parameters are not gauge invariant as gauge transformation alters the orientation when a single charge one instanton is considered. At higher charges, the relative orientations between charge one instanton constituents are preserved by gauge transformations. Hence the $3k$ gauge orientation parameters only exist for $k > 1$. This addition parameters along with position and size give the correct number of physical parameters as in the index theorem.

However, it is more convenient to study instanton parameter spaces and collective coordinates with a hyperkähler structure [28]. To remedy this, pure mathematicians also consider the 3 gauge orientation parameters of single charged instanton belong to the parameter space, hence bring the total number of parameters to $8k$, required by hyperkählerity. In this setting, we call the space of parameters with the additional gauge orientation as the moduli space and refer to the parameters as moduli.

Jackiw, Nohl and Rebbi [29] extended 't Hooft's ansatz using a conformal group transformation to obtain

$$\rho(x) = \sum_{i=1}^{K} \frac{\lambda_i^2}{|x - a_i|^2} , \tag{2.3.8}$$

which is referred to as JNR solution. In fact, solution (2.3.7) can be seen as a limiting case of (2.3.8) as $a_K \to \infty$, $\lambda_K \to \infty$, $\frac{\lambda_K^2}{a_K^2} = 1$ and $k = K - 1$. The JNR solution has $5k + 4$ parameters which does not correspond to the number of parameters by the index theorem. The discrepancy at $k = 1$ is due to the invariance of field potential with real multiplicative constant on $\rho(x)$, *i.e.* $\rho(x) \to \lambda_0 \rho(x)$. The "size" parameters of the instantons then are $\frac{\lambda_i}{\lambda_0}$, $i = 1 \ldots K$, hence reducing the parameter count to $5k + 3$ and at $k = 1$, agrees with index theorem. JNR at $k = 1$ gives the most general instanton solution but with $k > 1$, it is missing a factor of $3k$ which can be seen as the missing relative gauge orientation parameters for each instanton in $SU(2)$.

The ADHM construction by Atiyah *et al.* [6] is a algebraic construction of *all* solutions to self-duality equation (2.1.12) in Yang-Mills theory over compactified Euclidean 4-space $\mathbb{S}^4$ which corresponds to instanton solutions. The construction is generalised for all classical compact groups and it has been extended for instanton solutions in $\mathbb{T}^4$ [30]. We will not present the construction here but one can refer to [31] for the operational rules of ADHM.

Chapter 3

Magnetic Monopoles

The Dirac magnetic monopoles is a postulated particle with radial magnetic field $\mathbf{B}$ such that B_i, $i = 1, 2, 3$ is defined as

$$B_i = \frac{g\hat{r}_i}{4\pi r^2}, \tag{3.0.1}$$

where g is the magnetic charge and r is Euclidean distance from the original of $\mathbb{R}^3$. The study of magnetic monopoles started with Dirac [32] with the argument that if monopoles exist, charge must be quantised with

$$qg = 2\pi n\hbar, \quad n \in \mathbb{Z}, \tag{3.0.2}$$

where q and g are electric and magnetic charge respectively [33]. Schwinger [34] later shown the consistency of monopoles with quantum electrodynamics (QED) yet these particles remain undetected in nature. Due to the mass of monopoles, its production in particle accelerators is unfeasible. Astronomical searches *e.g.* in cosmological radiation also proved to be unfruitful [35]. An interesting development of monopoles is the insight into super-symmetry and search of multi-monopole solutions[36].

In this Chapter, we will first review the interpretation of topological solitons as magnetic monpoles with the gauge group $SU(2)$ in Section 3.1. We will then consider the Bogomolny-Prasad-Sommerfield (BPS)[37] limit of monopole energy and the solutions associated with it in Section 3.2. The general content of this chapter follows closely from the review by Weinberg *et al.* [38] and for more in depth discussion of the topic, please refer to [39, 38, 22, 21].

3.1 Monopole Formulation

In Dirac's original arguments, an $U(1)$ (Dirac) monopole (3.0.1) was constructed from Maxwell's equations to show the quantisation of magnetic charge. It contains a Dirac string with is a curve in $\mathbb{R}^3$ that originates from the centre of monopole and extends to infinity. Dirac's monopole is not a topological soliton as it is not singular along the Dirac string. An elementary argument[40] from bundle geometry shows that the Dirac string is, in fact, the result of the base manifold which the monopole exists in. Dirac monopole to exists in compactified $\mathbb{R}^3$, which we can denote $\mathbb{S}^3_\infty$ excluding the origin, which has the same homotopy class as $\mathbb{S}^2$. This manifold is non-trivial in the sense that more than one chart is recovered to completely cover it, thus creating more than one transition functions and trivialisation functions. The Dirac string corresponds exactly the overlapping of transition functions and it is not a gauge invariant quantity, it is a gauge artefact.

To find a class of monopoles without the Dirac string, 't Hooft considered the $U(1)$ gauge group of electromagnetism as a subgroup of a larger group with compact covering [41], e.g. $SU(2)$, the gauge we are interested in. In this formulation, regular solutions to the field equations are found which corresponds to magnetic monopoles without Dirac Strings as a pair of strings can be annihilated by forming monopole and anti-monopole pairs.

We consider the Yang-Mills-Higgs Lagrangian which couples a pure Yang-Mills field (2.1.1) with a Higgs. The gauge group is a large enough group whose symmetry can be spontaneously broken down to $U(1)$ by an (adjoint) Higgs field Φ. In the following discussion, we will focus on $SU(2)$ gauge group for simplicity of expressions. Note that in this chapter, we will revert to using Minkowski metric $(+,-,-,-)$ instead of Euclidean metric in Chapter 2. The Lagrangian is

$$\begin{aligned} L &= -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \mathrm{Tr}(D_\mu\Phi D^\mu\Phi) - V(\Phi) \\ &= -\frac{1}{4}(F^a_{\mu\nu}F^{a\mu\nu}) + \frac{1}{2}(D_\mu\Phi^a)(D^\mu\Phi^a) - V(\Phi^a), \end{aligned} \tag{3.1.1}$$

where $A_\mu = A^a_\mu T^a$ is the gauge potential with field strength $F_{\mu\nu} = F^a_{\mu\nu}T^a$ and $\Phi = \Phi^a T^a$ as the Higgs field in $\mathbb{R}^3$. In the $SU(2)$ gauge group setting, we write the generators of $\mathfrak{su}(2)$ using Pauli matrices $\{\tau_i\}$ to have a set of traceless Hermitian generators $T^a = \dfrac{\tau_a}{2}$. Thus $\{T^a\}$, $a \in \{1,2,3\}$ satisfies $\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^a_b$ and $[T^a, T^b] = i\epsilon_{abc}T^c$. In this convention, we shift gauge coupling constant e into the Lagrangian, then we have D_μ is the covariant derivative $D_\mu\Phi = \partial_\mu\Phi + ie\,[A_\mu, \Phi]$. We also have modified formula for field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie\,[A_\mu, A_\nu]$. Under the $SU(2)$ gauge group, we have the gauge transformation of gauge potential A_μ is the same as instantons gauge potential and Φ transforms in the same way as $F^{\mu\nu}$. One of the Higgs potential $V(\Phi)$ which corresponds to the symmetry breaking of $SU(2)$ to $U(1)$ is

$$V(\Phi) = \lambda(|\Phi|^2 - \eta^2)^2, \tag{3.1.2}$$

where $|\Phi|^2 = 2\mathrm{Tr}(\Phi^2) = \Phi^a\Phi^a$ and λ is a positive real constant.

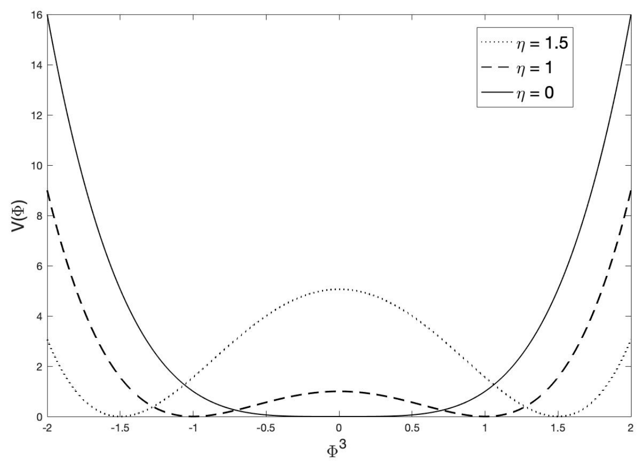

Figure 3.1: The function $V(\Phi)$ with $\lambda = 1$, $\Phi^1 = \Phi^2 = 0$ and varying η

In the rest of this chapter it is useful to consider $F_{\mu\nu}$ as a tensor whose components are composed of components of electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$. In natural units

($c = 1$), we have the following relation

$$B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}, \quad E_i = F_{0i}, \tag{3.1.3}$$

where E_i and B_i are components of electric and magnetic fields respectively.

Considering Lagrangian (3.1.1) as the difference of kinetic energy T and potential energy V then

$$T = \frac{1}{4}\int \mathrm{Tr}E_i^2 + \mathrm{Tr}(D_0\Phi)^2 \, d^3x, \tag{3.1.4}$$

and

$$V = -\frac{1}{4}\int \mathrm{Tr}B_i^2 + \mathrm{Tr}(D_i\Phi)^2 + V(\Phi) \, d^3x, \tag{3.1.5}$$

We can see the total energy is given by

$$E = \frac{1}{4}\int \mathrm{Tr}E_i^2 + \mathrm{Tr}B_i^2 + \mathrm{Tr}(D_\mu\Phi)^2 + V(\Phi) \, d^3x. \tag{3.1.6}$$

Assuming energy minima exists it is then obtained when

$$|\Phi|^2 = \Phi^a\Phi^a = \eta^2, \quad D_\mu\Phi^a = 0, \quad F^a_{\mu\nu} = 0. \tag{3.1.7}$$

This condition defines the vacuum solution. The set of Φ satisfying vacuum energy thus forms a sphere $\mathbb{S}^2$ in terms of generator coefficients Φ^a, $a = 1, 2, 3$, where each point on the sphere are related to each other by a $SU(2)$ gauge transformation[38]. For $\eta \neq 0$, the vacuum solutions belongs a family of degenerate solutions which only preserves the $U(1)$ subgroup, *i.e.* the $SU(2)$ symmetry is spontaneously broken down to $U(1)$ symmetry. If we choose the vacuum solution to be

$$\Phi = \eta\frac{\tau_3}{2}, \quad A_\mu = 0, \tag{3.1.8}$$

we have the $U(1)$ subgroup is orientated in one direction of $SU(2)$. In the case presented above, we have the $U(1)$ is aligned with one of the generators of $\mathfrak{su}(2)$, namely τ_3.

For finite energy solutions, the Higgs field Φ must tends to vacuum solution at spatial infinity. However, it is not necessary that all solutions at spatial infinity are aligned in the same arbitrary direction. Thus we will allow the Higgs field in spatial infinity limits to vary with direction, as long as it is accompanied with a gauge potential with suitable asymptotics.

Hence we can define a map Φ_∞ from $\mathbb{S}^2_\infty$, the two-sphere representing spatial infinity to the space of Higgs fields satisfying minimal energy condition (3.1.7). For general gauge group G with residual gauge group H after it is broken by Higgs field, we require the map from $\mathbb{S}^2_\infty$ to ${}^G/_H$. In case of $G = SU(2)$, $H = U(1)$ and ${}^{SU(2)}/_{U(1)} \cong \mathbb{S}^2$. Thus Φ_∞ is a map from $\mathbb{S}^2_\infty$ to $\mathbb{S}^2$ which can be classified by the homotopy group. It can be shown that $\pi_2(\mathbb{S}^2) \in \mathbb{Z}$ and we define the topological charge n (winding number) of a solution by its homotopy group $\pi_2(\mathbb{S}^2)$. The topological charge is given by

$$n = \frac{1}{8\pi}\epsilon_{ijk}\epsilon_{abc}\int_{\mathbb{S}^2_\infty} \hat{\Phi}^a\partial_j\hat{\Phi}^b\partial_k\hat{\Phi}^c \, d^2S_i, \tag{3.1.9}$$

where $\hat{\Phi}^a = {}^{\Phi^a}/_{|\Phi|}$ is the normalised Higgs field such that $|\hat{\Phi}| = \frac{1}{2}\mathrm{Tr}(\hat{\Phi})^2 = 1$. n is also closely related to the first Chern number, cf. instanton charge number which is the second

Chern number.

A classic example of non-trivial Higgs field configuration is pointed out by 't Hooft. Instead of taking the Higgs field to be aligned with the z axis, we can consider a unitary $SU(2)$ transformation defined by

$$\Omega(\theta,\varphi) = \cos(\frac{1}{2}\theta)\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} + \sin(\frac{1}{2}\theta)\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{3.1.10}$$

where the Higgs field is taken as (3.1.8) multiplied by $\Omega(\theta,\varphi)$ above. In this configuration of Φ (famously known as the "hedgehog" configuration), finite energy condition demands a new boundary condition which is dependent only on the radial distance at spatial infinity. 't Hooft and Polyakov gave the asymptotic behaviour of the radial Higgs field approaching spatial infinity [42] as

$$\Phi^a(x) \xrightarrow{r\to\infty} \eta\frac{x^a}{r}, \tag{3.1.11}$$

which has topological charge one. For general charge one monopole with arbitrary Higgs configuration, the asymptotics of Higgs field is required to be

$$\Phi^a(x) \xrightarrow{r\to\infty} \eta\frac{\Phi^a}{|\Phi|} = \eta\hat{\Phi}^a. \tag{3.1.12}$$

Compared to the configuration defined in (3.1.8) which has trivial topological charge, there is no global gauge transform between them. *i.e.* one can not deform one configuration smoothly to reach the other configuration which exactly corresponds to configurations belonging to different homotopy classes.

Standard calculus of variations yields the following Euler-Lagrange equations:

$$D_\mu D^\mu \Phi = -\lambda(|\Phi|^2 - \eta^2)\Phi, \tag{3.1.13a}$$
$$D_\mu F^{\mu\nu} = -ie\left[D^\nu\Phi, \Phi\right]. \tag{3.1.13b}$$

c.f the Euler-Lagrange equations (2.1.3), noting the dependence on gauge coupling constant e and i are result of scaling and generator conventions. A magnetic monopole as a topological soliton is defined as a smooth, finite energy solution to the equation of motion (3.1.13).

3.2 BPS Limit and BPS Solution

The BPS limit [37] is the $\lambda = 0$ limit of Yang-Mills-Higgs theory. Considering monopoles only, we set the electric field $E_i = 0$ [1] and we write the energy in $\lambda \to 0$ limit as

$$E = \frac{1}{4}\int \mathrm{Tr}B_i^2 + \mathrm{Tr}(D_\mu\Phi)^2 d^3x. \tag{3.2.1}$$

By completing the square similarly to completing the square of the instanton action to obtain

$$\begin{aligned} E &= \frac{1}{4}\int \mathrm{Tr}(B_i \mp D_i\Phi)^2 + \mathrm{Tr}(D_0\Phi)^2\, d^3x \pm \frac{1}{2}\int \mathrm{Tr}(B_i D_i\Phi)\, d^3x \\ &\geq \frac{1}{2}\int \mathrm{Tr}(B_i D_i \Phi)\, d^3x, \end{aligned} \tag{3.2.2}$$

[1]Similar analysis can be applied to Dyons[43] by setting $\mathbf{E} \neq 0$ and introducing a Dyon angle in the energy intergral.

Then the energy minima is achieved when the following equations are satisfied

$$B_i = \pm D_i\Phi, \qquad D_0\Phi = 0. \tag{3.2.3}$$

Considering static equations only, the second energy minima constraint is satisfied trivially and we have the Bogomolny equation remains

$$B_i = \pm D_i\Phi. \tag{3.2.4}$$

or alternatively written as [44]

$$\pm \frac{1}{2}\epsilon_{ijk}D_k\Phi = F_{ij}. \tag{3.2.5}$$

where the signs of solutions corresponds to monopole and anti monopole pairs similarly to instanton and anti instanton pairs. The minimum energy thus can be written in terms of the topological charge n as

$$E = \frac{1}{e}4\pi|n|\eta, \tag{3.2.6}$$

where the topological charge n in BPS limit takes the form

$$n = \frac{e}{8\pi\eta}\int \mathrm{Tr}(B_iD_i\Phi)\, d^3x. \tag{3.2.7}$$

In fact, solutions to the Bogomolny equation (3.2.5) are translational invariant solutions to self duality equation in pure Yang-Mills (2.1.12), where the invariance is under one component of A_μ and we can identify it to the Higgs field Φ [23]. We will here explicitly show the correspondence by assuming x_0 as the translational invariant coordinate, hence $\partial_0 = 0$. Self duality $F_{\mu\nu} = {}^\star F_{\mu\nu}$ implies

$$F_{10} = F_{23}, \qquad F_{20} = F_{31}, \qquad F_{30} = F_{12}. \tag{3.2.8}$$

Writing out in terms of gauge potential and taking $A_0 = \Phi$, we have

$$-\partial_1\Phi + ie\,[\Phi, A_1] = \partial_3A_2 - \partial_2A_3 + ie\,[A_2, A_3] = -B_1, \tag{3.2.9}$$

and similarly for other components. These are exactly the Bogomolny equations in the 3 indices. This observation is one of the key ingredients of the Nahm transform which will be discussed in more detail in Chapter 4.

Consider static solutions of the field equations. Static solutions are solutions $A_\mu(x)$ such that $\partial_0A_\mu(x) = 0$ for all $\mu \in \{0, 1, 2, 3\}$. Static fields obey

$$D_iD_i\Phi = -\lambda(|\Phi|^2 - \eta^2)\Phi, \tag{3.2.10a}$$
$$D_iF_{ij} = -ie\,[D_j\Phi, \Phi]\,, \tag{3.2.10b}$$

which follows directly from Euler Lagrange equation (3.1.13).

Seeking spherically symmetric solution as suggested by the hedgehog configuration leads to the ansatz

$$\Phi = h(r)\frac{x^a}{r}T^a, \tag{3.2.11a}$$
$$A_i = \epsilon_{iaj}\frac{x^j(1-k(r))}{r^2}T^a, \tag{3.2.11b}$$

where $h(r)$ and $k(r)$ are arbitrary functions of radius r. Substitution into equation of motion (3.2.10) to obtain ordinary differential equation for the arbitrary functions $h(r)$ and $k(r)$

$$\frac{d^2h}{dr^2} + \frac{2}{r}\frac{dh}{dr} = \frac{2k^2h}{r^2} - \lambda(\eta^2 - h^2)h, \tag{3.2.12a}$$

$$\frac{d^2k}{dr^2} = \frac{(k^2-1)k}{r^2} + e^2h^2k. \tag{3.2.12b}$$

The boundary conditions at $r = 0$ are $h(0) = 0$ and $k(0) = 1$ to avoid singularities. At $r \to \infty$, we require $h(\infty) = \eta$ and $k(\infty) = 1$ for finite energy.

First analytic solution to charge one monopole was found by Prasad *et al.* [45] with the simplification $\lambda = 0$ in (3.2.12) to arrive at

$$\frac{d^2h}{dr^2} + \frac{2}{r}\frac{dh}{dr} = \frac{2k^2h}{r^2}, \tag{3.2.13a}$$

$$\frac{d^2k}{dr^2} = \frac{(k^2-1)k}{r^2} + e^2h^2k. \tag{3.2.13b}$$

Then it can be solved by having

$$h(r) = \eta \coth(e\eta r) - \frac{1}{er}, \tag{3.2.14a}$$

$$k(r) = \frac{e\eta r}{\sinh(e\eta r)}. \tag{3.2.14b}$$

These solutions thus are called BPS monopoles where the topological charge is one.

Chapter 4

Nahm Transform

The Atiyah-Drienfeld-Hitchin-Manin-Nahm (ADHMN) construction or simply the Nahm transform[9, 8] is an extension of the ADHM method for construction of (anti) self-dual (BPS) monopoles. Nahm originally presented the relationship between solutions to the self-duality equation which are invariant under translation in one dimension with solutions that are invariant under translation in three dimensions. The generalisation of Nahm's construction is coined Nahm transform. It draws correspondence between solutions to (anti) self-dual equations which are invariant under translations in a subgroup of $\mathbb{R}^4$. *i.e.* solutions exist on generalised torus $\mathbb{T}^4$.

In this chapter, we will review the operational procedures of the Nahm transform on a four torus in Section 4.1 then review properties of Nahm transform in Section 4.2. We will then review the classic example of Nahm transform for BPS monopole in Section 4.3. Lastly, we will review the construction of doubly periodic instantons on $\mathbb{T}^2 \times \mathbb{R}^2$ using the Nahm transform in Section 4.4. More precisely, we will follow the presentation in [14] for the methodology and formula known in doubly periodic instanton in anticipation of applying them to our treatments of the Hitchin equations.

4.1 General Nahm Transform

Following presentation by Ford *et al.* [14], we start by considering subgroup Λ of translation in $\mathbb{R}^4$ which the solutions in invariant under, thus solutions will be defined on the quotient manifold $M = \mathbb{R}^4 \setminus \Lambda$. By defining Λ as four dimensional lattice with generators $(L_0, 0, 0, 0)$, $(0, L_1, 0, 0)$, $(0, 0, L_2, 0)$ and $(0, 0, 0, L_3)$, we have the solutions lie on the four torus $\mathbb{T}^4 = M = \mathbb{R}^4 \setminus \Lambda$ and L_μ are the four periods of $\mathbb{T}^4$. Consider a $SU(N)$ charge k field strength $F_{\mu\nu}$ with the corresponding gauge potential A_μ. $F_{\mu\nu}$ is defined on $\mathbb{T}^4$ when A_μ is periodic in all four coordinates of $\mathbb{T}^4$, modulo gauge transformation. *i.e.* $A(x_\mu) = A(x_\mu + L_\mu)$, where x_μ are the coordinates of base torus $\mathbb{T}^4$.

The fist step of the Nahm transform is to augment the $SU(N)$ gauge potential into $U(N)$ with the transformation $A_\mu \to A_\mu - iz_\mu$, where z_μ will become the coordinates of the dual torus $\hat{\mathbb{T}}^4$. Define the Weyl operator in $U(N)$ as

$$D_z(A) = \sigma_\mu D_z^\mu(A), \qquad D_z^\mu(A) = \frac{\partial}{\partial x^\mu} + A^\mu(x) - iz^\mu, \tag{4.1.1}$$

where $\sigma_\mu = \{1, i\tau_1, i\tau_2, i\tau_3\}$ and τ_i are the Pauli matrices. Existence of k orthonormal, normalisable (square integrable) zero modes of $D_z^\dagger(A)$, $\psi^i(x; z)$, $i = 1, 2 \ldots k$ is guaranteed by index theorems[46] and if we can explicit find the expressions, the Nahm potential is given by

$$\hat{A}^{ij}_{\mu} = \int_{\mathbb{T}^4} d^4x\, \psi^{i\dagger}(x;z)\frac{\partial}{\partial z^{\mu}}\psi^j(x;z), \tag{4.1.2}$$

where the Nahm potential corresponds to the an $U(k)$ instanton with topological charge N. The inverse Nahm transform recovers the original gauge potential which is given by

$$A^{pq}_{\mu} = \int_{\hat{\mathbb{T}}^4} d^4z\, \hat{\psi}^{p\dagger}(z;x)\frac{\partial}{\partial x^{\mu}}\hat{\psi}^q(z;x), \tag{4.1.3}$$

where $\hat{\psi}^p$ are the N normalisable zero modes of $D^{\dagger}_x(\hat{A}) = -\sigma^{\dagger}_{\mu}D^{\mu}_x(\hat{A})$, $D^{\mu}_x(\hat{A}) = \frac{\partial}{\partial z^{\mu}} + \hat{A}^{\mu}(z) - ix^{\mu}$. The field strength of original torus is given by

$$F^{ij}_{\mu\nu}(x) = \int_{\hat{\mathbb{T}}^4} d^4z \int_{\hat{\mathbb{T}}^4} d^4z' \hat{\psi}^{p\dagger}(z;x)\sigma\left(D^{\dagger}_x D_x\right)^{-1}(z,z';x)\sigma^{\dagger}\nu\hat{\psi}^q(z';x) - [\nu \leftrightarrow \mu]. \tag{4.1.4}$$

We can see that Nahm transform is most naturally defined on the 4-torus $\mathbb{T}^4$ as the manifold is compactified in all directions, thus the self dual property of gauge potentials are preserved during transformations, without the requirement $F_{\mu\nu} \to 0$ as $x_{\mu} \to \infty$ [24].

4.2 Properties of Nahm Transform

We will now discuss some of the uses of Nahm transform focusing on its behaviour on translation invariant solutions over $\mathbb{R}^4$ following the review by Jardim [47]. Let Λ denote a subgroup of translation group over $\mathbb{R}^4$, we define the dual translation group

$$\Lambda^* = \left\{\alpha \in (\mathbb{R}^4)^* | \, \alpha(\lambda) \in \mathbb{Z} \;\; \forall \lambda \in \Lambda\right\}. \tag{4.2.1}$$

Nahm transform thus draws correspondence between solutions over $M = \mathbb{R}^4 \setminus \Lambda$ and $\hat{M} = (\mathbb{R}^4)^* \setminus \Lambda^*$. Concrete examples of Λ and their corresponding soliton class are listed below:

- $\Lambda = \{0\}$, $M = \mathbb{R}^4$ gives instanton solutions in $\mathbb{R}^4$. This limit of the Nahm transform effectively becomes the ADHM construction. Thus we say that the Nahm transform encompasses the ADHM construction.
- $\Lambda = \mathbb{Z}$, $M = \mathbb{R}^3 \times \mathbb{S}$ gives rise to calorons which are periodic instantons in one dimension.
- $\Lambda = \mathbb{Z}^2$, $M = \mathbb{R}^2 \times \mathbb{T}^2$ gives us doubly periodic instantons which unsurprisingly, it is a class of instanton solutions that are periodic in two dimensions.
- $\Lambda = \mathbb{R}$, $M = \mathbb{R}^3$ gives us a monopole as the Nahm transform maps between the solutions to Nahm's equation (4.3.4) and spherical symmetric monopole in $\mathbb{R}^3$ (Most notably, the BPS monopole).
- $\Lambda = \mathbb{Z} \times \mathbb{R}$, $M = \mathbb{R}^2 \times \mathbb{S}$ gives rise to periodic monopoles.
- $\Lambda = \mathbb{Z}^2 \times \mathbb{R}$, $M = \mathbb{R} \times \mathbb{T}^2$ gives rise to doubly periodic monopoles.

We can see that the Nahm transform draws correspondence between solutions to self duality equations in different manifolds. More precisely, Nahm transforms from a $4-$ torus $\mathbb{T}^4$ have the effect of inverting the dimensional radii which is the direct consequence of the definition of dual translation group Λ^*. *i.e.* let (L_0, L_1, L_2, L_3) be the period of base torus

$\mathbb{T}^4$, through the Nahm transform the periods of dual torus $\hat{\mathbb{T}}^4$ are $({}^1\!/\!{}_{L_0}, {}^1\!/\!{}_{L_1}, {}^1\!/\!{}_{L_2}, {}^1\!/\!{}_{L_3})$. In the limits of non-compatified coordinates ($L_i \to \infty$) or "frozen" coordinates ($L_i \to 0$), the property of inverting periods extends to periods taking values of 0 and ∞. For example,

$$\mathbb{T}^2 \times \mathbb{R}^2 \underset{NT}{\longrightarrow} \hat{\mathbb{T}}^2, \qquad \mathbb{R}^3 \times \mathbb{S} \underset{NT}{\longrightarrow} \hat{\mathbb{S}}.$$

In the interesting case where $\Lambda = \mathbb{Z}^4$, the solutions are defined on the torus $\mathbb{T}^4$ while the transformed solutions exist in the dual torus $\hat{\mathbb{T}}^4$. Thus the $\Lambda = \mathbb{Z}^4$ case does not benefit from the dimensional reduction seen in the previous examples. Yet a remarkable theorem can be derived in this formulation

Theorem 1. *The Nahm transformation transforms between $SU(n)$ gauge potential A of charge k over manifold $\mathbb{R}^4 \setminus \Lambda$ and $SU(k)$ gauge potential $\hat{A}$ of charge n over manifold $(\mathbb{R}^4)^* \setminus \Lambda^*$.*

This mapping between gauge potential A in base manifold M and gauge potential $\hat{A}$ in dual manifold $\hat{M}$ suggests in cases that Nahm data in the dual manifold are known explicitly, we can use the Nahm transform to construct the solution in the base manifold. Since the Nahm potential $\hat{A}$ is part of the Nahm data, one would use the Nahm transform method in case the dual manifold $\hat{M}$ is of lower dimension than M as solving for $\hat{A}$ is more feasible than solving for A. The other interesting case would be for unity charge solutions in the base manifold, then by theorem above, the Nahm potential $\hat{A}$ would be abelian.

4.3 Nahm Transform for BPS Monopole

We will present the construction of $SU(2)$ BPS monopole via the Nahm transform following the review by Weinberg *et al.* [38]. The Nahm transform draws correspondence between charged k BPS monopoles with Nahm data T_μ, $\mu = 0, 1, 2, 3$ of $k \times k$ Hermitian matrices which satisfies the Nahm equation

$$0 = \frac{dT_i}{ds} + i\,[T_0, T_i] + \frac{i}{2}\epsilon_{ijk}\,[T_j, T_k]\,, \tag{4.3.1}$$

where the indices $i, j, k = 1, 2, 3$, the auxiliary variable s is the coordinate of the dual manifold $\hat{M}$. Through holonomy arguments and appropriate index theorems, $\hat{M}$ can be taken as a real interval $I = [-\frac{\eta}{2}, \frac{\eta}{2}] \subset \mathbb{R}$ and η is the vacuum Higgs expectation value cf. Lagrangian for Yang-Mills-Higgs theory (3.1.1). To recover the base gauge potential and Higgs field, we need to find the zero modes $\psi(s, \mathbf{r})$ of the Weyl operator

$$D^\dagger(s) = -\frac{d}{ds} - iT_0 \otimes \mathbb{1}_2 - T_i \otimes \sigma_i + r_i \mathbb{1}_k \otimes \sigma_i, \tag{4.3.2}$$

where $\mathbb{1}_k$ is the $k \times k$ identity matrix. It can be shown that for $SU(2)$, only two normalisable, linear independent solutions can be found and we denote them as ψ^i, ψ^j. The gauge potential and Higgs field then can be determined as

$$\Phi^{ij} = \int_{-\eta/2}^{\eta/2} \psi^{i\dagger}(s, \mathbf{r}) s \psi^j(s, \mathbf{r}) ds, \tag{4.3.3a}$$

$$A_\mu^{ij} = -i \int_{-\eta/2}^{\eta/2} \psi^{i\dagger}(s, \mathbf{r}) \partial_\mu \psi^j(s, \mathbf{r}) ds. \tag{4.3.3b}$$

This might appear to contradict previously quoted formula of the Nahm transform but it is the effect of scaling only. We can rescale our monopole Lagrangian (3.1.1) such that the gauge coupling is outside the definition of covariant derivative and field strength and through the uses of skew-Hermitian generators $\frac{\tau_a}{2i}$ to recover the expected transformation (4.1.3). In this convention, one would also need to modify the Nahm equation and Weyl operator accordingly. For the treatment of the construction in the this convention, please refer to [22].

It can be shown that due to gauge ambiguity on both manifolds, there is no bijective correspondence between Nahm data T_μ and solutions to the Bogomolny equation (3.2.4). *i.e.* gauge transformation (space time gauge transformation) of T_μ leaves A_μ invariant and $SU(2)$ gauge transformation of A_μ leaves T_μ invariant[38]. Thus we are free to set $T_0 = 0$ via an appropriate gauge transform and (4.3.1) simplify to

$$\frac{dT_i}{ds} = \frac{i}{2}\epsilon_{ijk}\left[T_j, T_k\right], \tag{4.3.4}$$

where indices $i, j, k = 1, 2, 3$ as before. In fact the Nahm equation is closely related to generalised Euler-Manakov Top equations which is an well known non-linear integrable system [48]. (4.3.4) is completely integrable in the sense that a Lax pair exists. The Weyl zero mode equation hence becomes

$$0 = \left[-\frac{d}{ds} - T_i \otimes \sigma_i + r_i \mathbb{1}_k \otimes \sigma_i\right]\psi(s, \mathbf{r}). \tag{4.3.5}$$

The equation appears to be formidable in its current form yet for the case of $k = 1$, it is rather simple and recovers the BPS monopole solution (3.2.14). We will outline its simple construction as follows. In the case $k = 1$, the Nahm Data T_i are just 1×1 matrices with trivial commutator relation $[T_i, T_j] = 0$, we will take $T_i = ic_i$ for some constant c_i. By translation invariance of T_i we can take $T_i(s) = 0$. The zero modes of Weyl operator satisfies

$$\left[\frac{d}{ds} - \mathbf{r}\cdot\boldsymbol{\sigma}\right]\psi(s, \mathbf{r}) = 0. \tag{4.3.6}$$

The two zero modes $\psi(s, \mathbf{r})$ can be solved for trivially by integration as there are no dependence on s to have

$$\psi(s, \mathbf{r}) = e^{s\mathbf{r}\cdot\boldsymbol{\sigma}} A(r), \tag{4.3.7}$$

where $A(r)$ is the r dependent normalisation factor. By choosing trivial orthonormal two vectors we write

$$\psi^i(s, \mathbf{r}) = \sqrt{\frac{r}{\sinh \eta r}} e^{s\mathbf{r}\cdot\boldsymbol{\sigma}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi^j(s, \mathbf{r}) = \sqrt{\frac{r}{\sinh \eta r}} e^{s\mathbf{r}\cdot\boldsymbol{\sigma}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{4.3.8}$$

which satisfy orthonormality conditions. Apply the identity $e^{s\mathbf{r}\cdot\boldsymbol{\sigma}} = \cosh sr + \sinh(sr)\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}$ to have the Higgs field as

$$\Phi^{ij} = \frac{1}{2}\left(\eta \coth \eta r - \frac{1}{r}\right)(\hat{\mathbf{r}}\cdot\boldsymbol{\sigma})^{ij}, \tag{4.3.9}$$

and through similar calculation to arrive at gauge potential A^{ij}. This recovers the BPS charge one monopole in (3.2.14).

4.4 Doubly Periodic Instantons

Doubly periodic instantons as the name suggests are instantons solutions periodic in two coordinates where the base manifold is taken as $\mathbb{T}^2 \times \mathbb{R}^2$ and the dual manifold is $\hat{\mathbb{T}}^2$.

We will follow closely in the construction in [14]. Considering said base manifold as deformation of four torus where Nahm transform is most naturally defined on, we will take two of the periods of $\mathbb{T}^4$ to be infinite. Without loss of generality, the periods L_0 and L_3 are taken to be infinite. Keeping the same convention as Chapter 4 where x^μ and z^μ are coordinates of base and dual manifolds respectively, we have the modified Nahm transform

$$\hat{A}_\mu^{ij}(z) = \int_{\mathbb{T}^2\times\mathbb{R}^2} d^4x\, \psi^{i^\dagger}(x;z)\frac{\partial}{\partial z^\mu}\psi^j(x;z), \quad \mu = 1,2, \tag{4.4.1a}$$

$$\hat{A}_\mu^{ij}(z) = \int_{\mathbb{T}^2\times\mathbb{R}^2} d^4x\, \psi^{i^\dagger}(x;z) i x_\mu \psi^j(x;z), \quad \mu = 0,3, \tag{4.4.1b}$$

where $\psi^i(x;z)$, $i = 1,2\ldots k$ are again zero modes of Weyl operator $D_z^\dagger(A)$. The inverse Nahm transform takes a similar form

$$A_\mu^{pq}(x) = \int_{\hat{\mathbb{T}}^2} d^2z\, \psi^{p\dagger}(z;x)\frac{\partial}{\partial x^\mu}\psi^q(z;x), \quad \mu = 0,1,2,3 \tag{4.4.2a}$$

where $\psi^p(z;x), r = 1\ldots N$ are the zero modes of Weyl operator $D_x^\dagger(\hat{A})$.

We can reduce the dimension of the self-duality equation to the Hitchin equations, then we have the following on $\hat{\mathbb{T}}^2$:

$$\left[D_y, \hat{\Phi}\right] = 0, \tag{4.4.3a}$$

$$\hat{F}_{y\bar{y}} = \left[\hat{\Phi}, \hat{\Phi}^\dagger\right] = \left[D_y, D_{\bar{y}}\right], \tag{4.4.3b}$$

where we introduce complex coordinates $y = z_1 + iz_2$ and covariant derivative $D_{\bar{y}}\hat{\Phi} = \partial_{\bar{y}} + \left[\hat{A}_{\bar{y}}, \hat{\Phi}\right]$. Here we defined complex Nahm potential $\hat{A}_y = \frac{1}{2}(\hat{A}_1 - i\hat{A}_2)$ and formed a complex "Higgs" field out of the remaining Nahm potential components $\hat{\Phi}(z) = \frac{1}{2}(\hat{A}_0 - i\hat{A}_3)$ corresponding to "frozen" coordinates z_0 and z_3.

We will also introduce a pair of complex coordinates in base manifold $\mathbb{T}^2 \times \mathbb{R}^2$ using x^μ such that

$$x_\perp = x_0 + ix_3, \quad x_\parallel = x_1 + ix_2, \tag{4.4.4}$$

where $x_\perp$ lies in the non-compact plane $\mathbb{R}^2$ whilst $x_\parallel$ lies in the plane of $\mathbb{T}^2$, together them form the complete coordinates for $\mathbb{R}^2 \times \mathbb{T}^2$. Thus, on the base manifold we can define complex gauge potentials $A_{x_\parallel} = \frac{1}{2}(A_1 - iA_2)$, $A_{x_\perp} = \frac{1}{2}(A_0 - iA_3)$ and their respective complex conjugates.

We start by recalling the problem of finding a charge one $SU(N)$ gauge potential can be mapped to finding a charge N $U(1)$ gauge potential on the dual manifold by Theorem 1. Abelian problems are generally more tractable than non-abelian problems and to recover the original solution we simply apply the inverse Nahm transform. For simplicity, we take the trivial solution $\hat{F}_{\mu\nu} = 0$ where $\hat{\Phi} = 0$ which corresponds the radially symmetric $\hat{A}$. By the Hitchin equations, we can take the Nahm potential $\hat{A}_y$ to have N simple poles located at ω_i, $i = 1\ldots N$. The ansatz is given be

$$\hat{A}_y = \partial_y\phi, \quad \hat{A}_{\bar{y}} = -\partial_{\bar{y}}\phi, \tag{4.4.5}$$

for some complex potential ϕ. This choice of $\hat{A}$ gives the dual field strength

$$\hat{F}_{y\bar{y}} = -2\partial_y\partial_{\bar{y}}\phi = 0. \tag{4.4.6}$$

As ϕ is the solution to Laplace's equation in $\mathbb{C}$, it is then harmonic with except at the N simple poles. We consider ϕ satisfying

$$-\nabla_z^2\phi = -2\pi\sum_{i=1}^{N}\kappa_i\delta^2(z-\omega_i), \tag{4.4.7}$$

where ω_i are the position of the poles and κ_i are N constants. Physically, we can consider this system as N Aharonov-Bohm fluxes threading through $\hat{\mathbb{T}}^2$ where the strength of fluxes are given by κ_i. In the case of doubly periodic instantons, we have the addition constraint that

$$\sum_{i=1}^{N}\kappa_i = 0, \tag{4.4.8}$$

to ensure periodicity of $\hat{A}$. We enforce $0 < \kappa_i < 1$ as it is possible to apply a gauge transformation to such that the condition holds. Unlike the case of periodic monopoles, the inverse Nahm transform does not completely remove the singularity of the fluxes in $\hat{A}_y$ and $\hat{A}_{\bar{y}}$. This means we need to align the poles of Higgs field Φ [49] in dual manifolds with the Nahm potential $\hat{A}_y$ such that only N poles exist, and a $SU(N)$ instanton is produced. In the simpler case of $SU(2)$, we enforce the poles of Higgs field and Nahm potential are proportional

$$\kappa\hat{\Phi} = \alpha\partial_{\bar{y}}\phi, \tag{4.4.9}$$

where α is a complex constant. Substitute these components into the Weyl operator to arrive at the Weyl operator

$$-\frac{i}{2}D_x^\dagger = \begin{pmatrix} \frac{1}{2}\bar{x}_\perp + i\alpha\kappa^{-1}\partial_{\bar{y}}\phi & \partial_y + \partial_y\phi - \frac{i}{2}\bar{x}_\parallel \\ \partial_{\bar{y}} - \partial_{\bar{y}}\phi - \frac{i}{2}x_\parallel & \frac{1}{2}x_\perp - i\hat{\alpha}\kappa^{-1}\partial_{\bar{y}}\phi \end{pmatrix}, \tag{4.4.10}$$

where we need to find 2 of its zero modes to construct a $SU(2)$ instanton corresponding to our Nahm potential (4.4.5). The $SU(2)$ gauge potential ansatz can be given by

$$A_{x_\perp} = -\frac{\tau_3}{2}\partial x_\perp \log\rho + 2\pi i(\tau_1 - i\tau_2)\kappa\rho\partial_{\bar{x}_\parallel}\frac{\nu^*}{\rho}, \tag{4.4.11}$$

and similarly for $A_{x_\parallel}$. In the case of radially symmetric solutions, $\alpha = 0$, self duality equation (2.1.12) can be written as

$$(\partial_{x_\parallel}\partial_{\bar{x}_\parallel} + \partial_{x_\perp}\partial_{\bar{x}_\perp})\log\rho + (4\pi\kappa\rho)^2\left(\partial_{x_\parallel}\frac{\nu}{\rho}\partial_{\bar{x}_\parallel}\frac{\nu^*}{\rho} + \partial_{x_\perp}\frac{\nu}{\rho}\partial_{\bar{x}_\perp}\frac{\nu^*}{\rho}\right) = 0, \tag{4.4.12a}$$

$$\partial_{x_\perp}\left(\rho^2\partial_{\bar{x}_\perp}\frac{\nu^*}{\rho}\right) + \partial_{x_\parallel}\left(\rho^2\partial_{\bar{x}_\parallel}\frac{\nu^*}{\rho}\right) = 0, \tag{4.4.12b}$$

where $\rho(|x_\perp|)$ is a real function of $|x_\perp|$ and $\nu(x_\parallel, |x_\perp|)$ is a complex function such that $x_\parallel$ contribute in phase only. We will write $\nu(x_\parallel, |x_\perp|) = \tilde{\nu}(x_\parallel)V(|x_\perp|) = e^{2i\omega\cdot x}V(|x_\perp|)$, where $V(|x_\perp|)$ is a real function or $|x_\perp|$. Considering in the plane defined by $x_\perp$ only, the functions V and ρ are radially symmetric. The full $\mathbb{T}^2\times\mathbb{R}^2$ gauge potential can be given in the form by Van Baal [50]:

$$A_{x_\perp} = -\frac{1}{2}\begin{pmatrix} \frac{\partial_{x_\perp}\rho}{\rho} & 0 \\ -8\pi i\kappa\rho\partial_{\bar{x}_\parallel}\frac{\nu^*}{\rho} & -\frac{\partial_{x_\perp}\rho}{\rho} \end{pmatrix}, \qquad A_{\bar{x}_\perp} = \frac{1}{2}\begin{pmatrix} \frac{\partial_{\bar{x}_\perp}\rho}{\rho} & 8\pi i\kappa\rho\partial_{x_\parallel}\frac{\nu}{\rho} \\ 0 & -\frac{\partial_{\bar{x}_\perp}\rho}{\rho} \end{pmatrix}, \tag{4.4.13}$$

and field strength as

$$F_{x_\perp\bar{x}_\perp} = \begin{pmatrix} \partial_{x_\perp}\partial_{\bar{x}_\perp}\log\rho + (4\pi\kappa\rho)^2\partial_{x_\parallel}\frac{\nu}{\rho}\partial_{\bar{x}_\parallel}\frac{\nu^*}{\rho} & -4\pi i\kappa\rho\partial_{x_\parallel}\partial_{x_\perp}\frac{\nu}{\rho} \\ 4\pi i\kappa\rho\partial_{\bar{x}_\parallel}\partial_{\bar{x}_\perp}\frac{\nu^*}{\rho} & -\partial_{x_\perp}\partial_{\bar{x}_\perp}\log\rho - (4\pi\kappa\rho)^2\partial_{x_\parallel}\frac{\nu}{\rho}\partial_{\bar{x}_\parallel}\frac{\nu^*}{\rho} \end{pmatrix}. \tag{4.4.14}$$

Chapter 5

Solutions to the Hitchin Equations

In this chapter, we will be considering solutions to the Hitchin equations using the Nahm transform following the same convention as Section 4.4. Solutions to the Hitchin equations can be considered as large $r = |x_\perp|$ limit of the doubly periodic instantons. This limit can be achieved equivalently by considering the limit of $L_1, L_2 \to 0$ where L_1, L_2 are periods of x_1, x_2 respectively. In this limit, the dependency of $x_\parallel$ only constitutes to a phase, $A_{x_\parallel}$ "becomes" the Higgs field and condition (4.4.8) is dropped. For this limit to exists, we need to perform gauge transformations such that $x_\parallel$ dependence vanishes and only constant constitution of the Higgs field remains. This formulation also can be reached starting from doubly periodic instantons by applying the $\mathbb{R}^2$ variant of the Nahm transformation where $x_\perp$ and $\bar{x}_\perp$ are the complex conjugate coordinates over the plane.

In the Hitchin case, we have the added interests that it is closely related to two other unsolved problems, the solutions to periodic monopoles and the Aharonov Bohm (AB) effect [19]. As we will see, the Weyl zero mode equation in constructing solutions to the Hitchin equations can be transformed via elementary functions to the Weyl zero mode equation of the periodic monopole. The Nahm datum of $SU(n)$ solutions to the Hitchin equations have n simple poles which correspond to exactly n perpendicular infinite solenoids in the dual manifold. Thus we can study the phase changes of charged particles moving in the dual manifold under quantum mechanical influences of these solenoids which is the AB effect.

We note that in the Hitchin case, the base manifold M and dual manifold $\hat{M}$ are both $\mathbb{R}^2$ which means we do not benefit from the dimensional reduction properties of the Nahm transform. As it is shown in [51], solutions to the Hitchin equations do not have finite energy, thus cannot be classified as topological solitons. However in the $\mathbb{R}^2 \to \mathbb{R}^2$ setting, the Nahm transformation transforms between $SU(n)$ gauge potential with k simple poles and $SU(k)$ gauge potential with n simple poles [16]. Thus to construct $SU(n)$ solutions with one simple pole, we take the Nahm potentials as abelian solutions with n simple poles on the dual manifold. This is exactly the same Nahm potential for the doubly instantons in Section 4.4.

Considering only in the $x_\perp$ plane, we drop the $\perp$ subscript and continue with complex variable x. Start with the Weyl operator

$$-\frac{i}{2}D_x^\dagger = \begin{pmatrix} \frac{1}{2}\bar{x} + i\alpha\kappa^{-1}\partial_{\bar{y}}\phi & \partial_y + \partial_y\phi \\ \partial_{\bar{y}} - \partial_{\bar{y}}\phi & \frac{1}{2}x - i\hat{\alpha}\kappa^{-1}\partial_{\bar{y}}\phi \end{pmatrix}, \tag{5.0.1}$$

and recall the Nahm data in dual manifold of charge one doubly periodic instantons. For clarity of notation we denote the base manifold as $\mathbb{R}^2$ and Nahm transformed (dual) manifold as $\hat{\mathbb{R}}^2$. In the complex plane setting, it is useful to explicitly write down the Nahm and inverse Nahm transform over $\mathbb{R}^2$ in terms of complex variables. Let $\hat{\psi}^p$ and ψ^i be zero modes of Weyl operators $D_x^\dagger(\hat{A})$ and $D_z^\dagger(A)$ respectively, we have

$$\hat{A}_y^{ij}(y) = \int_{\mathbb{R}^2} d^2x\, \psi^{i\dagger}(x;y)\frac{\partial}{\partial y}\psi^j(x;y), \tag{5.0.2a}$$

$$A_x^{pq}(x) = \int_{\hat{\mathbb{R}}^2} d^2y\, \hat{\psi}^{p\dagger}(y;x)\frac{\partial}{\partial x}\hat{\psi}^q(y;x), \tag{5.0.2b}$$

Recall that the Higgs field on the dual manifold $\hat{\mathbb{R}}^2$, $\hat{\Phi}$ is defined by $\hat{\Phi} = \frac{1}{2}\left(\hat{A}_0 - i\hat{A}_3\right)$. We can equally define a complex "Higgs"[1] in the base manifold $\mathbb{R}^2$ using the "frozen" coordinates x_1 and x_2 as $\Phi = \frac{1}{2}(A_1 - iA_2)$ which can be obtained through (4.4.1b) to have

$$\hat{\Phi}_y^{ij}(y) = \int_{\mathbb{R}^2} d^2x\, \psi^{i\dagger}(x;y) i\bar{x}\psi^j(x;y), \tag{5.0.3a}$$

$$\Phi_x^{pq}(x) = \int_{\hat{\mathbb{R}}^2} d^2y\, \hat{\psi}^{p\dagger}(y;x) i\bar{y}\hat{\psi}^q(y;x), \tag{5.0.3b}$$

where we call Φ and $\hat{\Phi}$ as base Higgs field and dual/Nahm Higgs field respectively for clarity.

This remainder of this chapter will be organised as follows, Section 5.1 and Section 5.2 will consider the cases of $U(1)$ the $SU(2)$ solution to the Hitchin equations under zero Nahm Higgs assumption respectively; Section 5.4 and Section 5.5 will consider case of non zero Nahm Higgs field and draw correspondence to the periodic monopoles.

5.1 The U(1) Solution

The simplest Nahm data for the $U(1)$ solution is a single flux $\phi = -\frac{1}{2}\kappa\log(y\bar{y})$ at origin of $\mathbb{R}^2$ with flux strength κ. Then the Nahm potentials are given by (4.4.5) as

$$\hat{A}_y = -\frac{\kappa}{2y}, \qquad \hat{A}_{\bar{y}} = \frac{\kappa}{2\bar{y}}. \tag{5.1.1}$$

The Weyl operator becomes

$$-\frac{i}{2}D_x^\dagger = \begin{pmatrix} \frac{1}{2}\bar{x} - i\alpha\kappa^{-1}\frac{\kappa}{2\bar{y}} & \partial_y - \frac{\kappa}{2y} \\ \partial_{\bar{y}} + \frac{\kappa}{2\bar{y}} & \frac{1}{2}x + i\hat{\alpha}\kappa^{-1}\frac{\kappa}{2y} \end{pmatrix}. \tag{5.1.2}$$

In the case of zero Nahm Higgs field, $\alpha = 0$ and we restore radial symmetry broken by said Higgs field. The Weyl operator thus becomes

$$-\frac{i}{2}D_x^\dagger = \begin{pmatrix} \frac{1}{2}\bar{x} & \partial_y - \frac{\kappa}{2y} \\ \partial_{\bar{y}} + \frac{\kappa}{2\bar{y}} & \frac{1}{2}x \end{pmatrix}. \tag{5.1.3}$$

Then the normalised zero mode ψ of above Weyl operator is, via Section A.1

$$\psi = \frac{|x|\sqrt{\sin(\pi\kappa)}}{\pi}\begin{pmatrix} K_\kappa(r|x|) \\ \frac{|x|}{x}e^{i\theta}K_{1-\kappa}(r|x|) \end{pmatrix}, \tag{5.1.4}$$

[1] Φ is in fact the $A_\parallel$ gauge potential when considering doubly periodic instantons.

where $y = re^{i\theta}$. Only one zero mode exists and it is unique up to a constant phase $e^{i\gamma}$. Note that the above formula recovers the known special case $\kappa = \frac{1}{2}$ [14], with a difference of constant phase where

$$K_{\frac{1}{2}}(z) = K_{-\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}, \tag{5.1.5a}$$

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\bigg|_{\kappa=\frac{1}{2}} = \frac{e^{-r|x|}}{\sqrt{2\pi|x|}}\begin{pmatrix} r^{-\frac{1}{2}}x \\ e^{i\theta}r^{-\frac{1}{2}}|x|\end{pmatrix}. \tag{5.1.5b}$$

To reconstruct the gauge potential in the base manifold $\mathbb{R}^2$, we use the inverse Nahm transform (5.0.2) in polar coordinates where $x = se^{i\vartheta}$ to have

$$\psi^\dagger(r,\theta;s,\vartheta)\frac{\partial}{\partial s}\psi(r,\theta;s,\vartheta) = \frac{s\sin\pi\kappa}{\pi^2}\big((1-\kappa)K_\kappa(rs)^2 + \kappa K_{1-\kappa}(rs)^2 - 2rsK_{1-\kappa}(rs)K_\kappa(rs)\big). \tag{5.1.6}$$

Direct calculation yields that

$$A_s = \int_{\hat{\mathbb{R}}^2}\psi^\dagger(r,\theta;s,\vartheta)\frac{\partial}{\partial s}\psi(r,\theta;s,\vartheta)\, r\, drd\theta = 0. \tag{5.1.7}$$

Consider the gauge potential component A_ϑ, we fix the gauge by fixing the phase in the zero mode solution to (5.1.4). Then,

$$\psi^\dagger(r,\theta;s,\vartheta)\frac{\partial}{\partial\vartheta}\psi(r,\theta;s,\vartheta) = -i\frac{\sin\pi\kappa}{\pi^2}s^2K_{1-\kappa}(rs)^2. \tag{5.1.8}$$

Direct calculations thus yields

$$A_\vartheta = \int_{\hat{\mathbb{R}}^2}\psi^\dagger(r,\theta;s,\vartheta)\frac{\partial}{\partial\vartheta}\psi(r,\theta;s,\vartheta)\, r\, drd\theta = -i(1-\kappa). \tag{5.1.9}$$

Translating to A_x and $A_{\bar{x}}$, gives gauge potential in complex coordinates as

$$A_x = -\frac{1-k}{2x}, \qquad A_{\bar{x}} = \frac{1-k}{2\bar{x}}. \tag{5.1.10}$$

This is simply a gauge transform of the expected gauge potential

$$A_x = \frac{\kappa}{2x}, \qquad A_{\bar{x}} = -\frac{\kappa}{2\bar{x}}, \tag{5.1.11}$$

which can be obtained by the same calculation with the zero mode $e^{i\vartheta}\psi$ or applying the $U(1)$ gauge transformation $g = e^{i\vartheta}$. Thus the arbitrariness of phase in (5.1.4) corresponds exactly to the $U(1)$ gauge freedom of the gauge potential.

Comparing to (5.1.1), we can see that the Nahm transformed gauge potential retain the form of Nahm potential. The introduction of minus sign in the gauge potential can be interpreted as the reversal of the flux strength $\kappa \to -\kappa$. It is worth noting that starting with Nahm potential $\hat{A}_y = -\frac{\kappa}{2(y-\omega)}$ where $\omega \in \mathbb{C}$ is an arbitrary constant, the resulting gauge potential is the same as (5.1.11). To obtain same correspondence of Nahm and gauge potential of arbitrarily positioned flux we require the modified Weyl operator with flux below

$$-\frac{i}{2}D_x^\dagger = \begin{pmatrix}\frac{1}{2}(\bar{x}-\bar{\omega}) & \partial_y - \frac{\kappa}{2y}\\ \partial_{\bar{y}} + \frac{\kappa}{2\bar{y}} & \frac{1}{2}(x-\omega)\end{pmatrix}, \qquad \phi = -\frac{1}{2}\kappa\log((y-\omega)(\bar{y}-\bar{\omega})), \tag{5.1.12}$$

where the constant ω corresponds to the flux location.

5.2 The SU(2) Solution

We will now focus on the case of $SU(2)$ solution to the Hitchin equations. The Nahm data for $SU(2)$ solutions requires two simple poles. This can be achieved by two fluxes; one of strength $-\kappa$ located at $y = \omega$ and the other of strength κ located at $y = -\omega$, where $\omega \in \mathbb{C}$. Then Nahm data ϕ satisfies

$$- \nabla_z^2 \phi = -2\pi\kappa(\delta^2(\omega + z) - \delta^2(z - \omega)), \tag{5.2.1}$$

where the solution can be found as

$$\phi = \frac{1}{2}\kappa \log(|y - \omega|^2) - \frac{1}{2}\kappa \log(|y + \omega|^2). \tag{5.2.2}$$

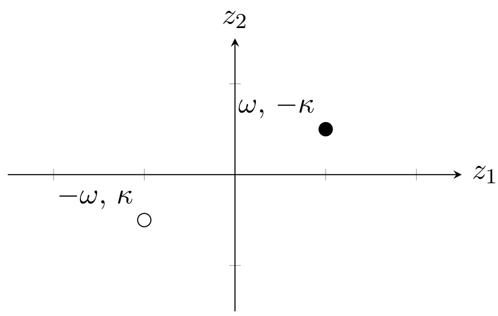

Figure 5.1: Positions $\pm\omega$ and flux strength $\mp\kappa$ of solenoids on the dual torus

There are in fact no analytic solution known to the zero modes of the Weyl operator with ϕ defined as above. Hence we consider the asymptotics of the Weyl operator $D_x^\dagger(\hat{A})$ as $|y|^2 \to \infty$ and approximate the zero modes with respective single solenoid zero modes. The zero mode of Weyl operator with a single positive flux $y = -\omega$ is $\psi^1 = \psi_\kappa(y + \omega; x)$ where

$$\psi_\kappa(y; x) = \frac{|x|\sqrt{\sin(\pi\kappa)}}{\pi} \begin{pmatrix} K_\kappa(\sqrt{y\bar{y}}|x|) \\ \frac{|x|}{x}\sqrt{\frac{y}{\bar{y}}} K_{1-\kappa}(\sqrt{y\bar{y}}|x|) \end{pmatrix}. \tag{5.2.3}$$

In the case of $\phi = \frac{1}{2}\kappa \log(y\bar{y})$, flux strength $-\kappa$, we have

$$-\frac{i}{2}D_x^\dagger = \begin{pmatrix} \frac{1}{2}\bar{x} & \partial_y + \frac{\kappa}{2y} \\ \partial_{\bar{y}} - \frac{\kappa}{2\bar{y}} & \frac{1}{2}x \end{pmatrix}, \tag{5.2.4}$$

and via similar calculation to Section A.1 we have the normalised solution is

$$\psi_{-\kappa}(y; x) = \frac{|x|\sqrt{\sin(\pi\kappa)}}{\pi} \begin{pmatrix} \frac{|x|}{\bar{x}}\sqrt{\frac{\bar{y}}{y}} K_{1-\kappa}(\sqrt{y\bar{y}}|x|) \\ K_\kappa(\sqrt{y\bar{y}}|x|) \end{pmatrix}, \tag{5.2.5}$$

which also extends to solution corresponding to a single flux at $y = \omega$ with $\psi^2 = \psi_{-\kappa}(y - \omega; x)$. Hence in large $|y|^2$ and $|x|^2$ limit, Weyl zero modes are approximated as ψ^1 and ψ^2. By introducing the set of coordinates definition

$$y_1 = y + \omega_1 + i\omega_2 = r_1 e^{i\theta_1}, \quad y_2 = y - \omega_1 - i\omega_2 = r_2 e^{i\theta_2}, \quad \omega = \omega_1 + i\omega_2, \quad \omega_1, \omega_2 \in \mathbb{R}, \tag{5.2.6}$$

we can check that orthonormality condition is satisfied via

$$\begin{aligned} \int_{\hat{\mathbb{R}}^2} \psi_1^\dagger \psi_2 \, dy d\bar{y} = \int_{\hat{\mathbb{R}}^2} \frac{|x|^3 \sin(\pi\kappa)}{\bar{x}\pi^2} \big(& e^{-i\theta_2} K_\kappa(r_1|x|) K_{1-\kappa}(r_2|x|) \\ & + e^{-i\theta_1} K_\kappa(r_2|x|) K_{1-\kappa}(r_1|x|) \big) \, dy d\bar{y} = 0, \end{aligned} \tag{5.2.7}$$

by the fact that the integrand is odd under the exchange of $y \to -y$.

Explicit value of off diagonal elements of A_x and $A_{\bar{x}}$ can be found as via calculations in Section A.2 as

$$A_x^{21} = \omega(\kappa - 1)e^{-2i\vartheta - 2s|\omega|}\sqrt{\frac{\pi}{s|\omega|}}\frac{\sin(\pi\kappa)}{\pi}, \tag{5.2.8a}$$

$$A_{\bar{x}}^{21} = \omega\kappa e^{-2s|\omega|}\sqrt{\frac{\pi}{s|\omega|}}\frac{\sin(\pi\kappa)}{\pi}. \tag{5.2.8b}$$

The opposite off diagonal elements A^{12} takes similar value where the full $\mathbb{R}^2$ gauge potential approximation for large s are

$$A_x = \begin{pmatrix} -\frac{1-\kappa}{2x} & \omega\kappa e^{-2s|\omega|}\sqrt{\frac{\pi}{s|\omega|}}\frac{\sin(\pi\kappa)}{\pi} \\ \omega(\kappa-1)e^{-2i\vartheta-2s|\omega|}\sqrt{\frac{\pi}{s|\omega|}}\frac{\sin(\pi\kappa)}{\pi} & \frac{1-\kappa}{2x} \end{pmatrix}, \tag{5.2.9a}$$

$$A_{\bar{x}} = \begin{pmatrix} \frac{1-\kappa}{2\bar{x}} & \omega(\kappa-1)e^{2i\vartheta-2s|\omega|}\sqrt{\frac{\pi}{s|\omega|}}\frac{\sin(\pi\kappa)}{\pi} \\ \omega\kappa e^{-2s|\omega|}\sqrt{\frac{\pi}{s|\omega|}}\frac{\sin(\pi\kappa)}{\pi} & -\frac{1-\kappa}{2\bar{x}} \end{pmatrix}. \tag{5.2.9b}$$

The field strength is then approximated by

$$\begin{aligned} F_{x\bar{x}}^{21} &= \partial_{\bar{x}}A_x^{21} - \partial_x A_{\bar{x}}^{21} + [A_x, A_{\bar{x}}]^{21} = \frac{\sin(\pi\kappa)}{\pi}\sqrt{\frac{\pi}{s|\omega|}}\omega e^{-i\vartheta-2s|\omega|}\left(|\omega| - \frac{1}{4s}\right) \\ &\approx \frac{\sin(\pi\kappa)}{\pi}\sqrt{\frac{\pi}{s|\omega|}}\omega e^{-i\vartheta-2s|\omega|}|\omega|, \\ F_{x\bar{x}}^{12} &= \partial_{\bar{x}}A_x^{12} - \partial_x A_{\bar{x}}^{12} + [A_x, A_{\bar{x}}]^{12} \approx \frac{\sin(\pi\kappa)}{\pi}\sqrt{\frac{\pi}{s|\omega|}}\omega e^{i\vartheta-2s|\omega|}|\omega|, \end{aligned} \tag{5.2.10}$$

for large s. Here we have $F_{x\bar{x}}^{12} = \left(F_{x\bar{x}}^{21}\right)^*$ and the diagonal elements of $F_{x\bar{x}}$ are

$$F_{x\bar{x}}^{11} = \partial_{\bar{x}}A_x^{11} - \partial_x A_{\bar{x}}^{11} + [A_x, A_{\bar{x}}]^{11} = (2\kappa-1)\omega^2 e^{-4s|\omega|}\frac{\pi}{s|\omega|}\frac{\sin^2(\pi\kappa)}{\pi^4} \approx O(e^{-4|\omega|s}), \tag{5.2.11}$$

and $F_{x\bar{x}}^{22} = -F_{x\bar{x}}^{11}$. Note that it vanishes in case $\kappa = \frac{1}{2}$.

Note that we can remove that phase of $e^{-i\vartheta}$ in the expression in $F_{x\bar{x}}$ using a double valued gauge transform

$$g = e^{\frac{1}{2}i\vartheta\tau_3} = \begin{pmatrix} e^{\frac{1}{2}i\vartheta} & 0 \\ 0 & e^{-\frac{1}{2}i\vartheta} \end{pmatrix}, \tag{5.2.12}$$

with the resulting gauge potential as

$$A_x = \begin{pmatrix} -\frac{1}{4x} + \frac{\kappa}{2x} & \omega\kappa e^{-i\vartheta-2s|\omega|}\sqrt{\frac{\pi}{s|\omega|}}\frac{\sin(\pi\kappa)}{\pi} \\ \omega(\kappa-1)e^{-i\vartheta-2s|\omega|}\sqrt{\frac{\pi}{s|\omega|}}\frac{\sin(\pi\kappa)}{\pi} & \frac{1}{4x} - \frac{\kappa}{2x} \end{pmatrix}, \tag{5.2.13}$$

and similarly for $A_{\bar{x}}$. Both application of gauge transform to the gauge potential and applying below transformation directly to the Weyl zero modes,

$$\psi^1 \to e^{\frac{1}{2}i\vartheta}\psi^1, \qquad \psi^2 \to e^{-\frac{1}{2}i\vartheta}\psi^2, \tag{5.2.14}$$

will transform away the effect of flux κ. *i.e.* the diagonal terms of gauge potential A_x will become 0 when A_x is evaluated for $\kappa = \frac{1}{2}$. However, this have the interesting effect of varying the constant phase in the coefficients of leading order term in non-diagonals of A_x, where the amount varied is the same as for the non-diagonal terms in $F_{x\bar{x}}$.

If we instead take large argument approximation of Bessel functions at the Weyl zero modes ψ_1 and ψ_2 directly, then carry through the same calculation we have different gauge potential $A^{'}$

$$A_x^{'21} = \int_{\hat{\mathbb{R}}^2} \psi^{2\dagger} \frac{\partial}{\partial x} \psi^1 \, d^2 z = -\omega e^{-2i\vartheta - 2s|\omega|} \sqrt{\frac{\pi}{s|\omega|}} \frac{\sin(\pi\kappa)}{2\pi}, \tag{5.2.15a}$$

$$A_{\bar{x}}^{'21} = \int_{\hat{\mathbb{R}}^2} \psi^{2\dagger} \frac{\partial}{\partial \bar{x}} \psi^1 \, d^2 z = \omega e^{-2s|\omega|} \sqrt{\frac{\pi}{s|\omega|}} \frac{\sin(\pi\kappa)}{2\pi}, \tag{5.2.15b}$$

which are the previous gauge potentials evaluated at $\kappa = \frac{1}{2}$. The field strength component $F_{x\hat{x}}^{21}$ remains the same.

Consider the Nahm transform for base Higgs field Φ, since our zero modes $\psi^{1,2}$ are displaced from the origin by $y = -\omega$ and $y = \omega$ respectively, we have

$$\Phi_x^{11} = \int_{\mathbb{R}^2} \psi^{1\dagger} \psi^1 \, i \left(r(\cos\theta + i \sin\theta) - \omega \right) \, r \, dr d\theta = -i\omega, \tag{5.2.16a}$$

$$\Phi_x^{22} = \int_{\mathbb{R}^2} \psi^{2\dagger} \psi^2 \, i \left(r(\cos\theta + i \sin\theta) + \omega \right) \, r \, dr d\theta = i\omega. \tag{5.2.16b}$$

Consider the self duality equations, we have ${}^\star F_{\mu\nu} = F_{\mu\nu}$ and more specifically in the Hitchin formulation,

$$F_{x\bar{x}} = \left[\Phi, \Phi^\dagger \right]. \tag{5.2.17}$$

Assume the base Higgs field takes the same form as A_x and using form of $F_{x\bar{x}}$ in (5.2.10), we can find

$$\Phi_x = \begin{pmatrix} -i\omega & i\omega e^{i\varphi - i\vartheta - 2s|\omega|} \sqrt{\frac{\pi}{s|\omega|}} \frac{\sin(\pi\kappa)}{4\pi} \\ -i\omega e^{-i\varphi + i\vartheta - 2s|\omega|} \sqrt{\frac{\pi}{s|\omega|}} \frac{\sin(\pi\kappa)}{4\pi} & i\omega \end{pmatrix}, \tag{5.2.18}$$

unique up to $SU(2)$ and $U(1)$ symmtries, where $\omega = |\omega| e^{i\varphi}$.

5.3 Successive Approximations

In view of finding exact analytic self dual solutions, we consider $F_{x\bar{x}}^{21}$ computed in the previous section as the first order approximations to the true solutions. In the radially symmetric case of $\alpha = 0$, Van Baal form of solution (4.4.14),(4.4.13) can be written in terms of radially symmetric functions $\nu(s)$ and $\rho(s)$. Since solutions are self dual, they also satisfies (4.4.12). In the following we will continue the notation of $s = |x| = |x_\perp|$, fix $F_{x\bar{x}}^{21}$ and A_x as in (5.2.10) and (5.2.9) respectively. Consider the simple first order ansatz for ρ

$$\rho(s) = \frac{C}{s^t}, \tag{5.3.1}$$

for arbitrary constant C and t. Substitution into the second self duality equation (4.4.12b) yields the differential equation corresponding to modified Bessel functions

$$\frac{d^2 V}{ds^2} + \frac{1}{s} \frac{dV}{ds} - V \left(\frac{t^2}{s^2} + 4|\omega|^2 \right) = 0, \tag{5.3.2}$$

where the solution can be taken as $V(s) = C_1 K_t(2s|\omega|)$ by square integrable condition with C_1 arbitrary constant. Choosing

$$\nu^* = \frac{1}{2}\sin(\pi\kappa)\pi^{-\frac{3}{2}}e^{2i\omega\cdot x}e^{-2x|\omega|}|\omega|^{-\frac{1}{2}}|x|^{-\frac{1}{2}}, \tag{5.3.3}$$

gives asymptotic form of $F_{x\bar{x}}^{21}$ in (4.4.14) given in (5.2.10). Compare to asymptotic of modified Bessel functions, we determine $C_1 = \frac{1}{\pi^2}$. Compare to the value of A_x^{11} in Van Baal form, we have the relation $t = 2(1-\kappa)$ where κ is the flux strength. The general, first order approximation is then

$$\rho(s) = \frac{C}{s^{2(1-\kappa)}}, \quad \nu^* = \frac{\sin(\pi\kappa)}{\pi^2}e^{2i\omega\cdot x}K_{2(1-\kappa)}(2s|w|). \tag{5.3.4}$$

For the remainder of the section we will focus on the case $\kappa = \frac{1}{2}$ as the index of modified Bessel functions in ν^* becomes integer which simplifies calculations. Physically, $\kappa = \frac{1}{2}$ gives additional symmetries in the dual manifold as gauge potential is invariant under Weyl reflection. In this specialism,

$$\rho(s) = \frac{C}{s}, \quad \nu^* = \frac{1}{\pi^2}e^{2i\omega\cdot x}K_1(2s|w|). \tag{5.3.5}$$

However this does not satisfies the first self-duality equation (4.4.12a). We can find the sub leading terms of the functions ρ and ν^* recursively using the method of successive approximation through appropriate substitutions to (4.4.12). We expect the solution takes the form

$$\rho(s) = \frac{C}{s}\left(1 + O\left(e^{-4|w|s}\right) + O\left(e^{-8|w|s}\right)\dots\right), \tag{5.3.6a}$$

$$\nu^* = \frac{1}{\pi^2}e^{2i\omega\cdot x}K_1(2s|w|)\left(1 + O\left(e^{-4|w|s}\right) + O\left(e^{-8|w|s}\right)\dots\right), \tag{5.3.6b}$$

and the second order approximation to $\rho(s)$ can be computed as

$$\rho(s) = \frac{C}{s}\left(1 + K_0\left(2|w|s\right)^2\right). \tag{5.3.7}$$

The explicit expression of $\rho(s)$ to this order is only possible with special values of flux strength κ such that index $2(1-\kappa)$ is either integer or half integer. These calculations are shown more explicitly in Section A.3.

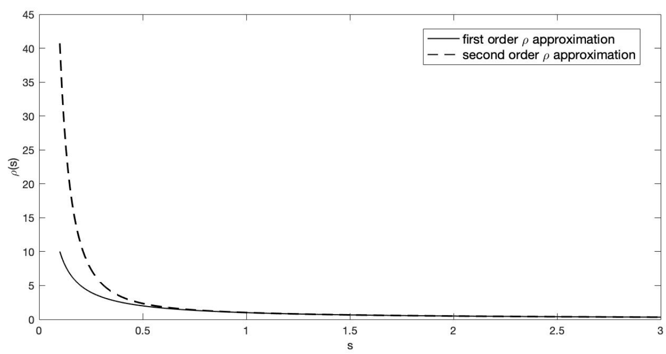

Figure 5.2: The function $\rho(s)$ where $C = |w| = 1$ in first and second order approximations

The addition of $K_0(2|w|s)^2$ term in second order approximation of $\rho(s)$ preserves the asymptotic behaviour due its exponential decay $\approx O(e^{-4|w|s})$ but it introduces a harsher singularity as $s \to 0$. We can apply the same operation to find $O(e^{-6|w|s})$ correction of ν^*. By considering the anstaz $\frac{1}{\pi^2}e^{2i\omega\cdot x}\left(K_1(2s|w|)+V(s)\right)$, we have second order ODE in $V(s)$

$$\frac{d^2V}{dr^2}+\frac{dV}{dr}\left(-\frac{1}{r}-4|w|\frac{K_0}{K_1}\right)+\frac{16|w|^2K_1^2}{\pi^2}+\frac{16|w|^2K_0^2}{\pi^2}+\frac{16|w|K_0K_1}{\pi^2 r}=0, \tag{5.3.8}$$

which have solution in integral form

$$V(s)=\int\frac{1}{sK_1^2}\left(\int-\frac{16}{\pi^2}\left(|w|^2sK_1^4+s|w|^2K_0^2K_1^2+|w|K_0K_1^3\right)\,ds\right)\,ds. \tag{5.3.9}$$

This integral however does not have simple solutions in terms of modified Bessel functions K_n with integer index. This suggests that the solutions should be written in terms of special functions encapsulating Bessel functions as a special case as K_n simply does not have the right structure to satisfy the equations. Since we are able to compute $\rho(s)$ and $\nu^*(s)$ to orders of $O(e^{-4|\omega|s})$ and $O(e^{-2|\omega|s})$ respectively, it is expected that the exact solution can be written down explicitly.

5.4 Non Zero Higgs Field

Return to the case of $U(1)$ solutions on $\mathbb{R}^2$ which has the corresponding Nahm potential of single flux through the origin of $\hat{\mathbb{R}}^2$, we consider the case of non-vanishing Nahm Higgs field. The Weyl operator is given by

$$-\frac{i}{2}D_x^\dagger=\begin{pmatrix}\frac{1}{2}\bar{x}-\frac{i\alpha}{2\bar{y}} & \partial_y-\frac{\kappa}{2y}\\ \partial_{\bar{y}}+\frac{\kappa}{2\bar{y}} & \frac{1}{2}x+\frac{i\bar{\alpha}}{2y}\end{pmatrix}, \tag{5.4.1}$$

where the Nahm Higgs field is introduced as the constant α parameter. Considering the operator

$$\begin{aligned}\frac{1}{4}D_x^\dagger D_x&=\begin{pmatrix}\frac{1}{2}\bar{x}-\frac{i\alpha}{2\bar{y}} & \partial_y-\frac{\kappa}{2y}\\ \partial_{\bar{y}}+\frac{\kappa}{2\bar{y}} & \frac{1}{2}x+\frac{i\bar{\alpha}}{2y}\end{pmatrix}\begin{pmatrix}\frac{1}{2}x+\frac{i\bar{\alpha}}{2y} & -\partial_y+\frac{\kappa}{2y}\\ -\partial_{\bar{y}}-\frac{\kappa}{2\bar{y}} & \frac{1}{2}\bar{x}-\frac{i\alpha}{2\bar{y}}\end{pmatrix}\\&=\left(\left(\frac{1}{2}x+\frac{i\bar{\alpha}}{2y}\right)\left(\frac{1}{2}\bar{x}-\frac{i\alpha}{2\bar{y}}\right)-\left(\partial_y-\frac{\kappa}{2y}\right)\left(\partial_{\bar{y}}+\frac{\kappa}{2\bar{y}}\right)\right)\begin{pmatrix}1&0\\0&-1\end{pmatrix}.\end{aligned} \tag{5.4.2}$$

The zero modes φ of the operator $D_x^\dagger D_x$ can be found by solving linear, second order PDE

$$\left[\left(\partial_y-\frac{\kappa}{2y}\right)\left(\partial_{\bar{y}}+\frac{\kappa}{2\bar{y}}\right)-\left(\frac{1}{2}x+\frac{i\bar{\alpha}}{2y}\right)\left(\frac{1}{2}\bar{x}-\frac{i\alpha}{2\bar{y}}\right)\right]\varphi=0, \tag{5.4.3}$$

with reconstruction equation to the zero modes of $D_x^\dagger$ as

$$\psi=D_x^\dagger\begin{pmatrix}\varphi\\0\end{pmatrix}. \tag{5.4.4}$$

Intuitively, we are expecting the gauge potential to take a similar form to (5.1.11). This is due to the coupling of a Higgs field to the Nahm potentials does not remove the singularity in $\mathbb{R}^2$. The singularity in $\mathbb{R}^2$ is expected to be of the same type as zero Higgs case. Hence, we expect the augmented flux κ' at the singularity to be more complex than a simple reversal of sign. However, we should be able to determine it via the interactions of Nahm Higgs field and flux in the dual manifold, which can be described in terms of κ and α.

Under the change of variable $\frac{1}{\bar{z}} = -\frac{i\alpha}{\bar{y}\bar{x}}$, $\frac{1}{z} = \frac{i\bar{\alpha}}{yx}$, we rescale and set the zero of second term of equation (5.4.3) to be at $z = -1$, this gives

$$\left(\partial_z - \frac{\kappa}{2z}\right)\left(\partial_{\bar{z}} + \frac{\kappa}{2\bar{z}}\right)\varphi = \frac{1}{4}|\alpha|^2\left(1 + \frac{1}{z}\right)\left(1 + \frac{1}{\bar{z}}\right)\varphi. \tag{5.4.5}$$

This equation appears to be a direct extension of the zero Nahm Higgs case (5.1.3), yet it proves to be extremely difficult to find a normalisable solution. In fact, it has direct correspondence to the another class of solutions known as periodic monopoles.

5.5 Outlook to Periodic Monopoles

Periodic monopoles as name suggests are solutions to the Bogomolny equation which is periodic in one its non static components, hence the solutions exist on $\mathbb{S}\times\mathbb{R}^2$ and the dual manifold is $\hat{\mathbb{S}}\times\mathbb{R}$. In consistency with the Nahm transform for the Hitchin equations, we set the non-compact coordinates to be x_0 and x_3. We also set x_1 to be zero to represent static solutions. Again defining complex coordinates $x_\perp = x_0 + ix_3$ on the base manifold and enforcing periodicity in x_2. On the dual manifold $\hat{\mathbb{S}}\times\mathbb{R}$ with coordinates (z_1, z_2), we define complex coordinates $y = z_1 + iz_2$ with gauge potential $\hat{A}_y = \frac{1}{2}(\hat{A}_{z_1} - i\hat{A}_{z_2})$ and dual Higgs field $\Psi = \frac{1}{2}(\hat{A}_{z_0} - i\hat{A}_{z_3})$. Then the Weyl Operator formed by taking $\hat{A}_y = 0$ becomes [49]

$$\frac{i}{2}D_x = \begin{pmatrix} -\frac{1}{2}x_\perp - \Psi(y) & \partial_y - \frac{a}{2} \\ \partial_{\bar{y}} + \frac{a}{2} & -\frac{1}{2}\bar{x}_\perp - \bar{\Psi}(y) \end{pmatrix}, \tag{5.5.1}$$

where a is x_2. For general $SU(2)$ periodic monopoles, the dual Higgs field $\Psi(y) = \lambda\sinh(\beta y)$[16] where $\lambda \in \mathbb{C}$ and β corresponds to the period of x_2 coordinate in the base manifold. In the case of abelian (Dirac) monopoles, we take the Higgs field to be $\Psi = \frac{1}{2}e^y$ by setting $\beta = 1$ and discard λ as it is not a required parameter in the Nahm data. Then we have

$$\frac{1}{4}D_x^\dagger D_x\varphi = \left(-\left(\partial_y - \frac{a}{2}\right)\left(\partial_{\bar{y}} + \frac{a}{2}\right) + \frac{1}{4}\left(e^y + x_\perp\right)\left(e^{\bar{y}} + \bar{x}_\perp\right)\right)\varphi = 0. \tag{5.5.2}$$

Considering the transformation $z = e^y$, $\bar{z} = e^{\bar{y}}$, we have the following zero mode equation

$$|z|^2\left(-\left(\partial_z - \frac{a}{2z}\right)\left(\partial_{\bar{z}} + \frac{a}{2\bar{z}}\right) + \frac{1}{4}\left(1 + \frac{x_\perp}{z}\right)\left(1 + \frac{\bar{x}_\perp}{\bar{z}}\right)\right)\varphi = 0. \tag{5.5.3}$$

Compare with equation (5.4.3), we see that they are almost identical. The relation of κ with a and α with $x_\perp$ suggest a correspondence between the position of periodic monopoles and strength of fluxes in the solutions of the Hitchin equations. Indeed on the line $x_\perp = 0$, normalisable solution to (5.5.2) can be found in similar form as (5.1.4). The physical Higgs field in $\mathbb{S}\times\mathbb{R}^2$ can also be found using the Nahm transform. For $x_\perp \neq 0$, the problem remains unsolved as solution to (5.4.3) would be simple corollary of solution to periodic monopoles.

Chapter 6

Conclusions

6.1 Summary

In this project, we considered solutions to the Hitchin equations using the Nahm transform where the Nahm potentials are taken as abelian in the dual manifold. In Chapter 2 and Chapter 3 we reviewed the topological and dynamical features of instantons and monopoles. Topological charge and self-duality equations (Bogomolny equations) are presented for both classes of topological solitons. The derivation of the Hitchin equations from the self-duality equation of the Pure Yang-Mills action is also presented. The central transformation of the project is the Nahm transform which is discussed in Chapter 4. Here we outline the operational rules of the Nahm transform, its property of inverting the periods of the base four torus, as well as the interchange of topological charge k and gauge group $SU(n)$. We also reviewed the doubly periodic instantons to motivate the study of the Hitchin equations using the Nahm transform.

In the Hitchin case, we have the added interest of other unsolved problems, for example, the AB effect of twin fluxes and the periodic monopoles. They are closely related to the Weyl zero modes equation in the inverse Nahm transform. The main results of this thesis are derived in Chapter 5. In Section 5.1, we considered $U(1)$ solutions of the Hitchin equations where Nahm potential corresponds to a single flux of arbitrary strength (modulo gauge transform) through the origin of $\mathbb{R}^2$, with zero Nahm Higgs. The expected form of the corresponding gauge potential is verified for arbitrarily shifted flux. We found that the gauge potential gains a minus sign from the corresponding Nahm potential which can be considered as the flipping of flux strength. In Section 5.2, we consider $SU(2)$ solutions of the Hitchin equations where the Nahm potentials correspond to two fluxes located ω and $-\omega$ away from the origin, again with zero Nahm Higgs. The two Weyl zero modes are approximated via their asymptotic behaviour at infinity to be zero modes of single flux translated to the two flux locations. The resultant gauge potential and field strength in this approximations are computed explicitly. We also note correspondences of symmetries and ambiguities in the base and dual manifold under Nahm transform. In Section 5.3, the method of successive approximation is used in attempts to find the exact $SU(2)$ gauge potentials corresponding to Nahm potentials defined in Section 5.2. Spherically symmetric functions ρ and ν introduced in the Van Baal form of gauge potential were approximated to the order of $O(e^{-4|\omega|s})$ and $O(e^{-2|\omega|s})$ respectively. Section 5.4 and Section 5.5 consider the case of non zero Nahm Higgs field by introducing the α parameter in the Weyl operator. We showed via elementary transformation the similarity of Weyl zero mode equations of the Hitchin case and periodic monopoles.

6.2 Applications to Doubly Periodic Instantons

Here we present an application of the results from the solutions to the Hitchin equations in Chapter 5. In the large $|x_\perp|$ limit of $SU(2)$ doubly periodic instantons considered in Section 4.4, gauge potential of the $x_\perp$ plane, $A_{x_\perp}$ can be approximated by the solutions to the Hitchin equations. Using the result (5.2.10) we can approximate the action density of doubly periodic instanton in large $|x_\perp|$ limit as

$$-\frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} \approx \frac{32|\omega|^3}{s\pi}e^{-4s|w|}\sin(\pi\kappa)^2, \tag{6.2.1}$$

where the effect of flux strength κ enters in terms of $\sin(\pi\kappa)^2$ coefficient in the exponential decay. A more accurate approximation of $A_{x_\perp}$ can be computed with radially symmetric functions ρ and ν^* in (5.3.4). We find action density under this approximation as

$$-\frac{1}{2}\mathrm{Tr}F^{\mu\nu}F_{\mu\nu} \approx \frac{128|\omega|^3}{\pi^2}K^2_{2(1-\kappa)}(2|\omega|s)\sin(\pi\kappa)^2. \tag{6.2.2}$$

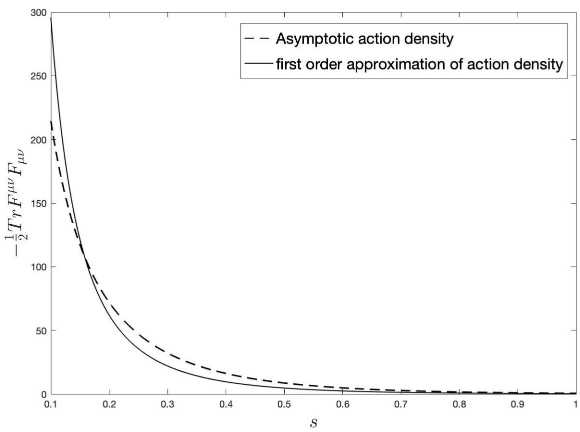

Figure 6.1: The action density $-\frac{1}{2}F^{\mu\nu}F_{\mu\nu}$ where $\kappa = \frac{1}{2}$, $|w| = 1$ in asymptotic (6.2.1) and first order (6.2.2) approximations

These two approximations retain the same asymptotics as expected from large argument limits of modified Bessel functions while the first order approximation produces stronger singularity as $s \to 0$. These action densities are more general formulas from that presented in [14] which has the specialism $\kappa = \frac{1}{2}$.

6.3 Remarks

Due to time constraints, we were unable to finish some of the more difficult calculations. Certain intriguing observations were not investigated fully. One of said observations that is worth investigating is the correspondence of Higgs flux parameter α in the Hitchin case with the absolute value of the position $|x_\perp|$ in the periodic monopoles. The moduli spaces of these classes of solutions are also worth investigation as they hold non-trivial topology and geometry mirroring the spaces of gauge potentials modulo gauge transformations. For calculation intractable in the physical system, one can turn to the approximation in the moduli spaces similar to the scattering equations [52, 53]. The same approach of using the Nahm transform can also be applied to other manifolds, for instance, the spatially periodic instantons[54] which exists on $\mathbb{T}^3 \times \mathbb{R}$ or doubly periodic monopoles. Nahm transform of these limits of $\mathbb{T}^4$ and their dual manifolds are generally known but the technicality of executing these calculations (finding zero modes of Weyl operator and performing the Nahm transform) can be extraordinarily difficult.

For an alternative treatment of solutions to the Hitchin equations, we direct the reader to a recent publication by Ward [55]. The asymptotics of the base Higgs field under consideration is non-constant; more precisely, determinant of Φ is a polynomial of degree $n > 0$ in terms of complex variable x of base manifold. For $n = 1, 2$, rotational symmetrical solutions, the associated Hitchin equations became can be solved in terms of Painlevé transcendents. $n = 3$ case was also investigated by Ward where the singularities of the solutions are approximated by asymptotic metric in the moduli spaces. It is not yet clear the connection of this thesis and the paper by Ward but it will be investigated in the future.

Appendix A

Appendix

A.1 U(1) Solution

To find zero mode of (5.1.3) we note that

$$\frac{1}{2}\bar{x}\psi_1 + D_y\psi_2 = 0, \qquad \frac{1}{2}x\psi_2 + D_{\bar{y}}\psi_1 = 0, \tag{A.1.1}$$

where $\psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$ is the zero mode of (5.1.3), $D_y = \partial_y - \frac{\kappa}{2y}$ and $D_{\bar{y}} = \partial_{\bar{y}} + \frac{\kappa}{2\bar{y}}$. We notice that D_y and $D_{\bar{y}}$ commutes and we have ψ_1 and ψ_2 satisfying the same second order differential equation

$$D_y D_{\bar{y}}\psi_{1,2} - \frac{1}{4}|x|^2\psi_{1,2} = 0, \tag{A.1.2}$$

with vanishing condition at spatial infinity. Considering the problem in complex polar coordinates where $y = re^{i\theta}$ and $\bar{y} = re^{-i\theta}$, we have differential operators in polar coordinates

$$\partial_y = \frac{e^{-i\theta}}{2}\left(\partial_r - \frac{i}{r}\partial_\theta\right), \qquad \partial_{\bar{y}} = \frac{e^{i\theta}}{2}\left(\partial_r + \frac{i}{r}\partial_\theta\right), \tag{A.1.3}$$

and

$$\left(\frac{1}{r}\partial_r + \partial_r^2 + \frac{1}{r^2}\partial_\theta^2 - \frac{2i\kappa}{r^2}\partial_\theta - \frac{\kappa^2}{r^2} - |x|^2\right)\psi_{1,2} = 0. \tag{A.1.4}$$

We can seek separable solutions in form of $\psi_{1,2} = A_{1,2}\Theta(\theta)R(r)$ where $\Theta(\theta)$ is periodic with period 2π. Take $\Theta(\theta) = e^{in\theta}$ with $n \in \mathbb{Z}$ and we have following ordinary differential equation for $R(r)$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - R\left(\frac{(n-\kappa)^2}{r^2} + |x|^2\right) = 0. \tag{A.1.5}$$

This has solution in term of modified Bessel functions $K_\nu(x)$ [56]

$$R = C_1 I_{(n-\kappa)}(r|x|) + C_2 K_{(n-\kappa)}(r|x|), \tag{A.1.6}$$

where

$$K_\kappa(x) = \frac{\pi}{2}\frac{I_{-\kappa}(x) - I_\kappa(x)}{\sin\pi\kappa}. \tag{A.1.7}$$

Since we are seeking normalisable solutions, we have $C_1 = 0$ and the solution becomes

$$\psi_{1,2}(r,\theta) = e^{in_{1,2}\theta}K_{(n_{1,2}-\kappa)}(r|x|), \tag{A.1.8}$$

where it satisfies (A.1.1). To find the particular solution, take $n_1 = 0$ for radially invariant solution and

$$\psi_1 = K_{-\kappa}(r|x|) = K_\kappa(r|x|). \tag{A.1.9}$$

ψ_2 satisfies $\left(\partial_{\bar y} + \frac{\kappa}{2\bar y}\right)\psi_1 + \frac{1}{2}x\psi_2 = 0$. Using the Bessel function identity

$$z\frac{d}{dz}K_\nu(z) + \nu K_\nu(z) = -zK_{\nu-1}(z), \tag{A.1.10}$$

to have

$$\psi = \begin{pmatrix} K_\kappa(r|x|) \\ \frac{|x|}{x}e^{i\theta}K_{1-\kappa}(r|x|) \end{pmatrix}, \tag{A.1.11}$$

as $\kappa \in (0,1)$ via gauge transform. Normalisation means that

$$\begin{aligned}\langle\psi,\psi\rangle &= \int_{\mathbb{R}\times\mathbb{R}} \left(|\psi_1|^2 + |\psi_2|^2\right)\, r\, drd\theta \\ &= 2\pi\int_0^\infty r\left(|K_\kappa(r|x|)|^2 + |K_{1-\kappa}(r|x|)|^2\right)dr \\ &= \frac{2\pi}{|x|^2}\left(\frac{\pi\kappa}{2\sin(\pi\kappa)} + \frac{\pi(1-\kappa)}{2\sin((1-\kappa)\pi)}\right) = \frac{\pi^2}{|x|^2\sin(\pi\kappa)},\end{aligned} \tag{A.1.12}$$

by the standard integral equality

$$\int_0^\infty xK_\nu(ax)K_\nu(bx)dx = \frac{\pi(ab)^{-\nu}(a^{2\nu} - b^{2\nu})}{2\sin(\nu\pi)(a^2 - b^2)}. \tag{A.1.13}$$

Or alternatively by below calculation. Noting

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{e^{-t-\frac{z^2}{4t}}}{t^{\nu+1}}dt. \tag{A.1.14}$$

Write

$$\int_0^\infty x^pK_\nu(x)^2dx = \frac{1}{4}\int_0^\infty \left(\frac{x}{2}\right)^{2\nu} x^pdx \int_0^\infty \frac{e^{-t-\frac{x^2}{4t}}}{t^{\nu+1}}dt \int_0^\infty \frac{e^{-s-\frac{x^2}{4s}}}{s^{\nu+1}}ds. \tag{A.1.15}$$

Substitution of $z = x^2\left(\frac{1}{4t} + \frac{1}{4s}\right)$ gives

$$\begin{aligned}\int_0^\infty x^pK_\nu(x)^2dx &= \frac{1}{4}^{\frac{2-p}{2}} \int_0^\infty z^{\nu+\frac{p}{2}-\frac{1}{2}}e^{-z}dz \int_0^\infty\int_0^\infty \frac{e^{-t-s}}{(st)^{\nu+1}}\left(\frac{st}{t+s}\right)^{\nu+\frac{p}{2}+\frac{1}{2}} dsdt \\ &= \frac{1}{4}^{\frac{2-p}{2}}\Gamma(\nu + \frac{p}{2} + \frac{1}{2})\int_0^\infty\int_0^\infty \frac{e^{-t-s}}{(st)^{\nu+1}}\left(\frac{st}{t+s}\right)^{\nu+\frac{p}{2}+\frac{1}{2}} dsdt.\end{aligned} \tag{A.1.16}$$

Use the substitution $x^2 = s$, $y^2 = t$, $x = r\cos\theta$ and $y = r\sin\theta$ to have

$$\begin{aligned}\int_0^\infty x^pK_\nu(x)^2dx &= \frac{1}{4}^{\frac{2-p}{2}}\Gamma(\nu + \frac{p}{2} + \frac{1}{2})\int_0^{\frac{\pi}{2}} d\theta \int_0^\infty dr\, 4r(r^2\cos\theta\sin\theta)^p r^{-2\nu-p-1}e^{-r^2} \\ &= \frac{1}{4}^{\frac{-p}{2}}\Gamma(\nu + \frac{p}{2} + \frac{1}{2})\int_0^{\frac{\pi}{2}} d\theta\, (\cos(\theta)\sin(\theta))^p \int_0^\infty r^{2p+1-2\nu-p-1}e^{-r^2} \\ &= \frac{1}{4}^{1-\frac{p}{2}}\Gamma(\nu + \frac{p}{2} + \frac{1}{2})\Gamma(\frac{p}{2} + \frac{1}{2} - \nu)B(\frac{p+1}{2}, \frac{p+1}{2}) \\ &= \frac{1}{4}^{1-\frac{p}{2}}\Gamma(\nu + \frac{p}{2} + \frac{1}{2})\Gamma(\frac{p}{2} + \frac{1}{2} - \nu)\left(\frac{\Gamma(\frac{p+1}{2})^2}{\Gamma(p+1)}\right),\end{aligned} \tag{A.1.17}$$

which is the desired result. A related, useful result via similar calculation is

$$\int_0^\infty x^pK_\nu(x)K_{\nu-1}(x)dx = \frac{1}{4}^{1-\frac{p}{2}}\Gamma(\nu + \frac{p}{2})\Gamma(\frac{p}{2} + 1 - \nu)\left(\frac{\Gamma(\frac{p}{2})\Gamma(\frac{p}{2}+1)}{\Gamma(p+1)}\right). \tag{A.1.18}$$

A.2 SU(2) Solution Asymptotics

For off diagonal elements, consider A_x^{21}, using ψ^1, ψ^2 as defined in Section 5.2 we have

$$\partial_x \psi_1^1 = \frac{e^{-i\vartheta}\sqrt{\sin(\pi\kappa)}}{2\pi}\left((1-\kappa)\,K_\kappa(r_1 s) - r_1 s K_{\kappa-1}(r_1 s)\right), \tag{A.2.1a}$$

$$\partial_x \psi_2^1 = \frac{e^{-2i\vartheta+i\theta_1}\sqrt{\sin(\pi\kappa)}}{2\pi}\left((\kappa-1)\,K_{1-\kappa}(r_1 s) - r_1 s K_\kappa(r_1 s)\right), \tag{A.2.1b}$$

and

$$A_x^{21} = \int_{\hat{\mathbb{R}}^2} \psi^{2\dagger}\frac{\partial}{\partial x}\psi^1 d^2 z = \frac{s e^{-2i\vartheta}\sin(\pi\kappa)}{2\pi^2}\left(I_1 + J_1\right), \tag{A.2.2}$$

where

$$I_1 = \int_{\hat{\mathbb{R}}^2} e^{i\theta_2} K_{1-\kappa}(r_2 s)\left((1-\kappa)\,K_\kappa(r_1 s) - r_1 s K_{\kappa-1}(r_1 s)\right)\, d^2 z, \tag{A.2.3a}$$

$$J_1 = \int_{\hat{\mathbb{R}}^2} e^{i\theta_1} K_\kappa(r_2 s)\left((\kappa-1)\,K_{1-\kappa}(r_1 s) - r_1 s K_\kappa(r_1 s)\right)\, d^2 z. \tag{A.2.3b}$$

Similarly for

$$A_{\bar{x}}^{21} = \int_{\hat{\mathbb{R}}^2} \psi^{2\dagger}\frac{\partial}{\partial \bar{x}}\psi^1 d^2 z = \frac{s\sin(\pi\kappa)}{2\pi^2}\left(I_2 + J_2\right), \tag{A.2.4}$$

where

$$I_2 = \int_{\hat{\mathbb{R}}^2} e^{i\theta_2} K_{1-\kappa}(r_2 s)\left((1-\kappa)\,K_\kappa(r_1 s) - r_1 s K_{\kappa-1}(r_1 s)\right)\, d^2 z, \tag{A.2.5a}$$

$$J_2 = \int_{\hat{\mathbb{R}}^2} e^{i\theta_1} K_\kappa(r_2 s)\left((\kappa+1)\,K_{1-\kappa}(r_1 s) - r_1 s K_\kappa(r_1 s)\right)\, d^2 z. \tag{A.2.5b}$$

Utilising

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}e^{-z}\left[\sum_{k=0}^{n-1}\frac{1}{(2z)^k}\frac{\Gamma(\nu+k+\frac{1}{2})}{\Gamma(\nu-k+\frac{1}{2})} + \theta_3\frac{\Gamma(\nu+n+\frac{1}{2})}{(2z)^n n!\,\Gamma(\nu-k+\frac{1}{2})}\right], \tag{A.2.6}$$

and considering asymptotic as $z\to\infty$, we have the following approximation

$$I_1 = \frac{\pi}{2s}\int_{\hat{\mathbb{R}}^2} e^{i\theta_2}e^{-s(r_2+r_1)}\left[(1-\kappa)\,r_1^{-\frac{1}{2}}r_2^{-\frac{1}{2}} - s r_1^{\frac{1}{2}} r_2^{-\frac{1}{2}}\right]\, d^2 z, \tag{A.2.7a}$$

$$J_1 = \frac{\pi}{2s}\int_{\hat{\mathbb{R}}^2} e^{i\theta_1}e^{-s(r_2+r_1)}\left[(\kappa-1)\,r_1^{-\frac{1}{2}}r_2^{-\frac{1}{2}} - s r_1^{\frac{1}{2}} r_2^{-\frac{1}{2}}\right]\, d^2 z. \tag{A.2.7b}$$

And similarly for I_2 and I_2. For large s, the integral of $e^{-s(\sqrt{y_1\bar{y}_1}+\sqrt{y_2\bar{y}_2})}$ over $\mathbb{C}$ of is localised in the line segment connecting $-\omega$ and ω. Hence integrals over $\mathbb{R}^2$ including these terms can be seen as integrals over the line segment $-\omega < z < \omega$. Assuming z_2 small,

$$e^{-s(\sqrt{y_1\bar{y}_1}+\sqrt{y_2\bar{y}_2})} \approx e^{-2s|a|}e^{-s\left(\frac{z_2^2}{2|z_1+a|}+\frac{z_2^2}{2|z_1-a|}\right)}. \tag{A.2.8}$$

Noting the Gaussian integral $\int_\infty^\infty e^{-s^2\omega}ds = \sqrt{\frac{\pi}{\omega}}$ and in the case $\omega\in\mathbb{R}$, we have

$$\begin{aligned} e^{-2s|\omega|}e^{-s\left(\frac{z_2^2}{2|z_1+\omega|}+\frac{z_2^2}{2|z_1-\omega|}\right)} &= e^{-2s|\omega|}e^{-\frac{s z_2^2|\omega|}{\omega^2-z_1^2}} \\ &\approx e^{-2s|\omega|}\delta(z_2)\sqrt{\frac{\pi(\omega^2-z_1^2)}{s|\omega|}}, \end{aligned} \tag{A.2.9}$$

under integration. Thus we apply affine transformation to the dual manifold $\hat{\mathbb{R}}^2$ coordinates such that $\omega \in \mathbb{R}$. Noting that the terms $r_1^{\frac{1}{2}} r_2^{-\frac{1}{2}} e^{\theta_2}$ and $r_1^{\frac{1}{2}} r_2^{-\frac{1}{2}} e^{\theta_1}$ takes opposite values for $z = z_1 + iz_2$ on the line segment joining the fluxes, we have

$$
\begin{aligned}
I_1 + J_1 &\approx \frac{\pi}{2s} \int_\ell e^{-2s|\omega|} \delta(z_2) \sqrt{\frac{\pi(\omega^2 - z_1^2)}{s|\omega|}} \left[e^{i\theta_2} (1-\kappa) r_1^{-\frac{1}{2}} r_2^{-\frac{1}{2}} + e^{i\theta_1} (\kappa - 1) r_1^{-\frac{1}{2}} r_2^{-\frac{1}{2}} \right] d^2 z \\
&= \frac{\pi}{2s} \int_{-\omega}^{\omega} e^{-2s|\omega|} \sqrt{\frac{\pi(\omega^2 - z_1^2)}{s|\omega|}} \left[(\kappa - 1) r_1^{-\frac{1}{2}} r_2^{-\frac{1}{2}} + (\kappa - 1) r_1^{-\frac{1}{2}} r_2^{-\frac{1}{2}} \right] dz_1 \\
&= \frac{\pi}{2s} e^{-2s|\omega|} \int_{-\omega}^{\omega} \sqrt{\frac{\pi(\omega^2 - z_1^2)}{s|\omega|}} \left[2(\kappa - 1)(\omega^2 - z_1^2)^{-\frac{1}{2}} \right] dz_1 \\
&= \frac{\pi}{s} e^{-2s|\omega|} \omega(\kappa - 1) \sqrt{\frac{\pi}{s|\omega|}},
\end{aligned}
\tag{A.2.10}
$$

which give the require result. One can employ similar technique to have explicit analytic form of $A_{\bar{x}}^{12}$ as

$$
A_{\bar{x}}^{12} = \int_{\hat{\mathbb{R}}^2} \psi^{1\dagger} \frac{\partial}{\partial x} \psi^2 \, r \, dr d\theta = \frac{s e^{i2\vartheta} \sin(\pi\kappa)}{2\pi^2} (I + J) \approx \omega(\kappa - 1) e^{2i\vartheta - 2s|\omega|} \sqrt{\frac{\pi}{s|\omega|}} \frac{\sin(\pi\kappa)}{\pi}, \tag{A.2.11a}
$$

$$
I = \int_{\mathbb{R}^2} e^{-i\theta_1} K_{1-\kappa}(r_1 s) \left((1-\kappa) K_\kappa(r_2 s) - r_2 s K_{\kappa - 1}(r_2 s) \right) r \, dr d\theta, \tag{A.2.11b}
$$

$$
J = \int_{\mathbb{R}^2} e^{-i\theta_2} K_\kappa(r_1 s) \left((\kappa - 1) K_{1-\kappa}(r_1 s) - r_2 s K_\kappa(r_2 s) \right) r \, dr d\theta. \tag{A.2.11c}
$$

A.3 Successive Approximations

Calculations for $O\left(e^{-4|w|r}\right)$ correction of $\rho(s)$ in (5.3.7). Consider (4.4.12a), we substitute for ρ the anstaz $\rho(s) = \frac{1}{s} + f(s)$ to give

$$
\begin{aligned}
\partial_{x_\perp} \partial_{\hat{x}_\perp} \log \rho &= -\frac{4\rho^2}{\pi^2} \left(\frac{s^2 K_1^2 |w|^2}{(1 + f(s))^2} + \partial_{x_\perp} \frac{s K_1 e^{-2iw\cdot x}}{(1 + f(s))^2} \partial_{\hat{x}_\perp} \frac{s K_1 e^{2iw\cdot x}}{(1 + f(s))^2} \right) \\
&= -\frac{4}{\pi^2 s^2} \big(s^2 K_1^2 |w|^2 + \frac{s^2}{4} \big(K_1^2 (f')^2 \\
&\qquad + 4|w|(f+1) f' K_0 K_1 + 4|w|^2 (f+1)^2 K_0^2 \big) \big).
\end{aligned}
\tag{A.3.1}
$$

Assuming $f(s) \approx O\left(e^{-4|w|s}\right)$, we ignore higher order terms and the equation reduces to

$$
\frac{1}{4} \left(\partial_r^2 + \frac{1}{r} \partial_r \right) \log \rho = -\frac{4}{\pi^2} \left(K_1^2 |w|^2 + K_0^2 |w|^2 \right). \tag{A.3.2}
$$

Solving this equation which is first order in $\partial_r(\log \rho)$ noting Bessel function identity (A.1.10) gives

$$
\log \rho = -\frac{2}{\pi^2} K_0^2 + C_1 \log s + C_2, \tag{A.3.3}
$$

which gives the desired answer with right degree of freedoms in its arbitrary constants.

In case of general $\kappa \in [0, 1]$ as in (5.3.4), same analysis gives

$$
\frac{1}{4} \left(\partial_r^2 + \frac{1}{r} \partial_r \right) \log \rho = -\frac{16\kappa^2 |\omega|^2 \sin^2(\pi\kappa)}{\pi^2} \left(K_{2\kappa - 1}^2 + K_{2\kappa}^2 \right), \tag{A.3.4}
$$

where the solution

$$\log\rho = \int \frac{32\kappa^2|\omega|\sin^2(\pi\kappa)}{\pi^2} K^2_{2\kappa-1}(2|\omega|s)K^2_{2\kappa}(2|\omega|s)\,ds + C_1\log s, \tag{A.3.5}$$

can be explicitly written down only if $\kappa = \frac{1}{2}$ by noting $\partial_x K^2_\kappa(x) = 2K_\kappa(x)(K_\kappa(x) - K_{\kappa-1}(x))(\kappa-1)$.

It is worthy noting that in case $\kappa = \frac{3}{4}$ (or $\kappa = \frac{1}{4}$ via a gauge transform) in (5.3.4), we have

$$\log\rho = \frac{1}{4\pi^2}\mathrm{Ei}(-4|\omega|s) + C_1\log s + C_2, \tag{A.3.6}$$

where $\mathrm{Ei}(x)$ is the exponential integral defined as

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty}\frac{e^{-t}}{t}\,dt, \tag{A.3.7}$$

which indeed has the correct exponential decaying factor. This specialism is not pursued further due to complications of working with special functions of these types.

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